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Comments on $N = 1$ Heterotic String Vacua

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Abstract

We analyze three aspects of $N = 1$ heterotic string compactifications on elliptically fibered Calabi-Yau threefolds: stability of vector bundles, five-brane instanton transitions and chiral matter. First we show that relative Fourier-Mukai transformation preserves absolute stability. This is relevant for vector bundles whose spectral cover is reducible. Then we derive an explicit formula for the number of moduli which occur in (vertical) five-brane instanton transitions provided a certain vanishing argument applies. Such transitions increase the holonomy of the heterotic vector bundle and cause gauge changing phase transitions. In a M-theory description the transitions are associated with collisions of bulk five-branes with one of the boundary fixed planes. In F-theory they correspond to three-brane instanton transitions. Our derivation relies on an index computation with data localized along the curve which is related to the existence of chiral matter in this class of heterotic vacua. Finally, we show how to compute the number of chiral matter multiplets for this class of vacua allowing to discuss associated Yukawa couplings.

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1 Introduction

One of the important issues in studying string compactifications is to understand the moduli space of four-dimensional $N = 1$ string vacua. The compactification of the heterotic string on a Calabi-Yau threefold X with a stable, holomorphic vector bundle V is one way to obtain such vacua.

In recent years, there has been tremendous progress in understanding the class of four-dimensional $N = 1$ string vacua obtained by compactification on elliptically fibered Calabi-Yau manifolds. This class has the double advantage to admit an explicit description of vector bundles in terms of spectral covers [1] and to allow a dual description in terms of F-theory.

To obtain a consistent heterotic string compactification on an elliptic Calabi-Yau threefold one has to include a number of five-branes which wrap the elliptic fibers [1]. It has been shown that these five-branes match precisely the number of space-time filling three-branes necessary for tadpole cancellation in F-theory compactified on elliptically fibered Calabi-Yau fourfolds Y [1]. Various aspects of the map between the geometrical moduli of the pair (X, V) and those of Y have been studied in [1, 2, 3, 4, 5, 6, 7]. The question of which pairs (X, V) are stable under world-sheet instanton corrections [8] has been recently reconsidered in [9, 10].

Besides an improved understanding of this map and the discussion of phenomenologically viable pairs (X, V), one would like to understand the behavior of (X, V) at singularities. Such singularities can be either associated to a degeneration of X or V. In general one expects there new non-perturbative effects, associated to the breakdown of world-sheet conformal field theory. A well known example is the small instanton singularity in heterotic string compactifications on a $K3$ surface [11]. The heterotic vector bundle degenerates to a torsion free sheaf with singularity locus of codimension at most two [12, 13]. Thus from a mathematical point of view, the bundle fails to be locally free at a finite number of points on the $K3$ surface. One can interpret [14] such small instantons as five-branes whose world volumes fill the six uncompactified directions and intersect the $K3$ surface at these points [15, 14, 16]. The observable effects are a change in the unbroken gauge symmetry and the number of tensor multiplets.

In passing to four-dimensional compactifications of the heterotic string a similar picture is expected to hold. In particular the class of elliptically fibered Calabi-Yau threefolds allows an explicit study of such transitions. For this class one has three possibilities of codimension-two bundle degeneration associated to curves in the base, the fibers or linear combinations of both.

Like in the six-dimensional situation these degenerations can be interpreted in terms of five-branes wrapping over these holomorphic curves. In addition to codimension-two degenerations one expects pointlike bundle singularities in codimension-three which have been studied in [17].

The codimension-two degenerations have in common that the heterotic vector bundle is associated (via fiberwise T-duality or relative Fourier-Mukai transformation) to the spectral data (C, L) with C being a reducible spectral cover [18] and L the spectral line bundle. The generic heterotic vector bundle associated with an irreducible spectral cover is stable. This leads to a stability question of V in the reducible case which we will tackle in section 3.

The observable effects in four dimensions depend on whether the bundle degenerates over base or fiber curves (or linear combinations of both). If the bundle degenerates over a base curve one observes (after smoothing out the singular gauge configuration) a change in the charged matter content [18]. This provides evidence for chirality changing phase transitions in four-dimensional string vacua [19]. As the second Chern class gets shifted by the cohomology class of the associated curve, one can effectively interpret such transition as five-brane instanton transition [18]. On the other hand, bundle degeneration associated to fiber curves do not change the net-amount of chiral matter (the third Chern class is left unchanged), however, they do change the structure group of the heterotic vector bundle and thus the unbroken gauge group in four dimensions.

Both transition types are expected to have a dual interpretation in terms of F-theory. In particular, one expects chirality changing transitions to be dual to a change in the F-theory four form flux [20, 18]. Gauge changing transitions are expected to be dual to three-brane instanton transitions [2]. More precisely, the number of five-branes which dissolve in such transition are supposed to match the precise number of three-branes in F-theory. One can also consider the transitions in heterotic M-theory [21, 22, 23]. The anomaly cancellation requires to include additional five-branes in the bulk space. The transition is then interpreted as 'collision' of a bulk five brane with one of the boundary fixed planes. Conditions under which the five-brane is attracted to the boundary fixed plane are discussed in [23].

The geometrical moduli of the pair (X, V) are given by the complex structure and Kähler deformations of X and the bundle moduli given by the dimension of $H^1(X, \mathrm{End}(V))$. One would like to know how the moduli of V are altered in chirality or gauge changing phase transitions. For chirality changing transitions the question has been studied in [21]. In particular, it was shown in [21] that the transition moduli (the difference of the moduli of the original and transition bundle) can be interpreted as moduli of the altered spectral cover restricted to the lift of the horizontal curve about which the five-brane wraps.

Our main interest in this paper is the question: *How do the bundle moduli change during a gauge changing phase transition?* As such transition is naturally associated to reducible spectral covers [1, 2, 18] and appears to be hard to study in general, we will adopt here the situation (first studied in [2]) of having a vector bundle $E = V \oplus \pi^* M$. Here M is a good vector bundle on the base of X whose second Chern class counts the five-branes dissolved in the transition. The associated spectral cover is the union of the spectral cover of V and the zero section $m\sigma$ (which carries the rank m vector bundle M). If one considers the structure group of the involved bundle one encounters the following situation: one starts with a stable $SU(n)$ vector bundle V on X whose structure group changes during the transition to $SU(n) \times SU(m)$ as discussed in [18].

It is known that the moduli of E decompose into four classes: the moduli of V, $\pi^* M$ and the moduli which 'measure' the deviation of E from being a direct sum. The main problem which occurs is that only the difference of the latter moduli can be obtained by an index computation on X [2] which will be reviewed in section 4.4. Now, two observations will help us to obtain information about the total number of the moduli. First, the mentioned index which one can evaluate on X is proportional to the net-generation number (which is one half of $c_3(V)$); second, chiral matter is localized along the intersection curve S of C_V and σ (first pointed out in [1] and later used in [24] and [3]). These observations and the fact that E can be obtained by a Fourier-Mukai transformation lead to the idea of reducing the index computation on X to an index computation on the intersection curve S as we will explain in section 4.5. This reduction will help to apply a vanishing argument and allows to obtain information about the total number of moduli.

Finally, in section 5, we will apply a similar argument to the computation of the number of chiral matter multiplets in heterotic compactifications on ellitically fibered Calabi-Yau threefolds X. As a result, we find an alternative derivation of the net-generation number as originally performed in [3] and show that if a vanishing argument applies, one can obtain the precise number of chiral matter multiplets. We find that chiral matter associated to $H^1(X, V)$ vanishes in heterotic string compactifications on elliptically fibered X with vector bundles constructed in the spectral cover approach and matter localized along a curve S of arithmetic genus $g(S) > 1$. This implies the vanishing of the corresponding Yukawa couplings.

Let us summarize the organization of this paper. In section 2, we review the spectral cover construction of vector bundles. In section 3, we prove that the Fourier-Mukai transformation preserves absolute stability. In section 4, we first review the necessary facts about the five-brane instanton transition. Then we work out a formula for the number of moduli which occur in a (vertical) five-brane instanton transition. We show how this moduli can be explicitly computed by applying a vanishing argument. In section 5, we reconsider the localization of chiral matter. We apply a similar vanishing argument to compute the number of chiral matter multiplets. The appendices contain all necessary calculations and proofs required for the sub-sections.

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2 Review of Vector Bundle Construction

We begin by recalling the construction of vector bundles on elliptic Calabi-Yau threefolds following the original construction given in [1] to which we refer for more details.

2.1 Spectral Cover Construction

One starts with a Calabi-Yau threefold X which is elliptically fibered over a complex two-dimensional base B and denotes by π the projection of X onto B. In addition one requires that X has a section σ. The construction of $SU(n)$ vector bundles V (fiberwise semistable and with $c_1(V) = 0$) on X using the spectral cover construction proceeds in two steps. First one describes bundles on the elliptic fiber and then uses global data in the base to "glue" them together to a bundle V on X.

More precisely, one starts on an elliptic fiber F given in the Weierstrass representation with distinguished reference point p. On the fiber, V decomposes as a direct sum of degree zero line bundles, each associated with a unique point on F. The condition that V is an $SU(n)$ bundle means that the product of the line bundles is trivial or equivalently that the points sum up to zero in the group law on F. For this n-tuple of points exists a meromorphic function vanishing to first order at the points and having a pole only at p.

When the reference point is globalized by the section σ the variation of the n points in the fiber leads to a hypersurface C embedded in X which is a ramified n-fold cover -the spectral cover- of the base given by

$$s = a_0 + a_2 x + a_3 y + ... + a_n x^{n/2} = 0 \tag{2.1}$$

here $a_r \in \Gamma(B, \mathcal{M} \otimes K_B^r)$, a_0 is a section of $\mathcal{M}$ and x, y sections of K_B^{-2} resp. K_B^{-3} in the Weierstrass model of X [1]. Note the last term eq. (2.1) for n-odd is $a_n x^{(n-3)/2}$. The pole order condition leads to s being a section of $\mathcal{O}(\sigma)^n$ which can in the process of globalization still be twisted by a line bundle $\mathcal{M}$ over B of $c_1(\mathcal{M}) = \eta$. Thus s can be actually a section of $\mathcal{O}(\sigma)^n \otimes \mathcal{M}$ and the cohomology class of C in X is

$$C = n\sigma + \pi^* \eta$$

So far we have recalled how to construct a spectral cover C by starting with a vector bundle V over X. The basic idea of the spectral cover construction is now to recover V from C! Therefore one starts with a suitable line bundle $\mathcal{R}^1$ on the n-fold cover $p\colon X \times_B C \to X$ and V will be induced as $V = p_* \mathcal{R}$. If one takes for $\mathcal{R}$ the Poincaré sheaf $\mathcal{P}$ on $X \times_B X$ (suitably modified in away that will made precise in the following section) and takes into account that the twist by a line bundle L over C leaves the fiberwise isomorphism class unchanged, one obtains

$$V = p_*(p_C^* L \otimes \mathcal{P}) \tag{2.2}$$

where p and p_C are the projections of the first and second factor of $X \times_B C$. The condition $c_1(V) = 0$ translates to a fixing of $\pi_* c_1(L)$ in $H^{1,1}(C)$ up to a class in $ker \pi_* \colon H^{1,1}(C) \to H^{1,1}(B)$.

2.2 Fourier-Mukai Transformation

The structure of V which occurs in eq. (2.2) makes transparent that V can be considered more generally as Fourier-Mukai transformation of the pair

[1] If C is not irreducible, $\mathcal{R}$ may be only a sheaf of rank one with no concentrated subsheaves

(C, L). For the description of the Fourier-Mukai transform it is appropriate instead of working on $X \times_B C$ to work on $X \times_B \tilde{X}$ where $\tilde{X}$ is the compactified relative Jacobian of X. $\tilde{X}$ parameterizes torsion-free rank 1 and degree zero sheaves of the fibers of $X \to B$ and it is actually isomorphic with X (see [25] or [26]) so that we will identify $\tilde{X}$ with X.

We have a diagram:

$$\begin{tikzcd} X \times_B X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "\pi_2"] \\ X \arrow[r, "\pi_1"] & B \end{tikzcd}$$

and the *Poincaré* sheaf

$$\mathcal{P} = \mathcal{O}(\Delta) \otimes \mathcal{O}(-p_1^*\sigma) \otimes \mathcal{O}(-p_2^*\sigma) \otimes q^* K_B^{-1}$$

normalized to make $\mathcal{P}$ trivial along $\sigma \times \tilde{X}$and $X \times \sigma$. Here σ is the fixed section, $q = \pi_1 \circ p_1 = \pi_2 \circ p_2$ and $\mathcal{O}(\Delta)$ is the dual of the ideal sheaf of the diagonal, which is torsion-free of rank 1.

The Fourier-Mukai transform and the inverse Fourier-Mukai transform are defined as functors of the derived categories $D(X)$ of complexes of coherent sheaves on X bounded from above. We have

$$\Phi\colon D^-(X) \to D^-(X)\,; \quad \Phi(\mathcal{G}) = Rp_{1*}(p_2^*(\mathcal{G}) \otimes \mathcal{P})\,,$$
$$\hat{\Phi}\colon D^-(X) \to D^-(X)\,; \quad \hat{\Phi}(\mathcal{G}) = Rp_{2*}(p_1^*(\mathcal{G} \otimes \hat{\mathcal{P}})$$

where

$$\hat{\mathcal{P}} = \mathcal{P}^* \otimes q^* K_B^{-1}\,.$$

We can also define the Fourier-Mukai functors Φ^i and $\hat{\Phi}^i$, $i = 0, 1$ in terms of single sheaves by taking $\Phi^i(\mathcal{F})$ and $\hat{\Phi}^i(\mathcal{F})$ as the i-th cohomology sheaves of the complexes $\Phi(\mathcal{F})$ and $\hat{\Phi}(\mathcal{F})$, we have

$$\Phi^i(\mathcal{F}) = R^i p_{1*}(p_2^*(\mathcal{F}) \otimes \mathcal{P})\,, \tag{2.3}$$
$$\hat{\Phi}^i(\mathcal{F}) = R^i p_{1*}(p_2^*(\mathcal{F}) \otimes \hat{\mathcal{P}})\,. \tag{2.4}$$

WIT$_i$ Sheaves

We can talk now about WIT$_i$ sheaves: they are those sheaves $\mathcal{F}$ for which $\Phi^j(\mathcal{F}) = 0$ for $j \neq i$, and we have the same notion for the inverse Fourier-Mukai transform.

Note also that the Fourier-Mukai transform and the inverse Fourier Mukai transform are only inverse functors up to a shift (see for instance [26], Lemma 2.6):

$$\Phi(\hat{\Phi}(\mathcal{G})) = \mathcal{G}[-1], \quad \hat{\Phi}(\Phi(\mathcal{F})) = \mathcal{F}[-1].$$

The -1 shift implies that if we have a single sheaf $\mathcal{F}$, then $\hat{\Phi}(\Phi(\mathcal{F})) = \mathcal{F}[-1]$ is a complex with only one cohomology sheaf, which is $\mathcal{F}$, but located at "degree1". If $\Phi^0(\mathcal{F}) = 0$ ($\mathcal{F}$ is WIT_1) then the unique Fourier-Mukai transform $\Phi^1(\mathcal{F})$ is WIT_0 for the inverse Fourier-Mukai transform such that $\hat{\Phi}^0(\Phi^1(\mathcal{F})) = \mathcal{F}$. In the same way, if $\Phi^1(\mathcal{F}) = 0$ ($\mathcal{F}$ is WIT_0) then $\Phi^0(\mathcal{F})$ is WIT_1 for the inverse Fourier-Mukai and $\hat{\Phi}^1(\Phi^0(\mathcal{F})) = \mathcal{F}$. Let us return to the spectral cover construction.

Given a relatively semistable vector bundle V of rank n on X, [2] then V is WIT_1 and the unique Fourier-Mukai transform $\Phi^1(V)$ is supported on a surface $i\colon C \to X$ inside X and its restriction to C is a pure dimension 1 and rank 1 sheaf L on C. That is, $\Phi^1(V) = i_*L$.

The surface C projects onto the base B as a $n:1$ cover, the *spectral cover* of V. Due to the invertibility of the Fourier-Mukai transform, i_*L is WIT_0 for the inverse Fourier-Mukai so that we can recover the bundle V in terms of the spectral data as

$$V = \hat{\Phi}^0(i_*L) = p_{1*}(p_2^*(i_*L) \otimes \hat{\mathcal{P}}).$$

We can follow the inverse road: take a surface $i\colon C \hookrightarrow X$ inside X flat over B, and a pure dimension 1 and rank 1 sheaf L on C (for instance a line bundle). Then i_*L as a sheaf on X is WIT_0 for the inverse Fourier-Mukai transform. Its inverse $V = \hat{\Phi}^0(i_*L)$ is a sheaf on X relatively torsion-free semistable and of degree 0 [1, 27, 28, 26].

The topological invariants of $\Phi(\mathcal{G})$ and $\hat{\Phi}(\mathcal{G})$ for an arbitrary object $\mathcal{G}$ of the derived category has been computed explicitly in terms of those of $\mathcal{G}$ in [27, 28].

3 Comments on Stability

For a line bundle L on an irreducible spectral cover C the Fourier-Mukai transform of i_*L is a stable vector bundle V on X. But sometimes we have to deal with vector bundles (or more generally torsion-free sheaves) whose

[2]More generally we can take V flat over B and torsion-free semistable of degree 0 on fibers.

spectral cover C is not irreducible such that L is no longer a line bundle on C but rather a torsion-free rank one sheaf. It follows L is not automatically stable. We will show that (for V semistable of degree zero on fibers) the stability of L (as a torsion-free rank one sheaf on the reducible spectral cover) is equivalent to the stability of V as a torsion-free sheaf on X. As a result, Simpson Jacobians of stable torsion-free rank one sheaves on spectral covers are isomorphic to open subsets of moduli spaces of stable sheaves on X.

3.1 Review of the Elliptic Surface Case

It is known that for elliptic surfaces the relative Fourier Mukai transform preserves not only fiberwise stability (see [25] for the case of positive degree on fibers and [26] for the case of degree zero on fibers) but also absolute stability in a certain sense. By fiberwise or relative stability (or semistability) we understand stability (or semistability) on fibers. A sheaf $\mathcal{F}$ on a fibration $X \to B$ is said to be fiberwise or relatively stable (or semistable) if it is flat over B and the restriction of $\mathcal{F}$ to every fiber of $X \to B$ is stable (or semistable) in the ordinary sense. Further, we need a relative polarization to speak about relative stability. Such polarization is given by a divisor on X that meets every fiber in a polarization of the fiber.

Relative stability is a very important concept, however, when we have our elliptic fibration $X \to B$ we need to consider "absolute stability" as well. Here we refer to stability on X with respect to a certain polarization; we then somehow forget the fibered structure and consider X just as a manifold. The reason we need absolute stability is that we want to consider moduli spaces of stable sheaves on X.

There is still one more thing to keep in mind. Stability (or semistability) used to be defined in terms of the slope (μ-stability) or the Hilbert polynomial (Gieseker stability) but only for torsion-free sheaves. That excluded sheaves concentrated on closed subvarieties or even sheaves defined on reducible singular varieties. This problem was circumvented by Simpson [29] who defined both μ-stability and Gieseker stability (along with the corresponding semistability notions) for "pure" sheaves on arbitrary projective varieties. For Simpson, a pure sheaf of dimension i is a sheaf $\mathcal{F}$ whose support has dimension i and that has no subsheaves concentrated on smaller dimension. This gives the more natural generalization of the notion of torsion-free.

Recall that the Euler characteristic of a vector bundle (or more general, coherent sheaf) $\mathcal{F}$ is given by $\chi(\mathcal{F}) = \sum(-1)^i \dim H^i(X, \mathcal{F})$ and for a fixed ample line bundle $\mathcal{O}(1)$ on X we have the so called Hilbert polynomial $P(\mathcal{F}, m)$ given by $m \to \chi(\mathcal{F} \otimes \mathcal{O}m)$. The Hilbert polynomial can be written as

$$P(\mathcal{F}, m) = \sum_{i=0}^{3} \alpha_i(\mathcal{F}) \frac{m^i}{i!}$$

with integral coefficients $\alpha_i(\mathcal{F})$ which are listed in the appendix. Once the Hilbert polynomial (and then the slope) is defined, Simpson definitions are very similar to the ordinary ones.

When the support Y of a pure sheaf $\mathcal{F}$ is irreducible and reduced so that the restriction $\mathcal{F}_{|Y}$ is torsion-free on Y, the stability of $\mathcal{F}$ as a sheaf on X in the sense of Simpson is equivalent to the stability of $\mathcal{F}_{|Y}$ as a torsion-free sheaf on Y in the ordinary sense. But when Y is reducible, we have to consider Simpson stability as the unique reasonable notion. This is not an uncommon situation, for instance, if we take an elliptic fibration $X \to B$ and a relatively semistable sheaf $\mathcal{F}$ on X of degree zero on fibers, then the Fourier-Mukai transform $\Phi^1(\mathcal{F})$ is concentrated on the spectral cover C that in many cases is reducible. We can thus only consider the possible stability of $\Phi^1(\mathcal{F})$ in the sense of Simpson.

By this reason, in what sequel stability will always mean μ-stability in the sense of Simpson.

As a warm up, we start the discussion about preservation of absolute stability in the case of an elliptic surface $X \to B$. Given a Cartier divisor $C \hookrightarrow \widehat{X}$ flat of degree n that we polarize with the intersection F_C of C with the fiber F of π, we have the following result: for every $a > 0$, there exists $b_0 \geq 0$ depending only on the topological invariants of C, such that for every $b \geq b_0$ and every sheaf L on C of pure dimension one, rank one, degree r and semistable with respect to μ_C, the unique Fourier-Mukai transform $\Phi^0(L)$ is semistable on X with respect to the polarization $a\sigma + bF$. Moreover, if L is stable on C, then $\Phi^0(L)$ is stable as well on X.

A certain converse is also true: let us fix a Mukai vector (n, Δ, s) with $\Delta \cdot F = 0$. For every $a > 0$, there exists b_0 such that for every $b \geq b_0$ and every sheaf V on X with Chern character (n, Δ, s) and semistable with respect to the polarization $a\sigma + bF$, the restriction of V to the generic fiber is semistable. In particular, V is WIT_1 so that it has a unique relative Fourier-Mukai transform $\hat{\Phi}^1(V)$. If we assume that the restriction of V to every fiber is semistable then $\hat{\Phi}^1(V)$ is of pure dimension one, rank one, degree r and semistable on the spectral cover $C(V)$. In other words the spectral cover $C(V)$ does not contain fibers. If V is stable on X, $\hat{\Phi}^1(V)$ is stable on $C(V)$

as well (see [26]). These properties mean that we can construct non-empty open subsets of components of the moduli space of stable sheaves on X in terms of the compactified Simpson Jacobians of the spectral covers [26].[3]

3.2 Generalization to Calabi-Yau Threefolds

We want to study a similar question for elliptic Calabi Yau threefolds $\pi\colon X \to B$. Since we require that π has a section (in addition to the smoothness of B and X), the base surface B has to be of a particular kind, namely B has to be Del Pezzo, Hirzebruch, Enriques or a blow-up of a Hirzebruch (see [22] or [6]).

To our knowledge the problem of preservation of absolute stability for elliptic Calabi-Yau threefolds has not been considered in the literature so far. We polarize X with $\tilde{H} = a\sigma + bH_B$, where $H_B = \pi^*(\bar{H}_B)$ and $\bar{H}_B$ is a polarization of B to be chosen later. In the following we will assume that there is a decomposition

$$H^{2i}(X) = \sigma p^* H^{2i-2}(B) \oplus p^* H^{2i}(B) \tag{3.1}$$

Let us consider a torsion-free sheaf V on X of rank n and degree zero on fibers and write its Chern characters as $ch(V) = (n, \tilde{S}, \sigma\eta + aF, s)$ with $\eta, \tilde{S} \in p_2^* H^2(B)$ according with [27, 28] and eq. (3.1). Assume that V is WIT_1. This happens, for instance, when V is relatively semistable (i.e., flat over B and semistable on fibers). The Hilbert polynomial of the unique Fourier-Mukai transform $\hat{\Phi}(V)$ of V is given by $P(\hat{\Phi}(V), m)$ with $\alpha_i(\hat{\Phi}(V))$ given in appendix A. Simpson slopes of V and $\Phi(V)$ can then be determined and are given by

$$\mu(V) = \frac{\tilde{S} \cdot \tilde{H}^2}{n \cdot \tilde{H}^3}, \quad \text{and} \quad \mu(\Phi(V)) = \frac{\alpha_1(\hat{\Phi}(V))}{\alpha_2(\hat{\Phi}(V))} \tag{3.2}$$

Let us make a further assumption, that the support C of $\Phi(V)$ is flat over B, that is, that it does not contain fibers of π. It follows that the support

[3] For an integral variety, the Jacobian parameterizes line bundles of a fixed degree. In that case, line bundles are automatically stable, regardless of the polarization. In the reducible case this is no longer true, we need to fix a polarization and we can have line bundles that are unstable. In this situation instead of parameterizing line bundles, we parameterize pure dimension n (the dimension of the space) rank one and fixed degree sheaves that are stable in the sense of Simpson. The corresponding moduli space is the compactified Simpson Jacobian. For an integral variety, this is a compactification of the ordinary Jacobian because it contains not only all line bundles, but also the torsion-free rank one sheaves.

of every subsheaf $\mathcal{F}$ of $\Phi(V)$ is contained in C and has no fibers as well, so that it is WIT_0 with respect to the inverse Fourier-Mukai transform and its transform is a WIT_1 subsheaf V' of V. Moreover, V' has degree zero on fibers again by [27] (2.33) so that eq. (3.2) is still true for V' (writing primes for the correspondent invariants).

For every a there exists $b > 0$, depending only on the topological invariants of V such that V is stable (resp. semistable) with respect to $\tilde{H} = a\sigma + bH_B$ if and only if $\Phi(V)$ is stable (resp. semistable) as well. Assume that V is stable and that $\Phi(V)$ is destabilized by a subsheaf $\mathcal{F}$. Then, as we said before, $\mathcal{F} = \Phi(V')$ for certain subsheaf V' of V of degree zero on fibers and we have

$$\frac{\alpha_1(\hat{\Phi}(V))}{\alpha_2(\hat{\Phi}(V))} \leq \frac{\alpha_1(\hat{\Phi}(V'))}{\alpha_2(\hat{\Phi}(V'))}$$

with the α_i's given in appendix A.

If we write this condition as a polynomial on b, we have

$$\begin{aligned} n'(\sigma H_B^2)(\sigma\tilde{S}H_B + \frac{1}{2}nc_1\sigma H_B)b^3 &+ \text{lower terms} \\ &\leq n(\sigma H_B^2)(\sigma\tilde{S}'H_B + \frac{1}{2}n'c_1\sigma H_B)b^3 + \text{lower terms}. \end{aligned}$$

Since the family of subsheaves of V is bounded, there is a finite number of possibilities for the Hilbert polynomial of V'. Then, the value of b one has to chose depends only on a and on the topological invariants of V. For $b \gg 0$ the destabilizing condition is equivalent to

$$n'(\sigma\tilde{S}H_B) \leq n(\sigma\tilde{S}'H_B)$$

On the other hand the stability of V gives $n(\tilde{S}' \cdot \tilde{H}^2) < n'(\tilde{S} \cdot \tilde{H}^2)$ and then

$$n'(\sigma c_1\tilde{S}) < n(\sigma c_1\tilde{S}')$$

which is a contradiction. Both the converse and the corresponding semistability statements are proven analogously.

For elliptically fibered Calabi-Yau threefolds X we proceed as in the elliptic surface case and basically use the above result to prove that there are open subsets of components of the moduli space of stable sheaves on X that are isomorphic to compactified Simpson Jacobians of a universal spectral cover.

A last comment: we have shown the preservation of absolute stability under Fourier Mukai transformation for sheaves of degree zero on fibers. In this paper, we shall focus only on fiber bundles with vanishing first Chern class $c_1(V) = 0$, because they are the ones relevant for the problems we are considering. Their Fourier-Mukai transforms $\Phi^1(V)$ have non-vanishing c_1 ($c_1(\Phi^1(V))$ which represents the spectral cover of V; because of this, the problem of preservation of stability is better studied in the more symmetric situation of degree zero on fibers, a property shared by V and $\Phi^1(V)$.

4 Five-Brane Instanton Transition

4.1 Anomaly Cancellation

Heterotic string compactifications on elliptic Calabi-Yau threefolds require a number of five-branes in order to cancel the anomaly [1]. These five-branes wrapping holomorphic curves in X whose cohomology class is determined by the heterotic anomaly cancellation condition

$$[W] = c_2(TX) - c_2(V_1) - c_2(V_2) \tag{4.1}$$

where $[W]$ is the cohomology class of the wrapped curves, $c_2(V_i)$ are the second Chern classes of the vector bundles on X, $c_2(TX)$ is the second Chern class of the tangent bundle and given by [1]

$$c_2(TX) = c_2 + 11c_1^2 + 12\sigma c_1 \,.$$

We use the notation $c_i = \pi^* c_i(B)$ and σ (satisfying $\sigma^2 = -c_1\sigma$) the class of a section of π. Due to our assumption eq. (3.1), $[W]$ may be decomposed as

$$[W] = \sigma C_1 + C_2$$

where C_1 maps to a divisor in B_2 to be embedded in X via σ and $C_2 = h[F]$ describes the five-branes wrapping the elliptic fiber of X. Following [4] we refer to five-branes which wrap curves in the base as horizontal five-branes and branes wrapping the fiber as vertical ones. We will also mention that five-branes could wrap skew curves, i.e. curves which have both fiber and base components.

4.2 The Transition

We will be interested in the situation when heterotic five-branes which wrap the elliptic fiber dissolve into gauge instantons resulting in a new heterotic vector bundle E. Let us follow how such transition might proceed thereby recalling results partly obtained in [2] and [18]. To simplify our discussion we will assume (until otherwise stated) that V_2 in eq. (4.1) is trivial. Under heterotic/F-theory duality this corresponds to an unbroken $G = G_1 \times E_8$ gauge group where G_1 is determined by the commutator of the structure group of V_1 in E_8. Further, we assume that $[W] = hF$, that is, we only consider 'vertical five-branes'.

On the level of anomaly cancellation one expects that a five-brane instanton transition causes a change in the second Chern-class of the vector bundle

$$c_2(TX) - (c_2(V) + kF) = [\tilde{W}]$$

assuming here $k < h$ so 'absorbing' part of the five-brane class into the vector bundle.

We can think of a five-brane at the transition point (a particular point on the Coulomb branch) as a pointlike instanton concentrated in codimension two in X which will be the elliptic fiber in our case. That is, a five-brane can be considered as singular gauge field configuration such that the curvature is zero everywhere except on a fiber where it has a singularity. In terms of Hermitian-Yang-Mills connections we can think of a connection on a vector bundle which is smooth except along a curve in the class C_2 where it has a delta function behavior. Mathematically, such a configuration is described by a singular torsion free sheaf. If the singular sheaf can be smoothed out to a vector bundle a five-brane instanton transition can occur.

If we recall that $c_2(V)$ is in $H^4(X)$ and as we are concerned with an elliptic fibration $\pi\colon X \to B$ we have a decomposition $H^4(X) = H^2(B)\sigma \oplus H^4(B)$ with σ being the section. For $c_2(V)$ one has $c_2(V) = \pi^*(\eta)\sigma + \pi^*(\omega)$ with $\eta, \omega \in H^2(B)$ resp. $H^4(B)$. Therefore it is expected [2] that the singular configuration can be smoothed out to a new bundle with $\int_B c_2(E) = \int_B c_2(V) + k$ assuming that kF can be represented by k separated fibers projecting to k distinct points on B. This suggests that we are looking for a vector bundle M (or sheaf) on B with $c_2(M) = k$.

Thus the actual transition proceeds in two steps [18]. First, one describes a singular torsion free sheaf $\tilde{M}$ on B. Second, one shows that it can be smoothed out to a stable bundle M on B which pulls back to a stable vector bundle over X with $c_2(\pi^*M) = kF$.

To summarize: after the transition a non-trivial gauge bundle M of rank m has developed on the zero section. The new bundle $E = V \oplus \pi^*M$ is smooth and reducible of rank $n + m$. The second Chern class of M counts the number of five-branes which have been dissolved in the transition. As the zero section σ is isomorphic to the base, one can think of M as being a vector bundle on B.

The spectral cover of $E = V \oplus \pi^* M$ can be easily described in terms of the spectral covers of V and $\pi^* M$. One notices that E is still WIT_1 and that its unique Fourier-Mukai transform is the direct sum $\Phi^1(E) = \Phi^1(V) \oplus \Phi^1(\pi^* M)$ of the Fourier-Mukai transforms of V and $\pi^* M$. We know ([26]) that the spectral cover is closed defined by the Fitting ideal of the Fourier-Mukai transform. Since the Fitting ideal which describes the spectral cover is multiplicative over direct sums (see [26]), the spectral cover C_E is the union of the spectral covers C_V and $C_{\pi^* M}$, that is

$$C_E = C_V + C_{\pi^* M}$$

as numerical classes. If we proceed as in [26][4] one obtains

$$\Phi^1(\pi^* M) = \pi^* M \otimes \Phi^1(\mathcal{O}_X) = \pi^* M \otimes \pi^* K_B \otimes \mathcal{O}_\sigma = \sigma_*(G)\,, \quad G = M(K_B) \tag{4.2}$$

Then $\Phi^1(\pi^* M) = \sigma_*(G)$ is concentrated on σ, but due to the multiplicativity of the Fitting ideal, the spectral cover of $\pi^* M$ is not σ but rather $C_{\pi^* M} = m\sigma$ and the restriction $L_M = \sigma_*(G)_{|m\sigma}$ of the Fourier-Mukai transform to the spectral cover is a pure dimension one rank one sheaf on the reducible surface $m\sigma$ (which is not a line bundle). We then have

$$C_E = C_V + m\sigma$$

as we expected and

$$\Phi^1(\pi^* M) = \sigma_*(G) = h_*(L_M)$$

where $h\colon m\sigma \hookrightarrow X$ is the immersion of $m\sigma$ into X.

Matter Curve S

It is known that chiral matter ($c_3(V)/2 \neq 0$) is localized along the intersection curve S of σ and C [1, 3].

In the following we will assume that

- $S = \sigma \cdot C$ is irreducible
- $g(S) > 1$.

[4] The Poincaré sheaf considered in [26] is the dual of the Poincaré sheaf considered both here and in [27, 28].

4.3 F-Theory Perspective

Let us recall how the five-brane instanton transition is viewed from the perspective of F-theory!

Recall that F-theory is defined as type-IIB super string theory with varying coupling constant [30]. A consistent F-theory compactification on an Calabi-Yau fourfold Y[5] requires a number $\chi(Y)/24$ of three-branes filling the transverse space time $\mathbb{R}^{3,1}$ [31]. This number agrees with the number of five-branes required for consistent F-theory compactification giving a non-trivial test for the expected heterotic/F-theory duality [1, 5] and the adiabatic argument.

By duality one expects that if a five-brane disappears on the heterotic side, a three-brane should disappear on the F-theory side. More precisely, a three-brane 'dissolves' into a finite size instanton, i.e. a background gauge bundle $\tilde{M}$ on the corresponding component on the seven-brane is turned on [2]. The instanton number of this bundle counts thereby the number of dissolved three-branes. In [2] it was then suggested that the two bundles M and $\tilde{M}$ should actually be identified.

Such a transition leads to a modification of the anomaly cancellation condition

$$\frac{\chi(Y)}{24} = n_3 + \sum_j k_j$$

here n_3 is the number of three-branes and $k_j = \int_{D_j} c_2(M_j)$ and j labels the respective component of the seven-brane partially wrapped over the discriminant locus[6] in the three dimensional base of Y.

Due to the presence of a non-trivial instanton bundle on the seven-brane part of the gauge group will be broken. Otherwise, the gauge group would be given by a degeneration of A-D-E type of the elliptic fiber over the compact part of the seven-brane. The gauge group which is left over after the breaking by $\tilde{M}$ should correspond on the heterotic side to the commutator of E in E_8. Further one expects that in the transition extra chiral matter occurs (if the original bundle V had non zero $c_3(V)$) [2] which is on the heterotic side related to the moduli we are aiming to evaluate.

[5]here assumed to be elliptically fibered over a three dimensional base B which is a $\mathbb{P}^1$ bundle over the same B_2 as considered on the heterotic string side

[6][2] argues that a three-brane can only dissolve on a multiple seven-brane

4.4 Computation of Moduli

To begin let us recall some known facts about the moduli of $E = V \oplus \pi^* M$. The number of moduli is given by the dimension of the deformations space $H^1(\mathrm{End}(E))$. This space can be decomposed into four parts (as already noticed in [2])

$$\begin{aligned} H^1(\mathrm{End}(E)) = & H^1(\mathrm{End}(V)) \oplus H^1(\mathrm{End}(\pi^* M)) \oplus \\ & H^1(Hom(V, \pi^* M)) \oplus H^1(\mathrm{Hom}(\pi^* M, V)) \end{aligned}$$

where the first two summands correspond to deformations of E that preserve the direct sum and deform V and $\pi^* M$ individual. The last two elements give the deformations of E that deform away from the direct sum.

Moduli of V

The number of moduli of V can be determined in two ways, depending on whether one works with V directly or with its spectral cover data (C, L) from which it is obtained. In the direct approach one is restricted to so called τ-*invariant* bundles and therefore to a rather special point in the moduli space, whereas the second approach is not restricted to such a point. However, after a brief review of the first approach which was originally introduced in [1] making concrete earlier observations in [32], we will explain how the τ-*invariance* is translated to the (C, L) data. The issue of τ-*invariance* has been also addressed in [33].

The first approach starts with the index of the $\bar\partial$ operator with values in $\mathrm{End}(V)$ which is the $\mathrm{index}(\bar\partial) = \sum_{i=0}^{3} (-1)^i \dim H^i(X, \mathrm{End}(V))$. As this index vanishes by Serre duality on the Calabi-Yau threefold, one has to introduce a further twist to get a non-trivial index problem. This is usually given if the Calabi-Yau space admits a discrete symmetry group [32]. In case of elliptically fibered Calabi-Yau manifolds one has such a group G given by the involution τ coming from the "sign flip" in the elliptic fibers. One assumes that this symmetry can be lifted to an action on the bundle at least at some point in the moduli space [1]. In particular the action of τ lifts to an action on the adjoint bundle $ad(V)$ which are the traceless endomorphisms of $\mathrm{End}(V)$. It follows that the index of the $\bar\partial$ operator generalizes to a character valued index where for each $g \in G$ one defines $\mathrm{index}(g) = \sum_{i=0}^{3} (-1)^{i+1} \mathrm{Tr}_{H^i(X, ad(V))}\, g$ where $\mathrm{Tr}_{H^i(X, ad(V))}$ refers to a trace in the vector space $H^i(X, ad(V))$. The particular form of this index for elliptic Calabi-Yau threefolds has been determined in [1] (with $g = 1 + \frac{\tau}{2}$) one finds $\mathrm{index}(g) = \sum_{i=0}^{3} (-1)^{i+1} \dim H^i(X, ad(V))_e$ where the subscript "e" indicates the projection onto the even subspace of $H^i(X, ad(V))$. One can compute this index using a fixed point theorem as shown in [1].

The second approach makes intuitively clear where the moduli of V are coming from, namely, the number of parameters specifying the spectral cover C and by the dimension of the space of holomorphic line bundles L on C. The first number is given by the dimension of the linear system $|C| = |n\sigma+\eta|$. The second number is given by the dimension of the Picard group $Pic(C) = H^1(C, \mathcal{O}_C^*)$ of C. One thus expects the moduli of V to be given by [21]

$$h^1(X, \mathrm{End}(V)) = \dim |C| + \dim Pic(C)$$

which can be explicitly evaluated making the assumption that C is an irreducible, effective, positive divisor in X.

If one computes the endomorphisms of V using the character valued index one assumes that V is invariant under the involution of the elliptic fiber, i.e.$V = V^\tau$. On the other hand the number of moduli derived from the pair (C, L) requires no such restriction. Thus the question occurs: *how is the condition $V = V^\tau$ translated to the spectral data (C, L)?*

To see this translation let us study the meaning of the condition $V^\tau = V$ with respect to the relative Fourier Mukai transformation Φ, that is to

$$\Phi\colon D(X) \to D(X)\,, \quad F \mapsto \Phi(F) = \pi_{2,*}(\pi_1^*(F) \otimes \mathcal{P})\,,$$

where $\mathcal{P}$ is the relative Poincaré sheaf on $X \times_B X$. As already mentioned, Φ is an equivalence of categories whose inverse functor is the Fourier Mukai transform $\hat{\Phi}$ with respect to $\mathcal{P}^* \otimes q^* K_B^{-1}$, where $q\colon X \times_B X \to B$ is the natural projection and K_B is the canonical sheaf on B. We write Φ^* for the Fourier Mukai transfrom with respect to $\mathcal{P}^*$.

Further let us write $\tau\colon X \to X$ for the elliptic involution on $X \to B$ so that we write $\tau^* F$ instead of F^τ for F in the derived category. We then have three involutions on $X \times_B X$:

$$\begin{aligned} \tau(x,y) &= (\tau(x), y) = (-x, y)\\ \hat{\tau}(x,y) &= (x, \tau(y)) = (x, -y)\\ \bar{\tau} &= \hat{\tau} \circ \tau = \tau \circ \hat{\tau}. \end{aligned}$$

One easily sees that $\hat{\tau}^* \mathcal{P} = \mathcal{P}^*$ and one has

$$\begin{aligned} \Phi(\tau^* F) &= \pi_{2,*}(\pi_1^*(\tau^* F \otimes \mathcal{P})) = \pi_{2,*}(\tau^*(\pi_1^* F \otimes \tau^* \mathcal{P}))\\ &= \pi_{2,*}(\hat{\tau}^* \bar{\tau}^*(\pi_1^* F \otimes \tau^* \mathcal{P})) = \hat{\tau}^* \pi_{2,*}(\bar{\tau}^*(\pi_1^* F \otimes \tau^* \mathcal{P}))\\ &= \hat{\tau}^* \pi_{2,*}(\pi_1^*(\tau^* F) \otimes \hat{\tau}^* \mathcal{P})) = \hat{\tau}^* \pi_{2,*}(\pi_1^* \tau^* F \otimes \mathcal{P}^*) = \hat{\tau}^* \Phi^*(\tau^* F)\,. \end{aligned}$$

If $\tau^*F = F$, one has

$$\Phi(F) = \hat{\tau}^* \Phi^*(F)\,. \tag{4.3}$$

If we assume that F reduces to a single stable, irreducible, holomorphic $SU(n)$ vector bundle over elliptic Calabi-Yau specified by a pair (C, L) via the inverse Fourier Mukai transform $\hat{\Phi}$. Then $\Phi(F)$ reduce to the sheaf $\Phi^1(F) = i_*(L)$, where $i\colon C \hookrightarrow X$ is the immersion and eq. (4.3) means that $\Phi^*(F)$ reduces to a single sheaf $\Phi^{*1}(F)$ as well and that $\Phi^{*1}(F) = \hat{\tau}^*(i_*(L))$. If $C^\tau = \tau(C)$, $L^\tau = \tau^* L$ and $j\colon C^\tau \to X$ is the immersion, it follows that

$$\Phi^{*1}(F) = j_*(L^\tau)\,.$$

On can say that if $\tau^*F = F$, then F can be specified by spectral data in two different ways, either by (C, L) via the inverse Fourier Mukai transform $\hat{\Phi}$ or by $(C^\tau, L^\tau \otimes (q^*K_B)_{|C})$ via the standard FM (with respect to $\mathcal{P}$).

Moduli of π^*M

The number of moduli of π^*M are given by $h^1(X, \pi^* \operatorname{End}(M))$. If one applies the Leray spectral sequence to the elliptic fibration of X and assumes that the moduli space of M over B is smooth then one can show [2] that all moduli of π^*M come from moduli of M on the base B. The moduli can be then evaluated using the Riemann-Roch index theorem (assuming M being a $SU(m)$ bundle and B a rational surface)

$$h^1(B, \operatorname{End}(M)) = 2mk - (m^2 - 1)\,.$$

What is Known About the Remaining Moduli?

First we note that

$$H^i(\operatorname{Hom}(V, \pi^*M)) = \operatorname{Ext}^i(V, \pi^*M)$$
$$H^i(\operatorname{Hom}(\pi^*M, V)) = \operatorname{Ext}^i(\pi^*M, V)$$

since V and π^*M are locally free sheaves on X. In particular, elements of the vector spaces $H^1(\operatorname{Hom}(V, \pi^*M))$ and $H^1(\operatorname{Hom}(\pi^*M, V))$ give non-trivial extensions $0 \to \pi^*M \to E_\mu \to V \to 0$ respectively $0 \to V \to E^\nu \to \pi^*M \to 0$. More information can be obtained by computing the index

$$I_X = \sum_{i=0}^{3} (-1)^i \dim H^i(\operatorname{Hom}(V, \pi^*M))\,. \tag{4.4}$$

We will later show that $\dim H^i(\mathrm{Hom}(V, \pi^*M)) = 0$ for $i = 0, 3$. We also note that $H^i(\mathrm{Hom}(V, \pi^*M)) = H^i(V^* \otimes \pi^*M)$ and $H^i(\mathrm{Hom}(\pi^*M, V)) = H^i((\pi^*M)^* \otimes V)$ and applying Serre duality we get $H^2(V^* \otimes \pi^*M) = H^1(V \otimes (\pi^*M)^*)$ using the fact that the canonical bundle K_X of X is trivial. Thus we get

$$\begin{aligned} I_X &= \dim H^1(\mathrm{Hom}(\pi^*M, V)) - \dim H^1(\mathrm{Hom}(V, \pi^*M)) \\ &= \dim \mathrm{Ext}^1(\pi^*M, V)) - \dim \mathrm{Ext}^1(V, \pi^*M) \end{aligned} \tag{4.5}$$

The left hand side of eq. (4.4) can be evaluated using the Riemann-Roch theorem

$$I_X = \int_X ch(V^*) ch(\pi^*M) \,\mathrm{Td}(X) = -\frac{1}{2} m c_3(V) \tag{4.6}$$

and is related to chiral matter for non-zero $c_3(V)$ as observed in [2]. The computation of $c_3(V)$ in the spectral cover and the parabolic bundle construction has been performed in [24] respectively [34].

4.5 Evaluation of the Remaining Moduli

We will proceed as follows. We first rewrite the index in terms of the spectral data using the so-called Parseval theorem and then restrict to S to evaluate the index.

The Parseval theorem for the relative Fourier-Mukai transform has been proved by Mukai in his original Fourier-Mukai transform for abelian varieties [35], but can be easily extended to any situation in which a Fourier-Mukai transform is an equivalence of categories.

Parseval Theorem

Assume that we have sheaves $\mathcal{F}$, $\bar{\mathcal{F}}$ that are respectively WIT_h and WIT_j for certain h, j; this means that they only have one non-vanishing Fourier-Mukai transform, the h-th one $\Phi^h(\mathcal{F})$ in the case of $\mathcal{F}$ and the j-th one $\Phi^j(\bar{\mathcal{F}})$ in the case of $\bar{\mathcal{F}}$. Parseval theorem says that one has

$$\mathrm{Ext}^i_X(\mathcal{F}, \bar{\mathcal{F}}) = \mathrm{Ext}^{h-j+i}_X(\Phi^h(\mathcal{F}), \Phi^j(\bar{\mathcal{F}}))\,, \tag{4.7}$$

thus giving then a correspondence between the extensions of $\mathcal{F}$, $\bar{\mathcal{F}}$ and the extensions of their Fourier-Mukai transforms. The proof is very simple, and relays on two facts. The first one is that for arbitrary coherent sheaves E, G the ext-groups can be computed in terms of the derived category, namely

$$\mathrm{Ext}^i(E, G) = \mathrm{Hom}_{D(X)}(E, G[i]) \tag{4.8}$$

The second one is that the Fourier-Mukai transforms of $\mathcal{F}$, $\bar{\mathcal{F}}$ in the derived category $D(X)$ are $\Phi(\mathcal{F}) = \Phi^h(\mathcal{F})[-h]$, $\Phi(\bar{\mathcal{F}}) = \Phi^j(\bar{\mathcal{F}})[-j]$. Since the Fourier-Mukai transform is an equivalence of categories, one has

$$\begin{aligned}\mathrm{Hom}_{D(X)}(\mathcal{F}, \bar{\mathcal{F}}[i]) &= Hom_{D(X)}(\Phi(\mathcal{F}), \Phi(\bar{\mathcal{F}}[i])) \\ &= \mathrm{Hom}_{D(X)}(\Phi^h(\mathcal{F})[-h], \Phi^j(\bar{\mathcal{F}})[-j+i]) \\ &= Hom_{D(X)}(\Phi^h(\mathcal{F}), \Phi^j(\bar{\mathcal{F}})[h-j+i])\end{aligned}$$

so that eq. (4.8) gives the Parseval theorem eq. (4.7).

In particular, if both $\mathcal{F}$ and $\bar{\mathcal{F}}$ are WIT_j for the same j, we obtain

$$\mathrm{Ext}^i_X(\mathcal{F}, \bar{\mathcal{F}}) = \mathrm{Ext}^i_X(\Phi^j(\mathcal{F}), \Phi^j(\bar{\mathcal{F}}))$$

for every $i \geq 0$.

We can apply Parseval theorem to our situation, because V and π^*M are WIT_1. Since their Fourier-Mukai transforms are respectively i_*L where $i\colon C \hookrightarrow X$ is the immersion, and $\sigma_*(G)$ where $\sigma\colon B \to X$ is the section and $G = M(K_B)$ eq. (4.2), we have

$$\mathrm{Ext}^i_X(V, \pi^*M) = \mathrm{Ext}^i_X(i_*L, \sigma_*(G))\,. \quad i \geq 0 \tag{4.9}$$

We note that, due to the fact that V and π^*M are vector bundles, we have

$$H^1(\mathrm{Hom}(V, \pi^*M)) = \mathrm{Ext}^1_X(V, \pi^*M) \tag{4.10}$$
$$H^1(\mathrm{Hom}(\pi^*M, V))^* = \mathrm{Ext}^2_X(V, \pi^*M) \tag{4.11}$$

as we are computing dimensions we have

$$\dim H^1(\mathrm{Hom}(\pi^*M, V))^* = \dim H^1(\mathrm{Hom}(\pi^*M, V))$$

so that we can actually rewrite the index I_X in terms of the spectral bundles

$$I_X = \sum_{i=0}^{3} (-1)^i \dim \mathrm{Ext}^i_X(i_*L, \sigma_*(G))\,.$$

Restriction to S

We proceed as in Section 6 of [27] and use the sequence of low terms of the spectral sequence associated to Grothendieck duality for the immersion i.

In its simpler form, Grothendieck duality for a smooth morphism is a sort of relative Serre duality, a Serre duality for flat families of smooth varieties. Grothendieck extended this notion to very general morphisms of algebraic varieties; his formulation requires derived categories and a notion of "dualizing complex"; this is an object of the derived category that plays (for a general morphism) the same role as the sheaf of r-forms on a smooth r-dimensional variety.

We apply Grothendieck duality for the closed immersion $i\colon C \hookrightarrow X$ and denote by $\tilde\sigma$ the restriction of σ to S. Grothendieck duality for the closed immersion $i\colon C \hookrightarrow X$ says that there is an isomorphism in the derived category

$$R\ \mathrm{Hom}_X(i_*L, \sigma_*G) = R\ \mathrm{Hom}_C(L, i^!(\sigma_*G))$$

where $i^!(\sigma_*G)$ is the "dualizing complex" for the immersion, and $i^!(\sigma_*G)$ is determined by the equation $i_*(i^!(\sigma_*G)) = R\,\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_C, \sigma_*G)$ where $\mathcal{H}om$ stands for the Hom-sheaf (see [36] Section §6). Let us consider the exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to i_*\mathcal{O}_C \to 0 \tag{4.12}$$

where $\mathcal{O}_X$ and $\mathcal{O}_C$ are the trivial bundles (structure sheaves) on X and C; $\mathcal{O}_X(-C)$ is the inverse of the tautologically defined line bundle $\mathcal{O}_X(C)$ on X that admits a holomorphic section s that vanishes precisely on C; also note the first map in eq. (4.12) is multiplication by s and the second is restriction to C. We need i_* to understand $\mathcal{O}_C$ as a sheaf on X, the sheaf that coincides with $\mathcal{O}_C$ on C and it is zero on $X - C$.

From eq. (4.12) we read that $R\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_C, \sigma_*G)$ is represented by the complex

$$\sigma_*G \xrightarrow{d=0} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(-C), \sigma_*G)$$

that is,

$$i^!(\sigma_*G) = \{\, \tilde\sigma_*(G_{|S}) \xrightarrow{d=0} \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})\,\}$$

in the derived category, where $N_{X/C}$ is the normal sheaf to C in X. Then, since L is a line bundle, we have

$$\begin{aligned} R\ \mathrm{Hom}_X(i_*L,\sigma_*G) &= R\ \mathrm{Hom}_C(L, \{\, \tilde\sigma_*(G_{|S}) \xrightarrow{d=0} \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})\,\}) \\ &= R\,\Gamma(C, \{\, L^{-1} \otimes \tilde\sigma_*(G_{|S}) \xrightarrow{d=0} L^{-1} \otimes \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})\,\})\,. \end{aligned}$$

The above equality in the derived category means that we can approach the cohomology groups $\mathrm{Ext}^i_X(i_*L, \sigma_*G)$ on the left hand side from a double complex of global sections of an acyclic resolution of $\{\, L^{-1} \otimes \tilde\sigma_*(G_{|S}) \xrightarrow{d=0} L^{-1} \otimes \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})\,\}$. We have

$$\begin{aligned} E_2^{p,0} &= \mathrm{Ext}^p_C(L, \tilde\sigma_*(G_{|S})) \\ E_2^{p,1} &= \mathrm{Ext}^p_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})) \\ E_2^{p,q} &= 0\,, \qquad \text{for } q > 1\,. \end{aligned}$$

We have the exact sequence of the low terms

$$0 \to E_2^{1,0} \to H^1(M) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2(M)\,.$$

Moreover, the spectral sequence is of "spherical fiber" type, that is, one has $E_2^{p,q} = 0$ for every p and $q \neq 0, 1$[7]. It is a standard fact (see for instance [37] paragraph 4.6), that we can complete the above sequence to get

$$\begin{aligned} 0 \to E_2^{1,0} \to H^1(M) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to H^2(M) \to \\ \to E_2^{1,1} \xrightarrow{d_2} E_2^{3,0} \to H^3(M) \to E_2^{2,1} \xrightarrow{d_2} E_2^{4,0}. \end{aligned}$$

Moreover, since $d = 0$ the first differential of this double complex is zero and then d_2 is zero as well. This implies that the above exact sequence breaks into short exact sequences. We then have

$$\begin{aligned} 0 \to \mathrm{Ext}^1_C(L, \tilde\sigma_*(G_{|S})) \to \mathrm{Ext}^1_X(i_*L, \sigma_*(G)) \to \\ \mathrm{Hom}_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})) \to 0 \end{aligned}$$

and isomorphisms

$$\begin{aligned} \mathrm{Ext}^2_X(i_*L, \sigma_*(G)) \simeq \mathrm{Ext}^1_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})), \\ \mathrm{Ext}^3_X(i_*L, \sigma_*(G)) \simeq \mathrm{Ext}^2_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})) \end{aligned}$$

due to the fact that $\mathrm{Ext}^i_C(L, \tilde\sigma_*(G_{|S})) = H^i(S, L^{-1}_{|S} \otimes G_{|S}) = 0$ for $i \geq 2$. But we have $\mathrm{Ext}^2_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})) = H^2(S, L^{-1}_{|S} \otimes G_{|S} \otimes (N_{X/C})_{|S}) = 0$ as well, so that

$$\mathrm{Ext}^3_X(i_*L, \sigma_*(G)) = 0. \tag{4.13}$$

On the other hand

$$\mathrm{Hom}_X(i_*L, \sigma_*(G)) = \mathrm{Hom}_B(\sigma^*(i_*L), G) = 0 \tag{4.14}$$

because $\sigma^*(i_*L)$ is concentrated on S and G is a vector bundle. In the following we set

$$\mathcal{F} = L^{-1}_{|S} \otimes G_{|S} \otimes (N_{X/C})_{|S}\,.$$

We have

$$\begin{aligned} \mathrm{Hom}_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})) = H^0(S, \mathcal{F}) \\ \mathrm{Ext}^1_C(L, \tilde\sigma_*(G_{|S} \otimes (N_{X/C})_{|S})) = H^1(S, \mathcal{F}) \end{aligned}$$

[7]Spectral sequences with $E_2^{p,q} = 0$ for every p and $q \neq 0, m$ are called of "spherical fiber" type, because when one has a bundle $Z \to Y$ on m-spheres, the Leray spectral sequence approaching the cohomology of Z in terms of the cohomology of X is of that kind.

where $N_{C/S}$ is the normal bundle to S in C. We set $\dim H^i(S,\mathcal{F}) = h^i(S,\mathcal{F})$ as usual. From eq. (4.13) and eq. (4.14) we find that the index I_X simplifies to

$$I_X = \sum_{i=1}^{2}(-1)^i \dim \mathrm{Ext}^i_X(i_*L, \sigma_*(G))$$

in agreement with eq. (4.5). Thus we get for the dimensions we were looking for

$$\begin{aligned} \dim \mathrm{Ext}^1_X(i_*L, \sigma_*(G)) &= -I_X + h^1(S,\mathcal{F})\,, \\ \dim \mathrm{Ext}^2_X(i_*L, \sigma_*(G)) &= h^1(S,\mathcal{F})\,. \end{aligned} \tag{4.15}$$

Since we know that the value of the index $I_X = -\frac{1}{2}mc_3(V)$ by eq. (4.6), we have reduced the computation of the dimensions of $H^1(\mathrm{Hom}(V,\pi^*M))$ and $H^1(\mathrm{Hom}(\pi^*M,V))$ to the computation of the dimension of the first cohomology group of the vector bundle $\mathcal{F}$ on the intersection curve S of C and σ. Thus the question remains: can we actually compute $h^1(S,\mathcal{F})$? Let us assume that S is irreducible. Something we can do is to compute another index, namely, we can use the Riemann-Roch theorem on S to compute

$$\chi(S,\mathcal{F}) = h^0(S,\mathcal{F}) - h^1(S,\mathcal{F})\,.$$

When S is smooth this index can be computed as $\int_S ch(\mathcal{F})\,\mathrm{Td}(S)$. As we are working on S, the computation is reduced to the evaluation of the first Chern-class of $\mathcal{F}$. The details of this computation are given in appendix C, the result is

$$c_1(\mathcal{F}) = \frac{1}{2}m(3C\sigma^2 + C^2\sigma) + \frac{1}{2}mc_3(V)\,.$$

Since $S = C\cdot\sigma$ and we are assuming that C is a Cartier divisor in X, it follows that S is a Cartier divisor in B so that it is a Gorenstein curve. This means that it has a canonical divisor K_S with all properties that the usual canonical divisor for a smooth curve has. In particular, we have both the Riemann-Roch theorem and Serre duality for S. The Riemann-Roch theorem for the curve S gives

$$h^1(S,\mathcal{F}) = h^0(S,\mathcal{F}) - mC\sigma^2 - \frac{1}{2}mc_3(V)$$

which reduces the problem either to compute the number of sections of $\mathcal{F}$ or, by Serre duality, to compute the number of sections of

$$\mathcal{F}^\vee = \mathcal{F}^*\otimes K_S = L_{|S}\otimes M^{-1}_{|S}\,.$$

In the next section we will give some vanishing arguments such that $\mathcal{F}^\vee$ has no sections.

4.6 Applying a Vanishing Theorem

We are always assuming in this subsection that our curve S has arithmetic genus $g(S) = h^1(S, \mathcal{O}_S) > 1$. When S is smooth, the genus of S is determined from the known formula $e(S) = 2-2g(S)$ where $e(S)$ denotes the topological Euler characteristic of S. To compute $g(S)$ one considers first the canonical bundles of C and B in X

$$K_C = K_{X|C} + N_{X/C} = N_{X/C}$$
$$K_B = K_{X|B} + N_{X/B} = N_{X/B}$$

where $N_{X/B}$ respectively $N_{X/C}$ denotes the normal bundles of C and B in X. One then considers the canonical divisor $K_S = K_{C|S} + N_{C/S}$ or equivalently $K_S = K_{B|S} + N_{B/S}$ with $N_{B/S} = C^2\sigma$ and $N_{C/S} = C\sigma^2$ and finds

$$2g(S) - 2 = C^2\sigma + C\sigma^2 \tag{4.16}$$

In general, a vector bundle on a projective variety has no sections if it is stable of negative degree. As we are working on a curve S, to prove that a stable vector bundle $\bar{\mathcal{F}}$ of rank greater than 1 on S has no sections, we need $c_1(\bar{\mathcal{F}}) \leq 0$ because a section gives a trivial subbundle $\mathcal{O} \subset \bar{\mathcal{F}}$.[8]

In some cases we can have generically the vanishing of the sections without having negative first Chern class. This happens, always for stable $\bar{\mathcal{F}}$, when one has $\chi(S, \bar{\mathcal{F}}) \leq 0$, because those sheaves which have sections define the so called Θ-divisor in the moduli space of such sheaves; this means that we can deform $\bar{\mathcal{F}}$ to a stable sheaf with no sections. But in our situation, we will actually find that $\bar{\mathcal{F}}$ is stable and $\chi(S, \bar{\mathcal{F}}) > 0$, and then the above genericity argument does not apply; we need $c_1(\bar{\mathcal{F}}) \leq 0$ to ensure that $\bar{\mathcal{F}}$ has no sections.

We want to prove that $\mathcal{F}^\vee$ has no sections. Before considering the stability question, we just make sure that $c_1(\mathcal{F}^\vee) \leq 0$, which together with stability gives the desired vanishing theorem.

First, let us recall that due to eq. (4.16), the condition $g(S) > 1$ is equivalent to

$$C^2\sigma + C\sigma^2 > 0\,.$$

We have

$$mC^2\sigma \leq \frac{1}{2} m c_3(V) \tag{4.17}$$

[8]For a line bundle L, conditions $c_1(L) \leq 0$ implies that L has no sections except if L is trivial

since $\frac{1}{2}mc_3(V) - mC^2\sigma = h^0(S, L_{|S}^{-1} \otimes G_{|S}) \geq 0$ (see appendix D). Then

$$c_1(\mathcal{F}^\vee) = mc_1(L_{|S}) = \frac{1}{2}m(C^2\sigma - C\sigma^2) - \frac{1}{2}mc_3(V) < mC^2\sigma - \frac{1}{2}mc_3(V) \leq 0 \tag{4.18}$$

as claimed (the expression for $c_1(L_{|S})$ is given in the appendix C).

We are now going to see that we can select our bundle M on B in such a way that the restriction of M to S is stable[9]. Since the dual of a stable vector bundle is stable and twisting a vector bundle by a line bundle does not affect the stability. The sheaf $\mathcal{F}^\vee$ will be stable as well. For this we recall that on a curve of $g > 1$ the general deformation of a vector bundle is stable [38]. Thus we need to make sure that $M_{|S}$, as object in the local moduli space $\mathrm{Def}(M_{|S})$ of bundles on S, can be deformed in arbitrary directions in its moduli space if we deform M in its local moduli space $\mathrm{Def}(M)$. Then we want that the restriction map $\tau\colon \mathrm{Def}(M_{|S}) \to \mathrm{Def}(M)$ be surjective (or more technically, that the map defined between the local deformation functors be surjective). Let us consider the exact sequence:

$$0 \to \mathrm{ad}\, M \otimes \mathcal{O}(-S) \to \mathrm{ad}\, M \to \mathrm{ad}\, M_{|S} \to 0$$

where $\mathrm{ad}\, M$ are the traceless endomorphisms of M. This gives rise to a long exact sequence

$$\begin{aligned} \to H^1(\mathrm{ad}\, M) \xrightarrow{d\tau} H^1(\mathrm{ad}\, M_{|S}) \to & H^2(\mathrm{ad}\, M \otimes \mathcal{O}_B(-S)) \to \\ & H^2(\mathrm{ad}\, M \otimes \mathcal{O}_B(-S)) \to H^2(\mathrm{ad}\, M) \to 0\,. \end{aligned}$$

Thus if $H^2(\mathrm{ad}\, M \otimes \mathcal{O}_B(-S)) = 0$ then $H^2(\mathrm{ad}\, M) = 0$ so that τ is surjective and deformations of M give a general deformation of $M_{|S}$ [38]. Serre duality on S gives $H^2(\mathrm{ad}\, M \otimes \mathcal{O}(-S)) = H^0(K_B \otimes \mathrm{ad}\, M \otimes \mathcal{O}(S))$, and then the above condition transforms to

$$H^0(K_B \otimes \mathrm{ad}\, M \otimes \mathcal{O}(S)) = 0\,. \tag{4.19}$$

Let us note, whenever eq. (4.19) is satisfied, we can deform M so that $M_{|S}$ is stable and thus $\mathcal{F}^\vee$ is stable as well. Since we already know by eq. (4.18) that $c_1(\mathcal{F}^\vee) < 0$, we get $h^0(S, \mathcal{F}^\vee) = 0$. By Serre duality,

$$\begin{aligned} h^1(S, \mathcal{F}) &= h^0(S, \mathcal{F}^\vee) = 0 \\ h^0(S, \mathcal{F}) &= \chi(S, \mathcal{F}) = mC\sigma^2 + \frac{1}{2}mc_3(V) \end{aligned}$$

[9]We would like to thank G. Hein for helpful discussions!

and then we can compute directly, via the Parseval equality eq. (4.9), equation eq. (4.15) and the computation eq. (4.9) of the index I_X, the number of moduli we were looking for:

$$\begin{aligned} h^1(\mathrm{Hom}(V, \pi^*M)) &= \dim \mathrm{Ext}^1_X(V, \pi^*M) = -I_X + h^1(S, \mathcal{F}) = \frac{1}{2} m c_3(V) \\ h^1(\mathrm{Hom}(\pi^*M, V)) &= \dim \mathrm{Ext}^2_X(V, \pi^*M) = h^1(S, \mathcal{F}) = 0 \,. \end{aligned} \tag{4.20}$$

Conditions for the Vanishing of $H^0(K_B \otimes \mathrm{ad}\, M \otimes \mathcal{O}(S))$

We will consider two situations where eq. (4.19) is true and we can apply the vanishing theorem to get the number of moduli eq. (4.20).

Case 1: Conditions on the Curve

Since $\mathrm{ad}\, M$ is stable with $c_1(\mathrm{ad}\, M) = 0$, we get $H^0(K_B \otimes \mathrm{ad}\, M \otimes \mathcal{O}(S)) = 0$ if

$$\deg(K_B \otimes \mathcal{O}(S)) \leq 0$$

is satisfied for an arbitrary ample H in $H^2(B)$.

Case 2: Conditions on the Bundle V

By Theorem 40 of [38], there is a constant k_0 (depending on B, the polarization considered in B and the curve S), such that for $c_2(M) = k \geq k_0$ the vanishing equation eq. (4.19) is true.[10] We are not completely free to choose k, because k is related to V since is the number of vertical branes we wanted to remove. That is, we are constrained to have

$$k \leq a_F$$

where a_F is the number of fibers contained in the class $[W] = c_2(TX) - c_2(V)$ (see also [18]). If we write $c_2(V) = \sigma\pi^*(\eta) + \pi^*(\omega)$, we have

$$a_F = \int c_2(B) - c_1(B)^2 - \omega$$

Since the base surface B is fixed, the Chern classes $c_i(B)$ are fixed as well, and we see that a_F just depends on the choice of ω.

[10] Theorem 41 of [38] can be applied as well to see directly that M can be deformed to have $M_{|S}$ stable on S

If we write $a = -\int_B \omega$, using (2.32) of [27], we have

$$a = -\frac{1}{6} n c_1(B)^2 + ch_3(i_* L)$$

(because $\Phi^1(V) = i_* L$), and Grothendieck Riemann-Roch gives

$$a = -\frac{1}{6} n c_1(B)^2 + \frac{1}{2} c_1(L)(c_1(L) - i^* C) + \frac{1}{12}(i^* C)^2$$

where the intersections are made inside C.

We then proceed in this way: we fix the spectral cover C, so that $S = C\sigma$ is an irreducible curve of arithmetic genus $g(S) > 1$. Then we take the corresponding aforementioned constant k_0, and an arbitrary $k \geq k_0$. We can then take a line bundle L on C so that $c_1(L)(c_1(L) - i^* C)$ is big enough to have $k \leq a_F$. This can be done as follows: take L' very ample and $q \gg 0$ in such a way that $q c_1(L') - i^* C$ is a very ample divisor; if $L = (L')^{\otimes q}$, then $c_1(L)(c_1(L) - i^* C)$ grows as q^2 so that for $q \gg 0$ L fulfills our requirements.

As $V = \hat{\Phi}^0(i_* L)$ is a vector bundle on X with spectral cover C and in this situation we can take a stable bundle M on B with $rk(M) = m$, $c_1(M) = 0$ and $c_2(M)$ so that $M_{|S}$ is stable and we have the vanishing theorem and the formulas eq. (4.20) for the number of moduli.

5 Comments on Localized Chiral Matter

In this section we will analyze the question: can we determine not only the net amount of chiral matter but also the matter multiplets individually? Let us first motivate this question from various perspectives.

As it is well known, the net amount of chiral matter is determined in heterotic string compactification on Calabi-Yau threefolds by $\frac{1}{2} c_3(V)$. This follows from the fact that chiral fermions in four dimensions are related to zero modes of the Dirac operator on X. The index can be written as (denoting as usual $h^i(X, V) = \dim H^i(X, V)$)

$$\operatorname{index}(D_V) = \sum_{i=0}^{3} (-1)^i h^i(X, V) = \int_X ch(V)\, \mathrm{Td}(X)$$

and for a stable bundle V with $c_1(V) = 0$ we have $h^0(X, V) = h^3(X, V) = 0$, then

$$-(h^1(X, V) - h^2(X, V)) = \frac{1}{2} c_3(V) \,.$$

For V an $SU(n)$ vector bundle the corresponding unbroken space-time gauge group is the maximal subgroup of E_8 which commutes with $SU(n)$. For instance taking V a general $SU(3)$ bundle the unbroken observable gauge group is E_6. The only charged ten-dimensional fermions are in the adjoint representation of E_8 thus we get four-dimensional fermions only from the reduction of the adjoint representation. In particular one has under $E_6 \times SU(3)$

$$\mathbf{248} = (\mathbf{78}, \mathbf{1}) + (\mathbf{27}, \mathbf{3}) + (\overline{\mathbf{27}}, \overline{\mathbf{3}}) + (\mathbf{1}, \mathbf{8})\,.$$

Therefore fermions that are in the $\mathbf{27}$ of E_6 are in the $\mathbf{3}$ of $SU(3)$ thus in the index the left handed $\mathbf{27}$'s can be assigned[11] to elements of the space $H^2(X, V) = H^1(X, V^*)$ and the left handed $\overline{\mathbf{27}}$'s would be assigned to elements of $H^1(X, V)$. In case of V being the tangent bundle TX one simply has $h^1(X, TX) = h^{1,2}(X)$ and $h^2(X, TX) = h^{1,1}(X)$. In addition one can analyze [32] the corresponding Yukawa couplings taking into account the number of tangent bundle moduli $h^1(X, \mathrm{End}(TX))$ associated to E_6 singlets. The resulting Yukawa couplings are: $\mathbf{27}^3$, $\overline{\mathbf{27}}^3$, $\mathbf{27} \cdot \overline{\mathbf{27}} \cdot \mathbf{1}$, $\mathbf{1}^3$. Now assuming that $h^1(X, \mathrm{End}(V)) = 0$ one would expect the $\mathbf{27}^3$, $\overline{\mathbf{27}}^3$ terms only [32]. Similarly if $H^1(X, V)$ or $H^1(X, V^*)$ would vanish one would expect the vanishing of the corresponding couplings.

We will now proceed as in section 4 to compute each of the groups[12] $H^i(X, V)$ and their dimensions, just by taking the bundle M as the trivial line bundle $\mathcal{O}_B$. We have

$$H^i(X, V) = \mathrm{Ext}^i_X(\mathcal{O}_X, V) = \mathrm{Ext}^{3-i}_X(V, \mathcal{O}_X) \tag{5.1}$$

so that the index $\mathcal{I} = \sum_{i=0}^3 (-1)^i \dim \mathrm{Ext}^i(V, \mathcal{O}_X)$ fulfills

$$\mathcal{I} = -\,\mathrm{index}(D_V) = -\frac{1}{2} c_3(V)$$

We also have a Parseval equation like eq. (4.9)

$$\mathrm{Ext}^i_X(V, \mathcal{O}_X) = \mathrm{Ext}^i_X(i_* L, \sigma_*(K_B)) \tag{5.2}$$

and the new equation that corresponds to eq. (4.15) is

$$\begin{aligned} \dim \mathrm{Ext}^1_X(i_* L, \sigma_*(K_B)) &= -\mathcal{I} + h^1(S, \mathcal{F}') \\ \dim \mathrm{Ext}^2_X(i_* L, \sigma_*(K_B)) &= h^1(S, \mathcal{F}') \end{aligned} \tag{5.3}$$

[11] Note the assignment is a matter of convention. In case of $V = TX$ one typically assigns the $\mathbf{27}$'s to elements of $H^1(X, V)$ as the Euler characteristic in typical examples turns out to be negative.

[12] The localization of $H^i(X, V)$ was originally suggested in [1] and worked out precisely in [3] using the Leray spectral sequence. We give here an alternative approach and extend the discussion by giving a vanishing argument similar to the one discussed in the previous section.

with $\mathcal{F}' = L_{|S}^{-1} \otimes K_S$. By Serre duality on S, $h^1(S, \mathcal{F}') = h^0(S, L_{|S})$; then, eq. (5.1) and eq. (5.3) lead to

$$h^2(X,V) = \dim \mathrm{Ext}^1_X(V, \mathcal{O}_X) = \dim \mathrm{Ext}^1_X(i_* L, \sigma_*(K_B)) = -\mathcal{I} + h^0(S, L_{|S})$$
$$h^1(X,V) = \dim \mathrm{Ext}^2_X(V, \mathcal{O}_X) = \dim \mathrm{Ext}^2_X(i_* L, \sigma_*(K_B)) = h^0(S, L_{|S}) \,.$$

We have, by Appendix C,

$$c_1(L_{|S}) = \frac{1}{2}(-\sigma^2 C + \sigma C^2) - \frac{1}{2} c_3(V) = 1 - g(S) + \sigma C^2 - \frac{1}{2} c_3(V) \,.$$

If we assume that $g(S) > 1$, and since $C^2\sigma - \frac{1}{2}c_3(V) \leq 0$ by eq. (4.17), we have

$$c_1(L_{|S}) < 0 \,.$$

It follows that $H^0(S, L_{|S}) = 0$. Thus we see that

$$h^2(X,V) = -\mathcal{I} = \mathrm{index}(D_V) = \frac{1}{2} c_3(V)$$
$$h^1(X,V) = 0 \,.$$

We conclude that chiral matter associated to $H^1(X,V)$ vanishes in heterotic string compactifications on elliptically fibered X with vector bundles constructed in the spectral cover approach and matter localized along the curve S of genus $g(S) > 1$. This implies the vanishing of the corresponding Yukawa couplings. For example, we would expect for $V = SU(3)$ the Yukawa couplings: $\mathbf{27}^3$ and $\mathbf{1}^3$.

A Hilbert Polynomial Coefficients

In this appendix we provide the coefficients required for section 2.3. The coefficients of $P(\mathcal{F}, m)$ are given by

$$\alpha_0(\mathcal{F}) = \mathrm{ch}_3(\mathcal{F}) + \mathrm{ch}_1(\mathcal{F}) \frac{c_2(TX)}{12}$$
$$\alpha_1(\mathcal{F}) = \mathrm{ch}_2(\mathcal{F}) \tilde{H} + \mathrm{ch}_0(\mathcal{F}) \frac{c_2(TX)}{12} \tilde{H}$$
$$\alpha_2(\mathcal{F}) = \mathrm{ch}_1(\mathcal{F}) \tilde{H}^2$$
$$\alpha_3(\mathcal{F}) = \mathrm{ch}_0(\mathcal{F}) \tilde{H}^3 \,.$$

The coefficients of $P(\hat{\Phi}(V), m)$ are given by

$$\alpha_1(\hat{\Phi}(V)) = ((\frac{1}{2} n c_1 + \tilde{S})\sigma - (s + \frac{1}{2}\eta c_1 \sigma) F) \tilde{H}$$
$$\alpha_2(\hat{\Phi}(V)) = (n\sigma - \eta) \tilde{H}^2 \,.$$

The coefficients of $P(\hat{\Phi}(V'), m)$ are given by

$$\begin{aligned}\alpha_1(\hat{\Phi}(V')) &= ((\frac{1}{2}n'c_1 + \tilde{S}')\sigma - (s' + \frac{1}{2}\eta' c_1\sigma)F)\tilde{H}\\ \alpha_2(\hat{\Phi}(V')) &= (n'\sigma - \eta')\tilde{H}^2\,.\end{aligned}$$

B A Simple Test

We want to compute the index $I_X = \sum_{i=0}^{3}(-1)^i \dim \mathrm{Ext}_X^i(i_*L, \sigma_*(G))$. The Riemann-Roch theorem gives

$$\begin{aligned}I_X &= \int_X \mathrm{ch}(i_*L^{-1})\,\mathrm{ch}(\sigma_*(G))\,\mathrm{Td}(X)\\ &= \int_X \mathrm{ch}_1(i_*L^{-1})\,\mathrm{ch}_2(\sigma_*(G)) + \mathrm{ch}_1(\sigma_*(G))\,\mathrm{ch}_2(i_*L^{-1})\end{aligned} \tag{B.1}$$

The relevant Chern characters of i_*L are given by [27]

$$\begin{aligned}\mathrm{ch}_1(i_*L^{-1}) &= n\sigma + \eta\\ \mathrm{ch}_2(i_*L^{-1}) &= \frac{1}{2}nc_1(B)\sigma - (\mathrm{ch}_3(V) - \frac{1}{2}\eta c_1(B)\sigma)\cdot F\end{aligned}$$

where F denotes the class of the fiber of $\pi\colon X \to B$. Further we have to obtain the relevant Chern characters of $\sigma_*(G)$. We can compute these using the Grothendieck-Riemann-Roch theorem for $\sigma\colon B \to X$, which is $\mathrm{ch}(\sigma_*G)\,\mathrm{Td}(X) = \sigma_*(\mathrm{ch}(G)\,\mathrm{Td}(B))$ giving

$$\begin{aligned}\mathrm{ch}_1(\sigma_*G) &= m\sigma\\ \mathrm{ch}_2(\sigma_*G) &= -\frac{mc_1\sigma}{2}\end{aligned}$$

Inserting these expressions into eq. (B.1) we get

$$I_X = -\frac{1}{2}mc_3(V)$$

as we expected in eq. (4.6). This gives us a simple test that our reduction to the spectral data leads to the same result as it should be.

C Computation of $c_1(\mathcal{F})$

The first step is to provide some additional information about the restriction of the spectral line bundle L to the intersection curve S! Following [1] we have

$$c_1(L) = \frac{1}{2}(c_1(B) - c_1(C)) + \gamma\,.$$

Let us restrict to $S = C\sigma$! As C and B are both divisors in X it follows from adjunction formula [39] that

$$\begin{aligned} c_1(C)_{|S} &= -C^2\sigma \\ c_1(B)_{|S} &= -C\sigma^2 \end{aligned}$$

$$c_1(L_{|S}) = \frac{1}{2}(-\sigma^2 C + \sigma C^2) + \gamma_{|S} \tag{C.1}$$

As mentioned before the condition $c_1(V) = 0$ translates to fixing of $\pi_* c_1(L)$ in $H^{1,1}(C)$ up to a class in $\ker \pi_* \colon H^{1,1}(C) \to H^{1,1}(B)$ which is $\gamma = \lambda(n\sigma - \eta + nc_1)$ as discussed in [1], so we get (note that λ must be half-integral)

$$\gamma_{|S} = -\lambda\eta C\sigma\,.$$

In order to better understand eq. (C.1) we will use a slightly different perspective by computing $i_*(c_1(L))$ in terms of the Chern classes of V using for instance (2.33) of [27] taking into account that $i_* L = \hat{\Phi}^1(V)$. Let us write

$$\mathrm{ch}_0(V) = n\,, \quad \mathrm{ch}_1(V) = 0\,, \quad \mathrm{ch}_2(V) = -\eta\sigma + aF$$

with $\eta \in p^* H^2(B)$; we put the minus sign in $\mathrm{ch}_2(V)$ so that $C = \mathrm{ch}_1(i_* L) = n\sigma + \eta$. Then,

$$\mathrm{ch}_2(i_* L) = \frac{1}{2} n c_1(B)\sigma - (ch_3(V) - \frac{1}{2}\eta c_1(B)\sigma) \cdot F$$

and Grothendieck Riemann-Roch theorem gives

$$\mathrm{ch}_2(i_* L) = i_*(c_1(L) - \frac{1}{2} c_1(N_{X/C})) = i_*(c_1(L)) - \frac{1}{2} C^2$$

so that

$$\begin{aligned} i_*(c_1(L))\sigma &= -\frac{1}{2} n c_1(B)^2\sigma + \frac{1}{2} c_1(B)\eta\sigma + \frac{1}{2} C^2\sigma - \mathrm{ch}_3(V) \\ &= \frac{(\;^{2}}{C}\sigma - C\sigma^2) - \frac{1}{2} c_3(V) \end{aligned}$$

Now, we want to compute $c_1(L_{|S}) \in H^2(S)$; if we understood this class as a number, this is the intersection number of the class $c_1(L) \in H^2(C)$ with the class of S in C. Since $S = C \cdot \sigma$, we can simply compute the intersection number in X, thus obtaining $c_1(L_{|S}) = i_*(c_1(L))\sigma$ as numbers. It follows that $c_3(V)/2 = \lambda\eta(\eta - nc_1)\sigma$ which is agreement with results in [24] and [3]. So we obtain

$$c_1(L_{|S}) = \frac{1}{2}(-\sigma^2 C + \sigma C^2) - \frac{1}{2} c_3(V)\,. \tag{C.2}$$

We recall that $c_1(M) = 0$ and $K_C = K_{X|C} + N_{X/C} = N_{X/C}$ such that $K_{C|S} = (N_{X/C})_{|S}$ and we find with eq. (C.2)

$$c_1(\mathcal{F}) = \frac{1}{2} m(3C\sigma^2 + C^2\sigma) + \frac{1}{2} m c_3(V)\,.$$

D Index and Sections

In this appendix we will show that

$$\begin{aligned} h^0(S, L_{|S}^{-1} \otimes G_{|S}) &= \frac{1}{2} m c_3(V) - mC^2\sigma \\ h^1(S, L_{|S}^{-1} \otimes G_{|S}) &= -mC\sigma^2 \end{aligned} \tag{D.1}$$

Therefore let us consider the index

$$I_X = -\dim Ext^1_C(L, \tilde{\sigma}_*(G_{|S})) - \chi(S, \mathcal{F})$$

and recall that $\chi(S, \mathcal{F}) = mC\sigma^2 + \frac{1}{2} m c_3(V)$ and we can write

$$-\dim \mathrm{Ext}^1_C(L, \tilde{\sigma}_*(G_{|S})) = \chi(S, L_{|S}^{-1} \otimes G_{|S}) - h^0(S, L_{|S}^{-1} \otimes G_{|S}).$$

Applying the Riemann-Roch theorem, we compute

$$\chi(S, L_{|S}^{-1} \otimes G_{|S}) = m(C\sigma^2 - C^2\sigma) + \frac{1}{2} m c_3(V)$$

using the fact that

$$c_1(L_{|S}^{-1} \otimes G_{|S}) = \frac{1}{2} m(3C\sigma^2 - C^2\sigma) + \frac{1}{2} m c_3(V)\,.$$

Thus we find eq. (D.1) from

$$h^0(S, L_{|S}^{-1} \otimes G_{|S}) = \chi(S, L_{|S}^{-1} \otimes G_{|S}) - \chi(S, \mathcal{F}) - I_X.$$

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