Appendix : Proof of Căldăraru's conjecture

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In this short note we show how to combine Yoshioka's recent results on moduli spaces of twisted sheaves on K3 surfaces with more or less standard methods to prove Căldăraru's conjecture on the equivalence of twisted derived categories of projective K3 surfaces. More precisely, we shall show

Theorem 0.1. *Let X and X' be two projective K3 surfaces endowed with B-fields $B \in H^2(X, \mathbb{Q})$ respectively $B' \in H^2(X', \mathbb{Q})$. Suppose there exists a Hodge isometry*

$$g : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$$

that preserves the natural orientation of the four positive directions. Then there exists a Fourier–Mukai equivalence

$$\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$$

such that the induced action $\Phi_*^{B,B'}$ on cohomology equals g.

Here, $\alpha := \alpha_B$ and $\alpha' := \alpha_{B'}$ are the Brauer classes induced by B respectively B'.

The twisted Hodge structures and the cohomological Fourier-Mukai transform (based on the notion of twisted Chern character), indispensable for the formulation of the conjecture, were introduced in [4]. For a complete discussion of the natural orientation of the positive directions and the cohomological Fourier–Mukai transform $\Phi_*^{B,B'}$ we also refer to [4]. Note that Căldăraru's conjecture was originally formulated purely in terms of the transcendental lattice. But, as has been explained in [4], in the twisted case passing from the transcendental part to the full cohomology is not always possible, so that the original formulation had to be changed slightly to the above one.

Also note that any Fourier–Mukai equivalence

$$\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha')$$

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induces a Hodge isometry as above, but for the time being we cannot prove that this Hodge isometry also preserves the natural orientation. In the untwisted case this is harmless, for a given orientation reversing Hodge isometry can always be turned into an orientation preserving one by composing with $-\mathrm{id}_{H^2}$. In the twisted setting this cannot always be guaranteed, so that we cannot yet exclude the case of Fourier–Mukai equivalent twisted K3 surfaces (X, α_B) and $(X', \alpha_{B'})$ which only admit an orientation reversing Hodge isometry $\widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$. Of course, this is related to the question whether any Fourier–Mukai equivalence is orientation preserving which seems to be a difficult question even in the untwisted case (see [3, 11]).

From Yoshioka's paper [12] we shall use the following

Theorem 0.2. (Yoshioka) *Let X be a K3 surface with a rational B-field $B \in H^2(X, \mathbb{Q})$ and $v \in \widetilde{H}^{1,1}(X, B, \mathbb{Z})$ a primitive vector with $\langle v, v\rangle = 0$. Then there exists a moduli space $M(v)$ of stable (with respect to a generic polarizations) α_B-twisted sheaves E with $\mathrm{ch}^B(E)\sqrt{td(X)} = v$ such that:*

i) *Either $M(v)$ is empty or a K3 surface. The latter holds true if the degree zero part of v is positive.*

ii) *On $X' := M(v)$ one finds a B-field $B' \in H^2(X', \mathbb{Q})$ such that there exists a universal family $\mathcal{E}$ on $X \times X'$ which is an $\alpha_B^{-1} \boxtimes \alpha_{B'}$-twisted sheaf.*

iii) *The twisted sheaf $\mathcal{E}$ induces a Fourier–Mukai equivalence*

$$\mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(X', \alpha_{B'}).$$

The existence of the moduli space of semistable twisted sheaves has been proved by Yoshioka for arbitrary projective varieties. Instead of considering twisted sheaves, he works with coherent sheaves on a Brauer–Severi variety. Using the equivalence between twisted sheaves and modules over Azumaya algebras, one can in fact view these moduli spaces also as a special case of Simpson's general construction [10]. (The two stability conditions are indeed equivalent.) In his thesis [5] M. Lieblich considers similar moduli spaces. (See also [2] for the rank one case.)

The crucial part for the application to Căldăraru's conjecture is i), in particular the non-emptiness. Yoshioka follows Mukai's approach, which also yields ii). Part iii) is a rather formal consequence of the usual criteria for the equivalence of Fourier–Mukai transforms already applied to the twisted case in [1].

In the last section we provide a dictionary between the different versions of twisted Chern characters and the various notions of twisted sheaves. Only parts of it is actually used in the proof of the conjecture.

The rest is meant to complement [4] and to facilitate the comparison of [4], [5], and [12].

Acknowledgements: It should be clear that the lion's share of the proof of Căldăraru's conjecture in the above form is in fact contained in K. Yoshioka's paper. We are grateful to him for informing us about his work and comments on the various versions of this note. Thanks also to M. Lieblich who elucidated the relation between the different ways of constructing moduli spaces of twisted sheaves. Our proof follows the arguments in the untwisted case, due to S. Mukai [6] and D. Orlov [8] (see also [3, 7]), although some modifications were necessary. During the preparation of this paper the second named author was partially supported by the MIUR of the Italian government in the framework of the National Research Project "Algebraic Varieties" (Cofin 2002).

§1. Examples

Let X and X' be projective K3 surfaces (always over $\mathbb{C}$) with B-fields $B \in H^2(X,\mathbb{Q})$ respectively $B' \in H^2(X',\mathbb{Q})$. We denote the induced Brauer classes by $\alpha := \alpha_B := \exp(B^{0,2}) \in H^2(X,\mathcal{O}_X^*)$ respectively $\alpha' := \alpha_{B'} \in H^2(X',\mathcal{O}_{X'}^*)$. We start out with introducing a few examples of equivalences between the bounded derived categories $\mathrm{D^b}(X,\alpha)$ respectively $\mathrm{D^b}(X',\alpha')$ of the abelian categories of α-twisted (resp. α'-twisted) sheaves.

i) Let $f : X \cong X'$ be an automorphism with $f^*\alpha' = \alpha$. Then $\Phi := f_* : \mathrm{D^b}(X,\alpha) \to \mathrm{D^b}(X',\alpha'), E \mapsto Rf_*E$ is a Fourier–Mukai equivalence with kernel $\mathcal{O}_{\Gamma_f}$ viewed as an $\alpha^{-1} \boxtimes \alpha'$-twisted sheaf on $X \times X'$.

If in addition $f_*(B) = B'$ then $\Phi_*^{B,B'} = f_*$.

ii) Let $L \in \mathrm{Pic}(X)$ be a(n untwisted) line bundle on X. Then $E \mapsto L \otimes E$ defines a Fourier–Mukai equivalence $L \otimes (\ ) : \mathrm{D^b}(X,\alpha) \cong \mathrm{D^b}(X,\alpha)$ with kernel i_*L considered as an $\alpha^{-1} \boxtimes \alpha$-twisted sheaf on $X \times X$. Here, $i : X \hookrightarrow X \times X$ denotes the diagonal embedding. The induced cohomological Fourier–Mukai transform $(L \otimes (\ ))_*^{B,B} : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X,B,\mathbb{Z})$ is given by multiplication with $\exp(\mathrm{c}_1(L))$.

iii) Let $b \in H^2(X,\mathbb{Z})$. Then $\alpha_B = \alpha_{B+b}$. The identity

$$\mathrm{id} : \mathrm{D^b}(X,\alpha_B) = \mathrm{D^b}(X,\alpha_{B+b})$$

descends to $\mathrm{id}_*^{B,B+b} : \widetilde{H}(X,B,\mathbb{Z}) \cong \widetilde{H}(X,B+b,\mathbb{Z})$ which is given by the multiplication with $\exp(b)$. This follows from the formula $\mathrm{ch}^{B+b}(E) = \mathrm{ch}^B(E) \cdot \mathrm{ch}^b(\mathcal{O}) = \mathrm{ch}^B(E) \cdot \exp(b)$ (see [4, Prop. 1.2]).

iv) Changing the given B-field B by a class $b \in H^{1,1}(X, \mathbb{Q})$ does not affect $\widetilde{H}(X, B, \mathbb{Z})$. Thus, the identity can be considered as an orientation preserving Hodge isometry $\widetilde{H}(X, B, \mathbb{Z}) = \widetilde{H}(X, B+b, \mathbb{Z})$.

As shall be explained in the last section, this can be lifted to a Fourier–Mukai equivalence. More precisely, there is an exact functor $\Phi : \mathbf{Coh}(X, \alpha_B) \cong \mathbf{Coh}(X, \alpha_{B+b})$, whose derived functor, again denoted by $\Phi : \mathrm{D^b}(X, \alpha_B) \cong \mathrm{D^b}(X, \alpha_{B+b})$, is of Fourier–Mukai type and such that $\Phi_*^{B,B+b} = \mathrm{id}$.

v) Let $E \in \mathrm{D^b}(X, \alpha)$ be a spherical object, i.e. $\mathrm{Ext}^i(E, E) = 0$ for all i except for $i = 0, 2$ when it is of dimension one. Then the twist functor T_E that sends $F \in \mathrm{D^b}(X, \alpha)$ to the cone of $\mathrm{Hom}(E, F) \otimes E \to F$ defines a Fourier–Mukai autoequivalence $T_E : \mathrm{D^b}(X, \alpha) \cong \mathrm{D^b}(X, \alpha)$. The kernel of T_E is given by the cone of the natural map

$$E^* \boxtimes E \longrightarrow \mathcal{O}_\Delta,$$

where $\mathcal{O}_\Delta$ is considered as an $\alpha^{-1} \boxtimes \alpha$-twisted sheaf on $X \times X$. The result in the untwisted case goes back to Seidel and Thomas [9]. The following short proof of this, which carries over to the twisted case, has been communicated to us by D. Ploog [7]. Consider the class $\Omega \subset \mathrm{D^b}(X, \alpha)$ of objects F that are either isomorphic to E or contained in its orthogonal complement $E^\perp$, i.e. $\mathrm{Ext}^i(E, F) = 0$ for all i. It is straightforward to check that this class is spanning. Since $T_E(E) \cong E[-1]$ and $T_E(F) \cong F$ for $F \in E^\perp$, one easily verifies that $\mathrm{Ext}^i(F_1, F_2) = \mathrm{Ext}^i(T_E(F_1), T_E(F_2))$ for all $F_1, F_2 \in \Omega$.

In other words, T_E is fully faithful on the spanning class Ω and hence fully faithful. By the usual argument, the Fourier–Mukai functor T_E is then an equivalence.

As in the untwisted case, one proves that the induced action on cohomology is the reflection $\alpha \mapsto \alpha + \langle \alpha, v^B(E) \rangle \cdot v^B(E)$. Here, $v^B(E)$ is the Mukai vector $v^B(E) := \mathrm{ch}^B(E)\sqrt{\mathrm{td}(X)}$.

Special cases of this construction are:

– Let $\mathbb{P}^1 \cong C \subset X$ be a smooth rational curve. As $H^2(C, \mathcal{O}_C^*)$ is trivial, its structure sheaf $\mathcal{O}_C$ and any twist $\mathcal{O}_C(k)$ can naturally be considered as α-twisted sheaves. The Mukai vector for $k = -1$ is given by $v(\mathcal{O}_C(-1)) = (0, [C], 0)$.

– In the untwisted case, the trivial line bundle $\mathcal{O}$ (and in fact any line bundle) provides an example of a spherical object. Its Mukai vector is $(1, 0, 1)$ and has, in particular, a non-trivial degree zero component. It is the latter property that is of importance for the proof in the untwisted case. So the original argument goes through if at least one spherical object of non-trivial rank can be found. Unfortunately, spherical object (in particular those of positive rank) might not exist at all in the twisted case. In fact, any spherical object E has a Mukai vector $v^B(E) \in \mathrm{Pic}(X, B)$ of square $\langle v^B(E), v^B(E)\rangle = -2$ and it is not difficult to find examples of rational B-fields $B \neq 0$ such that such a vector does not exist.

vi) Let $\ell \in \mathrm{Pic}(X)$ be a nef class with $\langle \ell, \ell\rangle = 0$. If $w = (0, \ell, s)$ is a primitive vector, then the moduli space $M(w)$ of α_B-twisted sheaves which are stable with respect to a generic polarization is non-empty. Indeed, in this case ℓ is a multiple $n \cdot f$ of a fibre class f of an elliptic fibration $\pi : X \to \mathbb{P}^1$. As $\gcd(n, s) = 1$, there exists a stable rank n vector bundle of degree s on a smooth fibre of π which yields a point in $M(w)$.

If ℓ is the fibre class of an elliptic fibration $X \to \mathbb{P}^1$, we can think of $M(w)$ as the relative Jacobian $\mathcal{J}^s(X/\mathbb{P}^1) \to \mathbb{P}^1$.

In any case, $M(w)$ is a K3 surface and the universal twisted sheaf provides an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(M(w), \alpha_{B'}) \cong \mathrm{D}^{\mathrm{b}}(X, \alpha_B)$ (for some B-field B' on $M(w)$) inducing a Hodge isometry $\Phi_*^{B',B} : \widetilde{H}(M(w), B', \mathbb{Z}) \cong \widetilde{H}(X, B, \mathbb{Z})$ that sends $(0,0,1)$ to w.

§2. The proof

Let $g : \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X', B', \mathbb{Z})$ be an orientation preserving Hodge isometry. The Mukai vector of $k(x)$ with $x \in X$ is $v^B(k(x)) = v(k(x)) = (0,0,1)$. We shall denote its image under g by $w := g(0,0,1) = (r, \ell, s)$.

1st step. In the first step we assume that $r = 0$ and $\ell = 0$, i.e. $g(0,0,1) = \pm(0,0,1)$, and that furthermore $g(1,0,0) = \pm(1,0,0)$. By composing with $-\mathrm{id}$ we may actually assume $g(0,0,1) = (0,0,1)$ and $g(1,0,0) = (1,0,0)$.

In particular, g preserves the grading of $\widetilde{H}$ and induces a Hodge isometry $H^2(X, \mathbb{Z}) \cong H^2(X', \mathbb{Z})$. Denote $b := g(B) - B' \in H^2(X, \mathbb{Q})$. As g respects the Hodge structure, it maps $\sigma + B \wedge \sigma$ to $\sigma' + B' \wedge \sigma'$ and, therefore $\langle \sigma, B\rangle = \langle \sigma', B'\rangle$. On the other hand, as g is an isometry, one has $\langle \sigma, B\rangle = \langle \sigma', g(B)\rangle$. Altogether this yields $\langle \sigma', b\rangle = 0$, i.e. $b \in H^{1,1}(X, \mathbb{Q})$.

Now compose g with the orientation preserving Hodge isometry given by the identity $\widetilde{H}(X', B', \mathbb{Z}) = \widetilde{H}(X', g(B) = B' + b, \mathbb{Z})$. As the latter can be lifted to a Fourier–Mukai equivalence $\mathrm{D}^{\mathrm{b}}(X', \alpha_{B'}) \cong \mathrm{D}^{\mathrm{b}}(X', \alpha_{g(B)})$ (see example iv)), it suffices to show that g viewed as a

Hodge isometry $\widetilde{H}(X,B,\mathbb{Z}) = \widetilde{H}(X,g(B),\mathbb{Z})$ can be lifted. So we may from now on assume that $B' = g(B)$.

As g is orientation preserving, its degree two component defines a Hodge isometry that maps the positive cone $\mathcal{C}_X \subset H^{1,1}(X)$ onto the positive cone $\mathcal{C}_{X'} \subset H^{1,1}(X')$.

If g maps an ample class to an ample class, then by the Global Torelli Theorem g can be lifted to an isomorphism $f : X \cong X'$ which in turn yields a Fourier–Mukai equivalence $\Phi := f_* : \mathrm{D^b}(X,\alpha_B) \cong \mathrm{D^b}(X',\alpha_{B'})$. Obviously, with this definition $\Phi_*^{B,B'} = g$ (use $f_*(B) = g(B) = B'$).

If g does not preserve ampleness, then the argument has to be modified as follows: After a finite number of reflections s_{C_i} in hyperplanes orthogonal to (-2)-classes $[C_i]$ we may assume that $s_{C_1}(\dots s_{C_n}(h(a))\dots)$ is an ample class. As the reflections s_{C_i} are induced by the twist functors $T_{\mathcal{O}_{C_i}(-1)} : \mathrm{D^b}(X',\alpha_{B'}) \cong \mathrm{D^b}(X',\alpha_{B'})$ (see the explanations in the last section), the Hodge isometry g is induced by a Fourier–Mukai equivalence if and only if the composition $s_{C_1} \circ \dots s_{C_n} \circ g$ is. Thus, we have reduced the problem to the case already treated above.

In the following steps we shall explain how the general case can be reduced to the case just considered.

2nd step. Suppose $g(0,0,1) = \pm(0,0,1)$ but $g(1,0,0) \neq \pm(1,0,0)$. Again, by composing with $-\mathrm{id}$ we may reduce to $g(0,0,1) = (0,0,1)$ and $g(1,0,0) \neq (1,0,0)$. Then $g(1,0,0)$ is necessarily of the form $\exp(b)$ for some $b \in H^2(X',\mathbb{Z})$. Hence, we may compose g with the Hodge isometry $\exp(-b) : \widetilde{H}(X',B',\mathbb{Z}) \cong \widetilde{H}(X',B'-b,\mathbb{Z})$ (that preserves the orientation) which can be lifted to a Fourier–Mukai equivalence according to example iii). This reduces the problem to the situation studied in the previous step.

3rd step. Suppose that $r > 0$. Using Theorem 0.2 one finds a K3 surface X_0 with a B-field $B_0 \in H^2(X_0,\mathbb{Q})$ such that over $X_0 \times X'$ there exists a universal $\alpha_{B_0}^{-1} \boxtimes \alpha_{B'}$-twisted sheaf parametrizing stable α'-twisted sheaves on X' with Mukai vector $v^{B'} = w$. In particular, $\mathcal{E}$ induces an equivalence $\Phi_{\mathcal{E}} : \mathrm{D^b}(X_0,\alpha_{B_0}) \cong \mathrm{D^b}(X',\alpha_{B'})$ and $\Phi_{\mathcal{E}*}^{B_0,B'}(0,0,1) = w$.

Thus, the composition $g_0 := (\Phi_{\mathcal{E}*}^{B_0,B'})^{-1} \circ g$ yields an orientation preserving (!) Hodge isometry $\widetilde{H}(X,B,\mathbb{Z}) \cong H(X_0,B_0,\mathbb{Z})$. (The proof that the universal family of stable sheaves induces an orientation preserving Hodge isometry is analogous to the untwisted case. This seems to be widely known [3, 11]. For an explicit proof see [4].) Clearly, g can be lifted to a Fourier–Mukai equivalence if and only if g_0 can. The latter follows from step one.

4th step. Suppose g is given with $r < 0$. Then compose with the orientation preserving Hodge isometry $-\mathrm{id}$ of $\widetilde{H}(X', B', \mathbb{Z})$ which is lifted to the shift functor $E \mapsto E[1]$. Thus, it is enough to lift the composition $-\mathrm{id} \circ g$ which can be achieved according to step three.

5th step The remaining case is $r = 0$ and $\ell \neq 0$. One applies the construction of example vi) in Section 1 and proceeds as in step 3. The class ℓ can be made nef by applying $-\mathrm{id}$ if necessary to make it effective (i.e. contained in the closure of the positive cone) and then composing it with reflections s_C as in step one.

§3. The various twisted categories and their Chern characters

Let $\alpha \in H^2(X, \mathcal{O}_X^*)$ be a Brauer class represented by a Čech cocycle $\{\alpha_{ijk}\}$.

1. The abelian category $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ of $\{\alpha_{ijk}\}$-twisted coherent sheaves only depends on the class $\alpha \in H^2(X, \mathcal{O}_X^*)$. More precisely, for any other choice of a Čech-cocycle $\{\alpha'_{ijk}\}$ representing α, there exists an equivalence

$$\mathbf{Coh}(X, \{\alpha_{ijk}\}) \xrightarrow{\Psi_{\{\lambda_{ij}\}}} \mathbf{Coh}(X, \{\alpha'_{ijk}\}), \quad \{E_i, \varphi_{ij}\} \longmapsto \{E_i, \varphi_{ij} \cdot \lambda_{ij}\},$$

where $\{\lambda_{ij} \in \mathcal{O}^*(U_{ij})\}$ satisfies $\alpha'_{ijk}\alpha_{ijk}^{-1} = \lambda_{ij} \cdot \lambda_{jk} \cdot \lambda_{ki}$. Clearly, $\{\lambda_{ij}\}$ exists, as $\{\alpha_{ijk}\}$ and $\{\alpha'_{ijk}\}$ define the same Brauer class, but it is far from being unique. In other words, the above equivalence $\Psi_{\{\lambda_{ij}\}}$ is not canonical. In order to make this more precise, choose a second $\{\lambda'_{ij}\}$. Then $\gamma_{ij} := \lambda'_{ij} \cdot \lambda_{ij}^{-1}$ can be viewed as the transition function of a holomorphic line bundle $\mathcal{L}_{\lambda\lambda'}$. With this notation one finds

$$\Psi_{\{\lambda'_{ij}\}} = (\mathcal{L}_{\lambda\lambda'} \otimes (\ \ )) \circ \Psi_{\{\lambda_{ij}\}}.$$

A very special case of this is the equivalence

$$\mathcal{L} \otimes (\ \ ) : \mathbf{Coh}(X, \{\alpha_{ijk}\}) \longrightarrow \mathbf{Coh}(X, \{\alpha_{ijk}\})$$

that is induced by the tensor product with a holomorphic line bundle $\mathcal{L}$ given by a cocycle $\{\gamma_{ij}\}$.

Despite this ambiguity in identifying these categories for different choices of the Čech-representative, $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ is often simply denoted $\mathbf{Coh}(X, \alpha)$.

2. Now fix a B-field $B \in H^2(X, \mathbb{Q})$ together with a Čech-representative $\{B_{ijk}\}$. The induced Brauer class $\alpha := \exp(B^{0,2}) \in H^2(X, \mathcal{O}_X^*)$ is represented by the Čech-cocycle $\{\alpha_{ijk} := \exp(B_{ijk})\}$.

In [4] we introduced

$$\mathrm{ch}^B : \mathbf{Coh}(X, \{\alpha_{ijk}\}) \longrightarrow H^{*,*}(X, \mathbb{Q}).$$

The construction makes use of a further choice of $\mathcal{C}^\infty$-functions a_{ij} with $-B_{ijk} = a_{ij} + a_{jk} + a_{ki}$, but the result does not depend on it. Indeed, by definition, $\mathrm{ch}^B(\{E_i, \varphi_{ij}\}) = \mathrm{ch}(\{E_i, \varphi_{ij} \cdot \exp(a_{ij})\})$. Thus, if we pass from a_{ij} to $a_{ij} + c_{ij}$ with $c_{ij} + c_{jk} + c_{ki} = 0$, then $\mathrm{ch}^B(\{E_{ij}, \varphi_{ij}\})$ changes by $\exp(\mathrm{c}_1(\mathcal{L}))$, where $\mathcal{L}$ is given by the transition functions $\{\exp(c_{ij})\}$. But by the very definition of the first Chern class, one has $\mathrm{c}_1(\mathcal{L}) = [\{c_{ij} + c_{jk} + c_{ki}\}] = 0$.

More generally, we may change the class B by a class $b \in H^2(X, \mathbb{Q})$ represented by $\{b_{ijk}\}$. Suppose $\alpha_{B+b} = \alpha_B \in H^2(X, \mathcal{O}_X^*)$. We denote the Čech-representative $\exp(B_{ijk} + b_{ijk})$ by α'_{ijk}. As before, we write $-B_{ijk} = a_{ij} + a_{jk} + a_{ki}$ and $-b_{ijk} = c_{ij} + c_{jk} + c_{ki}$.

The Chern characters ch^B and ch^{B+b} fit into the following commutative diagram

$$\begin{tikzcd}
\mathbf{Sh}(X, \{\alpha_{ijk}\}) \arrow[rr, "\Psi_{\{\exp(-c_{ij})\}}"] \arrow[dr, "\mathrm{ch}^B"'] & & \mathbf{Sh}(X, \{\alpha'_{ijk}\}) \arrow[dl, "\mathrm{ch}^{B+b}"] \\
& H^*(X, \mathbb{Q}). &
\end{tikzcd}$$

Unfortunately, we cannot replace **Sh** by **Coh**, for $\exp(c_{ij})$ are only differentiable functions. Nevertheless, there exist $\beta_{ij} \in \mathcal{O}^*(U_{ij})$, non-unique, with $\alpha'_{ijk} = \alpha_{ijk} \cdot (\beta_{ij} \cdot \beta_{jk} \cdot \beta_{ki})$. Using these one finds a commutative diagram

$$\begin{tikzcd}
\mathbf{Coh}(X, \{\alpha_{ijk}\}) \arrow[r, "\Psi_{\{\beta_{ij}\}}"] \arrow[d, "\mathrm{ch}^B"'] & \mathbf{Coh}(X, \{\alpha'_{ijk}\}) \arrow[d, "\mathrm{ch}^{B+b}"] \\
H^*(X, \mathbb{Q}) \arrow[r, "\exp(\mathrm{c}_1(\mathcal{L}))"'] & H^*(X, \mathbb{Q}).
\end{tikzcd}$$

Here, $\mathcal{L}$ is the line bundle given by the transition functions $\beta_{ij} \cdot \exp(c_{ij})$.

It is not difficult to see that $\mathrm{c}_1(\mathcal{L}) \in H^{1,1}(X)$ whenever one has $b \in H^{1,1}(X, \mathbb{Q})$. Indeed, $\mathrm{c}_1(\mathcal{L}) = \{d \log(\beta_{ij})\} + b$, which is of type $(1,1)$, as b is $(1,1)$ by assumption and the functions β_{ij} are holomorphic.

Thus, in this case there exists a holomorphic line bundle $\widetilde{\mathcal{L}}$ with $\mathrm{c}_1(\widetilde{\mathcal{L}}) = \mathrm{c}_1(\mathcal{L})$. Now consider the composition $\Phi := (\widetilde{\mathcal{L}}^* \otimes (\ )) \circ \Psi_{\{\beta_{ij}\}}$: $\mathbf{Coh}(X, \alpha_B := \{\alpha_{ijk}\}) \cong \mathbf{Coh}(X, \alpha_{B+b} := \{\alpha'_{ijk}\})$, which is an exact equivalence, and denote the derived one again by $\Phi : \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \cong \mathrm{D}^{\mathrm{b}}(X, \alpha_{B+b})$. Then the above calculation of the twisted Chern character implies that $\Phi_*^{B,B+b} = \mathrm{id}$.

3. Consider again the abelian category $\mathbf{Coh}(X, \{\alpha_{ijk}\})$. For any locally free $G = \{G_i, \varphi_{ij}\} \in \mathbf{Coh}(X, \{\alpha_{ijk}\})$ one defines an Azumaya algebra $\mathcal{A}_G := \mathcal{E}nd(G^\vee)$. The abelian category of left $\mathcal{A}_G$-modules will be denoted $\mathbf{Coh}(\mathcal{A}_G)$. An equivalence of abelian categories is given by

$$\mathbf{Coh}(X, \{\alpha_{ijk}\}) \longrightarrow \mathbf{Coh}(\mathcal{A}_G), \ E \longmapsto G^\vee \otimes E.$$

In [12] Yoshioka considers yet another abelian category $\mathbf{Coh}(X, Y)$ of certain coherent sheaves on a projective bundle $Y \to X$ realizing the Brauer class α. As is explained in detail in [12], one again has an equivalence of abelian categories $\mathbf{Coh}(X, Y) \cong \mathbf{Coh}(X, \{\alpha_{ijk}\})$. In order to define an appropriate notion of stability, Yoshioka defines a Hilbert polynomial for objects $E \in \mathbf{Coh}(X, Y)$. It is straightforward to see that under the composition

$$\mathbf{Coh}(X, Y) \xrightarrow{\sim} \mathbf{Coh}(X, \{\alpha_{ijk}\}) \xrightarrow{\sim} \mathbf{Coh}(\mathcal{A}_G)$$

his Hilbert polynomial corresponds to the usual Hilbert polynomial for sheaves $F \in \mathbf{Coh}(\mathcal{A}_G)$ viewed as $\mathcal{O}_X$-modules. The additional choice of the locally free object G in $\mathbf{Coh}(X, Y)$ or equivalently in $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ needed to define the Hilbert polynomial in [12] enters this comparison via the equivalence $\mathbf{Coh}(X, \{\alpha_{ijk}\}) \cong \mathbf{Coh}(\mathcal{A}_G)$. From here it is easy to see that the stability conditions considered in [10, 12] are actually equivalent.

We would like to define a twisted Chern character for objects in $\mathbf{Coh}(\mathcal{A}_G)$. Of course, as any $F \in \mathbf{Coh}(\mathcal{A}_G)$ is in particular an ordinary sheaf, $\mathrm{ch}(F)$ is well defined. In order to define something that takes into account the $\mathcal{A}_G$-module structure, one fixes $B = \{B_{ijk}\}$ and assumes $\alpha_{ijk} = \exp(B_{ijk})$. Then we introduce

$$\mathrm{ch}_G^B : \mathbf{Coh}(\mathcal{A}_G) \longrightarrow H^*(X, \mathbb{Q}), \ F \longmapsto \frac{\mathrm{ch}(F)}{\mathrm{ch}^{-B}(G^\vee)}.$$

Note that *a priori* the definition depends on B and G, but the dependence on the latter is well-behaved as will be explained shortly.

Here are the main compatibilities for this new Chern character:

i) The following diagram is commutative:

$$\begin{tikzcd}
\mathbf{Coh}(X,\{\alpha_{ijk}\}) \arrow[rr] \arrow[dr, "\mathrm{ch}^B"'] & & \mathbf{Coh}(\mathcal{A}_G) \arrow[dl, "\mathrm{ch}^B_G"] \\
& H^*(X,\mathbb{Q}). &
\end{tikzcd}$$

Indeed, $\mathrm{ch}(G^\vee \otimes E) = \mathrm{ch}^{-B}(G^\vee) \cdot \mathrm{ch}^B(E)$.

ii) Let H be a locally free coherent sheaf and $G' := G \otimes H \in \mathbf{Coh}(X,\{\alpha_{ijk}\})$. Then the natural equivalence $\mathbf{Coh}(\mathcal{A}_G) \to \mathbf{Coh}(\mathcal{A}_{G'})$, $F \mapsto H^\vee \otimes F$ fits in the commutative diagram

$$\begin{tikzcd}
\mathbf{Coh}(\mathcal{A}_G) \arrow[rr] \arrow[dr, "\mathrm{ch}^B_G"'] & & \mathbf{Coh}(\mathcal{A}_{G'}) \arrow[dl, "\mathrm{ch}^B_{G'}"] \\
& H^*(X,\mathbb{Q}). &
\end{tikzcd}$$

This roughly says that the new Chern character is independent of G.

iii) If $E_1, E_2 \in \mathbf{Coh}(X,\{\alpha_{ijk}\})$ and $F_i := G^\vee \otimes E_i \in \mathbf{Coh}(\mathcal{A}_G)$ then $\chi(E_1,E_2) := \sum(-1)^i \dim \mathrm{Ext}^i(E_1,E_2)$ is well-defined and equals $\chi(F_1,F_2) := \sum(-1)^i \dim \mathrm{Ext}^i_{\mathcal{A}_G}(F_1,F_2)$. Both expressions can be computed in terms of the twisted Chern characters introduced above and the Mukai pairing. Concretely,

$$\chi(F_1,F_2) = -\langle \mathrm{ch}^B_G(F_1) \cdot \sqrt{\mathrm{td}(X)}, \mathrm{ch}^B_G(F_2) \cdot \sqrt{\mathrm{td}(X)} \rangle.$$

Here $\langle\ ,\ \rangle$ denotes the generalized Mukai pairing and

$$\chi(F_1,F_2) := -\langle \mathrm{ch}^B(E_1) \cdot \sqrt{\mathrm{td}(X)}, \mathrm{ch}^B(E_2) \cdot \sqrt{\mathrm{td}(X)} \rangle.$$

(Be aware of the different sign conventions for K3 surfaces and the general case.)

4. There is yet another way to define a twisted Chern character which is implicitly used in [12]. We use the above notations and define $\mathrm{ch}_G : \mathbf{Coh}(\mathcal{A}_G) \to H^*(X,\mathbb{Q})$ by $\mathrm{ch}_G(F) := \frac{\mathrm{ch}(F)}{\sqrt{\mathrm{ch}(\mathcal{A}_G)}}$, where F and $\mathcal{A}_G$ are considered as ordinary $\mathcal{O}_X$-modules. Using the natural identifications explained earlier, namely $\mathbf{Coh}(X,Y) \cong \mathbf{Coh}(X,\{\alpha_{ijk}\}) \cong \mathbf{Coh}(\mathcal{A}_G)$, this Chern character can also be viewed as a Chern character on the other abelian categories.

Although the definition ch_G seems very natural, it does not behave nicely under change of G. More precisely, in general $\mathrm{ch}_{G\otimes H}(H^\vee \otimes F) \neq \mathrm{ch}_G(F)$.

Fortunately, the situation is less critical for K3 surfaces. Here, the relation between ch_G and ch_G^B can be described explicitly and using the results in **3.** one deduces from this a formula for the change of ch_G under $G \mapsto G \otimes H$. In fact, it is straightforward to see that the following diagram commutes:

$$\begin{tikzcd} & \mathbf{Coh}(\mathcal{A}_G) \arrow[dl, "\mathrm{ch}_G^B"'] \arrow[dr, "\mathrm{ch}_G"] & \\ H^*(X, \mathbb{Q}) \arrow[rr, "\exp(-B_G)"'] & & H^*(X, \mathbb{Q}). \end{tikzcd}$$

Here $B_G := \frac{\mathrm{c}_1^B(G)}{\mathrm{rk}(G)}$, where $\mathrm{c}_1^B(G)$ is the degree two part of $\mathrm{ch}^B(G)$. Note that B and B_G define the same Brauer class. In particular, the Hodge structures $\widetilde{H}^*(X, B, \mathbb{Z})$ and $\widetilde{H}^*(X, B_G, \mathbb{Z})$ are isomorphic.

This relation between ch_G^B and ch_G can be used to compare the two versions of the cohomogical Fourier–Mukai transform in [4] and [12]. With $v^B := \mathrm{ch}^B \cdot \sqrt{\mathrm{td}(X)}$ and $v_G := \mathrm{ch}_G \cdot \sqrt{\mathrm{td}(X)}$ and the implicit identification $\mathbf{Coh}(X, \alpha) = \mathbf{Coh}(\mathcal{A}_G)$ the following diagram is commutative:

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(X, \alpha_B) \arrow[rrr, "\Phi"] \arrow[dr, "v_G"] \arrow[dd, "v^B"'] & & & \mathrm{D}^{\mathrm{b}}(X', \alpha') \arrow[dl, "v_{G'}"'] \arrow[dd, "v^{B'}"] \\ & H^*(X, \mathbb{Q}) \arrow[r] & H^*(X', \mathbb{Q}) & \\ H^*(X, B, \mathbb{Z}) \arrow[ur, "\exp(-B_G)"] \arrow[rrr, "\Phi_*^{B,B'}"'] & & & H^*(X', B', \mathbb{Z}). \arrow[ul, "\exp(-B_{G'})"'] \end{tikzcd}$$

Here, the central isomorphism $H^*(X, \mathbb{Q}) \cong H^*(X', \mathbb{Q})$ is the correspondence defined by $v_{G^\vee \boxtimes G'}(\mathcal{E})$ with $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}(X \times X', \alpha_B^{-1} \boxtimes \alpha_{B'})$ the kernel defining Φ.

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