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Submanifolds with a non-degenerate parallel normal vector field in euclidean spaces

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Abstract.

We consider the class of submanifolds M in an euclidean space $\mathbb{R}^n$ which admit a non-degenerate parallel normal vector field ν. The image of the associated Gauss map $G_\nu : M \to S^{n-1}$ defines an immersed hyperspherical submanifold M^ν which has the following property: if M has a contact of Boardman type $\Sigma^{i_1,\dots,i_k}$ with a hyperplane, then M^ν has the same contact type with the translated hyperplane. In particular, for a space curve α in $\mathbb{R}^3$, the spherical curve α^ν has the same flattenings and we deduce an extension of the Four Vertex Theorem. For an immersed surface M in $\mathbb{R}^4$, it admits a local non-degenerate parallel normal vector field if and only if it is totally semi-umbilic and has non zero gaussian curvature K. Moreover, G_ν preserves the inflections and the asymptotic lines between M and M^ν. As a consequence, we deduce an extension for this class of surfaces of the classical Loewner and Carathéodory conjectures for umbilic points of analytic immersed surfaces in $\mathbb{R}^3$.

§1. Introduction

Let $\xi : \mathbb{R}^n \to S^n \hookrightarrow \mathbb{R}^{n+1}$ denote the inverse of the stereographic projection and let M be an m-dimensional submanifold of $\mathbb{R}^n$. It has been shown by Sedykh [16] and Romero-Fuster [12, 13] that the contacts of M with the hyperspheres of $\mathbb{R}^n$ are the same as the contacts of its hyperspherical image $\xi(M)$ with the hyperplanes of $\mathbb{R}^{n+1}$. Since many of the differential-geometric aspects of a submanifold M of an euclidean space $\mathbb{R}^n$ can be translated in terms of contacts between hyperspheres or hyperplanes, they use this fact in order to obtain interesting relations between some special points of the submanifold M and its image $\xi(M)$.

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For instance, suppose that $\alpha : I \to \mathbb{R}^2$ defines a regular smooth curve in the plane. Then ξ transforms the vertices (points where $\kappa' = 0$) of α into the flattenings (points with $\tau = 0$) of the spherical space curve $\xi \circ \alpha$ in $\mathbb{R}^3$. This is due to the fact that a vertex corresponds to a point of contact of type $\Sigma^{1,1,1}$ (using Thom-Boardman notation) between the curve and a circle in the plane, while a flattening of a space curve is a point of contact of the same type $\Sigma^{1,1,1}$ with the osculating plane. One consequence is that it is possible to translate the classical Four Vertex Theorem in terms of flattenings and spherical space curves: every closed regular and simple space curve contained in the sphere S^2 has at least four flattenings. In fact, this observation led to the conjecture by Scherk in 1936 that every convex closed and simple space curve with non vanishing curvature has at least four flattenings, which was proved by Sedykh [15].

In the case of a smooth immersed surface M in $\mathbb{R}^3$ we have a similar history with respect to umbilics and the Carathéodory conjecture. The classical Carathéodory conjecture states that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics, i.e., points where the two principal curvatures coincide. This conjecture has a stronger local version, known as the Loewner conjecture, which states that the index of the principal foliation at any isolated umbilic of an immersed smooth surface in $\mathbb{R}^3$ is always ≤ 1. Since the sum of the indices of the umbilics of a compact immersed surface is equal to its Euler-Poincaré characteristic (according to the Poincaré-Hopf formula) it follows that the Loewner conjecture implies the Carathéodory conjecture, not only for a convex embedding of a 2-sphere, but for any immersion (not necessarily convex). The Loewner conjecture is known to be true in the analytic case (although there is a big controversy about the correct proof, see for instance [17, 7]).

Now, it is also possible to characterize an umbilic of M as a point which presents a contact of type $\Sigma^{2,2}$ between M and a sphere of $\mathbb{R}^3$. It follows that the map ξ will give a point of $\Sigma^{2,2}$ contact between the hyperspherical surface $\xi(M)$ and some hyperplane of $\mathbb{R}^4$. But this type of contact corresponds to an inflection of the surface in the sense of Little [8] (that is, a point where the two fundamental forms are collinear). In particular, we have that any analytic immersed surface M in $S^3 \subset \mathbb{R}^4$, homeomorphic to S^2, has at least two inflections. Moreover, it was also observed by Little that ξ also takes the principal foliation of M into the asymptotic foliation of $\xi(M)$ and it is also possible to translate Loewner conjecture: for any analytic immersed surface M in $S^3 \subset \mathbb{R}^4$, the index of the asymptotic foliation at any isolated inflection is always ≤ 1.

Now, it is natural to ask whether these results can be extended to general analytic surfaces immersed in $\mathbb{R}^4$. Since the asymptotic foliation is only defined in the convex part of the surface, it is obvious that we have to restrict ourselves to locally convex surfaces (that is, at any point there is a hyperplane which locally supports the surface). A proof of the Carathéodory conjecture for generic locally convex surfaces in $\mathbb{R}^4$ can be found in [4] (in fact, for a generic locally convex surface M, the index of an isolated inflection is always $\pm 1/2$ and hence, it must have at least $2\chi(M)$ inflections). In [6], they give a proof of the Loewner conjecture for a locally convex surface in $\mathbb{R}^4$ which satisfies some non-degeneracy condition with respect to the Newton polyhedra. Some results about the index of an isolated inflection of an immersed surface in $\mathbb{R}^4$ can be also found in [3].

In this paper, we consider the class of smooth submanifolds M immersed in $\mathbb{R}^n$ which admit a non-degenerate parallel normal vector field ν. This class appears in the literature in the context of differential geometry of submanifolds (see for instance [10]). We show that the image of the associated Gauss map $G_\nu : M \to S^{n-1}$ defines an immersed hyperspherical submanifold M^ν which has the following property: if M has a contact of Boardman type $\Sigma^{i_1,\dots,i_k}$ with a hyperplane, then M^ν has the same contact type with the translated hyperplane. Thus, for instance, in the case of a space curve α in $\mathbb{R}^3$, the spherical curve α^ν has the same flattenings and we deduce an extension of the Four Vertex Theorem.

For an immersed surface M in $\mathbb{R}^4$, it admits a non-degenerate parallel normal vector field if and only if it is totally semi-umbilic and has non zero gaussian curvature K. The semi-umbilic condition means that at any point of the surface, there is a non-zero normal vector ν such that the ν-principal curvatures are equal and it has been studied recently by Romero-Fuster and Sánchez-Bringas (see [14]). The totally semi-umbilic surfaces in $\mathbb{R}^4$ with $K \neq 0$ are an intermediate class between the class of hyperspherical surfaces and the class of locally convex surfaces. Moreover, we show that the Gauss map G_ν preserves the inflections and the asymptotic lines between M and the hyperspherical image M^ν. As a consequence, we obtain that Loewner and Carathéodory conjectures are also true for analytic totally semi-umbilic surfaces with $K \neq 0$.

§2. Contact with hyperspheres and hyperplanes

In this section, we recall basic definitions and properties of contact between submanifolds of an ambient manifold and the relationship with

$\mathcal{K}$-equivalence of map germs due to Montaldi [9]. We begin with the notion of contact.

Definition 2.1. Let M, N, M', N' be smooth submanifolds of $\mathbb{R}^n$ and let $x_0 \in M \cap N$ and $x_0' \in M' \cap N'$. We say that *the contact of M and N at x_0 is of the same type as the contact of M' and N' at x_0'* if there are open neighbourhoods U of x_0 and U' of x_0' in $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to U'$ such that $\phi(U \cap M) = U' \cap M'$ and $\phi(U \cap N) = U' \cap N'$.

Now, we recall the concept of $\mathcal{K}$-equivalence between two smooth map germs.

Definition 2.2. Consider two smooth map germs $f : (M, x_0) \to (N, y_0)$ and $g : (M', x_0') \to (N', y_0')$ between smooth manifolds. We say that f, g are *$\mathcal{K}$-equivalent* if there exists a diffeomorphism

$$H : (M \times N, (x_0, y_0)) \to (M' \times N', (x_0', y_0'))$$

such that:

(1) $H(x, y) = (h(x), \theta(x, y))$ for some map germs $h : (M, x_0) \to (M', x_0')$ and $\theta : (M \times N, (x_0, y_0)) \to (N', y_0')$.
(2) $\theta(x, y_0) = y_0'$ for any x in a neighbourhood of x_0 in M.
(3) $H(x, f(x)) = (h(x), g(h(x)))$ for any x in a neighbourhood of x_0 in M.

In order to see the relationship between contact and $\mathcal{K}$-equivalence we need to introduce some notations. Let M, N, M', N' be smooth submanifolds of $\mathbb{R}^n$ and let $x_0 \in M \cap N$ and $x_0' \in M' \cap N'$. We assume that M, M' are locally given by the image of an embedding. That is, there are open neighbourhoods W_1 of x_0 in $\mathbb{R}^n$ and W_2 of x_0' in $\mathbb{R}^n$, open subsets $U_1, U_2 \subset \mathbb{R}^m$ and smooth embeddings $f_1 : U_1 \to \mathbb{R}^n$ and $f_2 : U_2 \to \mathbb{R}^n$ such that $f_1(U_1) = M \cap W_1$ and $f_2(U_2) = M' \cap W_2$ (here m is the dimension of M and M'). We also denote $f_1(u_0) = x_0$ and $f_2(u_0') = x_0'$.

For N, N' we assume that they are given locally in implicit forms. That is, there are smooth maps $g_1 : W_1 \to \mathbb{R}^p$ and $g_2 : W_2 \to \mathbb{R}^p$ such that $W_1 \cap N = g_1^{-1}(v_0)$, with v_0 a regular value of g_1 and $W_2 \cap N' = g_2^{-1}(v_0')$, with v_0' a regular value of g_2 (now, p is the codimension of N and N').

Theorem 2.3. [9] *With the above notation, it follows that the contact of M and N at x_0 is of the same type as the contact of M' and N' at x_0' if and only if the map germs $g_1 \circ f_1 : (\mathbb{R}^m, u_0) \to (\mathbb{R}^p, v_0)$ and $g_2 \circ f_2 : (\mathbb{R}^m, u_0') \to (\mathbb{R}^p, v_0')$ are $\mathcal{K}$-equivalent.*

Definition 2.4. The map $g_1 \circ f_1 : U \to \mathbb{R}^p$ is called the *contact map* of M, N. It follows that its $\mathcal{K}$-singularity type at each point u_0 determines the contact of M, N at $x_0 = f_1(u_0)$ and does not depend on the choice of maps f_1, g_1.

In the case that we have a submanifold in Euclidean space $M \subset \mathbb{R}^n$, the most interesting contacts are those of M with hyperplanes and hyperspheres of $\mathbb{R}^n$, since they determine some of the geometrical invariants of M.

Assume that the embedding $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ locally parametrizes the submanifold $M \subset \mathbb{R}^n$ in a neighbourhood of $x_0 = f(u_0)$. For any $v \in S^{n-1}$, we consider the *height function* $h_v : U \to \mathbb{R}$, given by $h_v(u) = \langle f(u), v \rangle$. Then, h_v is the contact map of M and the hyperplane $\pi(x_0, v)$ of $\mathbb{R}^n$ through x_0 perpendicular to v. It is obvious that $u_0 \in U$ is a singular point of h_v if and only if v belongs to the normal subspace of M at x_0, that is, $\pi(x_0, v)$ is tangent to M at x_0.

Analogously, given $p \in \mathbb{R}^n$, we can also consider the *distance squared function* $d_p : U \to \mathbb{R}$, given by $d_p(u) = \|f(u) - p\|^2$. Now, d_p is the contact map between M and the hypersphere $S(p, R)$ of $\mathbb{R}^n$ centered at p with radius $R = d_p(u_0)$. Again, $u_0 \in U$ is a singular point of d_p if and only if p is in the (affine) normal subspace of M at x_0, that is, $S(p, R)$ is tangent to M at x_0.

Now, we recall the Thom-Boardman symbols $\Sigma^{i_1,\dots,i_k}$, which are a generalization of the rank of a map taking into account higher order derivatives and provide a useful invariant for $\mathcal{K}$-equivalence.

Let us denote by $\mathcal{E}_{m,x_0}$ the local ring of smooth function germs from $(\mathbb{R}^m, x_0)$ to $\mathbb{R}$. Given a $p \times q$ matrix U with entries in $\mathcal{E}_{m,x_0}$, we denote by $I_t(U)$ the ideal in $\mathcal{E}_{m,x_0}$ generated by the t-minors of U (by convention, $I_t(U) = \{0\}$ if $t > \min(p, q)$). In particular, if $f : (\mathbb{R}^m, x_0) \to (\mathbb{R}^p, y_0)$ is a smooth map germ, $I_t(Df)$ is the ideal generated by the t-minors of the jacobian matrix $Df = (\partial f_i / \partial x_j)$.

Definition 2.5. Let $f : (\mathbb{R}^m, x_0) \to \mathbb{R}^p$ be a smooth map germ and let $\mathbf{i} = (i_1, \dots, i_k)$ be a k-tuple of non-negative integer numbers. We define the *iterated jacobian extension* of f by induction on k. If $k = 1$, then $J_{i_1}(f) = I_{n-i_1+1}(Df)$. For $k > 1$, suppose that $J_{i_1,\dots,i_{k-1}}(f) = \langle g_1, \dots, g_r \rangle$, then

$$J_{i_1,\dots,i_k}(f) = J_{i_1,\dots,i_{k-1}}(f) + I_{n-i_k+1}(D(f, g)),$$

where $(f, g) = (f_1, \dots, f_p, g_1, \dots, g_r)$.

We say that f has *Boardman type* (or *Boardman symbol*) Σ^{i} if f has rank $n-i_1$ at x_0 and for $k>1$, $(f_1,\dots,f_p,g_1,\dots,g_r)$ has rank $n-i_k$ at x_0, being $g_1,\dots,g_r$ generators of the ideal $J_{i_1,\dots,i_{k-1}}(f)$.

Example 2.6. Given a smooth function $f: U\subset\mathbb{R}^m\to\mathbb{R}$, it has Boardman type $\Sigma^{m,\dots,m}$ at x_0 (with m repeated k times) if and only if all the partial derivatives of f at x_0 are zero up to order k.

We include now a result that will be used in next section.

Lemma 2.7. *Let $f,g:(\mathbb{R}^m,x_0)\to\mathbb{R}$ be two smooth function germs such that $J_m(f)=J_m(g)$. Then f,g have the same Boardman symbol $\Sigma^{i_1,\dots,i_k}$, for any $k\geq 1$.*

Proof. The ideals $J_m(f)=I_1(Df)$ and $J_m(g)=I_1(Dg)$ in $\mathcal{E}_{m,x_0}$ are generated by the partial derivatives $\partial f/\partial x_i$ and $\partial g/\partial x_i$ respectively. The assumption $J_m(f)=J_m(g)$ means that

$$\frac{\partial f}{\partial x_i}=\sum_{j=1}^{m}a_{ij}\frac{\partial g}{\partial x_j},$$

for some $a_{ij}\in\mathcal{E}_{m,x_0}$, with $\det(a_{ij})\neq 0$. We have that f,g have the same rank at x_0 and hence, the first Boardman number i_1 is the same for f,g.

If f,g are regular, then $i_1=m-1$, $J_{m-1}(f)=I_2(Df)=\{0\}$ and $J_{m-1}(g)=I_2(Dg)=\{0\}$. In particular, the Boardman symbol is $i_2=\cdots=i_k=m-1$ for both f,g.

Assume now that f,g are singular and $i_1=m$. We will show by induction on k that f,g have the same Boardman symbol and the same iterated jacobian ideals. Assume that the Boardman numbers $i_1,\dots,i_{k-1}$ are equal for f,g and $J_{i_1,\dots,i_{k-1}}(f)=J_{i_1,\dots,i_{k-1}}(g)=\langle h_1,\dots,h_r\rangle$. The Boardman number i_k for f,g is determined in each case by the rank at x_0 of $(f,h_1,\dots,h_r)$ and $(g,h_1,\dots,h_r)$ respectively. We consider the matrices

$$A=\begin{pmatrix}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_m}\\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m}\\ \cdots & \cdots & \cdots\\ \frac{\partial h_r}{\partial x_1} & \cdots & \frac{\partial h_r}{\partial x_m}\end{pmatrix},\quad B=\begin{pmatrix}\frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_m}\\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m}\\ \cdots & \cdots & \cdots\\ \frac{\partial h_r}{\partial x_1} & \cdots & \frac{\partial h_r}{\partial x_m}\end{pmatrix}.$$

Since $\partial f/\partial x_i$ and $\partial g/\partial x_i$ are 0 at x_0, it follows that A,B have the same rank at x_0 and the Boardman number i_k is the same for f,g.

On the other hand, by definition, $J_{i_1,\dots,i_k}(f)=J_{i_1,\dots,i_{k-1}}(f)+I_t(A)$ and $J_{i_1,\dots,i_k}(g)=J_{i_1,\dots,i_{k-1}}(g)+I_t(B)$, where $t=n-i_k+1$. Let M be a t-minor of A. If M does not contain the first row of A, then M is a t-minor of B. Otherwise, if M contains the first row of A, then $M \in J_m(f) = J_m(g) \subset J_{i_1,\ldots,i_k}(g)$. This shows that $J_{i_1,\ldots,i_k}(f) \subset J_{i_1,\ldots,i_k}(g)$ and the opposite inclusion follows by symmetry. Q.E.D.

In general, it is not true that if $f, g : (\mathbb{R}^m, x_0) \to \mathbb{R}$ are two smooth function germs such that $J_m(f) = J_m(g)$, then they are $\mathcal{K}$-equivalent. For instance, consider $f, g : (\mathbb{R}^2, 0) \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 - y^2$. In this case, $J_2(f) = J_2(g) = \langle x, y\rangle$, they have Boardman symbol $\Sigma^{2,0}$, but they are not $\mathcal{K}$-equivalent.

One of the well known properties of the Boardman symbol is that it is $\mathcal{K}$-invariant. Hence, it can be associated with each contact class between submanifolds.

Lemma 2.8. *Let $f, g : (\mathbb{R}^m, x_0) \to \mathbb{R}^p$ be two smooth map germs which are $\mathcal{K}$-equivalent. Then f, g have the same Boardman symbol $\Sigma^{i_1,\ldots,i_k}$, for any $k \geq 1$.*

(See for instance [5] for a proof.)

Definition 2.9. Given submanifolds $M, N \subset \mathbb{R}^n$ and $x_0 \in M \cap N$, we say that they have *contact type* $\Sigma^{\mathbf{i}}$ if its contact map germ has Boardman type $\Sigma^{\mathbf{i}}$. The above lemma ensures that this definition does not depend on the choice of contact map germs.

Example 2.10. Here we present some known basic examples how the contacts of a submanifold in an euclidean space with hyperplanes or hyperspheres can be useful to characterize several special points in the differential geometry of curves and surfaces.

(1) Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve. Then α has a $\Sigma^{1,1}$ contact with a line $\pi(x_0, v)$ at $x_0 = \alpha(t_0)$ if and only if this point is an inflection (that is, $\kappa(t_0) = 0$) and v is the normal vector at such point.

(2) Analogously, α has a $\Sigma^{1,1,1}$ contact with a circle $S(p, R)$ at x_0 if and only if this point is a non-flat vertex (that is, $\kappa'(t_0) = 0$ and $\kappa(t_0) \neq 0$), and p and R are the centre and the radius of curvature of α at x_0 respectively. The case of a flat vertex (that is, $\kappa'(t_0) = \kappa(t_0) = 0$) corresponds to a $\Sigma^{1,1,1}$ contact with the tangent line.

(3) In the case of a regular space curve $\alpha : I \to \mathbb{R}^3$ with non vanishing curvature, α has a $\Sigma^{1,1,1}$ contact with a plane $\pi(x_0, v)$ at $x_0 = \alpha(t_0)$ if and only if this point is a flattening (that is, $\tau(t_0) = 0$) and v is the binormal vector at such point.

(4) Let M be a regular surface in $\mathbb{R}^3$. Then, M has a $\Sigma^{2,2}$ contact with a sphere $S(p, R)$ at x_0 if and only if this point is an non-flat

umbilic (that is, a point where the two principal curvatures are equal and distinct from zero) and p and R are the centre and the radius of principal curvature of M at x_0 respectively. The case of a flat umbilic (that is, when both principal curvatures are zero) corresponds to a $\Sigma^{2,2}$ contact with the tangent plane.

(5) Finally, we consider a regular surface M in $\mathbb{R}^4$. Then, M has a $\Sigma^{2,2}$ contact with a hyperplane $\pi(x_0, v)$ at x_0 if and only if this point is an inflection in the sense of Little [8] (that is, a point where the two second fundamental forms are linearly dependent) and v is the corresponding binormal vector.

We finish this section by showing that the contacts with hyperspheres and hyperplanes are related through the stereographic projection. Let $\xi : \mathbb{R}^n \to S^n \hookrightarrow \mathbb{R}^{n+1}$ denote the inverse of the stereographic projection, which is given by

$$\xi(x) = \frac{(2x, \|x\|^2 - 1)}{(\|x\|^2 + 1)}.$$

Since this map is conformal, it follows that it transforms any hypersphere $S(p, R)$ or hyperplane $\pi(x_0, v)$ of $\mathbb{R}^n$ into a $(n-1)$-sphere contained in S^n. We denote by $\pi_S(p, R)$ (respectively $\pi_S(x_0, v)$) the only hyperplane of $\mathbb{R}^{n+1}$ which has the property $\xi^{-1}(\pi_S(p, R)) = S(p, R)$ (respectively $\xi^{-1}(\pi_S(x_0, v)) = \pi(x_0, v)$).

It follows from the works by Romero Fuster [12, 13] and Sedykh [16] that the contact of a submanifold $M \subset \mathbb{R}^n$ and hyperplane $\pi(x_0, v)$ or hypersphere $S(p, R)$at $x_0 \in M$ is of the same type as the contact of $\xi(M)$ and $\pi_S(x_0, v)$ or $\pi_S(p, R)$ at $\xi(x_0)$ respectively. In fact, they show more, namely, that the family of distance squared functions of M in $\mathbb{R}^n$ is $\mathcal{K}$-equivalent to the family of height functions of $\xi(M)$ in $\mathbb{R}^{n+1}$.

Example 2.11. By looking at the examples of 2.10, we get some immediate consequences of this fact. For instance, if $\alpha : I \to \mathbb{R}^2$ is a regular plane curve, then $t_0 \in I$ is a vertex of α if and only if t_0 is a flattening of $\xi \circ \alpha : I \to S^2 \subset \mathbb{R}^3$.

In the case of a regular surface $M \subset \mathbb{R}^3$, it follows that $x_0 \in M$ is an umbilic if and only if $\xi(x_0)$ is an inflection of $\xi(M) \subset S^3 \subset \mathbb{R}^4$.

In general, which we can conclude is that in $\mathbb{R}^{n+1}$, the class of hyperspherical submanifolds (that is, submanifolds contained in some hypersphere of $\mathbb{R}^{n+1}$), presents the same contacts with hyperplanes of $\mathbb{R}^{n+1}$ as the submanifolds of $\mathbb{R}^n$ with respect to hyperspheres or hyperplanes.

Question 2.12. *Determine the submanifolds of $\mathbb{R}^{n+1}$ which have the same contacts with hyperplanes as the hyperspherical submanifolds.*

In the next section, we give a partial answer to this question, by considering submanifolds which admit a non-degenerate parallel normal vector field.

§3. Submanifolds with a non-degenerate parallel normal vector field

Let M be a smooth immersed m-dimensional submanifold in $\mathbb{R}^n$. We consider in M the riemannian metric induced by the euclidean metric of $\mathbb{R}^n$. Given a point $p \in M$, we have a decomposition $\mathbb{R}^n = T_pM \oplus T_pM^\perp$ and the corresponding orthogonal projections $\top : \mathbb{R}^n \to T_pM$ and $\perp : \mathbb{R}^n \to T_pM^\perp$. For vector fields X, Y tangent along M in a neighbourhood of p, we have

$$\nabla'_{X_p} Y = \top(\nabla'_{X_p} Y) + \perp(\nabla'_{X_p} Y),$$

where ∇' is the covariant derivative in $\mathbb{R}^n$. It follows that $\top(\nabla'_{X_p} Y) = \nabla_{X_p} Y$, where ∇ is the covariant derivative in M induced by the metric, while $\perp(\nabla'_{X_p} Y) = s(X_p, Y_p)$ is symmetric in X_p and Y_p (and independent of the extension Y of Y_p). This gives us the Gauss formula,

$$\nabla'_{X_p} Y = \nabla_{X_p} Y + s(X_p, Y_p).$$

Analogously, if ν is a normal vector field along M in a neighbourhood of p, we have a similar decomposition

$$\nabla'_{X_p} \nu = \top(\nabla'_{X_p} \nu) + \perp(\nabla'_{X_p} \nu).$$

The tangential component satisfies

$$\langle \top(\nabla'_{X_p} \nu), Y_p \rangle = \langle \nabla'_{X_p} \nu, Y_p \rangle = -\langle \nu_p, s(X_p, Y_p) \rangle,$$

and consequently, $\top(\nabla'_{X_p} \nu)$ depends only on X_p and ν_p. Now, for each normal vector $\nu_p \in T_pM^\perp$, we can define the self-adjoint linear map $A_{\nu_p} : T_pM \to T_pM$ by

$$A_{\nu_p}(X_p) = -\top(\nabla'_{X_p} \nu),$$

where ν is any normal vector field extending ν_p. We also define the second fundamental form II_{ν_p} as

$$\mathrm{II}_{\nu_p}(X_p, Y_p) = \langle A_{\nu_p}(X_p), Y_p \rangle = \langle s(X_p, Y_p), \nu_p \rangle, \quad X_p, Y_p \in T_pM.$$

For the normal component $\perp(\nabla'_{X_p} \nu)$, we will denote it by $D_{X_p} \nu$ so that it defines a connection on the normal bundle of M in $\mathbb{R}^n$ called the normal connection. With this notation, the decomposition of $\nabla'_{X_p}\nu$ can be written as

$$\nabla'_{X_p}\nu = -A_{\nu_p}(X_p) + D_{X_p}\nu,$$

which is called Weingarten equation.

Definition 3.1. We say that a normal vector field ν is *parallel* if $D_{X_p}\nu = 0$ for any $X_p \in T_pM$ and for any $p \in M$.

Definition 3.2. We say that a normal vector $\nu_p \in T_pM^\perp$ is *non-degenerate* if the self-adjoint linear map A_{ν_p} (or equivalently the second fundamental form II_{ν_p}) is non-degenerate. We say that a normal vector field ν is *non-degenerate* if it is non-degenerate at any point.

It follows from the definition that a parallel normal vector field ν has always constant length. Since D defines a connection on the normal bundle, we have

$$X_p\langle\nu,\nu\rangle = 2\langle\nu, D_{X_p}\nu\rangle = 0,$$

for any $X_p \in T_pM$. Hence $\langle\nu,\nu\rangle$ is a constant function.

If ν is a normal vector field on M with constant length, we can assume without loss of generality that it is unitary. Then, by translating the normal vector ν_p to the origin of $\mathbb{R}^n$ we have the Gauss map $G_\nu : M \to S^{n-1}$. That is, let $(x_1, \dots, x_n)$ be the standard coordinates of $\mathbb{R}^n$, so that $\nu = \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ for some smooth functions ν_i on M. Then G_ν is equal to the map $(\nu_1, \dots, \nu_n)$. We characterize the parallel and non-degenerate conditions in terms of the differential or tangent map $G_{\nu *} : T_pM \to T_{\nu_p}S^{n-1}$.

Lemma 3.3. *Let ν be a unit normal vector field on M. Then,*

(1) *ν is parallel if and only if $G_{\nu *}(T_pM) \subset T_pM$, for any $p \in M$.*
(2) *ν is parallel and non-degenerate if and only if $G_{\nu *}(T_pM) = T_pM$, for any $p \in M$.*

Proof. For any $p \in M$ and for any $X_p \in T_pM$, it is obvious that

$$G_{\nu *}(X_p) = \nabla'_{X_p}\nu = -A_{\nu_p}(X_p) + D_{X_p}\nu.$$

Then, (1) and (2) follow directly from the definitions. Q.E.D.

In fact, in the case of a hypersurface, the constant length condition is also sufficient for a normal vector field to be parallel. Since the normal bundle is 1-dimensional in this case, it is obvious that $\langle\nu, D_{X_p}\nu\rangle = 0$ implies $D_{X_p}\nu = 0$, for any $X_p \in T_pM$. Thus, any hypersurface has always a local parallel normal vector field. Moreover, if it is orientable, then there is a global parallel normal vector field. Finally, it is also non-degenerate if and only if the gaussian curvature K is not zero.

In the case of a Frenet curve in $\mathbb{R}^n$, there always exists a parallel normal vector field. Let $\alpha : I \to \mathbb{R}^n$ be a smooth curve such that $\alpha'(t), \ldots, \alpha^{(n-1)}(t)$ are linearly independent at any $t \in I$. We assume that α is parametrized by arc length and we denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the Frenet frame and $\kappa_1, \ldots, \kappa_{n-1}$ the curvatures. Let $\nu = \sum_{i=2}^n a_i \mathbf{e}_i$ be a normal vector field. Then, Frenet equations give

$$\begin{aligned}
\nu' =& \sum_{i=2}^{n-1} \big(a_i'\mathbf{e}_i + a_i(-\kappa_{i-1}\mathbf{e}_{i-1} + \kappa_i\mathbf{e}_{i+1})\big) + a_n'\mathbf{e}_n + a_n(-\kappa_{n-1}\mathbf{e}_{n-2}) \\
=& -a_2\kappa_1\mathbf{e}_1 + (a_2' - a_3\kappa_2)\mathbf{e}_2 + \sum_{i=3}^{n-1}(a_i' + a_{i-1}\kappa_{i-1} - a_{i+1}\kappa_i)\mathbf{e}_i \\
&+ (a_n' + a_{n-1}\kappa_{n-1})\mathbf{e}_n.
\end{aligned}$$

Thus, ν is parallel if and only if $a_2, \ldots, a_n$ are a solution of the following system of ordinary differential equations:

$$\text{(1)} \qquad \begin{array}{lcl}
a_2' - a_3\kappa_2 & = & 0, \\
a_3' + a_2\kappa_2 - a_4\kappa_3 & = & 0, \\
\cdots & & \\
a_{n-1}' + a_{n-2}\kappa_{n-2} - a_n\kappa_{n-1} & = & 0, \\
a_n' + a_{n-1}\kappa_{n-1} & = & 0.
\end{array}$$

Finally, note that if ν is parallel, then $\nu' = -a_2\kappa_1\mathbf{e}_1$. Hence, it is non-degenerate if and only if both a_2 and κ_1 are not zero (note that $\kappa_1 > 0$ if $n \geq 3$).

In general, if M is an immersed submanifold of dimension m in $\mathbb{R}^n$, with $1 < m < n$, a local parallel normal vector field does not always exist (see Section 5). However, it is obvious that if M is contained in a hyperplane $\pi(x_0, v)$ of $\mathbb{R}^n$, the constant normal vector field v is parallel. Analogously, if M is contained in a hypersphere $S(p, R)$ of $\mathbb{R}^n$, then the outward unit normal vector field of the hypersphere restricted to M is parallel and non-degenerate (in this case, G_ν is the inclusion map).

Corollary 3.4. *Let ν be a non-degenerate parallel unit normal vector field on M. Then the Gauss map $G_\nu : M \to S^{n-1}$ is an immersion whose image, $M^\nu = G_\nu(M)$, satisfies that $T_pM = T_{\nu_p}M^\nu$, for any $p \in M$.*

The condition $T_pM = T_{\nu_p}M^\nu$ can be seen as some kind of "parallelism" between M and the hyperspherical submanifold M^ν. Next proposition shows that this hyperspherical submanifold M^ν has the same contact type $\Sigma^{\mathbf{i}}$ with hyperplanes as the original submanifold M, thus giving a partial answer to Question 2.12.

Proposition 3.5. *Let ν be a non-degenerate parallel unit normal vector field on M. If M has contact type $\Sigma^{\mathbf{i}}$ with a hyperplane $\pi(p, v)$ at $p \in M$, then M^ν has the same contact type $\Sigma^{\mathbf{i}}$ with the translated hyperplane $\pi(\nu_p, v)$ at $\nu_p \in M^\nu$.*

Proof. Assume that M is locally parametrized in a neighbourhood of p by the immersion $g : U \subset \mathbb{R}^m \to \mathbb{R}^n$, with $g(u_0) = p$. Then, the contact between M and $\pi(p, v)$ is determined by the $\mathcal{K}$-class at u_0 of the height function $h_v : U \to \mathbb{R}$ given by $h_v(u) = \langle g(u), v \rangle$. Analogously, to study the contact between M^ν and $\pi(\nu_p, v)$ we consider $h_v^\nu : U \to \mathbb{R}$ given by $h_v^\nu(u) = \langle G_\nu(g(u)), v \rangle$.

Since ν is parallel, this means that

$$\frac{\partial G_\nu \circ g}{\partial u_i} = \nabla'_{\frac{\partial}{\partial u_i}} \nu \circ g = \sum_{j=1}^{m} a_{ij} \frac{\partial g}{\partial u_j},$$

for some smooth functions a_{ij}. Moreover, the fact that it is non-degenerate implies that $\det(a_{ij}) \neq 0$.

Hence, we also have that

$$\frac{h_v^\nu}{\partial u_i} = \sum_{j=1}^{m} a_{ij} \frac{h_v}{\partial u_j},$$

and the result is a consequence of Lemma 2.7, since this condition is equivalent to $J_m(h_v) = J_m(h_v^\nu)$ when considered as function germs from $(\mathbb{R}^m, u_0)$ to $\mathbb{R}$. Q.E.D.

In general, it is not true that M^ν has the same contact with hyperplanes as the original sumbanifold M. For instance, if $M \subset \mathbb{R}^3$ is a surface with gaussian curvature $K < 0$, then the corresponding height function is $\mathcal{K}$-equivalent to $x^2 - y^2$. However, the image of the Gauss map M^ν is an open subset of the sphere S^2 and the contact with the tangent plane is given by the height function $x^2 + y^2$.

§4. Curves in $\mathbb{R}^3$

Let $\alpha : I \to \mathbb{R}^3$ be a regular space curve with non-vanishing curvature, so that it has a well defined Frenet frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We also denote by κ, τ the curvature and the torsion of α respectively. Assume that α is parametrized by arc length. A normal unit vector field is given by $\nu = \cos\theta \mathbf{e}_2 + \sin\theta \mathbf{e}_3$.

Proposition 4.1. *A normal unit vector field* $\nu = \cos\theta \mathbf{e}_2 + \sin\theta \mathbf{e}_3$ *is parallel if and only if* $\theta' = -\tau$. *Moreover, it is also non-degenerate if and only if* $\cos\theta \neq 0$.

Proof. In this case, the system of differential equations (1) is

$$-\theta' \sin\theta - \tau \sin\theta = 0,$$
$$\theta' \cos\theta + \tau \cos\theta = 0,$$

which reduces to $\theta' = -\tau$. For the second part, just note that if ν is parallel, then $\nu' = -\kappa \cos\theta \mathbf{e}_1$, with $\kappa > 0$. Q.E.D.

As a consequence, we have that a parallel normal unit vector field always exists for a space curve and it is unique up to rotation in the normal plane. Moreover, since we can take the initial condition $\cos\theta_0 \neq 0$, we can choose the parallel vector to be non-degenerate in a neighbourhood of each point of the curve.

Assume that ν is parallel and non-degenerate. Then the "parallel" curve $\alpha^\nu : I \to S^2$ is nothing but the spherical indicatrix of ν. According to Proposition 3.5, α^ν has the same contact type $\Sigma^{\mathbf{i}}$ with planes than the original curve α. In particular, α has a $\Sigma^{1,1,1}$ contact with its osculating plane at a point if and only if α^ν has the same contact type $\Sigma^{1,1,1}$ with the translated plane at the corresponding point. Hence, $t \in I$ is a flattening (i.e., $\tau(t) = 0$) of α if and only if it is a flattening of α^ν.

The classical four vertex theorem for plane curves states that any regular closed and simple plane curve has at least four vertices. By taking stereographic projection this is equivalent to say that any regular closed and simple space curve contained in the sphere S^2 has at least four flattenings. This has been generalized in different ways by several authors (see [1, 2, 11, 15]) for convex space curves, although they do not use the same definition of convexity.

As a corollary of our computations we obtain one more different extension of the Four Vertex Theorem.

Corollary 4.2. *Let* $\alpha : I \to \mathbb{R}^3$ *be a regular, simple and closed space curve with non-vanishing curvature. Assume that the parallel curve* α^ν *is also regular, simple and closed. Then* α *has at least four flattenings.*

The regularity condition on α^ν is just the non-degeneracy of ν. Although we can choose ν so that it is non-degenerate in a neighbouhood of each point, it is not true that there always exists a parallel normal vector field which is globally non-degenerate. On the other hand, the condition that α^ν is closed is equivalent to the vanishing of the total torsion of α, that is, $\int_I \tau = 0$, which implies the existence of at least two flattenings.

§5. Totally semi-umbilic surfaces in $\mathbb{R}^4$

Let M be a smooth surface immersed in $\mathbb{R}^4$. Given a normal vector $\nu \in T_pM^\perp$, we define the ν-principal directions and the ν-principal curvatures to be the unit eigenvectors and corresponding eigenvalues for the self-adjoint linear map $A_\nu : T_pM \to T_pM$.

Definition 5.1. A point p of a smooth immersed surface M in $\mathbb{R}^4$ is said to be *semi-umbilic* if there is a non-zero normal vector $\nu \in T_pM^\perp$ such that the ν-principal curvatures are equal. We say that p is *umbilic* if the ν-principal curvatures are equal for any normal vector $\nu \in T_pM^\perp$. We say that M is *totally semi-umbilic* (respectively *totally umbilic*) if all its points are semi-umbilic (respectively umbilic).

Assume that M is locally parameterized as the image of a smooth immersion $\mathbf{x} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R}^2$ is an open set. We denote by u, v the coordinates in $\mathbb{R}^2$ and by $\mathbf{x}_u, \mathbf{x}_v$. the partial derivatives of $\mathbf{x}$ with respect to these coordinates. Then, the first fundamental form is given in local coordinates by

$$\mathrm{I} = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

Moreover, for any normal vector $\nu \in T_pM^\perp$, the second fundamental form can be expressed as

$$\mathrm{II}_\nu = a_\nu du^2 + 2b_\nu dudv + c_\nu dv^2,$$

with coefficients

$$a_\nu = \langle \mathbf{x}_{uu}, \nu \rangle, \quad b_\nu = \langle \mathbf{x}_{uv}, \nu \rangle, \quad c_\nu = \langle \mathbf{x}_{vv}, \nu \rangle.$$

Then, it follows that the ν-principal directions can be computed as the null directions of the quadratic form:

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_\nu & b_\nu & c_\nu \\ E & F & G \end{vmatrix}.$$

Example 5.2. Every hyperspherical surface M immersed in $\mathbb{R}^4$ is totally semi-umbilic. In fact, if M is contained in a hypersphere of $\mathbb{R}^4$ with center $p \in \mathbb{R}^4$ and radius $R > 0$ a simple computation shows that the principal curvatures with respect to some unit normal vector to the hypersphere are either both equal to $1/R$ or both equal to $-1/R$ (depending on the chosen normal vector).

Analogously, if M is contained in some hyperplane of $\mathbb{R}^4$, the principal curvatures with respect to any normal vector to the hyperplane are both equal to zero and hence, M is totally semi-umbilic.

Finally, note that there are semi-umbilic surfaces which are not contained in a hypersphere nor a hyperplane. For instance, consider two plane regular curves $\alpha : I \to \mathbb{R}^2$ and $\beta : J \to \mathbb{R}^2$. Then $\mathbf{x} = \alpha \times \beta : I \times J \to \mathbb{R}^4$ parameterizes a semi-umbilic surface. For simplicity, we assume that both α and β are parameterized by arc-length. Since $\mathbf{x}_u = (\alpha', 0)$ and $\mathbf{x}_v = (0, \beta')$, this implies that $E = G = 1$ and $F = 0$. Let us denote by $n_\alpha, n_\beta, \kappa_\alpha, \kappa_\beta$ the normal vectors and the curvatures of α, β respectively. If $\kappa_\alpha^2 + \kappa_\beta^2 > 0$, we consider $\nu = (\kappa_\beta n_\alpha, \kappa_\alpha n_\beta)$ so that both ν-principal curvatures are equal to $\kappa_\alpha\kappa_\beta$. Otherwise, if $\kappa_\alpha = \kappa_\beta = 0$, we consider $\nu = (n_\alpha, n_\beta)$ and the corresponding principal curvatures are both equal to zero.

We recall now the concept of curvature ellipse of an immersed surface M in $\mathbb{R}^4$. Given a point $p \in M$, we consider the unit circle in T_pM parameterized by the angle $\theta \in [0, 2\pi]$. Let γ_θ be the curve obtained by intersecting M with the hyperplane at p given by the direct sum of the normal plane $T_pM^\perp$ and the straight line in the tangent direction represented by θ. Such curve is called the *normal section* of M in the direction θ. The curvature vector $\eta(\theta)$ of γ_θ in p lies in $T_pM^\perp$. Varying θ from 0 to 2π, this vector describes an ellipse in $T_pM^\perp$, called the *curvature ellipse* of M at p. A tangent direction represented by the angle θ is called an *asymptotic direction* (or *conjugate direction* in the terminology of Little [8]) if $\eta(\theta)$ and $\frac{d\eta}{d\theta}(\theta)$ are collinear.

It is possible to characterize the asymptotic directions as those directions θ such that the line in $T_pM^\perp$ joining the origin p with $\eta(\theta)$ is tangent to the curvature ellipse at such point. Thus, if the curvature ellipse is not a radial segment, we can have three cases (Figure 1):

(1) The origin p lies outside the curvature ellipse. There are exactly two asymptotic directions and the point p is called *hyperbolic.*
(2) The origin p lies on the curvature ellipse. There is only one asymptotic direction and the point p is called *parabolic.*
(3) The origin p lies inside the curvature ellipse. There are no asymptotic directions and the point p is called *elliptic.*

Finally, in the case that the curvature ellipse degenerates to a radial segment, it follows that all the directions are asymptotic and the point

p is called an *inflection.* An inflection is said to be of real type when p belongs to the curvature ellipse and of imaginary type when it does not.

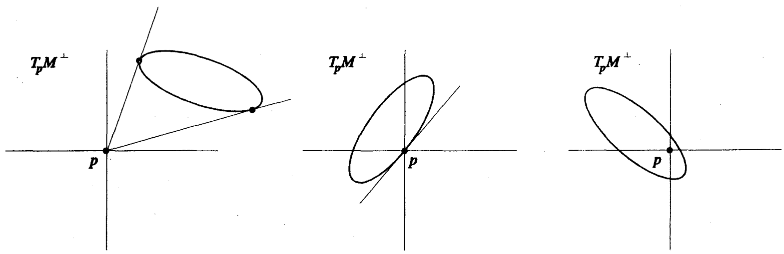

Fig. 1

In local coordinates, the asymptotic directions are computed by means of the quadratic equation

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_{\nu_1} & b_{\nu_1} & c_{\nu_1} \\ a_{\nu_2} & b_{\nu_2} & c_{\nu_2} \end{vmatrix} = 0,$$

being ν_1, ν_2 some orthonormal frame of the normal plane $T_pM^\perp$. Moreover, the inflections correspond to the singular points of the above differential equation, that is, the points where the matrix

$$\begin{pmatrix} a_{\nu_1} & b_{\nu_1} & c_{\nu_1} \\ a_{\nu_2} & b_{\nu_2} & c_{\nu_2} \end{pmatrix}$$

has rank ≤ 1.

The following theorem [14] gives characterizations of the semi-umbilic points of an immersed surface in $\mathbb{R}^4$ in terms of the curvature ellipse and asymptotic lines.

Theorem 5.3. *Let M be an immersed surface in $\mathbb{R}^4$ and let $p \in M$. The following are equivalent conditions:*

(1) *p is semi-umbilic.*
(2) *The curvature ellipse at p degenerates to a segment.*
(3) *There are two orthogonal asymptotic directions at p.*

Using this theorem, we give a characterization of a totally semi-umbilic surface in terms of the existence of a local parallel normal vector field.

Theorem 5.4. *Let M be an immersed surface in $\mathbb{R}^4$. Then M is totally semi-umbilic if and only if there is a parallel normal vector field defined in a neighbourhood of each point of M.*

Proof. Let $p \in M$ and assume that M is locally parameterized in a neighbourhood of p as the image of a smooth isothermal immersion $\mathbf{x} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R}^2$ is an open set. This means that $E = G$ and $F = 0$. Now we take ν_1, ν_2 an orthonormal frame of the normal plane at each point, so that $\mathbf{x}_u, \mathbf{x}_v, \nu_1, \nu_2$ give an orthogonal frame of $\mathbb{R}^4$.

In order to make computations, we need to take the following coefficients:

$$\begin{aligned}
\nu_{1,u} &= \lambda_{11}\mathbf{x}_u + \lambda_{12}\mathbf{x}_v + \lambda_{13}\nu_2,\\
\nu_{1,v} &= \lambda_{21}\mathbf{x}_u + \lambda_{22}\mathbf{x}_v + \lambda_{23}\nu_2,\\
\nu_{2,u} &= \mu_{11}\mathbf{x}_u + \mu_{12}\mathbf{x}_v + \mu_{13}\nu_1,\\
\nu_{2,v} &= \mu_{21}\mathbf{x}_u + \mu_{22}\mathbf{x}_v + \mu_{23}\nu_1.
\end{aligned}$$

Since $\langle \nu_1, \nu_2 \rangle = 0$, it follows easily that $\mu_{13} = -\lambda_{13}$ and $\mu_{23} = -\lambda_{23}$.

Let now $\nu = A\nu_1 + B\nu_2$ be a normal vector field. We have that if ν is parallel then $\langle \nu, \nu \rangle =$ constant. Hence, we can assume, without loss of generality, that $A^2 + B^2 = 1$. With this assumption, it follows that ν is parallel if and only if

$$\det(\nu_u, \nu, \mathbf{x}_u, \mathbf{x}_v) = \det(\nu_v, \nu, \mathbf{x}_u, \mathbf{x}_v) = 0.$$

By direct computation we get that

$$\begin{aligned}
\det(\nu_u, \nu, \mathbf{x}_u, \mathbf{x}_v) &= (A_uB - B_uA - \lambda_{13}) \det(\nu_1, \nu_2, \mathbf{x}_u, \mathbf{x}_v),\\
\det(\nu_v, \nu, \mathbf{x}_u, \mathbf{x}_v) &= (A_vB - B_vA - \lambda_{23}) \det(\nu_1, \nu_2, \mathbf{x}_u, \mathbf{x}_v),
\end{aligned}$$

so that ν is parallel if and only if

$$\begin{aligned}
\lambda_{13} &= A_uB - B_uA,\\
\lambda_{23} &= A_vB - B_vA.
\end{aligned}$$

Finally, it is not difficult to see that this system of PDE's has a solution A, B with $A^2 + B^2 = 1$ if and only if $\lambda_{13,v} = \lambda_{23,u}$. In the second part of the proof, we see that such condition is equivalent to the fact that the asymptotic directions are orthogonal, and hence that M is totally semi-umbilic, by the above theorem.

In fact, since $\lambda_{13} = \langle \nu_{1,u}, \nu_2 \rangle$ and $\lambda_{23} = \langle \nu_{1,v}, \nu_2 \rangle$, we get

$$\begin{aligned}
\lambda_{13,v} - \lambda_{23,u} =& \langle \nu_{1,uv}, \nu_2 \rangle + \langle \nu_{1,u}, \nu_{2,v} \rangle - \langle \nu_{1,uv}, \nu_2 \rangle - \langle \nu_{1,v}, \nu_{2,u} \rangle\\
=& \langle \nu_{1,u}, \nu_{2,v} \rangle - \langle \nu_{1,v}, \nu_{2,u} \rangle\\
=& E(\lambda_{11}\mu_{21} + \lambda_{12}\mu_{22} - \lambda_{21}\mu_{11} - \lambda_{22}\mu_{12}).
\end{aligned}$$

In order to simplify the notation we change the notation for the coefficients of the second fundamental forms of ν_1, ν_2 in the following way:

$$\begin{aligned} a &= a_{\nu_1}, \quad & b &= b_{\nu_1}, \quad & c &= c_{\nu_1}, \\ e &= a_{\nu_2}, \quad & f &= b_{\nu_2}, \quad & g &= c_{\nu_2}. \end{aligned}$$

Then, since $\langle \mathbf{x}_u, \nu_i \rangle = \langle \mathbf{x}_v, \nu_i \rangle = 0$, we deduce that

$$\begin{aligned} a &= \langle \mathbf{x}_{uu}, \nu_1 \rangle = -\langle \mathbf{x}_u, \nu_{1,u} \rangle = -E\lambda_{11}, \\ e &= \langle \mathbf{x}_{uu}, \nu_2 \rangle = -\langle \mathbf{x}_u, \nu_{2,u} \rangle = -E\mu_{11}, \\ b &= \langle \mathbf{x}_{uv}, \nu_1 \rangle = -\langle \mathbf{x}_u, \nu_{1,v} \rangle = -E\lambda_{21} = -\langle \mathbf{x}_v, \nu_{1,u} \rangle = -E\lambda_{12}, \\ f &= \langle \mathbf{x}_{uv}, \nu_2 \rangle = -\langle \mathbf{x}_u, \nu_{2,v} \rangle = -E\mu_{21} = -\langle \mathbf{x}_v, \nu_{2,u} \rangle = -E\mu_{12}, \\ c &= \langle \mathbf{x}_{vv}, \nu_1 \rangle = -\langle \mathbf{x}_v, \nu_{1,v} \rangle = -E\lambda_{22}, \\ g &= \langle \mathbf{x}_{vv}, \nu_1 \rangle = -\langle \mathbf{x}_v, \nu_{1,v} \rangle = -E\mu_{22}. \end{aligned}$$

Using this, we conclude that $\lambda_{13,v} = \lambda_{23,u}$ if and only if

$$(a-c)f = (e-g)b.$$

On the other hand, if we look at the differential equation of the asymptotic lines

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a & b & c \\ e & f & g \end{vmatrix} = \begin{vmatrix} b & c \\ f & g \end{vmatrix} dv^2 + \begin{vmatrix} a & c \\ e & g \end{vmatrix} dudv + \begin{vmatrix} a & b \\ e & f \end{vmatrix} du^2 = 0,$$

we see that there are two orthogonal asymptotic lines at each point if and only if

$$\begin{vmatrix} b & c \\ f & g \end{vmatrix} = - \begin{vmatrix} a & b \\ e & f \end{vmatrix},$$

which is in fact equivalent to the above condition $(a-c)f = (e-g)b$.
Q.E.D.

Now we see that if we also impose the condition that the gaussian curvature K of M is not zero, then we can choose the local parallel vector field to be non-degenerate.

Theorem 5.5. *Let M be a totally semi-umbilic surface immersed in $\mathbb{R}^4$ with $K \neq 0$. Then, there is a non-degenerate parallel normal vector field defined in a neighbourhood of each point of M.*

Proof. Assume that M is locally parametrized in a neighbourhood of p as the image of a smooth isothermal immersion $\mathbf{x} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R}^2$ is some open set. By Theorem 5.4, there is a parallel unit normal vector field ν and we take another unit normal vector field ξ so that ν, ξ is an orthonormal frame of the normal plane at each point and $\mathbf{x}_u, \mathbf{x}_v, \nu, \xi$ is an orthogonal frame of $\mathbb{R}^4$.

Since $\langle \nu, \xi \rangle = 0$ it follows that $\langle \xi_u, \nu \rangle = -\langle \xi, \nu_u \rangle = 0$ and $\langle \xi_v, \nu \rangle = -\langle \xi, \nu_v \rangle = 0$. This shows that ξ is also parallel. Therefore, we can write

$$\begin{aligned} \nu_u &= \lambda_{11}\mathbf{x}_u + \lambda_{12}\mathbf{x}_v, \\ \nu_v &= \lambda_{21}\mathbf{x}_u + \lambda_{22}\mathbf{x}_v, \\ \xi_u &= \mu_{11}\mathbf{x}_u + \mu_{12}\mathbf{x}_v, \\ \xi_v &= \mu_{21}\mathbf{x}_u + \mu_{22}\mathbf{x}_v, \end{aligned}$$

for some coefficients λ_{ij} and μ_{ij}. In the proof of Theorem 5.4 we showed that these coefficients are given by

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = -\frac{1}{E}\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} = -\frac{1}{E}\begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

where a, b, c and e, f, g are the coefficients of the second fundamental forms of the normal vectors ν and ξ respectively.

On the other hand, we can use the Gauss equation (see [8]) which gives the gaussian curvature K in terms of these coefficients:

$$K = \frac{1}{E^2}(ac - b^2 + eg - f^2).$$

In particular, if $K \neq 0$, it follows that either $ac - b^2 \neq 0$ or $eg - f^2 \neq 0$ and hence, either ν is an immersion or ξ is an immersion. Q.E.D.

Theorem 5.6. *Let M be a totally semi-umbilic surface immersed in $\mathbb{R}^4$ with $K \neq 0$. Assume that $G_\nu : W \to S^3$ is the Gauss map of a non-degenerate parallel unit normal vector field ν defined in a neighbourhood W of $p \in M$. Then G_ν preserves the inflections and the asymptotic directions between W and the image W^ν.*

Proof. We use the same notation as in the proof of the above theorem. Since ν is parallel, it follows that for any $q \in W$, the normal plane to W^ν at ν_q coincides with the normal plane to W at q. Thus, we can also take ν, ξ as an orthonormal frame of the normal plane of W^ν.

Remember that, according to the proof of the above theorem, we have

$$\begin{aligned}
\nu_u &= -\frac{1}{E}(a\mathbf{x}_u + b\mathbf{x}_v),\\
\nu_v &= -\frac{1}{E}(b\mathbf{x}_u + c\mathbf{x}_v),\\
\xi_u &= -\frac{1}{E}(e\mathbf{x}_u + f\mathbf{x}_v),\\
\xi_v &= -\frac{1}{E}(f\mathbf{x}_u + g\mathbf{x}_v),
\end{aligned}$$

where $\mathbf{x}_u, \mathbf{x}_v, \nu, \xi$ is the orthogonal frame adapted to the original surface M. Thus, it follows that the coefficients of the second fundamental form of W^ν with respect to ν are

$$\begin{aligned}
\langle \nu_{uu}, \nu\rangle &= -\langle \nu_u, \nu_u\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, a\mathbf{x}_u + b\mathbf{x}_v\rangle = -\frac{1}{E}(a^2+b^2),\\
\langle \nu_{uv}, \nu\rangle &= -\langle \nu_u, \nu_v\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, b\mathbf{x}_u + c\mathbf{x}_v\rangle = -\frac{1}{E}(ab+bc),\\
\langle \nu_{vv}, \nu\rangle &= -\langle \nu_v, \nu_v\rangle = -\frac{1}{E^2}\langle b\mathbf{x}_u + c\mathbf{x}_v, b\mathbf{x}_u + c\mathbf{x}_v\rangle = -\frac{1}{E}(b^2+c^2).
\end{aligned}$$

We compute now the coefficients of the second fundamental form with respect to ξ:

$$\begin{aligned}
\langle \nu_{uu}, \xi\rangle &= -\langle \nu_u, \xi_u\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, e\mathbf{x}_u + f\mathbf{x}_v\rangle = -\frac{1}{E}(ae+bf),\\
\langle \nu_{uv}, \xi\rangle &= -\langle \nu_u, \xi_v\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, f\mathbf{x}_u + g\mathbf{x}_v\rangle = -\frac{1}{E}(af+bg),\\
\langle \nu_{vv}, \xi\rangle &= -\langle \nu_v, \xi_v\rangle = -\frac{1}{E^2}\langle b\mathbf{x}_u + c\mathbf{x}_v, f\mathbf{x}_u + g\mathbf{x}_v\rangle = -\frac{1}{E}(bf+cg).
\end{aligned}$$

Now, it is easy to write down the differential equation for the asymptotic directions of W^ν,

$$\frac{1}{E^2}\begin{vmatrix} dv^2 & -dudv & du^2\\ a^2+b^2 & ab+bc & b^2+c^2\\ ae+bf & af+bg & bf+cg \end{vmatrix} = 0.$$

In the proof of Theorem 5.4, we showed that if M is semi-umbilic then $(a-c)f = (e-g)b$. By using this condition, it follows that

$$\frac{1}{E^2}\begin{vmatrix} dv^2 & -dudv & du^2\\ a^2+b^2 & ab+bc & b^2+c^2\\ ae+bf & af+bg & bf+cg \end{vmatrix} = \frac{ac-b^2}{E^2}\begin{vmatrix} dv^2 & -dudv & du^2\\ a & b & c\\ e & f & g \end{vmatrix}.$$

Finally, note that the fact that ν is non-degenerate implies that $ac-b^2 \neq 0$, which completes the proof. Q.E.D.

Finally, we consider the composition of the Gauss map of the parallel unit normal vector field ν with the stereographic projection from S^3 into $\mathbb{R}^3$. It follows that such a map transforms the asymptotic lines of M into the principal lines of its image in $\mathbb{R}^3$ and the inflections into the umbilics. Hence, we get as a direct consequence the following extension of the Loewner and Carathéodory conjectures for totally semi-umbilic analytic surfaces immersed in $\mathbb{R}^4$ with $K \neq 0$.

Corollary 5.7. *Let M be a totally semi-umbilic analytic surface immersed in $\mathbb{R}^4$ with $K \neq 0$. Then the index of the asymptotic foliation at an isolated inflection of M is always ≤ 1.*

Corollary 5.8. *Let M be a totally semi-umbilic analytic surface immersed in $\mathbb{R}^4$ with $K \neq 0$ and assume that M is homeomorphic to S^2. Then M has at least two inflections.*

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