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Thom polynomials

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Abstract.

By (generalized) Thom polynomials we mean universal cohomology characteristic classes that express Poincaré duals to the singularity loci appearing in various context: singularities of maps, hypersurface singularities, complete intersection singularities, Lagrange and Legendre singularities, multisingularities, etc. In these notes we give a short review of the whole theory with a special account of discoveries of last years. We discuss existence of Thom polynomials, methods of their computations, relation between Thom polynomials for different classifications. Some of the theorems announced here are new and their proofs are not published yet. Some of known results acquire a new interpretation.

§1. Introduction

Theorems of the global singularity theory relate global topological invariants of manifolds, bundles, etc. to the geometry of singularities of various differential geometry structures. The classical example is the Poncaré theorem that relates the Euler characteristic of a manifold to the singular points of a generic vector field on it. Many classical relations in algebraic geometry like Rieman-Hurwitz or Plücker formulas for algebraic curves can also be considered as theorems of global singularity theory.

As a separate theory, the global singularity theory appeared in the 60s after R. Thom's observation that the cohomology classes Poincaré dual to the cycles of singularities of smooth maps can be expressed as

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universal polynomials (called later Thom polynomials) in the Stiefel-Whitney classes of manifolds [39]. Though Thom used topological arguments, the computation of particular Thom polynomials have been accomplished in 60–70s using the algebraic geometry methods of blowups, residue intersections, etc. (see references in the review [4]).

The first step towards the general study of multisingularities has been done by S. Kleiman [24, 25]. His theory of multiple points has been constructed entirely in the framework of the intersection theory. For technical reasons, Kleiman's formulas can be applied if the map admits only singularities of corank 1 (or if the singularities of corank greater than 1 can be ignored by dimensional reasons) but even in this case Kleiman's theory found many interesting applications [6, 7].

Topological methods in the study of global properties of singularities have been developing independently by two groups. In Moscow, Vassiliev [41] inspired by the ideas of Arnold [2] has created the theory of characteristic classes for real Lagrange and Legendre singularities. He introduced the universal complex of singularity classes which allows one to chose those singularity classes in the real problems for which Poincaré dual cohomology class is well defined. The Vassiliev universal complex has been generalized by M. Kazarian [17, 18, 19, 20] to the characteristic spectral sequence which contains all cohomological information about adjacencies of singularities.

About the same time in Budapest A. Szücs developed his theory of cobordisms of maps with prescribed collections of allowed singularities [35, 36, 37]. He constructed classifying spaces for this type of cobordisms by gluing the classifying spaces of the symmetry groups of various singularities. Following Szücs, R. Rimányi [31, 32, 33] showed that the collection of allowed singularities can be extended at least to the set of all stable map germs. He noticed also that the gluing construction provides an effective method for computing Thom polynomials which does not require the detailed geometric study of the singularities. He demonstrated the efficiency of his restriction method by computing Thom polynomials for essentially all classified singularities.

The two approaches have been developing quite independently until the Oberwolfach Conference 2000 in Singularity theory where A. Szücs introduced the author to his theory. Since that time the two approaches combined providing a very strong counterpart to the methods of intersection theory. It is clear now that the global theory of multisingularities is related to the cobordism theory in the same way as the global theory of monosingularities is related to the theory of characteristic classes of vector bundles. The universal formulas for the classes of multisingularities have been obtained in [22]. These formulas allowed the author to solve in a unified form many enumerative problems that have not been solved by the methods of algebraic geometry.

The discoveries of the last years changed the face of the global singularity theory dramatically. Although the construction of the theory is not completed yet, its general pattern seems to be more or less clear. In these notes I present my own view of the modern state of the theory. Some of the theorems announced here are new and their proofs are not published yet. Some of known results acquire a new interpretation. I have tried to present main formulas in the form ready to be applied to specific enumerative geometric problems. They may provide rich experimental material for further research, even without rigorous justification.

The paper is organized as follows. In Section 2 we discuss the existence theorems for Thom polynomials in different context: singularities of maps (Sect. 2.1 and 2.2), Lie group action (Sect. 2.3), stable $\mathcal{K}$-singularities (Sect. 2.4), isolated hypersurface singularities (Sect. 2.5), and multisingularities (Sect. 2.6 and 2.7). In Sect. 2.8 we discuss some aspects of global singularity theory which are common for all these classifications.

Section 3 is devoted to the detailed study of the structure of Thom polynomials. We introduce the notion of a localized Thom polynomial which is the usual Thom polynomial written in a special additive basis well adjusted to the classification of singularities by corank. It allows one to single out the terms in the Thom polynomial for which closed formulae could be given. In Sect. 3.1 and 3.2 we present formulae for localized terms of Thom polynomials related to singularities of maps. In Sect 3.3 and 3.4 we extend this computation to the case of Lagrange, Legendre, and isolated hypersurface singularities.

Some terms of the Thom polynomials can be computed using the method of resolution of singularities discussed in Section 3. The simplest way to compute remaining terms is to apply the restriction method suggested by Rimányi. It is discussed in Section 4. The method itself is described in Sect. 4.1. In Sect. 4.2 we explain some details of the extension of this method to the study of multisingularities. The results of computations are presented in Sect. 4.3. More complete tables of computed Thom polynomials are available in [23].

§2. Existence of Thom polynomials

2.1. Characteristic classes of singularities

In the most general form the problems of the global singularity theory are often formulated in the following way. Suppose we are given a parameter (or moduli) space M whose points parameterize some geometric objects: varieties, maps, fields, configurations, etc. Generic parameter values correspond to non-degenerate objects. These values form an open subspace $M_0 \subset M$. The complement $M \setminus M_0$ consists of degenerate objects. It is stratified according to the possible degeneracy types. Local singularity theory studies local behavior of these degenerations, normal forms, adjacencies, etc. Denote by $M(\alpha) \subset M$ the closure of the locus of points with prescribed degeneracy type α. Then the problem is to find the cohomology class

$$[M(\alpha)] \in H^*(M)$$

Poincaré dual to the cycle $M(\alpha)$. For example, if $M(\alpha)$ consists of finite number of points, then the problem is just to find the number of these points.

The general answer to this problem suggested by singularity theory is as follows. To each classification problem $\mathcal{S}$ one associates an appropriate 'classifying space' $B\mathcal{S}$. The classifying space is equipped with the natural stratification. The strata of this stratification are labelled by various singularity classes of the classification.

Consider the underlying manifold M of a particular geometric problem. Assume that the degeneracies associated with the points of M are classified with respect to the given classification $\mathcal{S}$. Then one constructs the 'classifying map'

$$\varkappa : M \to B\mathcal{S}$$

such that the partition on M is induced from the partition on $B\mathcal{S}$ by the map $\varkappa$.

The cohomology ring $\mathcal{C}(\mathcal{S}) = H^*(B\mathcal{S})$ of the classifying space is considered as the ring of 'universal characteristic classes' associated with the classification $\mathcal{S}$. The characteristic homomorphism

$$\varkappa^* : \mathcal{C}(\mathcal{S}) \to H^*(M).$$

is a topological invariant of M. It usually can be computed independently of the study of the singularities of the stratification on M.

Then the general principle says:

each singularity type α determines a universal characteristic class $\mathrm{Tp}_\alpha \in \mathcal{C}(\mathcal{S})$ so that the class $[M(\alpha)]$ is given by this class evaluated at the given parameter space M:

$$[M(\alpha)] = \varkappa^*(\mathrm{Tp}_\alpha).$$

The class Tp_α expressed in terms of the multiplicative generators of the ring $\mathcal{C}(\mathcal{S})$ is referred to as the (generalized) *Thom polynomial* of the singularity α. It can be defined simply as the cohomology class $\mathrm{Tp}_\alpha \in H^*(B\mathcal{S}) = \mathcal{C}(\mathcal{S})$ Poincaré dual to the closure of the stratum of the singularity α on the classifying space.

Thus the solution of the initial problem consists of the following steps.

Step 1. Identify the singularity theory problem $\mathcal{S}$ that reflects the classification of points on M.

Step 2. Determine the ring of universal characteristic classes $\mathcal{C}(\mathcal{S})$ corresponding to this classification problem.

Step 3. Find the Thom polynomial $\mathrm{Tp}_\alpha \in \mathcal{C}(\mathcal{S})$ for a particular singularity type α.

Step 4. Compute the characteristic homomorphism $\varkappa^* : \mathcal{C}(\mathcal{S}) \to H^*(M)$ and the required cohomology class $[M(\alpha)] = \varkappa^*(\mathrm{Tp}_\alpha) \in H^*(M)$.

Every step in this program is usually non-trivial and can be done independently. The aim of these notes is to show how this program can be accomplished in various particular geometric problems of counting singularities and multisingularities of maps, complete intersection singularities, Lagrange and Legendre singularities, critical points of functions, etc.

It is known that the same types of local singularities can appear in a stable way in quite different situations. For example, the famous 'swallowtail' singularity A_k could appear in the context of critical point function singularities, complete intersection singularities, caustic, wave front singularities, and many others. Therefore, the choice of a classification problem is not well formalized and can vary according to the preferences of the author. The variety of known classifications in singularity theory is enormous. Some of them appearing in complex problems, in a sense, the basic ones, are listed in Table 1. One can notice that even for well-studied classifications the final answer for the topology of the classifying space is not evident at all. The detailed explanation of the entries of this table is discussed in the main body of the paper.

2.1. Example. Consider a nonsingular projective subvariety $V \subset \mathbb{C}P^d$, $\dim V = r$. We study the tangency singularities of V with respect to various s-dimensional projective subspaces. All these subspaces form the Grassmann manifold $N = G_{s+1,d+1}$. Denote by $M \subset V \times G_{s+1,d+1}$ the incidence subvariety formed by the pairs (x, λ) such that $x \in \lambda$ and let

$$f : M \to N, \quad (x, \lambda) \mapsto \lambda$$

be the natural projection to the second factor. If V satisfies certain genericity condition, then the map f possesses only standard singularities studied in singularity theory. Thus the singularity theory in

Table 1. **Characteristic classes in singularity theory**

Classification $\mathcal{S}$	Classifying space $B\mathcal{S}$	Characteristic classes, $H^*(B\mathcal{S})$
$(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$	$BU(m)\times BU(n)$	$\mathbb{Z}[c_1,...,c_m,c'_1,...,c'_n]$
Orbits of G-action on an affine space V	BG	Characteristic classes of G-bundles
Stable $\mathcal{K}_\ell$-classification of map germs, $\ell=n-m$	BU	$\mathbb{Z}[c_1,c_2,\dots]$
Classification of critical points; IHS	Stable Lagrange (Legendre) Grassmannian Λ (Λ^{leg})	Lagrange (Legendre) characteristic classes
Multisingularities	Classifying space of complex cobordisms $\Omega^{2m}MU(m+\ell)$, $m\to\infty$	Landweber-Novikov operations $U^{2\ell}(\cdot)\to H^*(\cdot)$
Any classification	'Generalized Pontryagin-Thom-Szücs construction'	Splitting $H^*(B\mathcal{S})=\bigoplus_\alpha H^*(BG_\alpha)$

question is the classification of map germs $(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$, where $m=\dim M=r+s(d-s)$ and $n=\dim N=(s+1)(n-d)$.

By a *singularity class* we mean any non-singular semialgebraic (not necessary closed) subvariety in the k-jet space of map germs $(\mathbb{C}^m,0)\to(\mathbb{C}^n,0)$ which is invariant with respect to the coordinate change of the source and the target manifolds, where k is some large integer. The singularity class may consist of a unique orbit of the group of left-right equivalence, or it may contain a continuous family of non-equivalent orbits.

The ring of characteristic classes for this classification is the ring of polynomials in two groups of variables $c_1, \dots, c_m$ and $c_1', \dots, c_n'$. The characteristic classes corresponding to these variables are the corresponding Chern classes of the manifolds M and N,

$$\varkappa^* c_i = c_i(M), \quad \varkappa^* c_j' = f^* c_j(N).$$

R. Thom formulated the following general statement.

2.2. Theorem ([39, 13]). *The cohomology class on $H^*(M)$ dual to the closure of a particular singularity α is given by a universal polynomial Tp_α in the classes $c_1(M), \dots, c_m(M), f^*c_1(N), \dots, f^*c_n(N)$.*

The proof and the computation of particular Thom polynomials is discussed below. The theorem above holds also in the real case with $\mathbb{Z}_2$-cohomology and the Stiefel-Whitney classes instead of Chern classes. It holds also in the algebraic situation for an arbitrary algebraically closed ground field and Chow groups instead of cohomology. It seems that the algebraic geometry proof that does not relay on topological arguments has never been published. Therefore, we present it here.

2.2. Proof of the existence theorem for Thom polynomials

The standard topological argument used in the proof of Theorem 2.2 is as follows. On the first step we notice that the class $[M(\alpha)]$ is a pull-back of the class dual to the corresponding singularity locus in the jet bundle space $J^k(M,N)$ under the jet extension map $j^k f : M \to J^k(M,N)$. The natural projection $J^k(M,N) \to M \times N$ has contractible fibers

$$V = J_0^k(\mathbb{C}^m, \mathbb{C}^n)$$

and the map $j^k f$ lifts the graph map $\Gamma_f = \mathrm{id} \times f : M \to M \times N$. Therefore, it suffices to prove that the Poincaré dual of the locus $J^k(M,N)(\alpha)$ in the cohomology ring $H^*(J^k(M,N)) = H^*(M \times N)$ is given by a universal polynomial in the Chern classes $p_1^* c_i(M)$ and $p_2^* c_j(N)$, where p_i, $i = 1, 2$, is the projection of $M \times N$ to the corresponding factor. Remark that as a result we have obtained a reformulation of the existence theorem for Thom polynomials that does not involve the original map f at all.

On the second step we notice that $J^k(M,N)$ forms a fiber bundle space over $M \times N$ whose structure group G of k-jets of left-right changes is homotopy equivalent to the group of linear changes,

$$G = J_0^k \mathrm{Diff}(\mathbb{C}^m) \times J_0^k \mathrm{Diff}(\mathbb{C}^n) \simeq GL(m, \mathbb{C}) \times GL(n, \mathbb{C}) \simeq U(m) \times U(n).$$

Therefore, this bundle is a pull-back of the corresponding classifying bundle BV over the classifying space BG of G-bundles:

$$\begin{tikzcd}
 & & J^k(M,N) \arrow[rr, "\varkappa"] \arrow[d, "V"'] & & BV \arrow[d, "V"'] \\
M \arrow[urr, "j^k f"] \arrow[rr, "\Gamma_f"] & & M \times N \arrow[rr] & & BG
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(M) & H^*(J^k(M,N)) \arrow[l, "j^k f^*"'] \arrow[d, equal] & H^*(BV) \arrow[l, "\varkappa^*"'] \arrow[d, equal] \\
 & H^*(M \times N) & H^*(BG)
\end{tikzcd}$$

$$\begin{tikzcd}
[M(\alpha)] & {[J^k(M,N)(\alpha)]} \arrow[l, mapsto, "j^k f^*"'] & \mathrm{Tp}_\alpha \arrow[l, mapsto, "\varkappa^*"']
\end{tikzcd}$$

Respectively, the cohomology class $[J^k(M,N)(\alpha)]$ under consideration is the pull-back of the corresponding class in $H^*(BV)$. Thus, Tp_α is a universal characteristic class

$$\mathrm{Tp}_\alpha \in H^*(U(m) \times U(n)) = \mathbb{Z}[c_1, \dots, c_m, c'_1, \dots, c'_n]$$

and Theorem follows.

Let us show how the topological argument above could be adjusted to the algebraic situation. We need to indicate explicitly an algebraic model for the classifying space BG and an algebraic replacement for the classifying map $\varkappa$.

Fix some K large enough and set $BV_K = G'_{m,K} \times G''_{K-n,K}$, where

—$G'_{m,K}$ is the variety of all k-jets of germs at the origin of non-singular m-dimensional submanifolds in $\mathbb{C}^K$;

—$G''_{n,k}$ is the variety of k-jets of germs at the origin of non-singular foliations in $\mathbb{C}^K$ with n-codimensional fibers.

It is easy to see that BV_K is a non-singular quasiprojective variety. There is a natural projection $BV_K \to G_{m,K} \times G_{K-n,K}$ sending an m-submanifold to its tangent space at the origin and a foliation to the tangent space of the fiber at the origin. This projection is a fibration whose fibers are affine spaces. Therefore,

$$H^*(BV_K) \simeq H^*(G_{m,K} \times G_{K-n,K}),$$

and this stabilizes to the polynomial ring $\mathbb{Z}[c_1, \dots, c_m, c_1', \dots, c_n']$ with the growth of K. Similar statement clearly holds for Chow groups as well.

To each point of BV_K one associates the singularity of the natural projection from the m-dimensional submanifold to the n-dimensional parameter space of the fibers of the foliation. More precisely, only k-jet of this singularity is well defined. Thus, for each singularity type α one associates the corresponding locus $BV_K(\alpha)$ in BV_K.

Let us *define* the Thom polynomial Tp_α as the class of the closure of $BV_K(\alpha)$ in the cohomology (or Chow) group of BV_K. By definition, the element Tp_α is independent of K if K is large enough and is expressed as certain polynomial in the generators c_i, c_j'.

To complete the proof of the algebraic version of Theorem 2.2 we need, for any two given non-singular quasiprojective varieties M and N, to construct a map $\varkappa$ from $J^k(M,N)$ to BV_K that classifies singularities. In the algebraic context such a map can not be constructed in general, but it can be constructed after a suitable modification of the source $J^k(M,N)$.

Denote by $\bar{J}^k_K(M,N)$ the variety whose points are parameterized by the tuples (x,y,j,p) where $x \in M$ and $y \in N$ are some points, j is the k-jet of a map germ $(M,x) \to (\mathbb{C}^K, 0)$, and p is the k-jet of a map germ $(\mathbb{C}^K, 0) \to (N,y)$. Since the choices for j and p form an affine space we get that the cohomology (or Chow) groups of $\bar{J}^k_K(M,N)$ are isomorphic to those of $M \times N$. Passing to the composition $p \circ j : (M,x) \to (N,y)$ determines a natural morphism $\bar{J}^k_K(M,N) \to J^k(M,N)$.

Now, denote by $J^k_K(M,N) \subset \bar{J}^k_K(M,N)$ the open subvariety formed by the tuples (x,y,j,p) such that *j is injective and p is surjective.* Remark that the complement $\bar{J}^k_K(M,N) \setminus J^k_K(M,N)$ has codimension growing to infinity together with K. Therefore, passing to $J^k_K(M,N)$ does affect the cohomology (or Chow groups) in any specified in advance finite range of dimensions, if K is large enough. Over $J^k_K(M,N)$ we have the evident classifying map $\varkappa : J^k_K(M,N) \to BV_K$. This map associates with the injection i and surjection p the (k-jet of the) submanifold $j(M)$ and the foliation formed by the fibers of p, respectively. Thus we get the diagram of mappings

$$M \times N \longleftarrow J^k(M,N) \longleftarrow \bar{J}^k_K(M,N) \longleftarrow J^k_K(M,N) \xrightarrow{\varkappa} BV_K$$

and the induced diagram of homomorphisms in cohomology. Since the first three arrows induce an isomorphism, the universality of the Thom polynomial Tp_α follows from functorial properties of the pull-back homomorphism.

2.3. Remark (cf. [40]). The concept of the classifying space is commonly used in topology. For example, any complex vector bundle $E \to M$ can be induced from the classifying one over BU by some continuous map $\varkappa : M \to BU$. Therefore, from the topological point of view the Chern classes are just pull-backs of certain properly chosen generators in the cohomology ring of the classifying space. In algebraic setting this definition can not be used directly and the Chern classes are usually introduced by means of a direct geometric construction, see e.g. [10]. Let us show that the idea of the classifying map can be applied to the algebraic situation as well.

Let $\bar{M}_K$ be the total space of the bundle $\mathrm{Hom}(E, \mathbb{C}^K)$ and $M_K \subset \bar{M}_K$ be the open submanifold formed by those maps $f_x : E_x \to \mathbb{C}^K$, $x \in M$, which are injective. Let $\pi : M_K \to M$ be the natural projection. The maps f_x provide an embedding of the pull-back bundle $E_K = \pi^* E$ over M_K to the trivial bundle $\mathbb{C}^K$. Thus the fibers of E_K can be treated as n-dimensional subspaces in $\mathbb{C}^K$, where $n = \mathrm{rk}\, E$. This provides the classifying map $\varkappa : M_K \to G_{n,K}$. As above, we get the diagram of maps

$$M \longleftarrow \bar{M}_K \longleftarrow M_K \xrightarrow{\varkappa} G_{n,K}$$

The first two arrows induce isomorphisms in cohomology for K large enough. Therefore, $\varkappa$ can be used to define Chern classes via the characteristic homomorphism

$$\varkappa^* : H^d(G_{n,K}) \to H^d(M_K) \simeq H^d(M), \quad K \gg d.$$

2.3. Classifying space for Lie group action

Many classification problems in singularity theory can be formulated as the classification of orbits of a smooth action of a given Lie group G on a given manifold V. For example, in the case of the classification of map germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ up to the right-left equivalence one has

$$V = J_0^k(\mathbb{C}^m, \mathbb{C}^n)$$

is the space of k-jets of map germs at the origin and

$$G = J_0^k \mathrm{Diff}(\mathbb{C}^m) \times J_0^k \mathrm{Diff}(\mathbb{C}^m)$$

is the group of k-jets of changes of coordinates in the source and target manifolds, respectively, where k is a fixed sufficiently large integer.

In what follows we always assume that *the manifold V is topologically trivial*, that is, as a manifold it is isomorphic to an affine space of appropriate dimension. The theory of linear representations provides a lot of examples of such actions. Remark that in the case of left-right equivalence of map germs the action is not linear.

The construction for the classifying space BV associated with the classification of G-orbits on V and its cohomology group $H^*(BV)$ repeats the well-known Borel's construction of G-equivariant cohomology $H_G^*(V)$. Since V is a contractible topological space, we have

$$H^*(BV) = H_G^*(V) \simeq H_G^*(\mathrm{pt}) = H^*(BG).$$

In other words, the group of characteristic classes associated with the classification of G-orbits on V is actually the group of characteristic classes of G-bundles.

In more details, consider the classifying principle G-bundle $EG \to BG$. It means that EG is a topologically trivial space equipped with the free G-action, and BG is the orbit space of this action. It is well known that the cohomology ring of the classifying space $H^*(BG)$ serves as the ring of characteristic classes of G-bundles. Consider the diagonal G-action on the product space $V \times EG$.

2.4. Definition. *The classifying space of* (G, V)*-action* is defined as the orbit space

$$BV = V \times_G BE = (V \times BE)/G.$$

The fibers of the natural projection $\pi : BV \to BG$ are isomorphic to V. Therefore, BV can be interpreted as the total space of the bundle over BG with the fiber V and the structure group G associated with the principle G-bundle $EG \to BG$. Since π has contractible fibers, it induces the mentioned above isomorphism $H^*(BV) \simeq H^*(BG)$.

The spaces EG, BG, BV are usually infinite-dimensional. In practice, it is more convenient to replace them by smooth finite-dimensional approximations. Namely, consider the sequence of smooth principle G-bundles $EG_K \to BG_K$ with $K \to \infty$, such that the manifold EG_K is K-connected. Then the manifolds $BV_K = V_K \times_G EG$ can be considered as finite-dimensional approximations for the classifying space BG and the isomorphisms

$$H^p(BV_K) \simeq H^p(BG_K) \simeq H^p(BV) \simeq H^p(BG)$$

hold in the stable range of dimensions (that is for any fixed p and large enough K).

The partition $V = \bigcup \alpha$ by the orbits determines the corresponding partition $BV = \bigcup B\alpha$ of the classifying space. If $\alpha \subset V$ is an orbit, we set

$$B\alpha = \alpha \times_G EG \subset BV.$$

2.5. Definition. The *symmetry group* G_α of the orbit $\alpha \subset V$ is the stationary subgroup of any representative $x \in \alpha$ (this group is independent, up to an isomorphism, of the point $x \in \alpha$).

In singularity theory the orbit α is considered as a singularity class of the given classification, and the fixed representative $x \in \alpha$ is its 'normal form'.

2.6. Lemma. *The stratum $B\alpha \subset BV$ corresponding to the orbit α is homotopy equivalent to the classifying space BG_α of its symmetry group.*

Proof. By definition, we have

$$B\alpha = (\alpha \times EG)/G = (\{x\} \times EG)/G_\alpha.$$

The group G_α acts free on $(\{x\} \times EG) \simeq EG$. Since this space is topologically trivial, it can serve as the total space of the principle classifying G_α-bundle. Therefore, $(\{x\} \times EG)/G_\alpha \simeq BG_\alpha$. □

The lemma can be reformulated by saying that *the classifying space BV is glued from the classifying spaces of symmetry groups of various orbits,*

$$BV = \bigcup_\alpha BG_\alpha. \tag{1}$$

2.7. Remark. In singularity theory one considers various *stable* classification problems which are not reduced to the study of a unique Lie group action. Among those are the stable classification of $\mathcal{K}$-singularities, the classification of critical points of functions with respect to the stable $\mathcal{R}$- or $\mathcal{K}$-equivalence, the classification of multisingularities etc. *With a proper definition of the symmetry group, the statement above about the gluing of the classifying space from the classifying spaces of symmetry groups remains true for all those classifications.* This assertion will be detailed in the subsequent sections.

2.4. Thom polynomials for stable classification of maps

Computations show that in many cases the Thom polynomial depends actually on certain combinations $c_i(f) = c_i(f^*TN - TM)$ of the classes $c_i(M)$, $f^*c_j(N)$, given by the formal expansion

$$1 + c_1(f) + c_2(f) + \cdots = \frac{1 + f^*c_1(N) + f^*c_2(N) + \ldots}{1 + c_1(M) + c_2(M) + \ldots.}$$

This observation can be reformulated as follows. Fix some integer $\ell \in \mathbb{Z}$ and consider the so called *stable $\mathcal{K}$-classification* of map germs $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ where m may vary. By definition, the $\mathcal{K}$-singularity class of the map germ f is the isomorphism class of the zero level germ $f^{-1}(0)$ equipped with the local algebra of functions on it

$$Q_f = \mathcal{O}_{\mathbb{C}^m,0}/f^*\mathfrak{m}_{\mathbb{C}^{m+\ell},0}.$$

For $\ell \geq 0$ the $\mathcal{K}$-classification is actually the classification of finite-dimensional local algebras while for $\ell \leq 0$ it is the classification of $(-\ell)$-dimensional ICIS's (isolated complete intersection singularities).

There is the following simple interpretation of the classifying space for this classification. Choose some large integers $k \gg 0$ and $d \gg m \gg 0$. Consider the manifold $\mathcal{G}(m, d, k)$ of k-jets of germs of m-dimensional submanifolds in $(\mathbb{C}^d, 0)$. The points of this manifold are classified according to the $\mathcal{K}$-singularities of the projection to the fixed coordinate subspace $\mathbb{C}^{m+\ell} \subset \mathbb{C}^d$. We define the classifying space of stable $\mathcal{K}$-singularities as the limit space of $\mathcal{G}(m, d, k)$ with $m \to \infty$, $(d - m) \to \infty$, and $k \to \infty$.

The projection sending a germ of a submanifold to its tangent space at the origin has contractible fibers. Therefore, the space $\mathcal{G}(m, d, k)$ is homotopy equivalent to the usual Grassmannian $G_{m,d}$. Thus the ring of universal characteristic classes associated with the stable $\mathcal{K}$-classification is isomorphic (for each $\ell \in \mathbb{Z}$) to the cohomology ring of the stable Grassmannian that is to the ring of polynomials in the variables $c_1, c_2, \ldots$. In other words, the existence theorem for Thom polynomials of $\mathcal{K}$-singularities can be formulated as follows.

2.8. Theorem ([8]). *The cohomology class dual to the cycle of a $\mathcal{K}$-singularity α of a generic holomorphic map f is given by a universal polynomial* Tp_α *in the classes $c_i(f)$.*

2.5. Thom polynomials for isolated hypersurface singularities

Two function germs $f_i : (\mathbb{C}^{m_i}, 0) \to (\mathbb{C}, 0)$, $i = 1, 2$, are called *stably $\mathcal{K}$-equivalent*, if after adding suitable nondegenerate quadratic forms in new variables and after the multiplication by non-vanishing functions they can be brought one to the other by a change of coordinates in the source space. By equivalence of hypersurfaces we mean $\mathcal{K}$-equivalence of functions providing the equations of these hypersurfaces. The classification of IHS's (isolated hypersurface singularities) is one of the most studied classification problem in singularity theory. Nevertheless the theory of characteristic classes associated with this classification has appeared only recently [21]. It turns out that the theory of characteristic classes related to the stable classification of IHS's is the theory of Legendre characteristic classes. For the first glance, the definition below looks unmotivated. The explanation will be given in the subsequent sections.

2.9. Definition. The ring $\mathcal{L}$ of universal *Legendre characteristic classes* is the quotient ring of polynomials in the variables u, a_1, a_2, $\dots$, $\deg u = 1$, $\deg a_i = i$, over the ideal of relations generated by the homogeneous components of the formal expansion

$$(2) \qquad (1 + a_1 + a_2 + \dots)\left(1 - \frac{a_1}{1+u} + \frac{a_2}{(1+u)^2} - \dots\right) = 1.$$

If we set formally $1 + a_1 + a_2 + \dots = c(U)$ for a virtual bundle U of virtual rank 0 and $u = c_1(I)$ for a line bundle I, then (2) can be written as

$$(3) \qquad c(U + U^* \otimes I) = 1.$$

The additive basis of $\mathcal{L}$ is formed by the monomials of the form $u^k a_1^{i_1} a_2^{i_2} \dots$ with $i_j \in \{0, 1\}$.

2.10. Theorem. *The ring of characteristic classes associated to the classification of IHS's is the ring of Legendre characteristic classes.*

This theorem means that whenever we have a manifold M whose points are classified according to various IHS's types, we have also a natural characteristic homomorphism $\varkappa^* : \mathcal{L} \to H^*(M)$ such that the cohomology class dual to the locus of a given singularity α is given by a universal Legendre characteristic class $\mathrm{Tp}_\alpha \in \mathcal{L}$ determined uniquely by α and evaluated on the homomorphism $\varkappa^*$.

2.11. Example. Consider the diagram

$$(4) \qquad H \hookrightarrow W \xrightarrow{\pi} N$$

where the first arrow is a smooth embedding of a hypersurface and the second one is a smooth locally trivial bundle. Denote by $M \subset H$ the locus formed by the tangency points of H with the fibers of π. If certain genericity condition holds, then M is smooth of dimension $\dim M = \dim N - 1$. The points of M are classified according to the singularities of the hypersurfaces cut out by H on the fibers of π.

2.12. Example. The singularities considered in the previous example are determined completely by the composition $H \to N$. More general, consider an arbitrary smooth map $f : H \to N$ such that *the dimension of the cokernel of its derivative is not greater than* 1 at any point. The fibers of this map have dimension $-\ell = \dim H - \dim N$ and the embedded dimension of their singularities is $-\ell + 1$ i.e. these are isolated hypersurface singularities. The parameter space M in this situation is the locus $M \subset H$ of all singular points of the fibers of f i.e. it is the critical set of f.

2.13. Example. In the situation of the previous example, with any point $x \in M$ one can associate the tangent hyperplane $f_* T_x H \subset T_{f(x)} N$. This gives an embedding $i : M \to PT^*N$. This embedding is *Legendrian*: the manifold $i(M)$ is tangent to the natural contact distribution on PT^*N. More general, consider arbitrary Legendrian submanifold $M \subset PT^*N$. A *Legendrian mapping* is the projection of a Legendrian submanifold $M \subset PT^*N$ to the base N of the projectivized cotangent bundle. Singularities of Legendrian mappings are classified according to the classes of stable $\mathcal{K}$-equivalence of functions [5]. Therefore, to each point of M there corresponds an equivalence class of IHS's.

The Legendre characteristic classes in all three examples above are cohomology classes in $H^*(M)$ defined by $u = c_1(I)$ and $a_i = c_i(f^*TN - TM - I)$, where $I \simeq \mathcal{O}_{PT^*N}(1)$ is the conormal bundle of the contact structure on PT^*M. In Example 2.11 the bundle I can also be defined as the restriction to M of the line bundle of the divisor $H \subset W$. Verification of identity (3) is a nice exercise in the theory of characteristic classes.

2.6. Characteristic classes of multisingularities

In this section, by a local singularity we mean a class of stable $\mathcal{K}$-singularity of map germs $(\mathbb{C}^*, 0) \to (\mathbb{C}^{*+\ell}, 0)$, where $\ell \in \mathbb{Z}$ is fixed. A *multisingularity* $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ is a finite collection of local singularities. In what follows we assume that the collection $\underline{\alpha}$ contains no classes of submersion (this condition implies an additional restriction for $\ell \leq 0$ only).

The classification of multisingularities can be considered as an independent problem of singularity theory with its own table of normal forms, adjacencies, bifurcation diagrams etc. This implies that the general approach discussed in Sect. 2.1 can be applied to the case of multisingularities as well. It turns out that the construction for the classifying space of multisingularities and its cohomology ring is related to the theory of cobordisms and cohomological operations [22]. The formulated below existence theorem of universal expressions for characteristic classes of multisingularities appears as a corollary of this construction.

To a given map $f : M \to N$ one associates a number of multisingularity loci. First, we define $M(\underline{\alpha}) \subset M^r$ as the closure of the locus r-tuples of pairwise different of points $(x_1, \ldots, x_r)$, such that $f(x_1) = \cdots =$

$f(x_r)$ and such that f acquires the singularity α_i at x_i for $i = 1, \dots, r$. This definition is applicable only to the maps satisfying certain genericity condition. For the general case the definition should be corrected. If the genericity condition holds, then $M(\underline{\alpha})$ is a subvariety of expected dimension

$$\dim M(\underline{\alpha}) = \dim M - (r-1)\ell - \sum \operatorname{codim} \alpha_i,$$

where the codimension $\operatorname{codim} \alpha_i$ of the local singularity α_i is counted in the jet space of map germs $(\mathbb{C}^*, 0) \to (\mathbb{C}^{*+\ell}, 0)$.

Consider the natural projections $p : M^r \to M$ to the first factor and $q = f \circ p_M : M^r \to N$ to N, respectively. If the multisingularity type $\underline{\alpha}$ has no classes of submersion, then the restriction to $M(\underline{\alpha})$ of these projections is finite and we can consider the corresponding multisingularity loci on M and N, respectively. Denote by $\overline{m}_{\underline{\alpha}}$, $\overline{n}_{\underline{\alpha}}$ the cohomology classes dual to these loci considered as singular varieties *equipped with their reduced structures*,

$$\overline{m}_{\underline{\alpha}} = [pM(\underline{\alpha})] \in H^*(M), \qquad \overline{m}_{\underline{\alpha}} = [qM(\underline{\alpha})] \in H^*(N).$$

If the symbol $(\alpha_1, \dots, \alpha_r)$ of the multisingularity $\underline{\alpha}$ contains repeating entries, then the projections p and q of the locus $M(\underline{\alpha})$ to its images are not one-to-one, and it is natural to consider the classes $\overline{m}_{\underline{\alpha}}$, $\overline{n}_{\underline{\alpha}}$ with their natural multiplicity given as the degree of the corresponding projection. Thus, we set

$$m_{\underline{\alpha}} = |\operatorname{Aut}(\underline{\alpha})|\, \overline{m}_{\underline{\alpha}} = p_*[M(\underline{\alpha})], \quad n_{\underline{\alpha}} = |\operatorname{Aut}(\underline{\alpha}')|\, \overline{n}_{\underline{\alpha}} = q_*[M(\underline{\alpha})],$$

where $\underline{\alpha}' = (\alpha_2, \dots, \alpha_r)$ and $|\operatorname{Aut}(\underline{\alpha})|$ is the order of the permutation subgroup $\operatorname{Aut}(\underline{\alpha}) \subset S(r)$ whose elements preserve the collection $\underline{\alpha}$. These definitions imply the equalities

$$f_* m_{\underline{\alpha}} = n_{\underline{\alpha}}, \qquad f_* \overline{m}_{\underline{\alpha}} = k_1\, \overline{n}_{\underline{\alpha}},$$

where k_1 is the number of appearances of α_1 in the collection $\underline{\alpha}$.

Recall that $f_* : H^*(M) \to H^*(N)$ is the *push-forward*, or *Gysin homomorphism*. It is defined as the composition of the Poincaré duality in M, usual homomorphism f_* in homology, and Poincaré duality in N. In order this homomorphism to be defined, one needs to assume that the map f is *proper*. The homomorphism f_* shifts the (complex) grading of the even-dimensional cohomology by ℓ. It is not multiplicative. Instead the usual *projection formula* holds,

$$f_*(f^*a) \smile b = a \smile f_* b, \quad a \in H^*(N) \quad b \in H^*(M).$$

In other words, f_* is a homomorphism of $H^*(N)$-modules, where the action of $H^*(N)$ on $H^*(M)$ is defined via f^*. Because of that, we often drop the indication on f^* in the notation of classes on M and instead of $f^*a \smile b$ we write often $a \smile b$ or just $a\,b$.

To any monomial $c^I(f) = c_1^{i_1}(f)c_2^{i_2}(f)\ldots$ in the relative Chern classes $c_i(f) = c_i(f^*TN - TM)$ we associate the push-forward *Landweber-Novikov class*

$$s_I(f) = f_*(c^I(f)) \in H^*(N).$$

Landweber-Novikov classes are well known in cobordism theory. In the original definition they take values in complex cobordisms, here we use their images in cohomology only. In the case $\ell \leq 0$ the class $s_{i_1,i_2,\ldots}(f)$ vanishes for $\sum k\,i_k < -\ell$ by dimensional reason. Besides, if $\sum k\,i_k = -\ell$, then $s_{i_1,i_2,\ldots}(f) \in H^0(N)$ is equal to the corresponding characteristic number of a generic fiber of f. Except these evident relations Landweber-Novikov classes are multiplicatively independent for different monomials c^I. It means that for any polynomial in the Landweber-Novikov classes there is a sample map for which this polynomial gives a non-trivial class.

Now we are ready to formulate the principal theorem in the theory of characteristic classes of multisingularities. Shortly, this theorem says that the Thom polynomial for a multisingularity is a polynomial in Landweber-Novikov classes. In this theorem, $\ell \in \mathbb{Z}$ is a fixed integer. If $\ell \leq 0$, we assume that the collection of local singularities α_i forming the given multisingularity type $\underline{\alpha}$ contains no classes of submersion.

2.14. Theorem ([22]). 1. *For every collection $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ of local singularities the cohomology class of the corresponding multisingularity $n_{\underline{\alpha}}$ in the target (respectively, the class $m_{\underline{\alpha}}$ in the source) manifold is given by a universal polynomial with rational coefficients in the Landweber-Novikov classes $s_I(f)$ of the map (respectively, in the relative Chern classes $c_i(f)$ and the pull-backs $f^*s_I(f)$ of the Landweber-Novikov classes).*

2. *The multisingularity polynomial has in fact the following specific form. To every multisingularity $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ there corresponds a universal polynomial $R_{\underline{\alpha}}$ (called* residue polynomial*) in the Chern classes $c_1, c_2, \ldots$ such that the multisingularity classes $m_{\underline{\alpha}}$, $n_{\underline{\alpha}}$ are determined by the residue polynomials $R_{\underline{\alpha}_J} = R_{\alpha_{j_1},\ldots,\alpha_{j_k}}$ of various subcollections $\underline{\alpha}_J = (\alpha_{j_1}, \ldots, \alpha_{j_k}) \subset (\alpha_1, \ldots, \alpha_r)$ forming the given multisingularity $\underline{\alpha}$*

by the following explicit formulas

$$m_{\underline{\alpha}} = \sum_{J_1 \sqcup \dots \sqcup J_k = \{1,\dots,r\}} R_{\underline{\alpha}_{J_1}} f^*f_* R_{\underline{\alpha}_{J_2}} \dots f^*f_* R_{\underline{\alpha}_{J_k}}, \tag{5}$$

$$n_{\underline{\alpha}} = f_* m_{\underline{\alpha}} = \sum_{J_1 \sqcup \dots \sqcup J_k = \{1,\dots,r\}} f_* R_{\underline{\alpha}_{J_1}} \dots f_* R_{\underline{\alpha}_{J_k}}, \tag{6}$$

where the polynomials $R_{\underline{\alpha}_J}$ are evaluated on the relative Chern classes $c_i = c_i(f) = c_i(f^*TN - TM)$. The sum is taken over all possible partitions of the set $\{1,\dots,r\}$ into a disjoint union of non-empty non-ordered subsets $\{1,\dots,r\} = J_1 \sqcup \dots \sqcup J_k$, $k \geq 1$. The subset containing the element $1 \in \{1,\dots,r\}$ is denoted by J_1.

Moreover, the residue polynomial $R_{\underline{\alpha}}$ is independent of the order of local singularities α_i forming the collection $\underline{\alpha} = (\alpha_1,\dots,\alpha_r)$.

For example, if the collection $\underline{\alpha} = \{\alpha\}$ contains only one element ($r = 1$), then $R_\alpha = m_\alpha$ is the corresponding Thom polynomial of the local singularity α.

Combining the terms on the right hand side expressions we arrive at the following recursive relations equivalent to (5–6).

$$m_{\underline{\alpha}} = R_{\underline{\alpha}} + \sum_{1 \in J \subsetneq \{1,\dots,r\}} R_{\underline{\alpha}_J} f^* n_{\underline{\alpha}_{\overline{J}}}, \tag{7}$$

$$n_{\underline{\alpha}} = f_* m_{\underline{\alpha}} = f_* R_{\underline{\alpha}} + \sum_{1 \in J \subsetneq \{1,\dots,r\}} f_* R_{\underline{\alpha}_J} n_{\underline{\alpha}_{\overline{J}}}, \tag{8}$$

where the sum is taken over all proper subsets $J \subsetneq \{1,\dots,r\}$ containing the element 1, and $\overline{J} = \{1,\dots,r\} \setminus J$.

The combinatorial expression (6) for the multisingularity classes $n_{\underline{\alpha}}$ can be rewritten in the following way by means of generation functions. Assume that we study multisingularities formed by the local singularities (perhaps, with repetitions) from a finite list $\alpha_1,\dots,\alpha_r$ of pairwise different ones. Then the following formal identity holds

(9)

$$1 + \sum_{k_1,\dots,k_r} n_{\alpha_1^{k_1}\dots\alpha_r^{k_r}} \frac{t_1^{k_1}}{k_1!} \cdots \frac{t_r^{k_r}}{k_r!} = \exp\left(\sum_{k_1,\dots,k_r} f_*(R_{\alpha_1^{k_1}\dots\alpha_r^{k_r}}) \frac{t_1^{k_1}}{k_1!} \cdots \frac{t_r^{k_r}}{k_r!} \right).$$

(The author is grateful to S. Lando for this remark.)

2.15. Example. Assume that the multisingularity $\underline{\alpha} = (\alpha,\dots,\alpha) = (\alpha^r)$ contains r copies of the same singularity α. Then the class $\overline{m}_{\alpha^r}$

of *multiple singularity* α can be determined by the following recursive formula

$$\overline{m}_{\alpha^r} = q_r + \sum_{k=1}^{r-1} q_{r-k}\overline{n}_{\alpha^k}, \quad f_*\overline{m}_{\alpha^r} = r\,\overline{n}_{\alpha^r}, \tag{10}$$

where $q_k = \frac{1}{(k-1)!}R_{\alpha^k}$ are certain polynomials in relative Chern classes of the map. Over $\mathbb{Q}$ this relation follows from (7). Conjecturally the polynomials q_k have always integer coefficients and (10) holds in the integer cohomology group.

At present, the first statement of Theorem is proved under certain restrictions by topological argument from cobordism theory, see [22] and the next section. It is a challenge to find an intersection theory proof of this statement and especially of its various conjectural generalizations formulated in [22].

Let us show that the second statement is a consequence of the first one. For simplicity we shell prove the relation (9) equivalent to (6). The relation (5) is proved by similar argument. Consider the generating series

$$\mathcal{N}(f) = 1 + \sum_{k_1,\dots,k_r} n_{\alpha_1^{k_1}\dots\alpha_r^{k_r}} \frac{t_1^{k_1}}{k_1!}\cdots\frac{t_r^{k_r}}{k_r!} = 1 + \sum_{k_1,\dots,k_r} \overline{n}_{\alpha_1^{k_1}\dots\alpha_r^{k_r}} t_1^{k_1}\dots t_r^{k_r}.$$

By the first statement of Theorem, each coefficient in this series is a polynomial in Landweber-Novikov classes. So the coefficients of the logarithm $\log(\mathcal{N}(f))$ are. We need to show that the coefficients of $\log(\mathcal{N}(f))$ depend *linearly* in the Landweber-Novikov classes.

The generating series $\mathcal{N}(f)$ satisfies the following remarkable property. Assume that M has two connected components, $M = M_1 \sqcup M_2$. Then denoting by f_i the restriction of f to M_i we have

$$\mathcal{N}(f) = \mathcal{N}(f_1)\mathcal{N}(f_2).$$

Indeed, every multisingularity locus of the map f consists of many components numbered by possible distributions of local singularities between M_1 and M_2. These components correspond one-to-one to the summands in the right hand side of the equality provided by the multiplication rule for generating functions.

The equality is applied as follows. Starting from the given map $f : M \to N$ we construct a series of new maps $f^{(d)} : M^{(d)} \to N$, $d = 1, 2, \dots$, in the following way. The source manifold $M^{(d)}$ of $f^{(d)}$ is the disjoint union of d copies of M, and the restriction of $f^{(d)}$ to each component of $M^{(d)}$ is defined to be a small perturbation of the original map f. The Landweber-Novikov classes of $f^{(d)}$ are given by $s_I(f^{(d)}) = d\, s_I(f)$. On the other hand, from the equality above we have

$$\mathcal{N}(f^{(d)}) = (\mathcal{N}(f))^d, \qquad \log(\mathcal{N}(f^{(d)})) = d\,\log(\mathcal{N}(f)).$$

It follows that $\log(\mathcal{N}(f))$ has no terms of order greater then 1 in Landweber-Novikov classes since such terms would contribute to the terms in $\log(\mathcal{N}(f^{(d)}))$ of order greater then 1 in d. Equality (9) is proved.

In the applications related to the enumeration of isolated multisingularities of hypersurfaces one needs a Legendrian version of Theorem 2.14. It is formulated in a similar way. The only difference is that the residue classes $R_{\underline{\alpha}}$ of the Legendre (or isolated hypersurface) multisingularity is an element of the ring $\mathcal{L}$ of universal Legendre characteristic classes (and thus it determines a cohomology class on the source manifold of a Legendre map).

2.16. Theorem ([22]). *To every collection $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ of stable isolated hypersurface singularity classes there corresponds a universal Legendre characteristic class $R_{\underline{\alpha}} \in \mathcal{L}$ such that for any generic proper holomorphic Legendre map $f : M \to PT^*N \to N$ the corresponding multisingularity classes $m_{\underline{\alpha}}$, $n_{\underline{\alpha}}$ are given by the formulas* (5–8).

2.7. Multisingularities and cobordisms

In this section we show that the classical Thom's construction for the classifying space of cobordisms is the best fit for the study of multisingularities. Strangely this fact have not been noticed for such a long time!

Consider a differentiable map $f : M \to N$ of *real* compact manifolds. This map can be treated as a representative of the cobordism class $[f] \in O^\ell(N)$, $\ell = \dim N - \dim M$. Therefore, it corresponds to a classifying map from N to the classifying space of cobordisms. Recall the Thom's construction for this map. First, represent f as a composition of an embedding and a projection

$$M \hookrightarrow \mathbb{R}^K \times N \to N.$$

The normal bundle of the embedding is classified by certain map $M \to BO(K+\ell)$. This map extends to a map $U \to EO(K+\ell)$ of the total spaces of the corresponding rank $K+\ell$ bundles, where we identified the total space of the normal bundle of M with a tubular neighborhood U of the submanifold $M \subset \mathbb{R}^K \times N$. Finally, denoting by $MO(K+\ell)$ the Thom space of the bundle $EO(K+\ell)$ we extend the constructed map to the whole $\mathbb{R}^K \times N$ by sending the complement of U to the marked point in $MO(K+\ell)$. Thus constructed map $h : \mathbb{R}^K \times N \to MO(K+\ell)$ can be treated as a family of maps

$$h_y : S^k \to MO(K+\ell)$$

parameterized by points $y \in N$ or as a map

$$\varkappa : N \to \mathbf{N}_\ell$$

to the corresponding iterated loop space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$. The limit $\lim_K \Omega^K MO(K+\ell)$ is called the *classifying space of ℓ-dimensional cobordisms.*

The cobordism class on N is determined uniquely by the homotopy class of $\varkappa$. For example, the source manifold M of the map can be recovered as the inverse image of the 'zero section' $BO(K+\ell) \subset MO(K+\ell)$ under the associated map $\mathbb{R}^K \times N \to MO(K+\ell)$. In other words, for any $y \in N$ the preimages $f^{-1}(y)$ are in one-to-one correspondence with the intersection points of $h_y(S^K)$ with the zero section. To any such intersection point $h_y(x)$ we associate its $\mathcal{K}$-singularity type, that is the singularity type of the 'projection of S^K to the fiber of $EO(K+\ell)$ along the zero section'. The following statement is almost evident.

2.17. Lemma. *The stable $\mathcal{K}$-singularity type of the intersection point $h_y(x)$ coincides with the $\mathcal{K}$-singularity type of the map f at the corresponding point $x \in M$.*

Remark that the usage of stable $\mathcal{K}$-classification is essential here: the manifolds participating in the two map germs $(M, x) \to (N, y)$ and $(S^K, x) \to (\mathbb{R}^{K+\ell}, 0)$ of the lemma have different dimensions (but equal relative dimension ℓ). Besides, the second map is not well defined and only its $\mathcal{K}$-singularity type can be determined.

The elements of the classifying space $\mathbf{N}_\ell$ are continuous maps $g : S^K \to MO(K+\ell)$. Without loss of generality we can replace the infinite dimensional space $BO(K+\ell)$ by its smooth finite dimensional approximation $G_{K+\ell,K_1}$, $K_1 \gg K+\ell$. Moreover, we may assume that the maps g forming the classifying space are differentiable in a neighborhood of the zero section. Thus, the classifying space $\mathbf{N}_\ell$ is classified by the multisingularity types of the intersection of $g(S^K)$ with the zero section. Therefore, the lemma can be reformulated by saying that the *classifying map $\varkappa : N \to \mathbf{N}_\ell$ preserves the partitions by the multisingularity types.* As a result, we arrive at the following conclusion:

2.18. Corollary. *The classifying space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$ of cobordisms serves also as the classifying space of multisingularities.*

The proof of the first assertion of Theorem 2.14 uses the complex version $\mathbf{N}_\ell^{\mathbb{C}}$ of the classifying space of multisingularities. Ignoring some technical difficulties we claim that the homotopy type of this space is given by

$$\mathbf{N}_\ell^{\mathbb{C}} = \Omega^{2K} MU(K+\ell), \quad K \gg 0,$$

which is the classifying space of complex cobordisms. The cohomology ring of this space can be computed explicitly, at least in the case of rational coefficients. It is a polynomial ring whose generators correspond to Landweber-Novikov classes. This leads to the formulation of Theorem 2.14, see [22].

2.8. Generalized Pontryagin-Thom-Szücs construction

The topological type of the classifying space $B\mathcal{S}$ of singularities depends heavily on the particular classification $\mathcal{S}$. However all these spaces, in particular those considered in the previous sections, have many common features. The most important one is the following splitting that we call the *generalized Pontryagin-Thom-Szücs construction*:

$$B\mathcal{S} = \bigcup_\alpha B\alpha, \quad B\alpha \sim BG_\alpha, \tag{11}$$

where G_α is the symmetry group of the corresponding singularity α.

The notion of a 'singularity theory classification' can be axiomatized as follows. By a *classification* $\mathcal{S}$ we mean a (finite or infinite) list of symbols α called 'singularity classes'. Every singularity class is assigned a number $\operatorname{codim}\alpha$ (its codimension) and, in addition, the following data:

- a *bifurcation diagram* of this singularity, that is the germ of a $(\operatorname{codim}\alpha)$-dimensional manifold T_α equipped with the partition into the strata labelled by the singularity classes of smaller codimensions;
- a *symmetry group* G_α acting on T_α preserving the partition into the strata.

These data must satisfy some natural compatibility conditions for adjacent singularities. For example, a normal slice to any stratum in T_α (together with the induced partition) must be diffeomorphic to the bifurcation diagram of the corresponding singularity. We do not formulate the compatibility conditions explicitly; they are always automatically satisfied for all 'natural' classifications.

The homotopy type of the classifying space $B\mathcal{S}$ is determined uniquely by the classification.

Indeed, the condition $B\alpha \sim BG_\alpha$ determines the topology of the strata; and the geometry of bifurcation diagrams determines the way how these strata are glued for adjacent singularities.

The classifying property of BS is formulated as follows. Assume that we are given a parameter space M whose points are classified according to the given classification $\mathcal{S}$. Consider some stratum $M(\alpha) \subset M$. The structure group of the normal bundle to this stratum is reduced to G_α. Therefore, it is classified by some map $M(\alpha) \to BG_a$. These maps glue together to provide a map

$$\varkappa : M \to B\mathcal{S}.$$

in other words, the *partition on M is induced from the classifying space $B\mathcal{S}$ by certain classifying map $\varkappa$.*

The detailed realization of the general picture formulated above meets evident technical difficulties. These difficulties have been overcame by A. Szücs in his theory of τ-maps developed in a series of papers [35, 36, 37]. By a τ-map we mean a differentiable map that admits only (multi)singularities from a given list τ of allowed ones. In one of the most general form this theory is described in the joint paper with R. Rimanyi [33]. In this theory the classifying space τY is constructed by gluing the classifying spaces of symmetry groups of multisingularities from τ. With small changes the same construction can be applied to any 'abstract' classification, not necessary related to singularities of maps. It is assumed in the paper [33] that the classification of singularities is discrete that is a neighborhood of any point intersects only finitely many orbits. This technical restriction is not essential. It can be dropped using the notion of a cellular classification introduced in [41].

On the other hand, in the previous sections we have used the alternative *a priori* constructions for the classifying spaces of particular classifications.

Both approaches to the construction of the classifying space are equivalent.

It a consequence of a 'general nonsense': the uniqueness of the classifying space is guarantied by its universality that can be verified under either approach.

Example. The classifying space τY for τ-maps can be obtained from the classifying space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$, $K \gg 0$, of cobordisms by selecting the strata in $\mathbf{N}_\ell$ corresponding to the allowed singularities. The classifying property is almost evident: consider a τ-map $M \to N$ and repeat the Thom's construction for this map without regarding its singularities; then Lemma 2.17 assures that the resulting classifying map $\varkappa : N \to \mathbf{N}_\ell$ automatically takes values in the union τY of required strata. This interpretation of τY allows one to avoid technical difficulties arising in the gluing construction. Another advantage of the *a priori* construction is that the space obtained in this way is smooth (although non-compact). It is quite hard to achieve this by the gluing construction. Besides, we obtain a clear answer to the question what is the limit of the spaces τY when τ contains *all* multisingularities.

Thus instead of 'gluing' $B\mathcal{S}$ from BG_α's we prefer to speak about 'cutting' $B\mathcal{S}$ into BG_α's. Moreover, the validity of the splitting (11) can be used to give the correct definition for the symmetry group. Here are a few examples.

- If $\mathcal{S}$ is the classification of orbits of some Lie group G action then G_α is the stabilizer of (any point of) the orbit.
- In the case of stable $\mathcal{K}_\ell$-classification the symmetry group is the (maximal compact subgroup of) the symmetry group of the k-jet of any stable representative $f_0 : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ *with the smallest possible* m (equal to the codimension of the singularity).
- The symmetry group of a stable class of critical points of functions is the symmetry group of any representative f_0 : $(\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *with the smallest possible* m (equal to the corank of the singularity).
- For the classification of multisingularities the symmetry group of the multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ is the semidirect product of the symmetry groups G_{α_i} of local $\mathcal{K}$-singularities α_i and the subgroup in $S(r)$ of automorphisms of the multi-index $\underline{\alpha}$.

The cohomological information on the topology the splitting (11) is formulated in terms of the characteristic spectral sequence. Assume for simplicity that there are finitely many singularity classes (the general case is considered in [18, 19]). Consider the open increasing filtration on $B\mathcal{S}$ whose pth term F_p is formed by the singularity strata of codimension at most p,

$$F_p = \bigcup_{\operatorname{codim}\alpha \le p} B\alpha \subset B\mathcal{S}.$$

The spectral sequence $E_r^{p,q}$ associated with this filtration converges to the cohomology of the classifying space $H^*(B\mathcal{S})$. It is called the *characteristic spectral sequence.* The complement $F_p \setminus F_{p-1}$ is a smooth (in general, non-closed) submanifold in $B\mathcal{S}$ of codimension p formed by p-codimensional singularities. It follows that the initial term $E_1^{p,*}$ of this sequence is the cohomology group of the Thom space of the normal bundle to this manifold. Using the Thom isomorphism we get the following

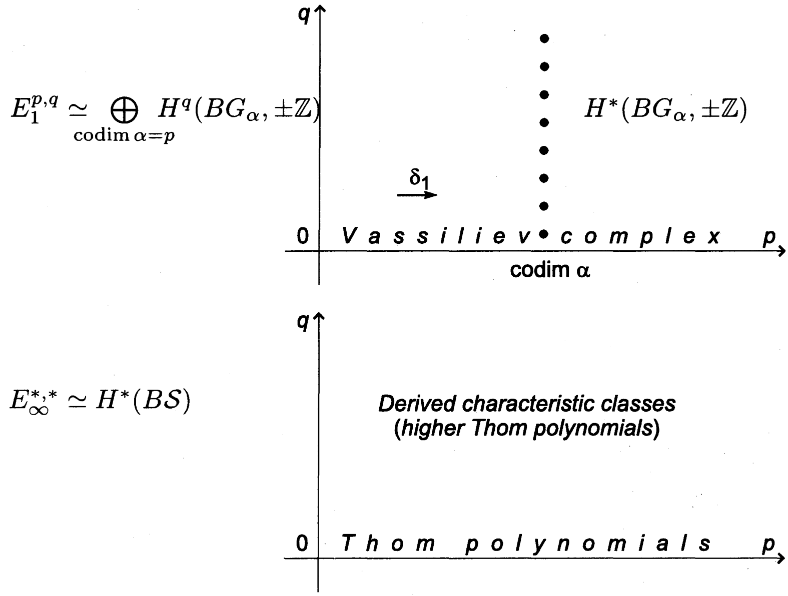

Fig. 1. **Characteristic spectral sequence**

description of the initial term:

$$E_1^{p,q} \simeq \bigoplus_{\operatorname{codim}\alpha=p} H^q(BG_\alpha, \pm\mathbb{Z}).$$

Here $\pm\mathbb{Z}$ is the coefficient system on $B\alpha = BG_\alpha$ that is locally isomorphic to $\mathbb{Z}$ and that is determined by the action of the group G_α on the orientation of the bifurcation diagram of the singularity α (for the details, see [18, 19]).

The *Vassiliev complex* is the row $(E_1^{*,0}, \delta_1)$ of the initial term. It allows one to select linear combinations of the strata for which the dual cohomology class is correctly defined.

The limit term $E_\infty^{*,*}$ defines a natural filtration on $H^*(B\mathcal{S})$. The pth term of this filtration is generated by the characteristic classes that can be represented by cycles supported on the union of strata of codimension greater than or equal to p.

The term $E_\infty^{p,0}$ corresponds to the fundamental cycles of strata of codimension p, that is, to Thom polynomials. The terms $E_\infty^{p,q}$ with $q > 0$ are *higher Thom polynomials*, or *derived characteristic classes* of singularities. They have the following meaning.

Let M be the parameter space of some geometric problem, and α be a singularity class of codimension p. The normal bundle to $M(\alpha)$ has the structure group G_α. Therefore, every characteristic class χ of the group G_α defines a cohomology class $\chi(\alpha) \in H^q(M(\alpha))$ on the locus $M(\alpha)$. Assume that $M(\alpha)$ is closed. Then the push-out class $i_*(\chi(\alpha)) \in H^{p+q}(M)$ is well defined, where i is the embedding. It corresponds to the term $E_1^{p,q}$ of the spectral sequence. The derived characteristic class is the result of an attempt to extend the definition of the class $i_*(\chi(\alpha))$ to the general situation. It is not always possible (only if all differentials vanish on this element of $E_1^{p,q}$). Even if this is possible, this extension is not unique (it is defined only modula higher terms of the filtration supported on the strata of codimension $> p$). Examples of derived characteristic classes are given in the subsequent sections.

The characteristic spectral sequence has especially simple description in the complex problems, where all topology is often concentrated in even dimensions and the sequence degenerates at the initial term by dimensional reason. In this case it implies the following splitting of the cohomology group of the classifying space,

$$H^n(B\mathcal{S}) \simeq \bigoplus_\alpha H^{n-\operatorname{codim}\alpha}(BG_\alpha). \tag{12}$$

This splitting implies an interesting relation between the Poincaré series of the cohomology groups $H^*(B\mathcal{S})$ and $H^*(BG_\alpha)$. The author used this relation many times to check various conjectures about classifications, the structure of symmetry groups and the topology of the classifying spaces.

§3. Localized Thom polynomials

A localized Thom polynomial is the Thom polynomial written in a special additive basis well adjusted to the classification of singularities by corank. It allows one to single out the terms in the Thom polynomial for which closed formulae could be given. It provides also the correspondence between the Thom polynomials for singularities with the same name appearing in different classifications. Finally, the concept of localized Thom polynomials is crucial in the application of the restriction method to the computation of the residue classes of complete intersection multisingularities.

3.1. Porteous-Thom classes and their derived classes

One of the historically first examples of the computed Thom polynomials are those for the so called Porteous-Thom singularities. Let M

be the source manifold of a generic holomorphic map

$$f : M \to N.$$

Denote by $\Sigma^r = \Sigma^r(f) \subset M$ the locus of points where the derivative of the map f has at least r-dimensional kernel, $r \geq \max(0, \ell)$, where $\ell = \dim N - \dim M$. More generally, let E, F be two complex vector bundles over the same base M and $\varphi : E \to F$ be a generic morphism. Then one can define the locus Σ^r for his morphism by similar conditions. In the case when M is the source of a holomorphic map f we can set $E = TM$, $F = f^*TN$, and $\varphi = f_* : TM \to f^*TN$ is the derivative map. If the genericity condition holds, then Σ^r is a subvariety of (complex) codimension

$$\operatorname{codim} \Sigma^r = r\,(r + \ell), \qquad \ell = \operatorname{rk} F - \operatorname{rk} E.$$

By Theorem 2.8, the dual of the locus Σ^r is expressed as a universal polynomial in the classes $c_i = c_i(f) = c_i(F - E)$.

3.1. Theorem ([27]). *The cohomology class Poincaré dual to the locus Σ^r is given by the following determinant*

$$[\Sigma^r] = \det \|c_{r+\ell-i+j}\|_{i,j=1,\ldots,r}, \qquad c_i = c_i(f) = c_i(F - E).$$

Assume for a moment that the locus Σ^{r+1} is empty. Then (provided the genericity condition holds) the locus Σ^r is smooth. Denote by $p_r : \Sigma^r \to M$ the embedding. The Gysin homomorphism $p_{r*} : H^*(\Sigma^r) \to H^*(M)$ allows us to push-forward to M cohomology classes defined on Σ^r. Over Σ^r one has the natural kernel bundle K and cokernel bundle Q of ranks r and $r+\ell$, respectively. These bundles form the exact sequence (defined on Σ^r only)

$$0 \to K \to E \xrightarrow{\varphi} F \to Q \to 0.$$

Let $R(v, u)$ be arbitrary polynomial in formal variables $v_1, \ldots, v_r$, $u_1, \ldots, u_{r+\ell}$. Set $v_i = c_i(K)$, $u_j = c_j(Q)$.

3.2. Proposition. *If Σ^{r+1} is empty, then the push-forward class $p_{r*}R(c(K), c(Q))$ can be expressed as a universal polynomial (determined by ℓ, r, and R) in the relative Chern classes $c_i = c_i(f) = c_i(F - E)$.*

The polynomial representing the class $p_{r*}R(c(K), c(Q))$ is called the *derived Thom polynomial* of the singularity Σ^r. Since it is a polynomial in the classes $c_i(F - E)$, it can be considered for any map *not necessary satisfying the condition* $\Sigma^{r+1} = \varnothing$. The derived Thom polynomial is not defined uniquely but only up to a class that can be represented by a cycle supported on Σ^{r+1}. This ambiguity can be fixed, for example, by the following geometric construction. Consider the Grassmann bundle $G_r(E)$ formed by all r-dimensional subspaces λ in the fibers of the bundle $E \to M$.

3.3. Definition. The *standard resolution* $\widetilde{\Sigma}^r$ *of the singularities* of the locus Σ^r is the submanifold in the space of the Grassmann bundle $G_r(E)$ formed by all pairs of the form (x, λ), $x \in M$, $\lambda \subset E_x$, such that $\lambda \subset \ker f$.

If the genericity condition for the morphism $\varphi : E \to F$ holds, then $\widetilde{\Sigma}^r$ is smooth. Denote by $p_r : \widetilde{\Sigma}^r \to M$ the natural projection.

Denote by K the restriction to $\widetilde{\Sigma}^r$ of the tautological rank r bundle over $G_r(E)$. Denote also by Q the (virtual) bundle $Q = F - E + K$. In the case when $\Sigma^{r+1} = \varnothing$ the map p carries $\widetilde{\Sigma}^r$ isomorphically to Σ^r and the bundles K, Q over $\widetilde{\Sigma}^r$ correspond to similar bundles over Σ^r under this isomorphism. This justifies our notation.

Denoting $v_i = c_i(K)$, $u_j = c_j(Q)$ we see that $R(v, u)$ can be considered as a cohomology class on $\widetilde{\Sigma}^r$. This extends the definition of the class $p_{r*}R(v, u) \in H^*(M)$ to the case of arbitrary morphism $E \to F$ not necessary satisfying the condition $\Sigma^{r+1} = \varnothing$.

The explicit form of the class $p_{r*}R(v, u)$ can be obtained as follows. From the definition of Q, we have $c(Q) = c(F-E)\, c(K)$, or

$$u_k = \sum_{i+j=k} c_i\, v_j, \qquad c_i = c_i(F-E). \tag{13}$$

In view of the projection formula it remains to compute the push-forward class $p_{r*}R(v, u)$ in the case when R depends on the variables v_i only. According to the splitting principle we set formally

$$c(K) = 1 + v_1 + \cdots + v_r = \prod_{i=1}^{r}(1 - t_i) \tag{14}$$

and express the polynomial R in terms of $t_1, \ldots, t_r$ using these relations.

3.4. Theorem. *The homomorphism* $p_{r*} : H^*(\widetilde{\Sigma}^r) \to M$ *is given on the monomials in* t_i *by the formula*

$$p_{r*}t_1^{s_1} \ldots t_r^{s_r} = \det \|c_{r+\ell-i+s_i+j}(F - E)\|_{i,j=1,\ldots,r}. \tag{15}$$

Some versions of this formula can be found in [15, 12] The relation of this theorem should be understood formally since the classes t_i are not defined on $\widetilde{\Sigma}^r$. The determinantal expression on the right hand side is known as the *Schur polynomial.*

3.2. Localized Thom polynomials

Relations (13–15) have the following formal treatment. Consider the polynomial rings of universal characteristic classes

$$H^*(BU) = \mathbb{Z}[c_1, c_2, \dots],$$
$$H^*(BU(r) \times BU(r+\ell)) = \mathbb{Z}[v_1, \dots, v_r, u_1, \dots, u_{r+\ell}].$$

We consider the grading on these rings by setting $\deg c_i = \deg v_i = \deg u_i = i$ so that for a homogeneous cohomology class a one has $a \in H^{2\deg a}(\cdot)$. These rings are related by the natural multiplicative homomorphism

$$p_r^* : H^*(BU) \longrightarrow H^*(BU(r) \times BU(r+\ell))$$

given on the generators c_i by the formal expansion

$$p_r^* : 1 + c_1 + c_2 + \dots \longmapsto \frac{1 + u_1 + \dots + u_{r+\ell}}{1 + v_1 + \dots + v_\ell}.$$

Besides, we consider the homomorphism of $H^*(BU)$-modules

$$p_{r*} : H^*(BU(r) \times BU(r+\ell)) \longrightarrow H^*(BU)$$

given by the explicit formulae (13–15). (The action of $H^*(BU)$ on $H^*(BU(r) \times BU(r+\ell))$ is determined via p_r^*.)

3.5. Theorem. *The homomorphisms p_{r*}, $r \geq \max(0, -\ell)$, provide a natural splitting*

$$H^*(BU) \simeq \bigoplus_r H^*(BU(r) \times BU(r+\ell)). \tag{16}$$

In other words, every polynomial P in variables c_i has a unique presentation in the form

$$P = \sum_r p_{r*} R^{(r)}, \tag{17}$$

where $R^{(r)}$ is a polynomial of degree $\deg R^{(r)} = \deg P - r\,(r+\ell)$ in variables $v_1, \dots, v_r, u_1, \dots, u_{r+\ell}$.

The right hand side of (17) is called the *localized form* of the polynomial P. Its terms $p_{r*}(R^{(r)})$ are determined uniquely by P and by the number ℓ. These terms have the following meaning. Consider a morphism of vector bundles $\varphi : E \to F$ over some base M with $\operatorname{rk} F - \operatorname{rk} E = \ell$. Then the term $p_{r*}R^{(r)} \in H^*(BU) = \mathbb{Z}[c_1, c_2, \dots]$ evaluated on the relative Chern classes $c_i = c_i(F - E)$ can be represented by a cycle supported on the locus Σ^r of the morphism φ. For example, if $P = \mathrm{Tp}_\alpha$ is the Thom polynomial of some singularity α having the kernel rank r, then the polynomials $R^{(i)} = R^{(i)}_\alpha$ vanish for $i < r$.

The localized form clarifies also the structure of the residue polynomials of multisingularities. The following assertion is verified on hundreds of computed examples. However, we still have no formal proof of this fact.

3.6. Conjecture. *For any multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ the residue cohomology class $R_{\underline{\alpha}} \in H^*(M)$ of a map $f : M \to N$ can be represented by a cycle supported on the intersection $\bigcap_{i=1}^r M(\alpha_i) \subset M$. In particular, the number k_0 of the first non-zero term in the localized residue polynomial $R_{\underline{\alpha}} = \sum_{k \geq k_0} p_{k*} R^{(k)}_{\underline{\alpha}}$ is equal to the biggest kernel rank of the singularities α_i.*

The splitting of the Theorem is a particular case of the splitting (12). Consider linear maps forming the space $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^{m+\ell})$ with ℓ fixed and $m = 1, 2, \dots$. The classification of such maps can be considered as an independent classification problem. The singularity classes for this classification are the classes Σ^r of maps of kernel rank r. By the *symmetry group* G_{Σ^r} of the class Σ^r we mean the stationary subgroup of any representative $x \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^{m+\ell})$ *with the smallest possible* m (that is, with $m = r$). It is clear that such a representative is exactly the zero map $x = 0 \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r+\ell})$. The stationary group for this element contains all linear transformations of the source and the target space. Therefore,

$$G_{\Sigma^r} \sim U(r) \times U(r + \ell),$$

and the splitting (16) follows from (12). □

The construction for the homomorphisms p_{r*} has some variations. For example, we could consider the resolution of the locus Σ^r using the Grassmann bundle $G_{r+\ell}(F)$ or even combine the two methods. This would lead to another choice for the homomorphism $p_{r*} : H^*(BU(r) \times BU(r+\ell)) \to H^*(BU)$ such that the difference of the two choices is supported on Σ^{r+1}. More formally, consider the decreasing filtration on $H^*(BU)$ whose rth term is formed by polynomials in Chern classes c_i that can be represented by cycles supported on Σ^r. Then the right hand side of (16) represents the adjoint graded space of this filtration and the equality (17) provides a particular splitting of this filtration. It follows that the first nonzero term in the localized form of a polynomial P has more invariant meaning with respect to the other terms.

3.7. Example. The term $p_{0*} R^{(0)}_{\underline{\alpha}}$ can be non-trivial only in the case when every local singularity α_i of the multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$

is the class of *immersion* (that is, the class A_0 of non-singular map germs). In the latter case one has

$$R^{(0)}_{A_0^r} = (r-1)!\,(-1)^{r-1}u_\ell^{r-1}, \qquad p_{0*}R^{(0)}_{A_0^r} = (r-1)!\,(-1)^{r-1}c_\ell^{r-1}.$$

This equality together with (10) is equivalent to the known *Herbert-Ronga formula for the classes of multiple points of immersions*,

$$\overline{m}_r = f^*\overline{n}_{r-1} - c_\ell\overline{m}_{r-1},$$

where we denote by $\overline{m}_r = \frac{1}{(r-1)!}m_{A_0^r}$ and $\overline{n}_r = \frac{1}{r!}n_{A_0^r}$ the classes of the corresponding reduced cycles of multiple points.

3.8. Example. There has been a number of papers studying Thom polynomials for *corank* 1 *maps* that is for maps such that the kernel rank of the derivative does not exceed 1 at any point, see [27, 24, 25, 6, 30, 26, 1]. The results of these papers can be interpreted as the study of the term $p_{1*}R^{(1)}$ of these Thom polynomials. Indeed, if the map has no points with singularities of corank greater, then 1 then $\Sigma^r = \varnothing$ for $r \geq 2$ and all terms of the localized Thom polynomial except the first one vanish for such a map.

Our findings on the Thom polynomials for such maps can be summarized as follows (all necessary ingredients for obtaining formulas of Theorems 3.9 and 3.12 below are contained implicitly in [24, 25, 6]). Let $\ell \geq 0$. Denote by A_k the $\mathcal{K}$-singularity class of maps with local algebra isomorphic to $\mathbb{C}[x]/x^{k+1}$ i.e. the singularity class of the Thom-Boardman type $\Sigma^{1,\dots,1}$ (k units). The polynomial $R^{(1)}_{A_k}$ of the first term in the localized Thom polynomial $\mathrm{Tp}_{A_k} = \sum p_{r*}R^{(r)}_{A_k}$ depends on the variables $v_1, u_1, \dots, u_{\ell+1}$. Set $t = -v_1$.

3.9. Theorem. *The term $R^{(1)}_{A_k}$ of the localized Thom polynomial of the singularity A_k is given by*

$$R^{(1)}_{A_k} = \sigma_2\sigma_3\dots\sigma_k,$$

where

$$\sigma_p = u_{\ell+1} + p\,t\sum_{i=0}^{\ell} p^i t^i u_{\ell-i} = c_{\ell+1} + (p-1)t\sum_{i=0}^{\ell} p^i t^i c_{\ell-i}, \qquad t = -v_1.$$

For applications of this theorem remark that the homomorphism p_{1*} has especially simple form:

$$p_{1*}t^s = c_{s+\ell+1}.$$

3.10. Corollary. *If $f : M \to N$ is a generic corank 1 map, then the cohomology class dual to the singularity locus A_k can be obtained as follows. One should expand all brackets in the product*

$$\prod_{p=2}^{k}\left(c_{\ell+1}+(p-1)t\sum_{i=0}^{\ell}p^i t^i c_{\ell-i}\right)$$

and formally replace any occurrence of t^s with $s \geq 0$ by $c_{\ell+1+s} = c_{\ell+1+s}(f^*TN - TM)$.

In the case $\ell = 0$ this assertion is proved in [26].

The residue polynomials of multisingularities of corank one maps can also be written in a closed form. To describe these polynomials we introduce the following notation. Consider the ring homomorphism

$$\rho : \mathbb{Z}[c_1, c_2, \dots] \to \mathbb{Z}[t, c_1, \dots, c_{\ell+1}]$$

given on the generators by $\rho(c_i) = c_i$ for $i \leq \ell+1$ and $\rho(c_{\ell+1+j}) = c_{\ell+1}t^j$. The following lemma can be formally derived from the formula of Theorem 3.4. Let P be arbitrary polynomial in the variables c_k. Consider its localized form $P = p_{*0}R^{(0)} + p_{*1}R^{(1)} + \dots$.

3.11. Lemma. *The homomorphism ρ vanishes on the terms $p_{s*}R^{(s)}$ for $s \geq 2$. Moreover, the terms $p_{*0}R^{(0)} + p_{*1}R^{(1)}$ are completely determined by the image $\rho(P)$ of this homomorphism.*

In what follows we set

$$Q_{p_1,\dots,p_r} = \rho(R_{A_{p_1-1},\dots,A_{k_r-1}}).$$

Due to Lemma 3.11, the polynomial $Q_{p_1,\dots,p_r}$ describes the initial terms of the residue polynomial for the multisingularity $(A_{p_1-1},\dots,A_{k_r-1})$. With this notation Theorem 3.9 asserts that for the case of a monosingularity ($r = 1$) this polynomial is given by

$$Q_p = \rho(R_{A_p-1}) = \sigma_1\sigma_2\dots\sigma_{p-1}. \tag{18}$$

3.12. Theorem. *The terms $p_{0*}R^{(0)}_{\underline{\alpha}} + p_{1*}R^{(1)}_{\underline{\alpha}}$ of the localized residue polynomial of a given multisingularity $\underline{\alpha} = (\alpha_1,\dots,\alpha_r)$ can be nontrivial only if every singularity α_i has the form A_k for some $k \geq 0$. For the multisingularity $(A_{p_1-1},\dots,A_{p_r-1})$ the corresponding localized terms are given by the following formula*

(19)

$$Q_{p_1,\dots,p_r} = \frac{1}{t^{r-1}}\sum_{\{1,\dots,r\}=J_1\sqcup\dots\sqcup J_k}(-1)^{r-k}(k-1)!\,\sigma_0^{k-1}\,Q_{|\underline{p}_{J_1}|}\cdots Q_{|\underline{p}_{J_1}|}.$$

In this relation, $|\underline{p}_J|$ denotes $\sum_{i\in J} p_i$. One can verify that the sum on the right hand side is divisible by t^{r-1} that is $Q_{p_1,\dots,p_r}$ is indeed a polynomial in variables $t, c_1, \dots, c_{\ell+1}$.

3.13. Corollary. *For a generic corank one map, the residue class of the multisingularity $(A_{p_1-1}, \dots, A_{p_r-1})$ can be obtained from the polynomial $Q_{p_1,\dots,p_r}$ in the following way. One should expand all brackets, replace any occurrence of $c_{\ell+1}t^k$ by $c_{\ell+1+k}$, and finally replace c_i by the relative Chern class $c_i(f) = c_i(f^*TN - TM)$.*

3.14. Example. For $\ell \le 0$ the first term of the localized Thom polynomial (with $r = 1-\ell$) has also a special meaning. If $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ is a map germ of kernel rank $1-\ell$ (i.e. of cokernel rank 1), then the fiber $f^{-1}(0)$ is the germ of ICIS of embedded dimension $1-\ell$. It means that this fiber is actually the germ of an IHS (isolated hypersurface singularity). The IHS's admit a stabilization allowing to compare hypersurfaces of different dimensions, see Sect. 2.5. The theory of characteristic classes associated with the stable classification of IHS's is the theory of Legendre characteristic classes. Thus the term $R_\alpha^{1-\ell}$ of the localized Thom polynomial for a given cokernel rank 1 ICIS α is determined by the Thom polynomial for the corresponding IHS. The same is applied to the term $R_{\underline{\alpha}}^{1-\ell}$ of the residue polynomial for arbitrary multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$. Recall that the ring $\mathcal{L}$ of universal Legendre characteristic classes is generated by the classes u, a_i which are subject to relations (2).

3.15. Theorem. *The polynomial $R_{\underline{\alpha}}^{(1-\ell)}(v_1, \dots, v_{1-\ell}, u_1)$ of the first localized term in the residue polynomial of the complete intersection multisingularity $\underline{\alpha}$ is nontrivial only if all singularities α_i forming the multisingularity $\underline{\alpha}$ are hypersurface singularities. If this is true, then this polynomial can be obtained from the Legendre residue polynomial of the hypersurface multisingularity $\underline{\alpha}$ by the change of variables determined by $u = u_1$ and*

$$1 + a_1 + a_2 + \cdots = \frac{(1+u_1)^{1-\ell} - (1+u_1)^{-\ell} v_1 + \cdots \pm v_{1-\ell}}{1 + v_1 + \cdots + v_{1-\ell}}.$$

Remark that the stabilization of IHS's does not extend to the complete intersection singularities of cokernel rank greater than 1: for different $\ell \le 0$ the classifications of $(-\ell)$-dimensional ICIS's are quite different.

3.3. Symmetric and Lagrange degeneracy loci

In this section we summarize some results on symmetric and Lagrange degeneracy loci. These results can be considered as symmetric analogues of the results on Porteous-Thom classes. Most of the relations of this section are known, see [14, 16, 10, 11, 12, 29]. Nevertheless, our presentation of these relations is quite different.

Consider the following problem. Assume we are given a complex vector bundle $V \to M$ over some smooth base and a generic self-conjugate morphism $\varphi : V \to V^*$. We may consider φ as a family of quadratic forms on the fibers of V or as a section of the bundle $\operatorname{Sym}^2 V^*$. The problem is to determine the cohomology classes dual to the locus $\Omega^r \subset M$ formed by the points at which φ has at least r-dimensional kernel.

The most efficient solution to this problem uses the language of symplectic geometry. Recall that the *symplectic structure* on a vector space E of even dimension $2n$ is a non-degenerate skew-symmetric bilinear form. The standard example is the space of the form $E = V \oplus V^*$ where the value of the symplectic form on the vectors $\xi \oplus \eta$, $\xi' \oplus \eta'$ is given by $\langle \xi', \eta \rangle - \langle \xi, \eta' \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between vectors and covectors.

A subspace $L \subset E$ of the middle dimension n is called *Lagrangian* if it is isotrope i.e. if $L^\perp = L$ where the orthogonal complement is considered with respect to the symplectic structure. All Lagrange subspaces of the fixed symplectic space E form the *Lagrange Grassmannian* Λ_n. Remark that a linear map $V \to V^*$ is self-adjoint iff its graph is Lagrangian. Thus the Lagrange Grassmannian can be considered as the natural compactification of the space of quadratic forms. In particular, $\dim \Lambda_n = n\,(n+1)/2$.

Now, consider more general problem formulated as follows. Consider a vector bundle E of even rank $2n$ over some smooth base M. Assume that the fibers of E are equipped with a symplectic structure smoothly depending on the point of the base. Let V, W be two Lagrange subbundles of E i.e. subbundles whose fibers are Lagrangian. We look for the cohomology class dual to the locus $\Omega^r \subset M$ formed by the points $x \in M$ at which the fibers V_x, W_x have at least r-dimensional intersection. Following [41] we call

$$[\Omega^r] \subset H^*(M)$$

Arnold-Fuks classes.

The problem on a self-adjoint map $\varphi : V \to V^*$ is a particular case of this one: for the symplectic bundle E one should take $E = V \oplus V^*$ and for Lagrange subbundles one should take the bundle $V \oplus \{0\}$ and the graph of the morphism φ, respectively.

3.16. Definition. The ring $\mathcal{L}^{\mathrm{Lag}}$ of *universal Lagrange characteristic classes* is the quotient ring of polynomials in variables $a_1, a_2, \ldots,$

$\deg a_i = i$, modulo the ideal generated by the relations:

$$a_k^2 - 2a_{k+1}a_{k-1} + 2a_{k+2}a_{k-2} - \cdots \pm 2a_{2k} = 0. \tag{20}$$

The ring $\mathcal{L}^{\mathrm{Lag}}$ is the cohomology ring of the stable Lagrange Grassmannian $\Lambda = \lim_n \Lambda_n$. The generators a_i correspond (up to a sign) to the Chern classes of the tautological rank n bundle U over Λ_n, namely, we set $a_i = c_i(U^*) = (-1)^i c_i(U)$. The relations can be written also in the form

$$(1 + a_1 + a_2 + \dots)(1 - a_1 + a_2 - \dots) = 1 \quad \text{or} \quad c(U + U^*) = 1.$$

In this form they immediately follow from the natural isomorphism $E/U \simeq U^*$ provided by the symplectic structure. The relations allow one to expand all powers of the variables a_i. The additive basis of $\mathcal{L}^{\mathrm{Lag}}$ is formed by the monomials of the form $a_1^{i_1}a_2^{i_2}\dots$ with $i_j \in \{0, 1\}$.

The Lagrange analogue of Schur polynomials are the so called *Schur Q-polynomials.* These are certain elements $Q_{\lambda_1,\dots,\lambda_r} \in \mathcal{L}^{\mathrm{Lag}}$ defined for any sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers by the following conditions.

- if $r = 1$, we set $Q_k = a_k$;
- if $r = 2$, we set

$$Q_{k,l} = a_k^2 - 2a_{k+1}a_{l-1} + 2a_{k+2}a_{l-2} - 2a_{k+3}a_{l-3} + \dots;$$

- for any even $r \geq 4$ we set

$$Q_{\lambda_1,\dots,\lambda_r} = \mathrm{Pf}\,|Q_{\lambda_i,\lambda_j}|_{1\leq i,j\leq r};$$

- for any odd $r \geq 3$ we set

$$Q_{\lambda_1,\dots,\lambda_r} = \sum_{k=1}^{r}(-1)^{k-1}a_{\lambda_k}\,Q_{\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1},\dots,\lambda_r}$$

Here Pf is the Pfaffian. Recall that the Pfaffian of a skew-symmetric matrix $\omega = \|\omega_{i,j}\|$ of even order $2n$ is given, by definition, by the equality

$$\mathrm{Pf}\,\|\omega_{i,j}\| = \sum \pm\omega_{i_1,i_2}\cdots\omega_{i_{2n-1},i_{2n}},$$

where the sum runs over all $(2n-1)!!$ ways to represent $\{1, 2, \dots, 2n\}$ as a union of n pairs $\{i_1, i_2\} \cup \dots \cup \{i_{2n-1}, i_{2n}\}$ and $\pm$ is the sign of the permutation $(1, 2, \dots, 2n) \mapsto (i_1, i_2, \dots, i_{2n})$.

The equality $Q_{k,l} = -Q_{l,k}$ for $k + l$ odd follows directly from the definition, and for $k + l$ even it follows from the identity (20). Moreover, the polynomial $Q_{\lambda_1,\dots,\lambda_r}$ depends skew-symmetrically on the indices λ_i. This follows from the fact that the Pfaffian is skew-symmetric with respect to simultaneous permutations of rows and columns of the matrix. In particular, $Q_{\lambda_1,\dots,\lambda_r}=0$ if one has $\lambda_i=\lambda_j$ for some $i\neq j$.

3.17. Remark. The distinction between the cases of even and odd r is apparent. In fact, there is the following explicit formula due to V. Kryukov:

$$Q_{\lambda_1,\dots,\lambda_r}=\sum_{i_1,\dots,i_r} w_{i_1,\dots,i_r}\,a_{\lambda_1+i_1}\dots a_{\lambda_r+i_r}, \tag{21}$$

where the coefficients $w_{i_1,\dots,i_r}$ do not depend on $\lambda_1,\dots,\lambda_r$ and are given by the formal expansion

(22)

$$\begin{aligned}\sum w_{i_1,\dots,i_r}\,\tau_1^{i_1}\dots\tau_r^{i_r} &= \prod_{1\le i<j\le r}\frac{\tau_j-\tau_i}{\tau_j+\tau_i}=\prod_{1\le i<j\le r}\frac{1-\tau_i\tau_j^{-1}}{1+\tau_i\tau_j^{-1}}\\ &=\prod_{1\le i<j\le r}(1-2\,\tau_i\tau_j^{-1}+2\,\tau_i^2\tau_j^{-2}-2\,\tau_i^3\tau_j^{-3}+\dots).\end{aligned}$$

Equation (22) is considered in the ring of infinite series in generators $\tau_1/\tau_2,\dots,\tau_{r-1}/\tau_r$, or, which is equivalent, in the completion of the ring of Loran polynomials with respect to an auxiliary grading such that the degree of the monomial $\tau_1^{i_1}\dots\tau_r^{i_r}$ is equal to $\sum_{k=1}^r k\,i_k$.

We are able now to formulate principle results on Lagrange and symmetric degeneracy problems. Let M be a manifold. Consider a symplectic vector bundle E and two its Lagrange subbundles V,W as at the beginning of this section.

3.18. Definition. *Lagrange characteristic classes of the triple* (E,V,W) are the Chern classes $a_i=c_i(E-V-W)=c_i(V^*-W)=c_i(W^*-V)$.

The identity (20) follows immediately from the equalities $E/V\simeq V^*$, $E/W\simeq W^*$ provided by the non-degeneracy of the symplectic structure.

For a self-adjoint map $V\to V^*$ one has $V\simeq W$ so that the Lagrange characteristic classes in this case are $a_i=c_i(V^*-V)$.

3.19. Theorem. *The Arnold-Fuks class $[\Omega^r]\in H^*(M)$ of the triple (E,V,W) is a universal Lagrange characteristic class given by an appropriate Schur Q-polynomial:*

$$[\Omega^r]=Q_{r,r-1,\dots,1}.$$

The *derived Arnold-Fuks classes* can also be defined. The singularity locus of the variety Ω^r coincides with Ω^{r+1}. Consider the *standard resolution* $\widetilde{\Omega}^r$ of Ω^r defined as the subvariety of $G_r(E)$ formed by all pairs (x, K_x), where $x \in M$ and K_x is an r-dimensional subspace of the intersection $V_x \cap W_x \subset E_x$. Denote by K the restriction to $\widetilde{\Omega}^r$ of the tautological rank r bundle over $G_r(E)$ and by $p_r : \widetilde{\Omega}^r \to M$ the natural projection.

Let R be an arbitrary polynomial in variables $v_1, \dots, v_r$. Denote by $R(c(K)) \in H^*(\widetilde{\Omega})$ the cohomology class obtained by setting $v_i = c_i(K)$.

3.20. Theorem. *The push-forward class $p_{r*}R(c(K))$ is expressed as a universal Lagrange characteristic class uniquely determined by r, R and evaluated for the given triple (E, V, W).*

More explicitly, set formally $c(K) = \prod_{i=1}^r (1 - t_i)$, substitute the corresponding symmetric functions in $-t_i$ to R and expand all brackets. Then the homomorphism p_{r_*} is given on the resulting monomials in the variables t_i by the following explicit formula

$$p_{r*}t_1^{s_1} \dots t_r^{s_r} = Q_{r+s_1, r-1+s_2, \dots, 1+s_r}.$$

3.21. Theorem. *The collection of homomorphisms p_{r*} provides the universal splitting*

$$\mathcal{L}^{\mathrm{Lag}} = \bigoplus_r H^*(BU(r)).$$

In other words, any universal Lagrange characteristic class $P \in \mathcal{L}^{\mathrm{Lag}}$ can be presented uniquely in the form

$$P = \sum_r p_{r*} R^{(r)},$$

where $R^{(r)}$ is a polynomial of degree $\deg R^{(r)} = \deg P - r\,(r+1)/2$ *in the variables $v_1, \dots, v_r$.*

The splitting of the Theorem is a particular case of the splitting (12). Namely, consider the classification of quadratic forms *in arbitrary number of variables*, where the forms $Q(x)$ and $Q'(x, y) = Q(x) + y^2$ are considered as stably equivalent, where $x \in \mathbb{C}^n$, $y \in \mathbb{C}$. The classifying space for this classification is the stable Lagrange Grassmannian $\Lambda = \lim_n \Lambda_n$. The singularity classes for this classification are the classes Ω^r of forms with kernel rank r. By the *symmetry group* G_{Ω^r} of the class Ω^r we mean the stationary subgroup of any quadratic form $x \in \Omega^r$ depending on *the smallest possible n number of variables.* It is clear that this representative is exactly the zero form and $n = r$. The stationary group for this element contains all linear transformations of the space $\mathbb{C}^r$. Therefore,

$$G_{\Omega^r} \sim U(r)$$

and the splitting of Theorem follows from (12). □

3.4. Twisted Lagrange degeneracy loci and localized Legendre characteristic classes

In applications, instead of Lagrange and symmetric degeneracy loci, one meets more often their twisted analogues that are called Legendre degeneracy loci.

Let $V \to M$ be a complex vector bundle. Consider a family φ of quadratic forms on the fibers of V that take values not in numbers but in the fibers of a supplementary line bundle $I \to M$. One can treat φ as a self-adjoint morphism $V \to V^* \otimes I$ or as a section of the bundle $\operatorname{Sym}^2 V^* \otimes I$.

Similarly, one can consider a vector bundle $E \to M$ equipped with the symplectic form on its fibers that takes values in the fibers of a line bundle I. Lagrange subbundles of this twisted symplectic bundle are defined similarly to the non-twisted case. If V, W are two Lagrange subbundles in E, then one defines in a similar way the degeneracy loci Ω^r and the corresponding Arnold-Fuks class $[\Omega^r] \in H^*(M)$.

Moreover, similarly to the non-twisted case one can consider the resolution subvariety $\widetilde{\Omega}^r \subset G_r(E)$, the tautological rank r bundle K over $\widetilde{\Omega}^r$ and the derived Arnold-Fuks class $p_{r*}R \in H^*(M)$ where R is an arbitrary polynomial in $v_i = c_i(K)$.

The twisted analogue of the Lagrange characteristic classes are Legendre ones. Recall (see Sect 2.5) that the ring $\mathcal{L}$ of Legendre characteristic classes is generated by the generators $u, a_1, a_2, \ldots$ that are subject to relations (2).

Lagrange characteristic classes can be obtained from Legendre ones by setting $u = 0$. Conversely, one can show that over $\mathbb{Q}$ as well as over any ring containing $1/2$ there is an isomorphism

$$\mathcal{L}[\tfrac{1}{2}] \simeq \mathbb{Z}[u] \otimes \mathcal{L}^{\mathrm{Lag}}[\tfrac{1}{2}]. \tag{23}$$

On should remark that this splitting does not hold over integers. This remark is especially important in the real problems where the Chern classes are replaced by the Stiefel-Whitney classes, all coefficients are reduced modulo 2 and the division by 2 is forbidden.

3.22. Definition. Legendre characteristic classes associated with the twisted symplectic bundle E and its Lagrange subbundles V, W are the Chern classes $u = c_1(I)$ and

$$a_i = c_i(E - V - W) = c_i(V^* \otimes I - W) = c_i(W^* \otimes I - V).$$

Respectively, the Legendre characteristic classes associated with a twisted self-adjoint morphism $V \to V^* \otimes I$ are $u = c_1(I)$ and $a_i = c_i(V^* \otimes I - V)$.

The identity (3) for these classes follows immediately from the isomorphisms $E/V \simeq V^* \otimes I$, $E/W \simeq W^* \otimes I$ implied by the non-degeneracy of the symplectic structure.

The twisted version of theorems of the previous section holds also true. One should replace only Lagrange characteristic classes by their Legendre analogues.

3.23. Theorem. *Every twisted Arnold-Fuks class as well as any its derived class is expressed as a universal Legendre characteristic class. Moreover, the collection of homomorphisms p_{r*} provides the universal splitting*

$$\mathcal{L} \simeq \bigoplus_r H^*(BU(r) \times BU(1)).$$

In other words, any Legendre characteristic class P has a unique representation in the following localized form

$$P = \sum_r p_{r*} R^{(r)},$$

where $R^{(r)}$ are polynomials in the classes $v_i = c_i(K)$, $i = 1, \dots, r$, and $u = c_1(I)$ of degree $\deg R^{(r)} = \deg P - r(r+1)/2$.

The importance of the splitting of this theorem is in the fact that for any Legendre or twisted symmetric degeneracy problem the term $p_{r*} R^{(r)}$ is represented by a cycle supported on the corresponding locus Ω^r.

The explicit formulae for the homomorphisms p_{r*} follow from the isomorphism (23). Namely, on can use the following trick borrowed from [14]. First consider the case when $I = J^{\otimes 2}$, where J is another line bundle with $c_1(J) = c_1(I)/2 = u/2$. In this case the twisted symplectic structure on E induces the non-twisted symplectic structure on $E' = E \otimes J^*$. The subbundles V, W of E induce the Lagrange subbundles $V' = V \otimes J^*$ and $W' = W \otimes J^*$ of E'. Moreover, the degeneracy locus Ω^r and its resolution $\widetilde{\Omega}^r$ for the triple (E, V, W) coincide with those for the triple (E', V', W').

Therefore we can apply the formulas of the previous section to find the direct images of the characteristic classes of the tautological bundle

K'. Since the Chern classes $c_i(K) = c_i(K' \otimes J)$ are expressed as polynomials in the classes $c_i(K')$ and $u/2 = c_1(J)$, this allows us to compute the direct image of any polynomial in the classes $c_i(K)$. The formulas obtained in this way can be applied to arbitrary line bundle I since they are universal. Remark that the intermediate steps in the derivation of these formulas use the division by 2 but the final expression for the derived classes has only integer coefficients since the group of Legendre characteristic classes is torsion free.

There are Legendre analogues of Theorems 3.6, 3.9, 3.12. In particular, there is an explicit formula for the initial terms of the localized form of a residue polynomial for a Lagrange multisingularity. These terms can be obtained from the trivial observation that the case $\ell = 0$ satisfies both $\ell \leq 0$ and $\ell \geq 0$. Namely, the terms $p_{0*}R^{(0)}_{\underline{\alpha}} + p_{1*}R^{(1)}_{\underline{\alpha}}$ for a Lagrange multisingularity $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ are non-trivial only if $\alpha_i = A_{p_i}$ for $i = 1, \ldots, r$ and some $p_i \geq 1$. By Theorem 3.15, these terms are determined by the corresponding formulas for the 0-dimensional complete intersection multisingularities $(A_{p_1}, \ldots, A_{p_r})$ that is by the formulas of Theorem 3.12 with $\ell = 0$.

§4. Computation of Thom polynomials

4.1. Restriction method

The Porteous-Thom singularities are determined by the 1-jet of the map germ. The Thom polynomial for these singularities have been computed in [27] by resolving these singularities and applying the known formulas for the Gysin homomorphism, see Sect. 3.1. The resolution method can be applied also for certain singularity classes determined by higher order jets, see eg. [28, 34, 21]. Nevertheless, for more complicated singularities finding an appropriate resolution is not easy and the method meets serious technical difficulties.

Quite recently R. Rimányi [31] suggested a much more simple indirect method that uses the following idea. Since the existence of the Thom polynomial is established, it remains to find the coefficients of this polynomial. For that it is sufficient to consider a number of examples for which both the Chern classes of the map and the classes dual to the singularity loci are known. Every such example provides linear relations on the coefficients of the Thom polynomial. With an appropriate choice of the examples these relations could determine the polynomial completely. Rimányi has shown that this method can be efficiently applied to compute Thom polynomials for essentially all classified classes of complex singularities. In the real problems the method can also be applied though with less efficiency [9].

For the collection of the test maps one can use the following ones. Let $f_0 : \mathbb{C}^m \to \mathbb{C}^n$ be the 'normal form' of a certain singularity class α. The symmetry group G_α of this singularity acts on the source and the target spaces $\mathbb{C}^m$, $\mathbb{C}^n$, respectively. Denote by BG_α the classifying space of the group G_α (or some of its smooth finite-dimensional approximations). Denote by M and N the total spaces of the vector bundles $E \to BG_\alpha$ and $F \to BG_\alpha$ with the fibers $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, corresponding to these actions. Finally, let $f : M \to N$ be the fibred map that coincides with f_0 on each fiber.

The relative Chern classes $c(f) = c(TN)/c(TM) = c(F)/c(E)$ of the test map can usually be easily computed using the splitting principle. Some information is known also on the classes dual to the singularity loci. For example, the locus of the singularity α is the zero section of the bundle $E \to BG_\alpha$ and its dual is the Euler characteristic class $e(E) = c_m(E)$ of the bundle. Besides, for any class β which is not adjacent to α the corresponding singularity locus is empty and so the dual cohomology class is equal to zero.

Relations arising from these test examples is usually sufficient to compute the Thom polynomials of all necessary singularities. To see when this method can lead to the desired answer let us turn back to the splitting (1). Assume that we know the complete classification of singularities up to a given codimension p. Assume that the classification problem under consideration is a complex one and that there are only finitely many singularity classes of codimension below p. In this case the splitting (12) holds and every test example allows us to compute the corresponding summand of the Thom polynomial provided by this splitting. This argument explains the applicability of the method in complex problems.

As an example let us show the computation of the Thom polynomial for the 'pleat' singularity $\Sigma^{1,1}$ of a map between two manifolds of equal dimension. Let $\xi \to B$ be a line bundle over some smooth base B, say, the tautological line bundle over $B = \mathbb{C}P^n$ for some $n \geq 2$. Set $t = c_1(\xi) \in H^2(B)$. Consider the following quasihomogeneous normal forms of the simplest singularities Σ^1 and $\Sigma^{1,1}$:

$$x \mapsto x^2, \qquad (x, y) \mapsto (x^3 + xy, y).$$

These formulas can be interpreted as the fibred maps $\xi \to \xi^{\otimes 2}$ and $\xi \oplus \xi^{\otimes 2} \to \xi^{\otimes 3} \oplus \xi^{\otimes 2}$, respectively, of the total spaces of the corresponding vector bundles. The source and the target manifolds are homotopy equivalent to B so we can identify $H^*(M) \simeq H^*(N) \simeq H^*(B)$. With these identifications, the relative Chern classes of the test maps are

$$\frac{1+2t}{1+t} = 1 + t - t^2 + \dots, \quad \frac{(1+3t)(1+2t)}{(1+t)(1+2t)} = 1 + 2t - 2t^2 + \dots,$$

respectively. Besides, the Euler class of the bundle $\xi \oplus \xi^{\otimes 2}$ is equal to $2t^2$. It follows that the unknown coefficients a, b of the desired Thom polynomial $a\, c_1^2 + b\, c_2$ satisfy the relations

$$0 = a\, t^2 + b\,(-t^2), \qquad 2\, t^2 = a\,(2t)^2 + b\,(-2t^2).$$

These equations lead to the unique solution $a = b = 1$ which means that the Tom polynomial of the singularity $\Sigma^{1,1}$ is equal to $c_1^2 + c_2$.

4.2. Computation of the residue polynomials for multisingularities

The restriction method considered above can be applied to the study of the characteristic classes of multisingularities. To apply the formulas of the section 2.6 we need every test map to be *proper* (this restriction is not needed for the study of monosingularities). The maps of the previous section satisfy this condition if *the relative dimension* $\ell = \dim N - \dim M$ *is non-negative.* Thus the direct application of the restriction method provides the computation of the residue polynomials of multisingularities with $\ell \geq 0$.

A new feature in this computation is that of the Gysin homomorphism $f_* : H^*(M) \to H^*(N)$ for the test maps. Assume that M and N are the total spaces of the vector bundles E and F, respectively, over some smooth base B. Let $f : M \to N$ be a fibred map whose restriction to each fiber coincides with the standard proper quasihomogeneous map $f_0 : \mathbb{C}^m \to \mathbb{C}^n$ in some coordinates. Because of the isomorphism $H^*(M) \simeq H^*(N) \simeq H^*(B)$ we can consider the homomorphism f_* as acting in the cohomology group $H^*(B)$. Due to the projection formula this homomorphism acts as the multiplication by the class $f_*(1) \in H^{2\ell}(B)$. This class can by found using the following lemma.

4.1. Lemma. *The class $f_*(1)$ satisfies the relation*

$$f_*(1)\, c_m(E) = c_n(F).$$

In applications, the cohomology ring $H^*(B)$ has no zero divisors, therefore, the relation of the lemma determines the class $f_*(1)$ and the homomorphism f_* uniquely.

Proof. The top Chern classes $c_m(E)$ and $c_n(F)$ are the cohomology classes Poincaré dual to the zero sections of the bundles E and F, respectively. In other words, $c_m(E) = i_*(1)$ and $c_n(F) = j_*(1)$, where $i : B \to M$ and $j : B \to N$ are the zero section embeddings. Since $j = f \circ i$, the relation of the lemma follows from the identity

$$f_*\, i_* = j_*.$$

Let now $\ell < 0$. Let the holomorphic map $f : M \to N$ be one of the test maps of the previous section used for the computation of the Thom polynomials. Then all level sets of f have positive dimensions. Therefore, this map cannot be proper, the homomorphism f_* is not defined, and the formulas of the section 2.6 cannot be applied directly. To extend the restriction method to this case we use the following observation. Set $\ell = \dim N - \dim M$. Fix some integer $s \geq \max(0, -\ell)$. Let $\Sigma^s = \Sigma^s(f) \subset M$ be the locus of the corresponding Porteous-Thom singularity. In what follows we make the following weaker assumption on the map f:

4.2. Assumption. The restriction of the map f to the locus Σ^s is proper.

If this assumption holds, then the homomorphism f_* is well defined on those classes that can be represented by cycles supported on Σ^s. More precisely, let $p_s : \widetilde{\Sigma}^s \to M$ be the natural resolution of Σ^s from Sect. 3.1. Then the map $f \circ p_s : \widetilde{\Sigma}^s \to N$ is proper and the homomorphism $(f \circ p_s)_* : H^*(\widetilde{\Sigma}^s) \to H^*(N)$ is well defined. In other words, *the homomorphism f_* is well defined on the image of the homomorphism $p_{s*} : H^*(\widetilde{\Sigma}^s) \to H^*(M)$.*

In particular, consider some element $P \in H^*(M)$ that is represented as a polynomial in the relative Chern classes $c_i = c_i(f)$. Assume that the localized form of Sect. 3.2 for this polynomial $P = \sum_k p_{k*} R^{(k)}$ has nontrivial terms with $k \geq s$ only. Then the homomorphism f_* is well defined on such a class. Namely, we set

$$f_* P = \sum_k (f p_k)_* R^{(k)}.$$

This assumption suggests the following conjectural sharpening of Conjecture 3.6.

4.3. Conjecture. *Let $f : M \to N$ be a holomorphic map, not necessary proper, that satisfies Assumption* 4.2. *Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ be a multisingularity type such that each individual singularity α_i of this collection has corank greater than or equal to s. Then every residue polynomial in the right hand side expressions of* (5–6) *is supported on*

Σ^s, *the homomorphism* f_* *is well defined on these polynomials and relations* (5–6) *hold.*

Now we observe that for $\ell < 0$ the test maps considered in the previous section satisfy Assumption 4.2 with $s = -\ell + 1$. This allows us to apply the restriction method to finding the residue polynomial of multisingularities of maps with $\ell < 0$.

4.3. Tables of computed polynomials

In this section we present some of the results on the computation of Thom polynomials for local singularities and residue polynomials for multisingularities. More complete tables occupy many pages. They are available on [23]. The polynomials are represented both in the localized form and in terms of the Chern classes. In the localized form, we skip the expressions for the terms given by the explicit formulas of Theorems 3.9, 3.12, and 3.15. The tables below allow the reader to estimate to what extend these formulas determine the residue polynomials.

The standard notation for the singularity classes is taken mostly from [3, 4]. All singularities in the studied range of dimensions are simple (have no modula in the normal form). Remark that the stable classification of $\mathcal{K}$-singularities is independent for different values ℓ of the relative dimension of the map. Therefore, one should not confuse with similar notation of different singularity classes appearing for different ℓ. Similarly, the homomorphism p_{r*} of Sect. 3.2 is different for different ℓ. By the codimension of a multisingularity we mean its complex codimension in the source manifold, that is, the degree of the corresponding residue polynomial.

Table 2 represents the residue polynomials of Legendre multisingularities (or hypersurface singularities) up to codimension 4. Up to codimension 6 the formulas can be found in [23].

Table 2: **Residue polynomials for Legendre multisingularities**

$$
\begin{aligned}
R_{A_1} &= p_{0*}(1) = 1\\
R_{A_2} &= p_{1*}(1) = a_1\\
R_{A_1^2} &= -p_{0*}(u) - 3\,p_{1*}(1) = -u - 3\,a_1\\
R_{A_3} &= p_{1*}(u - 3\,v_1) = u\,a_1 + 3\,a_2\\
R_{A_1A_2} &= -6\,p_{1*}(u - 2\,v_1) = -6\,(u\,a_1 + 2\,a_2)\\
R_{A_1^3} &= 2\,p_{0*}(u^2) + 2\,p_{1*}R^{(1)}_{A_1^3} = 2\,(u^2 + 19\,a_1u + 30\,a_2)\\
R_{A_4} &= p_{1*}R^{(1)}_{A_4} + 3\,p_{2*}(1) = a_1u^2 + 4\,a_2u + 3\,a_1a_2 + 6\,a_3\\
R_{D_4} &= p_{2*}(1) = -u\,a_2 + a_1a_2 - 2\,a_3
\end{aligned}
$$

$$
\begin{aligned}
R_{A_2^2} &= p_{1*}R^{(1)}_{A_2^2} - 21\,p_{2*}(1) = -3\,(3\,a_1u^2 + 8\,a_2u + 7\,a_1a_2 + 6\,a_3)\\
R_{A_1A_3} &= p_{1*}R^{(1)}_{A_1A_3} - 24\,p_{2*}(1) = -4\,(2\,a_1u^2 + 5\,a_2u + 6\,a_1a_2 + 3\,a_3)\\
R_{A_1^2A_2} &= p_{1*}R^{(1)}_{A_1^2A_2} + 144\,p_{2*}(1) = 24\,(3\,a_1u^2 + 7\,a_2u + 6\,a_1a_2 + 3\,a_3)\\
R_{A_1^4} &= -6\,p_{0*}(u^3) + p_{1*}R^{(1)}_{A_1^4} - 1026\,p_{2*}(1)\\
&= -6\,(u^3 + 111\,a_1u^2 + 239\,a_2u + 171\,a_1a_2 + 78\,a_3)\\
R_{A_5} &= p_{1*}R^{(1)}_{A_5} + p_{2*}(16\,u - 27\,v_1)\\
&= a_1u^3 - 4\,a_2u^2 + 16\,a_1a_2u - 12\,a_3u + 27\,a_1a_3 + 6\,a_4\\
R_{D_5} &= 2\,p_{2*}(2\,u - 3\,v_1) = -2\,(2\,a_2u^2 - 2\,a_1a_2u + 7\,a_3u - 3\,a_1a_3 + 6\,a_4)\\
R_{A_2A_3} &= p_{1*}R^{(1)}_{A_2A_3} - 6\,p_{2*}(28\,u - 39\,v_1)\\
&= -6\,(2\,a_1u^3 - 10\,a_2u^2 + 28\,a_1a_2u - 39\,a_3u + 39\,a_1a_3 - 18\,a_4)\\
R_{A_1A_4} &= p_{1*}R^{(1)}_{A_1A_4} - 70\,p_{2*}(2\,u - 3\,v_1)\\
&= -10\,(a_1u^3 - 4\,a_2u^2 + 14\,a_1a_2u - 16\,a_3u + 21\,a_1a_3 - 6\,a_4)\\
R_{A_1D_4} &= -4\,p_{2*}(5\,u - 6\,v_1) = 4\,(5\,a_2u^2 - 5\,a_1a_2u + 16\,a_3u - 6\,a_1a_3 + 12\,a_4)\\
R_{A_1A_2^2} &= p_{1*}R^{(1)}_{A_1A_2^2} + 18\,p_{2*}(74\,u - 95\,v_1)\\
&= 18\,(7\,a_1u^3 - 20\,a_2u^2 + 74\,a_1a_2u - 96\,a_3u + 95\,a_1a_3 - 50\,a_4)\\
R_{A_1^2A_3} &= p_{1*}R^{(1)}_{A_1^2A_3} + 18\,p_{2*}(76\,u - 99\,v_1)\\
&= 2\,(56\,a_1u^3 - 220\,a_2u^2 + 684\,a_1a_2u - 951\,a_3u + 891\,a_1a_3 - 522\,a_4)\\
R_{A_1^3A_2} &= p_{1*}R^{(1)}_{A_1^3A_2} - 2400\,p_{2*}(5\,u - 6\,v_1)\\
&= -48\,(28\,a_1u^3 - 55\,a_2u^2 + 250\,a_1a_2u - 318\,a_3u + 300\,a_1a_3 - 180\,a_4)\\
R_{A_1^5} &= 24\,p_{0*}(u^4) + p_{1*}R^{(1)}_{A_1^5} + 72\,p_{2*}(1621\,u - 1830\,v_1)\\
&= 24\,(u^4 + 671\,a_1u^3 - 701\,a_2u^2 + 4863\,a_1a_2u\\
&\qquad - 5844\,a_3u + 5490\,a_1a_3 - 3420\,a_4)
\end{aligned}
$$

The classification of $\mathcal{K}$-singularities of maps of relative dimension $\ell = -1$, that is, of one-dimensional complete intersection singularities starts with the classification of plane curve singularities. Up to codimension 5, the two classifications coincide and every residue polynomial of the complete intersection multisingularity is determined due to Theorem 3.15 by the residue polynomial of the corresponding hypersurface multisingularity. In codimension 6, there appears the simplest space curve singularity, S_5. The residue polynomials of the codimension 5 multisingularities are presented in Table 3. Up to codimension 8, these polynomials are available in [23].

Table 3: **Residue polynomials for multisingularities with $\ell = -1$**

$$\begin{aligned}
R_{A_5} &= p_{2*}R^{(2)}_{A_5} + 6\,p_{3*}(1)\\
&= 2\,(15\,c_1^6 + 5\,c_2c_1^4 + 25\,c_3c_1^3 - 26\,c_2^2c_1^2 - c_4c_1^2 - 15\,c_2c_3c_1\\
&\qquad - 6\,c_5c_1 + 3\,c_2^3 + 3\,c_3^2 - 3\,c_2c_4)\\
R_{D_5} &= p_{2*}R^{(2)}_{D_5} - 4\,p_{3*}(1)\\
&= 2\,(2\,c_1^6 - c_2\,c_1^4 - 9\,c_3c_1^3 + 6\,c_2^2c_1^2 - 3\,c_4c_1^2 + 6\,c_2c_3c_1\\
&\qquad + c_5c_1 - 2\,c_2^3 - 2\,c_3^2 + 2\,c_2c_4)\\
R_{S_5} &= p_{3*}(1) = c_2^3 - 2\,c_1c_3c_2 - c_4c_2 + c_3^2 + c_1^2c_4\\
R_{A_2A_3} &= p_{2*}R^{(2)}_{A_2A_3} - 12\,p_{3*}(1)\\
&= -6\,(50\,c_1^6 - 13\,c_2c_1^4 + 23\,c_3c_1^3 - 34\,c_2^2c_1^2 - 7\,c_4c_1^2\\
&\qquad - 14\,c_2c_3c_1 - 7\,c_5c_1 + 2\,c_2^3 + 2\,c_3^2 - 2\,c_2c_4)\\
R_{A_1A_4} &= p_{2*}R^{(2)}_{A_1A_4} - 20\,p_{3*}(1)\\
&= -20\,(13\,c_1^6 - c_2c_1^4 + 11\,c_3c_1^3 - 14\,c_2^2c_1^2 - 3\,c_4c_1^2 - 5\,c_2c_3c_1\\
&\qquad - 2\,c_5c_1 + c_2^3 + c_3^2 - c_2c_4)\\
R_{A_1D_4} &= p_{2*}R^{(2)}_{A_1D_4} + 8\,p_{3*}(1)\\
&= -4\,(5\,c_1^6 - 4\,c_2c_1^4 - 21\,c_3c_1^3 + 15\,c_2^2c_1^2 + 6\,c_2c_3c_1\\
&\qquad + c_5c_1 - 2\,c_2^3 - 2\,c_3^2 + 2\,c_2c_4)\\
R_{A_1A_2^2} &= p_{2*}R^{(2)}_{A_1A_2^2} + 108\,p_{3*}(1)\\
&= 36\,(71\,c_1^6 - 35\,c_2c_1^4 + 33\,c_3c_1^3 - 42\,c_2^2c_1^2 - 9\,c_4c_1^2\\
&\qquad - 15\,c_2c_3c_1 - 6\,c_5c_1 + 3\,c_2^3 + 3\,c_3^2 - 3\,c_2c_4)\\
R_{A_1^2A_3} &= p_{2*}R^{(2)}_{A_1^2A_3} + 72\,p_{3*}(1)\\
&= 2\,(1260\,c_1^6 - 545\,c_2c_1^4 + 425\,c_3c_1^3 - 692\,c_2^2c_1^2 - 173\,c_4c_1^2\\
&\qquad - 214\,c_2c_3c_1 - 97\,c_5c_1 + 36\,c_2^3 + 36\,c_3^2 - 36\,c_2c_4)\\
R_{A_1^3A_2} &= p_{2*}R^{(2)}_{A_1^3A_2} - 864\,p_{3*}(1)\\
&= -48\,(501\,c_1^6 - 332\,c_2c_1^4 + 215\,c_3c_1^3 - 241\,c_2^2c_1^2 - 56\,c_4c_1^2\\
&\qquad - 78\,c_2c_3c_1 - 27\,c_5c_1 + 18\,c_2^3 + 18\,c_3^2 - 18\,c_2c_4)\\
R_{A_1^5} &= p_{2*}R^{(2)}_{A_1^5} + 9408\,p_{3*}(1)\\
&= 24\,(10368\,c_1^6 - 8561\,c_2c_1^4 + 5045\,c_3c_1^3 - 4285\,c_2^2c_1^2 - 1125\,c_4c_1^2\\
&\qquad - 1456\,c_2c_3c_1 - 379\,c_5c_1 + 391\,c_2^3 + 393\,c_3^2 - 390\,c_2c_4 - c_6)
\end{aligned}$$

For the case of multisingularities of maps of equally dimensional manifolds ($\ell = 0$) and maps of relative dimension $\ell = 1$, the residue polynomials are given in Tables 4 and 5, respectively. These polynomials are computed up to codimension 8, see [23]. In the tables, we present these polynomials up to codimension 5 and 6, respectively. By A_k, $I_{a,b}$, and J_6 we denote singularity classes with local algebras isomorphic to $\mathbb{C}[x]/x^{k+1}$, $\mathbb{C}[x,y]/(x\,y,\ x^a+y^b)$, and $\mathbb{C}[x,y]/(x^2,\ x\,y,\ y^2)$, respectively.

Table 4: **Residue polynomials for multisingularities with $\boldsymbol{\ell = 0}$**

$$
\begin{aligned}
R_{A_1} &= p_{1*}(1) = c_1\\
R_{A_2} &= p_{1*}(u_1 - 2\,v_1) = c_1^2 + c_2\\
R_{A_1^2} &= -2\,p_{1*}(2\,u_1 - 3\,v_1) = -2\,(2\,c_1^2 + c_2)\\
R_{A_3} &= p_{1*}((u_1 - 3\,v_1)\,(u_1 - 2\,v_1)) = c_1^3 + 3\,c_2c_1 + 2\,c_3\\
R_{A_1A_2} &= -6\,p_{1*}((u_1 - 2\,v_1)^2) = -6\,(c_1^3 + 2\,c_2c_1 + c_3)\\
R_{A_1^3} &= 8\,p_{1*}(5\,u_1^2 - 17\,v_1u_1 + 15\,v_1^2) = 8\,(5\,c_1^3 + 7\,c_2c_1 + 3\,c_3)\\
R_{A_4} &= p_{1*}R^{(1)}_{A_4} + 2\,p_{2*}(1) = c_1^4 + 6\,c_2c_1^2 + 9\,c_3c_1 + 2\,c_2^2 + 6\,c_4\\
R_{I_{2,2}} &= p_{2*}(1) = c_2^2 - c_1c_3\\
R_{A_2^2} &= p_{1*}R^{(1)}_{A_2^2} - 12\,p_{2*}(1) = -3\,(3\,c_1^4 + 12\,c_2c_1^2 + 13\,c_3c_1 + 4\,c_2^2 + 8\,c_4)\\
R_{A_1A_3} &= p_{1*}R^{(1)}_{A_1A_3} - 8\,p_{2*}(1) = -4\,(2\,c_1^4 + 9\,c_2c_1^2 + 11\,c_3c_1 + 2\,c_2^2 + 6\,c_4)\\
R_{A_1^2A_2} &= p_{1*}R^{(1)}_{A_1^2A_2} + 48\,p_{2*}(1) = 24\,(3\,c_1^4 + 10\,c_2c_1^2 + 10\,c_3c_1 + 2\,c_2^2 + 5\,c_4)\\
R_{A_1^4} &= p_{1*}R^{(1)}_{A_1^4} - 288\,p_{2*}(1) = -48\,(14\,c_1^4 + 37\,c_2c_1^2 + 33\,c_3c_1 + 6\,c_2^2 + 15\,c_4)\\
R_{A_5} &= p_{1*}R^{(1)}_{A_5} + 2\,p_{2*}(5\,u_1 - 11\,v_1)\\
&= c_1^5 + 10\,c_2c_1^3 + 25\,c_3c_1^2 + 10\,c_2^2c_1 + 38\,c_4c_1 + 12\,c_2c_3 + 24\,c_5\\
R_{I_{2,3}} &= 2\,p_{2*}(u_1 - 2\,v_1) = -2\,(c_3c_1^2 - c_2^2c_1 + c_4c_1 - c_2c_3)\\
R_{A_2A_3} &= p_{1*}R^{(1)}_{A_2A_3} - 72\,p_{2*}(u_1 - 2\,v_1)\\
&= -12\,(c_1^5 + 7\,c_2c_1^3 + 13\,c_3c_1^2 + 6\,c_2^2c_1 + 17\,c_4c_1 + 6\,c_2c_3 + 10\,c_5)\\
R_{A_1A_4} &= p_{1*}R^{(1)}_{A_1A_4} - 60\,p_{2*}(u_1 - 2\,v_1)\\
&= -10\,(c_1^5 + 8\,c_2c_1^3 + 17\,c_3c_1^2 + 6\,c_2^2c_1 + 22\,c_4c_1 + 6\,c_2c_3 + 12\,c_5)\\
R_{A_1I_{2,2}} &= -2\,p_{2*}(5\,u_1 - 8\,v_1) = 2\,(5\,c_3c_1^2 - 5\,c_2^2c_1 + 3\,c_4c_1 - 3\,c_2c_3)\\
R_{A_1A_2^2} &= p_{1*}R^{(1)}_{A_1A_2^2} + 72\,p_{2*}(7\,u_1 - 13\,v_1)\\
&= 18\,(7\,c_1^5 + 40\,c_2c_1^3 + 65\,c_3c_1^2 + 28\,c_2^2c_1 + 76\,c_4c_1 + 24\,c_2c_3 + 40\,c_5)
\end{aligned}
$$

$$\begin{aligned}
R_{A_1^2A_3} &= p_{1*}R^{(1)}_{A_1^2A_3} + 48\,p_{2*}(9\,u_1 - 17\,v_1)\\
&= 8\,(14\,c_1^5 + 88\,c_2c_1^3 + 157\,c_3c_1^2 + 54\,c_2^2c_1 + 179\,c_4c_1 + 48\,c_2c_3 + 90\,c_5)\\
R_{A_1^3A_2} &= p_{1*}R^{(1)}_{A_1^3A_2} - 192\,p_{2*}(19\,u_1 - 34\,v_1)\\
&= -48\,(28\,c_1^5 + 139\,c_2c_1^3 + 211\,c_3c_1^2 + 76\,c_2^2c_1\\
&\qquad + 221\,c_4c_1 + 60\,c_2c_3 + 105\,c_5)\\
R_{A_1^5} &= p_{1*}R^{(1)}_{A_1^5} + 768\,p_{2*}(41\,u_1 - 71\,v_1)\\
&= 384\,(42\,c_1^5 + 176\,c_2c_1^3 + 244\,c_3c_1^2 + 82\,c_2^2c_1\\
&\qquad + 236\,c_4c_1 + 60\,c_2c_3 + 105\,c_5)
\end{aligned}$$

Table 5: **Residue polynomials for multisingularities with $\ell = 1$**

$$\begin{aligned}
R_{A_0} &= p_{0*}(1) = 1\\
R_{A_0^2} &= -p_{0*}(u_1) = -c_1\\
R_{A_1} &= p_{1*}(1) = c_2\\
R_{A_0^3} &= 2\,p_{0*}(u_1^2) + 2\,p_{1*}(1) = 2\,(c_1^2 + c_2)\\
R_{A_0A_1} &= 2\,p_{1*}(-u_1 + 2\,v_1) = -2\,(c_1c_2 + c_3)\\
R_{A_0^4} &= -6\,p_{0*}(u_1^3) - 6\,p_{1*}(3\,u_1 - 5\,v_1) = -6\,(c_1^3 + 3\,c_1c_2 + 2\,c_3)\\
R_{A_2} &= p_{1*}R^{(1)}_{A_2} = c_2^2 + c_1c_3 + 2\,c_4\\
R_{A_0^2A_1} &= p_{1*}R^{(1)}_{A_0^2A_1} = 2\,(3\,c_1^2c_2 + 2\,c_2^2 + 7\,c_1c_3 + 6\,c_4)\\
R_{A_0^5} &= 24\,p_{0*}(u_1^4) + p_{1*}R^{(1)}_{A_0^5} = 24\,(c_1^4 + 6\,c_1^2c_2 + 2\,c_2^2 + 9\,c_1c_3 + 6\,c_4)\\
R_{A_1^2} &= 2\,p_{1*}(-u_1 + 3\,v_1)\,(2\,u_2 - 3\,u_1v_1 + 6\,v_1^2)\\
&= -2\,(2\,c_1c_2^2 + c_1^2c_3 + 4\,c_2c_3 + 5\,c_1c_4 + 6\,c_5)\\
R_{A_0A_2} &= p_{1*}R^{(1)}_{A_0A_2} = -3\,(c_1c_2^2 + c_1^2c_3 + 2\,c_2c_3 + 4\,c_1c_4 + 4\,c_5)\\
R_{A_0^3A_1} &= p_{1*}R^{(1)}_{A_0^3A_1} = -24\,(c_1^3c_2 + 2\,c_1c_2^2 + 4\,c_1^2c_3 + 3\,c_2c_3 + 8\,c_1c_4 + 6\,c_5)\\
R_{A_0^6} &= -120\,p_{0*}(u_1^5) + p_{1*}R^{(1)}_{A_0^6}\\
&= -120\,(c_1^5 + 10\,c_1^3c_2 + 10\,c_1c_2^2 + 25\,c_1^2c_3 + 12\,c_2c_3 + 38\,c_1c_4 + 24\,c_5)\\
R_{A_3} &= p_{1*}R^{(1)}_{A_3} + p_{2*}(1)\\
&= c_2^3 + 3\,c_1c_2c_3 + c_3^2 + 2\,c_1^2c_4 + 7\,c_2c_4 + 10\,c_1c_5 + 12\,c_6\\
R_{J_6} &= p_{2*}(1) = c_3^2 - c_2c_4
\end{aligned}$$

$$
\begin{aligned}
R_{A_0A_1^2} &= p_{1*}R^{(1)}_{A_0A_1^2} + 24\,p_{2*}(1) \\
&= 8\,(2\,c_1^2c_2^2 + c_2^3 + c_1^3c_3 + 9\,c_1c_2c_3 + 3\,c_3^2 + 8\,c_1^2c_4 \\
&\qquad + 9\,c_2c_4 + 21\,c_1c_5 + 18\,c_6) \\
R_{A_0^2A_2} &= p_{1*}R^{(1)}_{A_0^2A_2} + 18\,p_{2*}(1) \\
&= 6\,(2\,c_1^2c_2^2 + c_2^3 + 2\,c_1^3c_3 + 10\,c_1c_2c_3 + 3\,c_3^2 + 13\,c_1^2c_4 \\
&\qquad + 11\,c_2c_4 + 30\,c_1c_5 + 24\,c_6) \\
R_{A_0^4A_1} &= p_{1*}R^{(1)}_{A_0^4A_1} + 408\,p_{2*}(1) \\
&= 24\,(5\,c_1^4c_2 + 20\,c_1^2c_2^2 + 5\,c_2^3 + 30\,c_1^3c_3 + 67\,c_1c_2c_3 \\
&\qquad + 17\,c_3^2 + 103\,c_1^2c_4 + 55\,c_2c_4 + 178\,c_1c_5 + 120\,c_6) \\
R_{A_0^7} &= 720\,p_{0*}(u_1^6) + p_{1*}R^{(1)}_{A_0^7} + 12240\,p_{2*}(1) \\
&= 720\,(c_1^6 + 15\,c_1^4c_2 + 30\,c_1^2c_2^2 + 5\,c_2^3 + 55\,c_1^3c_3 + 79\,c_1c_2c_3 \\
&\qquad + 17\,c_3^2 + 141\,c_1^2c_4 + 55\,c_2c_4 + 202\,c_1c_5 + 120\,c_6)
\end{aligned}
$$

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