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Numerical characterisations of hyperquadrics

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Abstract.

Smooth quadric hypersuraces in $\mathbb{P}^{n+1}(\mathbb{C})$ are numerically characterised as the smooth Fano n-folds of length n, *i.e.*, a smooth Fano n-fold X is isomorphic to a hyperquadric if and only if the minimum of the intersection number $(C, -K_X)$ is n, where C runs through the rational curves on X.

Introduction

This article is a supplement to the author's joint paper [2], where we characterised projective n-space as a unique smooth Fano n-fold of length $n+1$, the largest value possible. The purpose of this article is to characterise smooth hyperquadrics as Fano manifolds of the the second largest length n.

Given a Fano manifold X [resp. a pair (X, x_0) of a Fano manifold X and a closed point x_0 on it], we define the (global) *length* $l(X)$ of X [resp. the *local lenghth* $l(X, x_0)$ of (X, x_0)] to be the positive integer

$$\min_{C \subset X} \{(C, -K_X)\},$$

where C runs through the set of the rational curves contained in X [resp. the set of the rational curves such that $x_0 \in C \subset X$].

The local lenght $l(X, x_0)$ is a lower semiconitinuous function in x_0 and the global lenghth $l(X)$ is by definition equal to $\inf_{x_0 \in X} l(X, x_0)$. For a given closed point $x_0 \in X$, it is known that $l(X, x_0) \leq \dim X + 1$, the equality holding if and only if X is projective space [2].

In terms of the notions above, our main result is the following

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Theorem 0.1. *Let X be a smooth Fano variety of dimension $n \geq 3$ defined over an algebraically closed field k of characteristic zero. Then the following three conditions are equivalent:*

(1) *X is isomorphic to a smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$.*
(2) *The global length $l(X)$ is n.*
(3) *$\rho(X) = 1$ and $l(X, x_0) = n$ for a sufficiently general point $x_0 \in X$, where $\rho(X)$ stands for the Picard number.*[1]

This simple numerical result involves the preceding characterisations due to Brieskorn [1], Kobayashi-Ochiai [6], and Cho-Sato [3][4] as immediate corollaries. Namely

Theorem 0.2. *For a smooth X Fano n-fold ($n \geq 3$) over $\mathbb{C}$, the three conditions in* (0.1) *are also equivalent to the following four:*

(4) *There is a homotopy equivalence between X and Q_n such that the inducedcohomology isomorphism $\mathrm{H}^2(Q_n, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$ identifies the anticanonical classes.*
(5) *The anticanonical class $c_1(X)$ is divisible by n in* $\mathrm{Pic}(X)$.
(6) *The tangent bundle Θ_X is not ample, but $\wedge^2\Theta_X$ is ample.*[2]
(7) *There is a surjective morphism $Q_m \to X$, $m \geq n$, and $X \not\simeq \mathbb{P}^n$.*

Let us briefly outline our strategy to the proof of Theorem 0.1, the essential part of which is the implication (3) $\Rightarrow$ (1) proved in §3.

Assume that a smooth Fano n-fold X satisfies the condition (3). Because smooth Fano 3-folds with Picard number one are completely classified by Iskovskih [5], we may assume that $n \geq 4$ (this assumption is of course of purely technical nature). Pick up two general points $x_+, x_- \in X$. We consider an (arbitrary) irreducible component $W\langle x_+, x_- \rangle$ of the closed subset

$$\{C \subset X \,|\, C \text{ is a connected union of rational curves, } C \supset \{x_+, x_-\}, (C, -K_X) = 2n\}$$

of the Chow scheme $\mathrm{Chow}(X)$.

Under our hypothesis, it is easy to show that $\dim W\langle x_+, x_- \rangle = n - 1$. Each closed point $w \in W\langle x_+, x_- \rangle$ represents either an irreducible rational curve $C \subset X$ or a connected union of two irreducible rational curves $L_+ \cup L_- \subset X$ with $L_\pm \ni x_\pm$, $L_\pm \not\ni x_\mp$.

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[1]The condition on the Picard number is essential; see Remark 4.2 below.

[2]A differential-geometric analogue of this condition (positivity of the holomorphic bisectional curvature with one-dimensional degeneracy) is given in [8].

Let $V\langle x_+, x_-\rangle \subset W\langle x_+, x_-\rangle \times X$ be the associated incidence variety with natural surjective projection $\mathrm{pr}_X\colon V\langle x_+, x_-\rangle \to X$. Let $\overline{V}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$ and $\overline{\mathrm{pr}}_X\colon \overline{V}\langle x_+, x_-\rangle \to X$ the induced projection. The inverse image of $x_\pm$ via this projection determines a distinguished section $\sigma_\pm \subset \overline{V}\langle x_+, x_-\rangle$ over the normalisation $\overline{W}\langle x_+, x_-\rangle$ of $W\langle x_+, x_-\rangle$.

Given a smooth curve T and a morphism $f\colon T \to \overline{W}\langle x_+, x_-\rangle$, the fibre product $T \times_{\overline{W}\langle x_+, x_-\rangle} \overline{V}\langle x_1, x_2\rangle$ is a very special conic bundle over T, the properties of which are studied in §2. With the aid of the results obtained in §2, we show that $\overline{\mathrm{pr}}_X$ lifts to an isomophism between $\overline{V}\langle x_+, x_-\rangle$ and the two-point blowup $\mathrm{Bl}_{\{x_+, x_-\}}X$ of X, inducing isomorphisms

$$\overline{W}\langle x_+, x_-\rangle \simeq \sigma_\pm \simeq E_\pm \simeq \mathbb{P}^{n-1},$$

where $E_\pm \subset \mathrm{Bl}_{\{x_+, x_-\}}X$ is the exceptional divisor over $x_\pm \in X$. The pullback $\tilde{H}_0 = \mathrm{pr}^*_{\overline{W}}L$ of the hyperplane divisor $L \subset \overline{W}\langle x_+, x_-\rangle \simeq \mathbb{P}^{n-1}$ is a semiample divisor on $\overline{V}\langle x_+, x_-\rangle \simeq \mathrm{Bl}_{\{x_1, x_2\}}X$. Then we show that $\tilde{H}_0$ contracts to an ample divisor H_0 on X and that the complete linear system $|H_0|$ defines an isomorphism from X to a hyperquadric in $\mathbb{P}^{n+1}$.

The parameter space $W\langle x_+, x_-\rangle$ eventually turns out to be the dual projective space of the complete liner system $|\mu^*H_0 - E_+ - E_-| \simeq \mathbb{P}^{n-1}$ on $\mathrm{Bl}_{\{x_+, x_-\}}X$, which is viewed as the sublinear system $|H_0(-x_+ - x_-)| \subset |H_0| \simeq \mathbb{P}^{n+1}$ on X. To be more explicit, for each $n-1$-dimensional linear subspace Λ of

$$\mathrm{H}^0(X, \mathcal{I}_{x_+}\mathcal{I}_{x_-}(H_0)) \subset \mathrm{H}^0(Q_n, \mathcal{O}(1)),$$

we associate $[C] \in W\langle x_+, x_-\rangle$, where C is the plane conic cut out of Q_n by the $n-1$ hyperplanes $\in \Lambda$ through x_+, x_-.

Convention: In what follows, every scheme is defined over the complex number field. Schemes are often identified with the set of their complex points, regarded as analytic spaces with Euclidean topology.

For mathematical notation, we basically follow the convention in [2], to which we refer the reader for technical details as well.

§1. Review of basic facts

In this section, we review several elementary facts and some basic results of [2] concerning unsplitting family of rational curves.

Given a projective variety X, the *Chow scheme* $\mathrm{Chow}(X)$ and the *Hilbert scheme* $\mathrm{Hilb}(X)$ are defined as the parameter spaces of effective cycles and closed subschemes, respectively. They are known to exsist as disjoint union of projective schemes. An effective cycle (or a closed subscheme) $\Gamma \subset X$ will be denoted by $[\Gamma]$ when viewed as a point in $\mathrm{Chow}(X)$ (or of $\mathrm{Hilb}(X)$).

For two projecive varieties X, Y, the morphisms from Y to X form a locally closed (and hence quasiprojective) subset $\mathrm{Hom}(Y, X)$ of $\mathrm{Hilb}(Y \times X)$. When X is smooth and Y is a curve, we have the local dimension estimate

$$\chi(Y, f^*\Theta_X) \leq \dim_{[f]} \mathrm{Hom}(Y, X) \leq \dim \mathrm{H}^0(Y, f^*\Theta_X)$$

at a given closed point $[f]$. The second inequality becomes equality if and only if $\mathrm{Hom}(Y, X)$ is smooth at $[f]$.

The following is an immediate consequence of well known Sard's theorem.

Proposition 1.1. *Let X be a projective variety, M a smooth scheme of finite type* [resp. *locally of finite type*] *and let $h\colon M \to \mathrm{Hom}(\mathbb{P}^1, X)$ a morphism. Assume that the naturally induced morphism $\Phi_h\colon M \times \mathbb{P}^1 \to X$ is dominant (i.e. the image contains a nonempty open subset of X). Choose a general* [resp. *sufficiently general*] *nonsingular closed point $x_0 \in X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$ and take an arbitrary closed point $y \in M$. Then the natural $\mathbb{C}$-linear differential map*

$$\Theta_{M,y} \oplus \Theta_{\mathbb{P}^1,p} \to \Theta_{X,x_0}$$

is surjective at any closed point $(y, p) \in \Phi^{-1}(x_0)$. Specifically when h is a locally closed embedding of M into $\mathrm{Hom}(\mathbb{P}^1, X)$ with $y = [f] \in \mathrm{Hom}(\mathbb{P}^1, X)$, $f(\mathbb{P}^1) \subset X_{\mathrm{reg}}$, the natural evaluation map gives a surjection from $\Theta_{M,[f]} \subset \mathrm{H}^0(\mathbb{P}^1, f^*\Theta_X)$ onto Θ_{X,x_0} (under the condition that $f(\mathbb{P}^1)$ passes through the (sufficiently) general closed point $x_0 \in X$).

Let $\mathcal{U} \subset \mathrm{Chow}(X)$ be a locally closed subset. The *incidence variety* attached to $\mathcal{U}$ is the closed subset $\mathcal{G} \subset \mathcal{U} \times X$ defined by

$$\mathcal{G} = \{([Y], x) |\ [Y] \in \mathcal{U}, x \in Y \subset X\}.$$

We let $\mathrm{pr}_{\mathcal{U}}$ and pr_X denote the natural projections from the incidence variety to $\mathcal{U}$ and to X, respectively.

Corollary 1.2. *Let $A \subset X$ be an arbitrary finite set of closed points on a smooth projecive variety X. Let $\mathcal{U}\langle A\rangle$ be the locally closed* *subset* $\subset \mathrm{Chow}(X)$ *of finite type consisting of irreducible, reduced, smooth rational curves which contain* A, *and let* $\mathcal{G}\langle A\rangle \subset \mathcal{U}\langle A\rangle \times X$ *denote the associated incidence variety. If the projection* $\mathrm{pr}_X : \mathcal{G}\langle A\rangle \to X$ *is dominant and* $x_0 \in X$ *is a general closed point, then, for each element* $[C]$ *of the closed subset*

$$\mathcal{U}\langle A, x_0\rangle = \{[C] \in \mathcal{U}\langle A\rangle \mid C \ni x_0\},$$

the sheaf $\Theta_X \otimes_{\mathcal{O}_X} \mathcal{O}_C(-A)$ *is generated by global sections.*[3] *In particular,* $\mathcal{U}\langle A, x_0\rangle$ *is smooth, with Zariski tangent space* $\mathrm{H}^0(C, \mathcal{N}_{X/C}(-A - x_0))$ *at* $[C]$.

Proof. Since $\mathcal{U}\langle A\rangle$ consists of smooth rational curves on X, it is thought of as a locally closed subscheme of $\mathrm{Hilb}(X)$ in an obvious way, with $\mathcal{G}\langle A\rangle$ being the associated universal family. Its Zariski tangent space at $[C]$ is naturally identified with $\mathrm{H}^0(C, \mathcal{N}_{C/X}(-A))$. By assumption, the universal family $\mathcal{G}\langle A\rangle$ dominates X so that the differential $\Theta_{\mathcal{G}\langle A\rangle} \to \mathrm{pr}_X^* \Theta_X$ is onto at anly point $p \in \mathcal{G}\langle A\rangle$ over the general point $x_0 \in X$. This differential naturally induces homomorphisms

$$\begin{aligned} \Theta_{\mathcal{G}\langle A\rangle/\mathcal{U}\langle A\rangle}|_{\{[C]\}\times C} &\to \Theta_C, \\ \mathrm{pr}_{\mathcal{U}\langle A\rangle}^* \Theta_{\mathcal{U}\langle A\rangle}|_{\{[C]\}\times C} &\to \mathcal{N}_{X/C}. \end{aligned}$$

The second homomorphism is generically surjective whenever $C \ni x_0$. In particular, $\mathrm{H}^0(C, \mathcal{N}_{X/C}(-A))$ generically generates $\mathcal{N}_{X/C}(-A)$, meaning that

$$\mathcal{N}_{X/C}(-A) \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(d_i), \quad d_i \geq 0.$$

We have therefore

$$\mathrm{H}^1(C, \mathcal{N}_{X/C}(-A)) = \mathrm{H}^1(C, \mathcal{N}_{X/C}(-A - x_0)) = 0$$

and hence $\mathcal{U}\langle A\rangle$ and $\mathcal{U}\langle A, x\rangle$ are both smooth at $[C] \in \mathcal{U}\langle A, x_0\rangle$.

So far, we have been dealing with general families of rational curves. From now on, we will exclusively treat rational curves of low degree.

Let X be a smooth, projective, uniruled variety with an ample divisor H and $x_0 \in X$ a closed point. Define the *minimum degree* $\mathrm{Mindeg}(X, x_0, H)$ *of the rational curves through* x_0 to be the minimum of the intersection numbers (C, H), C running through the irreducible

[3] Here A is viewed as an effective divisor on the nonsingular curve C.

rational curves containing x_0 in X. (Of course $\mathrm{Mindeg}(X, x_0, H) = l(X, x_0)$ when X is Fano and $H = -K_X$, the case we are interested in.) If a rational curve $C \ni x_0$ satisfies $(C, H) = \mathrm{Mindeg}(X, x_0, H)$, call C a *rational curve of minimum degree through* x_0. The rational curves of minimum degree through the base point x_0 form a closed (and hence projective) subscheme of finite type of $\mathrm{Chow}(X)$, and so does its arbitrary irreducible component $S\langle x_0\rangle \subset \mathrm{Chow}(X)$. The associated incidence variety

$$F\langle x_0\rangle = \{([C], x) |\ [C] \in S\langle x_0\rangle, x \in C \subset X\}$$

is naturally a closed subscheme of $S\langle x_0\rangle \times X$ with two projections pr_S and pr_X to $S\langle x_0\rangle$ and X.

The family $\mathrm{pr}_S \colon F\langle x_0\rangle \to S\langle x_0\rangle$ (of rational curves of minimum degree through x_0) is an *unsplitting family* of rational curves, *i.e.*, every closed fibre $C = F_s$ is a reduced, irreducible rational curve on X.

Proposition 1.3. *In the above notation, assume that the base point x_0 is general in X and that $\dim S\langle x_0\rangle \geq 2$. Let Y be the image $\mathrm{pr}_X(F\langle x_0\rangle)$. Let $\overline{S}\langle x_0\rangle$, $\overline{F}\langle x_0\rangle$ and $\overline{Y}$ be the normalisations of $S\langle x_0\rangle$, $F\langle x_0\rangle$ and Y, and denote by $\overline{\mathrm{pr}}_{\overline{S}} \colon \overline{F}\langle x_0\rangle \to \overline{S}\langle x_0\rangle$ and $\overline{\mathrm{pr}}_{\overline{Y}} \colon \overline{F}\langle x_0\rangle \to \overline{Y}$ the naturally induced morphism. Then we have*

(1) *If $[L] \in S\langle x_0\rangle$ is a general member, then the rational curve $L \subset X$ is smooth $\mathbb{P}^1$ and its normal bundle $\mathcal{N}_{L/X}$ in X is isomorphic to $\mathcal{O}(1)^{\oplus r} \oplus \mathcal{O}^{\oplus n-r-1}$, where $2 \leq r = \dim Y - 1 = (C, -K_X) - 2 \leq n - 1$.*

(2) *Only finitely many members L of $S\langle x_0\rangle$ can have singulrities at the base point x_0.*

(3) *Only finitely many members L of $S\langle x_0\rangle$ can have cuspidal singularities and no member has a cuspidal singularity at the base point x_0.*

(4) *The first projection $\overline{\mathrm{pr}}_{\overline{S}} \colon \overline{F}\langle x_0\rangle \to \overline{S}\langle x_0\rangle$ is a $\mathbb{P}^1$-bundle.*

(5) *The scheme theoretic inverse image $\overline{\mathrm{pr}}_Y^*(x_0) \subset \overline{F}\langle x_0\rangle$ of the base point x_0 via the second projection $\overline{\mathrm{pr}}_Y \colon \overline{F}\langle x_0\rangle$ is a disjoint union of a specified section σ_0 and a (zero-dimensional) closed subscheme τ away from σ_0. In particular, locally around the Cartier divisor σ_0, the projection $\overline{\mathrm{pr}}_{\overline{Y}}$ naturally lifts to a morphism $\tilde{\mathrm{pr}}_{\tilde{Y}} \colon \overline{F}\langle x_0\rangle \to \tilde{Y}$, where $\tilde{Y}$ is the normalisation of the one-point blowup $\mathrm{Bl}_{x_0}Y$ of Y at x_0.*

(6) *The second projection $\overline{\mathrm{pr}}_{\overline{Y}}$ is unramified over $\overline{Y} \setminus (\mathrm{Sing}(\overline{Y}) \cup \{\overline{y}_0\})$, where $\overline{y}_0 = \overline{\mathrm{pr}}_{\overline{Y}}(\sigma_0) \in \overline{Y}$ is a point over $x_0 \in Y$. In particular, the induced morphism from a small open neighbourhood*

of σ_0 in $\overline{F}\langle x_0\rangle$ to a neighbourhood of the exceptional divisor E_{x_0} in $\tilde{Y}$ is unramified in codimension one.

Proof. The statements (1) through (5) are proved in [2, §§2 – 3]. In order to prove (6), we blowup the zero-dimensional subscheme τ and eliminate the indeterminacy to get a morphism $\mathrm{Bl}_\tau\overline{F}\langle x_0\rangle \to \tilde{Y}$. (Note that $\mathrm{Bl}_\tau\overline{F}\langle x_0\rangle$ is normal with only A-type rational double points as singularities by [2, 4.2, Step 2].) It is easy to show that the strict transform of a general member L of $S\langle x_0\rangle$ in $\tilde{Y}$ is a smooth rational curve lying on the nonsingular locus of $\tilde{Y}$ and has trivial normal bundle. Then we can photocopy the proof of [2, Theorem 4.2].

§2. Conic bundles

While in [2] we relied on special properties of $\mathbb{P}^1$-bundles over curves, the key ingredient in the present paper is the theory of two-dimensional conic bundles, *i.e.*, one-parameter families of plane conics. To be more precise, a flat projective family $\pi\colon \mathcal{C} \to T$ over a smooth curve T is said to be a (two-dimensional) *conic bundle* if

(1) a general fibre of π is a smooth $\mathbb{P}^1$, and
(2) there exists an étale open covering[4] $\{p_\alpha\colon U_\alpha \to T\}$ of T and a family of vector bundles $\mathcal{E}_\alpha$ of rank three on U_α such that $\mathcal{C}_\alpha = U_\alpha \times_T \mathcal{C}$ is isomorphic to a hypersurface $\in |2\mathbf{L}_{\mathcal{E}_\alpha}|$ in the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}_\alpha)$ with tautological line bundle $\mathbf{L}_{\mathcal{E}_\alpha}$.

A singular fibre of a conic bundle is either a union of two lines meeting at a single point or a double line (a non-reduced fibre). The singular loci of the fibres $\mathcal{C}_t$ form a closed subset $\mathrm{Cr}(\mathcal{C}) \subset \mathcal{C}$, called the *critical locus.*

Let $\hat{\pi}\colon \hat{\mathcal{C}} \to T$ be a projective morphism from an irreducible (possibly singular) surface onto a smooth curve. Let $\mathcal{C}$ denote the normalisation of $\hat{\mathcal{C}}$, and $\pi\colon \mathcal{C} \to T$ the morphism naturally induced by $\hat{\pi}$.

Lemma 2.1. *In the above notation, assume that*

(a) *for each cosed point $t \in T$, the effective Cartier divisor $\hat{\mathcal{C}}_t = \hat{\pi}^*(t)$ is reduced and contains at most two irreducible components, and that*
(b) *a general fibre $\hat{\mathcal{C}}_t$ is smooth $\mathbb{P}^1$.*

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[4]alternatively, an analytic open covering

Then

(1) *the fibration $\pi: \mathcal{C} \to T$ is a conic bundle without non-reduced fibres, and*

(2) *$\mathcal{C}$ has at worst* A*-type Du Val points as singularities. Any singular point of $\mathcal{C}$ is contained in the unique intersection point of the two components of some reducible fibre of π.*

Proof. Pick up an arbitrary closed point $t \in T$. Since the base T is smooth and the reduced closed fibre $\hat{\mathcal{C}}_t$ is smooth outside a finite set $\Sigma_t \subset \hat{\mathcal{C}}_t$, we see that $\hat{\mathcal{C}}$ is smooth along $\hat{\mathcal{C}}_t \setminus \Sigma_t$. Therefore $\mathcal{C}$ and $\hat{\mathcal{C}}$ are isomorphic in codimension one, so that the closed fibre $\mathcal{C}_t \subset \mathcal{C}$ is also reduced having at most two irreducible components.

Take the minimal resolution $\mu: \tilde{\mathcal{C}} \to \mathcal{C}$. The smooth surface $\tilde{\mathcal{C}}$ is flat over T, and we have $\tilde{\mathcal{C}}_t K_{\tilde{\mathcal{C}}} = -2$. Furthermore, $\tilde{\mathcal{C}}$ is obtained as a blowup of a suitable $\mathbb{P}^1$-bundle over T. Each irreducible component E of $\tilde{\mathcal{C}}_t$ is thus smooth $\mathbb{P}^1$ with nonpositive self intersection, and $E^2 = 0$ if and only if $E = \tilde{\mathcal{C}}_t$. By the adjunction formula, $EK_{\tilde{\mathcal{C}}} = -2 - E^2 \geq -1$ unless $E = \tilde{\mathcal{C}}_t$. If E is contracted to a point on $\mathcal{C}$, then $EK_{\tilde{\mathcal{C}}} \geq 0$ because our resolution is minimal.

We have two cases:

Case 1. *$\mathcal{C}_t$ is irreducible.* In this case, we have a unique component $\tilde{\mathcal{C}}_t^0$ of $\tilde{\mathcal{C}}_t$ which surjects onto $\mathcal{C}_t$. Any other component is contracted to a point and has non-negative intersection with $K_{\tilde{\mathcal{C}}}$, while $\tilde{\mathcal{C}}_t K_{\tilde{\mathcal{C}}} = -2$. This implies that $\tilde{\mathcal{C}}_t^0 K_{\tilde{\mathcal{C}}} \leq -2$, so that $\tilde{\mathcal{C}}_t^0 = \tilde{\mathcal{C}}_t$ or, equivalently, $\mathcal{C}_t$ is a smooth fibre.

Case 2. *$\mathcal{C}_t$ is the union of two irreducible components $\mathcal{C}_{t\pm}$.* In this case, there are at most two irreducible components with $EK_{\tilde{\mathcal{C}}} = -1$ and all the other compoents have nonnegative intersection with $K_{\tilde{\mathcal{C}}}$, while the sum of the intersection numbers is -2. This means that the two strict transforms $\tilde{\mathcal{C}}_{t\pm}$ of $\mathcal{C}_{t\pm}$ are (-1)-curves and the other components are (-2)-curves. If we write

$$\tilde{\mathcal{C}}_t = \tilde{\mathcal{C}}_{t+} + \tilde{\mathcal{C}}_{t-} + \sum_i a_i E_i,$$

then

$$1 = -(\tilde{\mathcal{C}}_{t+})^2 = \tilde{\mathcal{C}}_{t+}\tilde{\mathcal{C}}_{t-} + \sum_i a_i \tilde{\mathcal{C}}_{t+} E_i,$$

meaning that $\tilde{\mathcal{C}}_{t+}$ meets with a single reduced irreducible component E_+. If E_+ is $\tilde{\mathcal{C}}_{t-}$, then, by symmetry, $E_- = \tilde{\mathcal{C}}_{t+}$ is the unique component which meets $\tilde{\mathcal{C}}_{t-}$, so that $\mathcal{C}_t = \tilde{\mathcal{C}}_t = \tilde{\mathcal{C}}_{t+} + \tilde{\mathcal{C}}_{t-}$. If E_+ is one of the (-2)-curves, then the blowdown of $\tilde{\mathcal{C}}_{t+}$ affects the single component E_+ to produce a new (-1)-curve, and we get a similar situation, $\tilde{\mathcal{C}}_{t+}$ being replaced with the image of E_+. Reiterating the same process, we arrive at the situation where $E_+ = \tilde{\mathcal{C}}_{t-}$. Thus $\tilde{\mathcal{C}}_t$ is a single chain

$$\tilde{\mathcal{C}}_{t+} + E_1 + \cdots + E_m + \tilde{\mathcal{C}}_{t-},$$

of which the two ends are the (-1)-curves. Since the intermediary curves form a chain of (-2)-curves, we can contract the chain to an A_m-singularity. After contracting all such chains on $\tilde{\mathcal{C}}$, we get a normal surface $\mathcal{C}^*$. By construction, the resolution $\mu\colon \tilde{\mathcal{C}} \to \mathcal{C}$ factors through $\mathcal{C}^*$, which is finite over $\mathcal{C}$. Hence, by Zariski's Main Theorem, $\mathcal{C}^* = \mathcal{C}$.

The relative anticanonical divisor $-K_{\mathcal{C}/T}$ gives a closed embedding of $\mathcal{C}$ into the projective bundle $\mathbb{P}\big(\mathrm{pr}_{T*}\mathcal{O}_{\mathcal{C}}(-K_{\mathcal{C}/T})\big)$, defining a standard conic bundle structure on $\mathcal{C}$.

When it has an A_m-singularity (a smooth point is considered as an A_0-singularity) on a reducible fibre $\mathcal{C}_t$, the normal surface $\mathcal{C}$ is locally defined by the equation $\xi_1\xi_2 = \tau^{m+1}$ in $T \times \mathbb{P}^2$, where τ is a local parameter of T and ξ_0, ξ_1, ξ_2 are homogeneous coordinates of $\mathbb{P}^2$.

Proposition (2.1) determines the rational Néron-Severi group of the conic bundle $\mathcal{C}$. In fact we have the following

Corollary 2.2. *Let the notation and assumptions be as in* (2.1). *Let $\mathcal{C}^\circ$ denote the non-critical locus $\mathcal{C}\setminus\mathrm{Cr}(\mathcal{C})$. Then there exists a section $\sigma\colon T \to \mathcal{C}^\circ \subset \mathcal{C}$ of the projection π. The surface $\mathcal{C}$ is $\mathbb{Q}$-factorial, i.e., every Weil divisor is Cartier if multiplied by a suitable positive integer. The $\mathbb{Q}$-Néron-Severi group $\mathrm{NS}(\mathcal{C})_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{NS}(\mathcal{C})$ is a $\mathbb{Q}$ vector space freely generated by σ, $\mathfrak{f} = [\mathcal{C}_t]$ and δ_i, $i = 1, \ldots, r$, where $\delta_i = [\mathcal{C}_{t_i+}] - [\mathcal{C}_{t_i-}]$ and the $\mathcal{C}_{t_i} = \mathcal{C}_{t_i+} + \mathcal{C}_{t_i-}$, $i = 1, \ldots, r$ are the decomposition of the singular fibres such that $\sigma\mathcal{C}_{t_i+} = 1$. If $\mathcal{C}$ has an A_{m_i}-singularity at $\mathcal{C}_{t_i+} \cap \mathcal{C}_{t_i-}$, we have the following intersection table:*

$$\begin{aligned}
&\mathfrak{f}^2 = \mathfrak{f}\delta_i = \delta_i\delta_j = 0, \quad i \neq j,\\
&\delta_i^2 = -\frac{4}{m_i+1},\\
&\sigma\mathfrak{f} = \sigma\delta_i = 1.
\end{aligned}$$

Proof. Let $\mu\colon \tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution and E_{ik} a (-2)-curve over the singualar point on $\mathcal{C}_{t_i}$. Denoting $\tilde{\mathcal{C}}_{t_i+}$ denote the strict transform of $\mathcal{C}_{t_i+}$, we can write $\mu^*\mathcal{C}_{t_i+} = \tilde{\mathcal{C}}_{t_i+} + \sum_k a_k E_{ik}$, while $E_{ik}\mu^*\mathcal{C}_{t_i+} = 0, k = 1, \dots, m_i$. This determines the coefficients a_k, yielding

$$\mu^*\mathcal{C}_{t_i+} = \tilde{\mathcal{C}}_{t_i+} + \frac{1}{m_i+1}\sum_k (m_i + 1 - k)E_{ik}.$$

Then the above intersection table follows from simple computation.

Definition 2.3. Let $\pi\colon \mathcal{C} \to T$ be a normal conic bundle over a smooth projective curve T and B a nef and big Cartier divisor on $\mathcal{C}$. The fibre space $\pi\colon \mathcal{C} \to T$ (or the total space $\mathcal{C}$, by abuse of terminology) is said to be an *B-symmetric* conic bundle if $B\mathcal{C}_{t+} = B\mathcal{C}_{t-}$ whenever a closed fibre $\mathcal{C}_t$ is a union of two components $\mathcal{C}_{t+}$, $\mathcal{C}_{t-}$.

Assume that $\pi\colon \mathcal{C} \to T$ has two distinct sections σ_+, σ_-. The triple $(\mathcal{C}; \sigma_+, \sigma_-)$ is said to be *strongly B-symmetric* if the following four conditions are satisfied:

(a) $\mathcal{C}$ is B-symmetric;
(b) σ_+ and σ_- are mutually disjoint divisors contained in the non-critical locus $\mathcal{C}^\circ = \mathcal{C} \setminus \mathrm{Cr}(\mathcal{C})$;
(c) $B\sigma_+ = B\sigma_-$;
(d) For any reducible fibre $\mathcal{C}_t = \mathcal{C}_{t+} + \mathcal{C}_{t-}$, we have

$$\sigma_+\mathcal{C}_{t+} = \sigma_-\mathcal{C}_{t-} = 1,$$
$$\sigma_+\mathcal{C}_{t-} = \sigma_-\mathcal{C}_{t+} = 0,$$

(possibly after suitable reindexing of the irreducible components $\mathcal{C}_{t\pm}$).

Proposition 2.4. *Let $\pi\colon \mathcal{C} \to T$ be a normal conic bundle over a smooth projective curve with a nef big divisor B and two sections σ_+, σ_-. Assume that $(\mathcal{C}; \sigma_+, \sigma_-)$ is strongly B-symmetric and let s denote the number of the singular fibres. Let $\mu\colon \tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution. Then we have:*

(1) *$B \approx d(\sigma_+ + \sigma_-) + a\mathfrak{f}$, $d \in \mathbb{N}$, $a \in \mathbb{Q}$.*
(2) *$\sigma_- \approx \sigma_+ + \sum \frac{m_i+1}{2}\delta_i$.*
(3) *Let $\sigma \subset \mathcal{C}$ be a section of π and $\tilde{\sigma} \subset \tilde{\mathcal{C}}$ its strict transform. Let*

$$\tilde{\mathcal{C}}_{t_i} = \sum_{k=0}^{m_i+1} E_{ik} = \tilde{\mathcal{C}}_{t_i-} + E_{i1} + \cdots + E_{im_i} + \tilde{\mathcal{C}}_{t_i+}$$

be the irreducible decomposition of a singular fibre of $\tilde{\pi}\colon \tilde{\mathcal{C}} \to T$ over t_i and let $E_{i\kappa_i}$ be the unique component which meets $\tilde{\sigma}$.

Then

$$\tilde{\sigma} = \mu^*\sigma - \sum_i \Big(\sum_{k=1}^{\kappa_i} \frac{k(m_i+1-\kappa_i)}{m_i+1} E_{ik} + \sum_{k=\kappa_i+1}^{m_i} \frac{(m_i+1-k)\kappa_i}{m_i+1} E_{ik}\Big).$$

(4) $\sigma_+^2 = \sigma_-^2 = -\frac{e}{2} \leq 0$, *where*

$$e = \sum_i (m_i + 1),$$

the sum being taken over the the reducible fibres $\mathcal{C}_{t_i}$, on which $\mathcal{C}$ has singularities of type A_{m_i} (of course we define $m_i = 0$ if $\mathcal{C}$ is nonsingular near $\mathcal{C}_{t_i}$).

(5) *If its strict transform $\tilde{\sigma} \subset \tilde{\mathcal{C}}$ has negative self intersection, then a section $\sigma \subset \mathcal{C}$ coincides with one of the two specified sections $\sigma_\pm$. In particular, σ is one of the $\sigma_\pm$ once a section $\sigma \subset \mathcal{C}$ satisfies $\sigma^2 < 0$. If $\sigma \neq \sigma_\pm$ and its strict transform $\tilde{\sigma}$ satisfies $\tilde{\sigma}^2 = 0$, then σ is disjoint with $\sigma_\pm$. If, furthermore, $\sigma^2 = 0$, then it is away from* $\mathrm{Sing}(\mathcal{C})$.

(6) *If there are two sections $\sigma_1, \sigma_2 \neq \sigma_\pm \subset \mathcal{C}$ such that the strict transforms $\tilde{\sigma}_1, \tilde{\sigma}_2$ are mutually disjoint in $\tilde{\mathcal{C}}$, then $\sigma_1 \cup \sigma_2$ is away from $\sigma_+ \cup \sigma_-$.*

Proof. The first three statements are direct consequences of the intersection table in (2.2) and we leave the proof to the reader.

Take the minimal resolution $\tilde{\mathcal{C}}$ of $\mathcal{C}$. Let $\tilde{\sigma}_\pm \subset \tilde{\mathcal{C}}$ denote the strict transforms of the sections $\sigma_\pm$. Starting from $\tilde{\mathcal{C}}$, we can find a series of blowdowns

$$\tilde{\mathcal{C}} = \mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_e$$

to reach a $\mathbb{P}^1$-bundle $\mathcal{C}_e$. The number of the blowdowns is computed by

$$e = \sum_{t\in T} \big((\text{the number of the components of } \tilde{C}_t) - 1\big) = \sum_{i=1}^{s} (m_i + 1).$$

We denote by $\sigma_{\pm k}$ the image of $\tilde{\sigma}_\pm = \sigma_{\pm 0}$ in $\mathcal{C}_k$.

The choice of blowdowns is not unique. Our choice is inductively made in such a way that at each step the (-1)-curve to be contracted must intersect σ_{+k} (or, equivalently, that the (-1)-curve does not touch σ_{-k}). In such a (unique) choice of blowdowns, we can easily see that the two divisors $\sigma_{\pm e}$ are still disjoint on the $\mathbb{P}^1$-bundle $\mathcal{C}_e$. A $\mathbb{P}^1$-bundle with two disjoint sections is canonically a projective bundle $\mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-)$, the two direct summand corresponding to the two sections. Thus we have $\sigma_{+e}^2 = -\sigma_{-e}^2 = \deg \mathcal{L}_+\mathcal{L}_-^{-1}$. By construction, $\sigma_{-e}^2 = \tilde{\sigma}_-^2 = \sigma_-^2 = \sigma_+^2$, while $\sigma_{+e}^2 \geq \sigma_{+0} = \tilde{\sigma}_+^2 = \tilde{\sigma_+}^2 = \sigma_-^2$, the equality holding if and only if $e = 0$, $\mathcal{C}_e = \tilde{\mathcal{C}} = \mathcal{C}$. Therefore $\sigma_+^2 + e = -\sigma_-^2 = -\sigma_+^2$, whence follows (4).

We trace back the blowdown procedure by starting from the $\mathbb{P}^1$-bundle $\mathcal{C}_e$ with two disjoint sections $\sigma_{\pm e}$ and by successively blowing up points on the strict transforms σ_{+k} on $\mathcal{C}_k$, eventually to reach $\tilde{C} = \mathcal{C}_0$.

Let $\sigma \subset \mathcal{C}$ be a section different from $\sigma_\pm$. Its strict transform $\tilde{\sigma}$ in $\tilde{\mathcal{C}}$ is mapped to a section σ_e on the $\mathbb{P}^1$-bundle $\mathcal{C}_e$. Putting $a = \sigma_e\sigma_{-e} \geq 0$, we have $\sigma_e\sigma_{+e} = e + a \geq e$, $\sigma_e^2 = e + 2a$. Let $\mathcal{C}_{t_ie} \simeq \mathbb{P}^1$ be the strict transform in $\mathcal{C}_e$ of the singular fibre $\mathcal{C}_{t_i} \subset \mathcal{C}$. Let κ_i denote the local intersection number $(\sigma_e, \sigma_{+e})_{\mathrm{loc}}$ at the single point $\mathcal{C}_{t_ie} \cap \sigma_{+e}$, with the obvious inequality $\sum_{i=1}^s \kappa_i \leq \sigma_e\sigma_{+e} = e + a$. By the description of the blowing up $\mathcal{C}_0 \to \mathcal{C}_e$, the selfintersection $\tilde{\sigma}$ is computed by $e + 2a - \sum_i \kappa_i \geq a \geq 0$, the equalities are attained if and only if $a = 0, \sum \kappa_i = e + a = e$, meaning that $\tilde{\sigma}$ is disjoint with $\tilde{\sigma}_\pm$ in this case. These facts in mind, we readily deduce (5) and (6) from the easy inequality $\sigma^2 \geq \tilde{\sigma}^2$, the equality holding if and only if σ does not pass through the singular points.

If none of the two sections $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ coincides with $\tilde{\sigma}_\pm$, the both divisors are necessarily nef with non-negative selfintersection. When one of them has positive self-intersection, they must intersect by Hodge index theorem. If both have self intersection zero, then they cannot meet $\tilde{\sigma}_\pm$ by (6) (recall that $\sigma_\pm$ is not affected by the resolution).

Corollary 2.5. *Let X be a projective variety with an ample divisor H and let $\pi\colon \mathcal{C} \to T$ be a normal conic bundle over a smooth curve. Let $f\colon \mathcal{C} \to X$ be a morphism with two-dimensional image such that its restriction to each fibre $\mathcal{C}_t$ is finite. Assume that π admits two sections $\sigma_\pm$ such that*

(1) *$f(\sigma_\pm)$ is a single point $x_\pm \in X$, $x_+ \neq x_-$,*
(2) *$(\mathcal{C}_t, f^*H) = 2\,\mathrm{mindeg}(X, x_1; H) = 2\mathrm{mindeg}(X, x_2; H)$ for each closed fibre $\mathcal{C}_t$ of π, and that*
(3) *no irreducible component of a singlular fibre $\mathcal{C}_{t_i}$ of π simultaneously meets both σ_+ and σ_-.*

Then $\mathcal{C}$ is a strongly f^*H-symmetric conic bundle and f is finite over $X \setminus \{x_+, x_-\}$.

Proof. The first statement follows from the condition (3) plus the equalities $\sigma_\pm f^*H = 0$ and $f(\mathcal{C}_{t+})H = f(\mathcal{C}_{t-})H = \mathrm{mindeg})(X, x_i; H)$ for a reducible fibre $\mathcal{C}_t = \mathcal{C}_{t+} \cup \mathcal{C}_{t-}$. In order to prove the second statement, assume that there is a curve $\sigma \subset \mathcal{C}$ which is contracted to a point by f. By considering a suitable base change if necessary, we may assume that σ is a section without loss of generality. Then, by the equality $\sigma f^*H = 0$ and the Hodge index theorem, we infer that $\sigma^2 < 0$, contradicting (2.4).

Proposition 2.6. *Let $\pi\colon \mathcal{C} \to T$ be a two-dimensional normal conic bundle and $f\colon \mathcal{C} \to X$ a morphism with two-dimensional image as in* (2.5)*. Assume that*

(1) *$\mathcal{C}$ is f^*H-symmetric, that*
(2) *There are two sections $\sigma_\pm$ such that $f(\sigma_\pm)$ is a single point $x_\pm \in X$, $x_+ \neq x_-$, and that*
(3) *there is a third section $\sigma \subset \mathcal{C}$ such that $f(\mathcal{C}_t)$ has a cuspidal singularity at $f(\sigma \cap \mathcal{C}_t)$ for each irreducible fibre $\mathcal{C}_t$.*

Then σ is away from one of the $\sigma_\pm$.

Proof. Let $\mathcal{I}_\sigma \subset \mathcal{O}_\mathcal{C}$ denote the ideal sheaf of the closed subscheme $\sigma \subset \mathcal{C}$. Let $R \subset \mathbb{C}(T)\mathcal{O}_\mathcal{C} \subset \mathbb{C}(\mathcal{C})$ be the $\mathbb{C}(T)$-subalgebra generated by 1, $\mathcal{I}_\sigma^2$, $\mathcal{I}_\sigma^3$. We define the $\mathcal{O}_T$-subalgebra $\mathcal{O}_\mathcal{G} \subset \mathcal{O}_\mathcal{C}$ by $\mathcal{O}_\mathcal{G} = R \cap \mathcal{O}_\mathcal{C}$. $\mathcal{O}_\mathcal{G}$ determines a family $\hat{\pi}\colon \mathcal{G} \to T$ of singular rational curves, which factors $f\colon \mathcal{C} \to X$ into the natural projection $\mathcal{C} \to \mathcal{G}$ and $g\colon \mathcal{G} \to X$. Let $\mathrm{Pic}(\mathcal{G}/T) = \coprod_d \mathrm{Pic}^d(\mathcal{G}/T)$ be the relative Picard group scheme, $\mathrm{Pic}^d(\mathcal{G}/T)$ consisting of the equivalence classes of line bundles of degree d on each fibre.

If a closed fibre $\mathcal{G}_t$ is an irreducible cuspidal curve, then $\mathrm{Pic}^0(\mathcal{G}/T)_t = \mathrm{Pic}^0(\mathcal{G}_t)$ is naturally isomorphic to $\mathbb{G}_a \simeq \mathbb{A}^1$. The line bundle $\mathcal{O}_\mathcal{G}(g^*H)$ determines a global section of $\mathrm{Pic}^d(\mathcal{G}/T) \to T$, and, at a generic point $t \in T$, there is a unique section λ such that $\lambda^{\otimes d} \sim \mathcal{O}(g^*H)$, determining a unique rational (and hence holomorphic) section $\sigma^*\colon T \to \mathcal{C}$ such that $\sigma^*(t) \in \mathcal{C}_t \setminus \sigma(t) \simeq \mathcal{G}_t \setminus \sigma(t)$ and that $\mathcal{O}(\sigma^*(t)) \sim \lambda(t)$ for general $t \in T$.

Take the minimal resolution $\tilde{\mathcal{C}} \to \mathcal{C}$ and let $\tilde{\sigma}, \tilde{\sigma}^*$, etc. be the strict transforms in $\tilde{\mathcal{C}}$ of σ, σ^*, etc. $\subset \mathcal{C}$. Let us check that $\tilde{\sigma}^* \subset \tilde{\mathcal{C}}$ is away from $\tilde{\sigma}$.

By construction, $\tilde{\sigma}^*$ does not meet $\tilde{\sigma}$ outside the singular fibres.

The local structure of $\tilde{\sigma}^*$ around a singular fibre $\tilde{\mathcal{C}}_{t_i}$ is also very simple. Let $\tilde{\mathcal{C}}_{t_i} = \tilde{\mathcal{C}}_{t_i-} + E_{i1} + \cdots + E_{im_i} + \tilde{\mathcal{C}}_{t+}$ be the irreducible decomposition of the singular fibre, a chain of smooth $\mathbb{P}^1$'s. For the strict transform $\tilde{\sigma} \subset \tilde{\mathcal{C}}$ of $\sigma \subset \mathcal{C}$, let $E_{i\kappa_i}$ denote the unique component which meets $\tilde{\sigma}$ (we set $E_{i0} = \tilde{\mathcal{C}}_{t_i-}, E_{i,m_i+1} = \tilde{\mathcal{C}}_{t+}$, by convention). As we have seen in (2.4.3), there is a unique solution $(y_{ik}) \in \mathbb{Q}^e$ (actually

$\in \mathbb{Z}^e$), $e = \sum_i (m_i + 1)$, which satisfies the linear equations

$$(\tilde{f}^* H + \sum_{i,k} y_{ik} E_{i+1}) E_{ik} = d\delta_{k\kappa_i}.$$

Noting that there are two (-1)-curves as two ends of the chain $\tilde{C}_{t_i}$, we can blow down $\tilde{\mathcal{C}}$ to a smooth $\mathbb{P}^1$-bundle $\mathcal{C}^\dagger$ in such a way that all the components of $\tilde{C}_{t_i} \setminus E_{i\kappa_i}$ are contracted to points. Then the divisor $\tilde{f}^* H + \sum_{i,k} y_{ik} E_{i+1}$ is a pull-back of a divisor $H^\dagger$ on $\mathcal{C}^\dagger$. Let $\sigma^\dagger$ denote the image of $\tilde{\sigma}$ on $\mathcal{C}^\dagger$. Starting from $\mathcal{C}^\dagger$ and $\sigma^\dagger$, we can easily construct a family of cupidal plane cubics $\mathcal{G}^\dagger \to T$ which coincides with $\mathcal{G} \to T$ over a general point t. The divisor $H^\dagger$ is a global section of $\mathrm{Pic}^d(\mathcal{G}^\dagger/T)$ and we find a unique section

$$\sigma^{*\dagger} \subset \mathrm{Pic}^1(\mathcal{G}^\dagger/T) \simeq \mathcal{G}^\dagger \setminus \sigma^\dagger$$

such that $\sigma^{\dagger \otimes d} \sim H^\dagger$ on $\mathcal{G}_t^\dagger$. The section $\tilde{\sigma}^*$ on $\tilde{\mathcal{C}}$ is then the strict transform of $\sigma^{*\dagger} \subset \mathcal{C}^\dagger$, and in particular is off $\tilde{\sigma}$, the strict transform of $\sigma^\dagger$. If $\tilde{\sigma}^*$ is one of the $\tilde{\sigma}_\pm$, say $\tilde{\sigma}_+$, then its image σ does not meet σ_+ on $\mathcal{C}$ (because the resolution $\tilde{\mathcal{C}} \to \mathcal{C}$ does not affect $\sigma_\pm$). If $\tilde{\sigma}^* \subset \tilde{\mathcal{C}}$ is not one of the $\tilde{\sigma}_\pm$, then, by (2.4.6), σ does not intersect $\sigma_\pm$.

§3. Fano n-manifolds with Picard number one and local length n

In this section, we prove the essential part of Theorem 0.1, the implication (3) $\Rightarrow$ (1). Recall that Theorem 0.1 is known for Fano 3-folds.

Throughout the section, we assume:

(a) X is a Fano manifold of dimension $n \geq 4$ with Picard number one.

(b) The two closed points $x_+, x_- \in X$ are general.

(c) $l(X, x_\pm; -K_X) = n$.

Consider an irreducible component W of the closed subset $\mathcal{W} \subset \mathrm{Chow}(X)$ which consists of the connected rational curves C with $(C, -K_X) = 2n$. Let $\mathrm{pr}_W : V \to W$ be the associated incidence variety. Let $D \subset W$ denote the *descriminant locus*, the locus consisting of the reducible rational curves and the non-reduced curves. The induced subfamily of curves over D is denoted by V_D.

The symbol $W\langle x_+, x_- \rangle$ [resp. $D\langle x_+, x_- \rangle$] stands for the closed subset of the curves $\in W$ [resp. $\in D$] passing through the two points

x_+, x_-. The associated incidence varieties are denoted by $V\langle x_+, x_-\rangle$ and $V_D\langle x_+, x_-\rangle$.

By our construction, the following assertion is immediate.

Proposition 3.1. *The fibre of* pr_W *over a point* $w \in D\rangle x_+, x_- \subset W$ *is either a connected union of two irreducible, reduced rational curves or a non-reduced rational curve of generic multiplicity two. Given a smooth curve* T *and a non-constant morphism* $T \to W\langle x_+, x_-\rangle \subset W$ *of which the image is not contained in* $D\langle x_+, x_-\rangle$*, the normalisation of the fibre product* $T \times_W V$ *is a symmetric conic bundle over* T.

Proposition 3.2. *In the above notation, we have*

(1) $\dim D\langle x_+, x_-\rangle = n-2$ *and the image of the projection* $\mathrm{pr}_X : V_D\langle x_+, x_-\rangle \to X$ *is a divisor* Y *on* X.

(2) *The divisor* Y *a union of two divisors* Y_+, Y_- *such that* x_i *is contained in* Y_j *if and only if* $i = j$ $(i, j = +, -)$.

(3) *An arbitrary element* $[C] \in D\langle x_+, x_-\rangle$ *is a reduced reducible curve* $L_+ \cup L_-$ *with* $L_\pm \ni x_\pm$, $L_\pm \subset Y_\pm \subset X$.

(4) $\dim(\mathrm{Sing}(Y_+) \cap Y_-) \leq n-3$ *and a general member* $L_+ \cup L_-$ *does not pass through this set.*

(5) *If* $L_+ \cup L_-$ *is a general member of* $D\langle x_+, x_-\rangle$*, then* $L_\pm$ *is smooth with normal bundle* $\simeq \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

(6) *A general member* $L_+ \cup L_-$ *of* $D\langle x_+, x_-\rangle$ *deforms to an irreducible rational curve* C *such that* $C \supset \{x_+, x_-\}$*. More precisely, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}_0 & \longrightarrow & L_+ \cup L_- \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & V\langle x_+, x_-\rangle \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & W\langle x_+, x_-\rangle \end{array}$$

where

$$\mathcal{C} = \{((x:y:z), t); xy = t\} \subset \mathbb{P}^2 \times \Delta$$

is a nonsingular conic bundle over a small disk Δ *with reducible central fibre* $\mathcal{C}_0$.

Proof. The rational curves L with $L(-K_X) = n$ form a family F parametrised by a variety S of dimension $\geq 2n-3$. By our assumption (c), the closed subfamily $F\langle x_\pm\rangle \to S\langle x_\pm\rangle$ consisting of the members through a general base point $x_\pm$ is a non-empty unsplitting family parametrised by $S\langle x_\pm\rangle$. Then we apply (1.) to a general member L of $S\langle x_\pm\rangle$, to deduce that

a) the parameter space $S\langle x_\pm\rangle$ has dimension $n-2$, that
b) the projection $\mathrm{pr}_X : F\langle x_\pm\rangle \to X$ is finite over $X\setminus\{x_\pm\}$ and that
c) $Y_\pm = \mathrm{pr}_X(F\langle x_\pm\rangle)$ is a divisor.

By our genericity condition, $x_\pm \in X \setminus Y_\mp$, and so $S\langle x_+\rangle \cap S\langle x_-\rangle = \emptyset$. In particular, any member of $D\langle x_+, x_-\rangle$ is a reduced, reducible curve $L_+ \cup L_-$ such that $L_\pm \ni x_\pm$.

If $[L_-] \in S\langle x_-\rangle$, then $L_- \not\subset Y_+$ and $L_- \cap Y_+ \neq \emptyset \subset X$ because $\rho(X) = 1$, meaning that we can find a curve $[L_+] \in S\langle x_+\rangle$ so that L_+ meets L_-, *i.e.*, $[L_+ \cup L_+] \in D\langle x_+, x_-\rangle$. Furthermore, since $L_- \cap Y_+$ is a finite set, we have only finitely many choices of such L_+ (because $F\langle x_+\rangle$ is an unsplitting family of rational curves). Put in another way, the projection $D\langle x_+, x_-\rangle \to S\langle x_-\rangle$, $[L_+ \cup L_-] \mapsto [L_-]$ is surjective and finite (and so is the other projection $D\langle x_+, x_-\rangle \to S\langle x_+\rangle$ by symmetry).

In particular, $\dim D\langle x_+, x_-\rangle = \dim S\langle x_+\rangle = n-2$. If $[L_+ \cup L_-]$ is a general point in $D\langle x_+, x_-\rangle$, then so is $[L_\pm]$ in $S\langle x_\pm\rangle$ and

$$\Theta_X|_{L_\pm} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}.$$

It is clear that $Y = \mathrm{pr}_X(F_D\langle x_+, x_-\rangle$ is the union of the divisors $Y_\pm = \mathrm{pr}_X(F\langle x_\pm\rangle)$.

The members L_- that meet $\mathrm{Sing}(Y_+)$ form a closed subset of $S\langle x_-\rangle$. Suppose that this subset is the whole space $S\langle x_-\rangle$. Then it follows that there is an $n-2$-dimensional irreducible component Σ of $\mathrm{Sing}(Y_+)$ such that every member L_- of $S\langle x_-\rangle$ passes through Σ. If follows that, for a general closed point $x_0 \in \Sigma$, there is a member L_- of $S\langle x_0, x_-\rangle$. On the other hand, $\dim F\langle x_0\rangle$ is $n-1$ near L_-, so that $\mathrm{pr}_X(F\langle x_0\rangle)$ is a divisor on X. If we replace x_- by another general point $\notin \mathrm{pr}_X(F\langle x_0\rangle)$, we cannot find L_- which connects x_0 and x_-, meaning that $\mathrm{pr}_X(F\langle x_-\rangle) \cap \Sigma \neq \Sigma$ for a generic choice of x_-.

We have so far checked the statements (1) – (5). In order to prove (6), consider a nonsingular conic bundle

$$\mathcal{C} = \{((x:y:z),t); xy = t\} \subset \mathbb{P}^2 \times \Delta$$

over a small disk Δ with reducible central fibre $\mathcal{C}_0$. Choose a general member $L_+ \cup L_-$ of D and fix a birational map $f\colon \mathcal{C}_0 \to L_+ \cup L_- \subset X$. The graph Γ_f of this map is a locally complete intersection in $\mathcal{C} \times X$ with normal bundle $\mathcal{N} \simeq f^*\Theta_X \oplus \mathcal{O}$. Hence $\dim \mathrm{H}^0(\Gamma_f, \mathcal{N}) = 3n+1$,

$\mathrm{H}^1(\Gamma_f, \mathcal{N}) = 0$. On the other hand, the deformation of $f\colon \mathcal{C}_0 \to X$ has dimension $3n$ by the splitting type of $\Theta_X|_{L_i}$. This shows that, locally around $D\langle x_+, x_-\rangle$, we have the dimension estimate $\dim W \geq \dim D + 1$.

Recall that $W\langle x_+, x_-\rangle$ [resp. $D\langle x_+, x_-\rangle$] is naturally identified with the inverse image of (x_+, x_-) via the natural projection $V^{(2)} \to X \times X$ [resp. $V_D^{(2)} \to X \times X$]. Then elementary dimension count gives the following equalities:

$$\dim W\langle x_+, x_-\rangle = \dim W + 2 - 2n,$$
$$\dim D\langle x_+, x_-\rangle = \dim D + 2 - 2n,$$

so that $\dim W\langle x_+, x_-\rangle \geq \dim D\langle x_+, x_-\rangle + 1$. In other words, there is an irreducible member $C \supset \{x_+, x_-\}$ which is a deformation of $L_+ \cup L_-$.

The above deformation argument also shows that W is smooth at a general point of D, and so is $W\langle x_+, x_-\rangle$ at a general point of $D\langle x_+, x_-\rangle$. It is easy to show that $\overline{V}\langle x_+, x_-\rangle$ is smooth along a general member $L_+ \cup L_-$ of $D\langle x_+, x_-\rangle$. (Analytically-locally, it looks like $\mathcal{C} \times D\langle x_+, x_-\rangle$).

We list below a few corollaries of Proposition 3.2.

Corollary 3.3. *Let T be a smooth curve and $f\colon T \to W\langle x_+, x_-\rangle$ a non-constant morphism with image not contained in $D\langle x_+, x_-\rangle$. Then the normalisation of the fibre product $T \times_{W\langle x_+, x_-\rangle} V\langle x_+, x_-\rangle$ is a strongly $\mathrm{pr}_X^* H$-symmetric conic bundle over T.*

Proof. By construction.

Corollary 3.4. *Let V_t be an arbitrary irreducible fibre of the family $V\langle x_+, x_-\rangle \to W\langle x_+, x_-\rangle$. Then $C = \mathrm{pr}_X(V_t)$ is not contained in the divisor $Y \subset X$. Given any non-empty irreducible closed subset $R \subset W\langle x_+, x_-\rangle \backslash D\langle x_+, x_-\rangle$ of dimension $\leq n-2$ and the associated subfamily $V_R\langle x_+, x_-\rangle \to R$, the image $\mathrm{pr}_X(V_R\langle x_+, x_-\rangle) \subset X$ can neither contain any irreducible component of $Y_\pm$ nor be contained in $Y_\pm$.*

Proof. If C is contained in Y, then C must be contained in one of Y_+, Y_- because C is irreducible. Then C cannot pass one of the $x_\pm$, which is absurd. In particular, the irreducible constructible set $\mathrm{pr}_X(V_R\langle x_+, x_-\rangle)$ of dimension $\leq n-1$ cannot be contained in any of the irreducible component of the divisor $Y_\pm$ and and so cannot contain any component of $Y_\pm$.

Consider the fibre product $V^{(3)} = V \times_W V \times_W V$ with the natural projection $\mathrm{pr}_X^{(3)}\colon V^{(3)} \to X \times X \times X$.

Corollary 3.5. *In the above notation, we have*

(1) $\dim W = 3n-3$ *(near $W\langle x_+, x_-\rangle$). and the projection* $\mathrm{pr}_X^{(3)}: V^{(3)} \to X \times X \times X$ *is dominant.*

(2) *A general element $[C] \in W$ is irreducible and if $f: \mathbb{P}^1 \to C \subset X$ is the normalisation,* $f^*\Theta_X \simeq \mathcal{O}(2)^{\oplus n}$.

Proof. Everything is considered around $W\langle x_+, x_-\rangle$.

By the condition $C(-K_X) = 2n$, $[C] \in W$, we have the inequality $\dim W \geq 3n-3$, so that $\dim V^{(3)} \geq 3n$. Hence (1) follows if we check that the inverse image $(\mathrm{pr}_X^{(3)})^{(-1)}(x_+, x_-, x_0)$ of a general point (x_+, x_-, x_0) of $\mathrm{pr}_X^{(3)}(V^{(3)})$ is finite. The inverse image of $\{(x_+, x_-)\} \times X$ is naturally identified with $V\langle x_+, x_-\rangle$, family of rational curves passing through x_+, x_-, parametrised by the closed subset $W\langle x_+, x_-\rangle \subset W$. By (2.5), the projection $V\langle x_+, x_-\rangle \to X$ is finite over $X \setminus \{x_+, x_-\}$, which in particular means that the inverse image of x_0 is finite.

Take a general nonsingular point $[C] \in W\langle x_+, x_-\rangle$ and let $\overline{V}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$. $\overline{V}\langle x_+, x_-\rangle$ is locally a $\mathbb{P}^1$ bundle over a small smooth neighbourhood of $[C]$. Since the projection $V\langle x_+, x_-\rangle \to X$ is dominant, the natural map $\Theta_{\overline{V}\langle x_+, x_-\rangle} \to \overline{\mathrm{pr}}_X^*\Theta_X$ is surjective at a general point of $\overline{C}$. This means that $\mathrm{H}^0(\overline{C}, f^*\Theta_X(-x_+ - x_-))$ and $\Theta_{\overline{C}} \simeq \mathcal{O}(2)$ generates a subsheaf of rank n of $f^*\Theta_X$. It follows that the direct sum decomposition $f^*\Theta_X \simeq \oplus\mathcal{O}(d_i)$ satisfies $d_i \geq 2$, while $\sum d_i = C(-K_X) = 2n$, whence follows (2).

Let $\overline{V}\langle x_+, x_-\rangle$ and $\overline{W}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$ and $W\langle x_+, x_-\rangle$, and let

$$\overline{\mathrm{pr}}_{\overline{W}}: \overline{V}\langle x_+, x_-\rangle \to \overline{W}\langle x_+, x_-\rangle,\ \overline{\mathrm{pr}}_X: \overline{V}\langle x_+, x_-\rangle \to X$$

be the natural projections. Let $\overline{D}\langle x_+, x_-\rangle \subset \overline{W}\langle x_+, x_-\rangle$ denote the inverse image of $D\langle x_+, x_-\rangle$. It is known that $\overline{V}\langle x_+, x_-\rangle$ is a $\mathbb{P}^1$-bundle if restricted over the open subset

$$\overline{W}^\circ\langle x_+, x_-\rangle = \overline{W}\langle x_+, x_-\rangle \setminus \overline{D}\langle x_+, x_-\rangle.$$

Let $R \subset \overline{V}^\circ\langle x_+, x_-\rangle$ be the ramification locus of $\overline{\mathrm{pr}}_X|_{\overline{V}^\circ\langle x_+, x_-\rangle}$.

The following statement follows from standard deformation theory:

Proposition 3.6. *In the notation above, R is the union of σ_+, σ_- and $\overline{\mathrm{pr}}_{\overline{W}}^{-1}(B)$, where $B \subset \overline{W}^\circ\langle x_+, x_-\rangle$ is the closed subset*

$$\{s \in \overline{W}^\circ\langle x_+, x_-\rangle;\ \overline{\mathrm{pr}}_X^*\Theta_X|_{\overline{V}_s} \not\simeq \mathcal{O}(2)^{\oplus n}\}.$$

Combined with (3.4) and (3.5), this means

Corollary 3.7. *The closure of* $\overline{\mathrm{pr}}_X(R) \subset X$ *does not contain* $Y_\pm = \mathrm{pr}_X(F\langle x_\pm\rangle)$. *In particular, if* $[L_+ \cup L_-]$ *is a general member of* $D\langle x_+, x_-\rangle$, *any closed subset* Γ *of the inverse image* $\overline{\mathrm{pr}}_X^{-1}(L_+) \subset \overline{V}\langle x_+, x_-\rangle$ *is not contained in the closure of* R *as long as* Γ *surjects onto* L_1.

Let $\Gamma \subset \overline{V}\langle x_+, x_-\rangle$ be an irreducible curve which surjects onto $L_+ \subset X$, where $[L_+ \cup L_-]$ is a general point of $D\langle x_+, x_-\rangle$. There are three cases:

- **A.** Γ is contained in a fibre of $\overline{\mathrm{pr}}_W$ (in this case, Γ is simply the normalisation of the first irreducible component L_+ of the fibre $L_+ \cup L_- \subset V\langle x_+, x_-\rangle$).
- **B.** $\overline{\mathrm{pr}}_W(\Gamma)$ is a curve on $\overline{D}\langle x_+, x_-\rangle$.
- **C.** $\overline{\mathrm{pr}}_W(\Gamma)$ is a curve not contained in $\overline{D}\langle x_+, x_-\rangle$.

Lemma 3.8. *In Case* **B**, Γ *does not intersect* σ_+, σ_-.

Proof. It is trivial that $\Gamma \not\ni \sigma_-$ because $L_+ \not\ni x_-$. The fibre space $\overline{V}_D\langle x_+, x_-\rangle \to \overline{D}\langle x_+, x_-\rangle$ is a union of two irreducible components $F^*\langle x_+\rangle$ and $F^*\langle x_-\rangle$, and Γ must be a curve on $F^*\langle x_+\rangle$. $F^*\langle x_+\rangle \to \overline{D}\langle x_+, x_-\rangle$ is the base change of the fibre space $F\langle x_+\rangle \to S\langle x_+\rangle$ given by the finite morphism $D\langle x_+, x_-\rangle \to S\langle x_+\rangle$. In particular, $\Gamma \subset F^*\langle x_+\rangle$ comes from a curve $\Gamma_0 \subset \overline{F}\langle x_+\rangle$.

$\overline{F}\langle x_+\rangle$ is a $\mathbb{P}^1$-bundle over $\overline{S}\langle x_+\rangle$ and surjects onto the divisor Y_+. Furthermore $\overline{\mathrm{pr}}_X(\Gamma_0) = L_+$ passes through a general point of Y_+. This means that the differential homomorphism $\Theta_{\overline{F}\langle x_+\rangle} \to \overline{\mathrm{pr}}_X^*\Theta_X$ is of rank $n-1$ at a general point of Γ_0.

Put $\Delta = \overline{\mathrm{pr}}_{\overline{S}}(\Gamma_0) \subset \overline{S}\langle x_+\rangle$. Given a finite morphism $\tilde{\Delta} \to \Delta$, let $\mathcal{F} \to \tilde{\Delta}$ denote the induced $\mathbb{P}^1$-bundle over $\tilde{\Delta}$ with the natural finite-to-one morphisms $h\colon \mathcal{F} \to \overline{F}\langle x_+\rangle$ and generically finite-to-one morphism $f = \overline{\mathrm{pr}}_X h\colon \mathcal{F} \to X$. $\mathcal{F}$ carries the specified section $\tilde{\sigma}_+ = h^{-1}(\sigma_+)$. If $\tilde{\Delta}$ is suitably chosen, the inverse image of Γ_0 is a union of sections σ_i. By construction,

$$\Theta_{\mathcal{F}}|_{\sigma_i} \subset h^*\Theta_{\overline{F}\langle x_+}|_{\sigma_i} \subset f^*\Theta_X|_{\sigma_i},$$

inducing an injection $\mathcal{N}_{\sigma_i/\mathcal{F}} \hookrightarrow f^*\mathcal{N}_{L_+/X}$.

Recalling the isomorphism $\mathcal{N}_{L_+/X} \simeq \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$, we infer that the self intersection number σ_i^2 of the effective divisor $\sigma_i \subset \mathcal{F}$ is bounded from above by a_i, where a_i is the mapping degree of the surjection $\sigma_i \to L_+$.

Let H be an ample divisor on X and set $d = L_+H$. Then, for each fibre $\mathcal{F}_s$, $s \in \tilde{\Delta}$, we have three equalities $F_s f^*H = d$, while $\tilde{\sigma}_+ f^*H = 0$, $\sigma_i f^*H = a_i d$. Since the Néron-Severi group of $\mathcal{F}$ is generated by $\tilde{\sigma}_+$ and the fibre $\mathcal{F}_s$, the first two equalities yield the numerical equivalence $f^*H \approx d(\tilde{\sigma}_+ + e\mathcal{F}_s)$, where $e = -\tilde{\sigma}_+^2 > 0$. Similarly, if we put $\sigma_i \approx \tilde{\sigma}_+ + a_i'\mathcal{F}_s$ for a suitable integer a_i', the third equality gives

$$a_i d = \sigma_i f^*H = (\tilde{\sigma}_+ + a_i'\mathcal{F}_s)f^*H = a_i'\mathcal{F}_s f^*H = a_i'd,$$

so that $a_i' = a_i$. Then the inequality $\sigma_i^2 \le a_i$ shown above is rewritten into

$$a_i \ge \sigma_i^2 = (\tilde{\sigma}_+ + a_i F_s)^2 = -e + 2a_i,$$

or, equivalently, $a_i \le e$, and we get the inequality $\sigma_i\tilde{\sigma}_+ = -e + a_i \le 0$. By our assumption $\sigma_i \ne \tilde{\sigma}_+$, this means that σ_i does not meet $\tilde{\sigma}_+$ for every i. In other words, Γ is off σ_+.

Lemma 3.9. *In Case* **C**, Γ *does not intersect* σ_+, σ_-.

Proof. In this case, $\hat{\Delta} = \overline{\mathrm{pr}}_{\overline{W}}(\Gamma)$ is not contained in $B \subset W\langle x_+, x_-\rangle$, where $R = \overline{\mathrm{pr}}_{\overline{S}}^{-1}(B)$ is the ramification locus of $\overline{\mathrm{pr}}_X$. By taking a suitable covering $\tilde{\Delta} \to \hat{\Delta}$, we get a conic bundle $\mathcal{C} = \tilde{\Delta} \times_{\overline{W}\langle x_+, x_-\rangle} \overline{V}\langle x_+, x_-\rangle \to \tilde{\Delta}$, on which the inverse image of Γ is a union of sections Γ_i. Let $\tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution, $\tilde{\Gamma}_i$ and $\tilde{\sigma}_\pm$ being the strict transforms of Γ_i and $\sigma_\pm$. (The $\tilde{\sigma}_\pm$ are also the total transforms because $\sigma_\pm$ lies on the nonsingular locus of $\mathcal{C}$.) In this situation, what we have to show is that the divisor $\tilde{\Gamma}_i$ on $\tilde{\mathcal{C}}$ is away from the specified sections $\tilde{\sigma}_\pm$. By (2.4.5) and (2.4.6), this will follow from the inequality $\tilde{\Gamma}_i^2 \le 0$.

In order to establish this inequality, we start with the following observation.

Let $G \supset \Gamma$ be an irreducible component of the closed subset $\overline{\mathrm{pr}}_X^{-1}(Y_+) \subset \overline{V}\langle x_+, x_-\rangle$. G is a divisor on $\overline{V}\langle x_+, x_-\rangle$ which surjects onto $\overline{W}\langle x_+, x_-\rangle$. In particular, G is a multi-section of the (generically) conic fibration $\overline{V}\langle x_+, x_-\rangle \to \overline{W}\langle x_+, x_1\rangle$. At a general closed point of Γ (which is also a general closed point of G), we have local isomorphisms

$$\Theta_{\overline{V}\langle x_+, x_-\rangle} \simeq \overline{\mathrm{pr}}_X^*\Theta_X$$

$$\overline{\mathrm{pr}}_{\overline{W}}^*\Theta_{\overline{W}\langle x_+, x_-\rangle} \simeq \Theta_G \simeq \overline{\mathrm{pr}}_X^*\Theta_{Y_+},$$

implying that the composite of natural homomorphisms

$$\begin{aligned}\Theta_{\overline{V}\langle x_+, x_-\rangle/\overline{W}\langle x_+, x_-\rangle}|_\Gamma &\to \Theta_{\overline{V}\langle x_+, x_-\rangle}|_\Gamma \to (\overline{\mathrm{pr}}_X|_\Gamma)^*(\Theta_X|_{L_+}) \\ &\to (\overline{\mathrm{pr}}_X|_\Gamma)^*\big((\Theta_X|_{L_+}/(\Omega^1_{Y_+}|_{L_+})^*\big)/(\mathrm{torsion}) \simeq \mathcal{O}_\Gamma\end{aligned}$$

is non-zero (and hence injective) at a general point of Γ. (Here we used the fact that $\Theta_X|_{L_+} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2} \oplus \mathcal{O}$, and its ample part of rank $n-1$ exactly corresponds to Θ_{Y_+} at a general point of L_+.)

Going back to the conic bundle $\tilde{\mathcal{C}} \to \tilde{\Delta}$ with the section $\tilde{\Gamma}_i$, this observation tells us that, at a general point of Γ_i, the natural homomorphism $\mathcal{N}_{\tilde{\Gamma}_i,\mathcal{C}} \simeq \Theta_{\tilde{\mathcal{C}}/\tilde{\Delta}}|_{\Gamma_i} \to \mathcal{O}_{\tilde{\Gamma}_i}$ is non-zero. This shows that $\deg \mathcal{N}_{\tilde{\Gamma}_i/\tilde{\mathcal{C}}} = \tilde{\Gamma}_i^2 \leq 0$.

Lemma 3.10. *No member C of $W\langle x_+, x_-\rangle$ has a cuspidal singularity at x_+ or x_-.*

Proof. Let $W\langle\text{cusp}\rangle \subset W$ denote the closure of the locus of irreducible cuspidal curves. Let $V\langle\text{cusp}\rangle$ be the associated family and $\Sigma \subset V\langle\text{cusp}\rangle$ the locus of the cuspidal points of the fibres. What we are going to show is that the natural projection $\Sigma \times_{W\langle\text{cusp}\rangle} V\langle\text{cusp}\rangle \to X \times X$ is not surjective, meaning that there is no member of W which has a cusp at x_+ and passes through x_- when (x_+, x_-) are general.

For simplicity of the notation, we put

$$\begin{aligned} Z &= V\langle\text{cusp}\rangle \times_{W\langle\text{cusp}\rangle} V\langle\text{cusp}\rangle, \\ \Sigma_1 &= \Sigma \times_{W\langle\text{cusp}\rangle} V\langle\text{cusp}\rangle \\ \Sigma_2 &= V\langle\text{cusp}\rangle \times_{W\langle\text{cusp}\rangle} \Sigma. \end{aligned}$$

Suppose that Σ_1 and Σ_2 dominate $X \times X$ via the natural projection $Z \to X \times X$. Let $Z \xrightarrow{g} Y \xrightarrow{h} X \times X$ be the Stein factorisation: namely, h is finite and the fibre Z_y of g over a general point $y \in Y$ is an irreducible variety. Our hypothesis amounts to the condition $\dim Z_y \cap \Sigma_i = a \geq 0$, so that $\dim Z_y = a + 1 \geq 1$. Hence we can find an irreducible curve $f\colon T \to Z_y \subset Z = V\langle\text{cusp}\rangle \times_{W\langle\text{cusp}\rangle} V\langle\text{cusp}\rangle$ such that

($\clubsuit$) *$f(T)$ is not contained in $\Sigma_1 \cup \Sigma_2$ but connects these two divisors.*

Let T' be the image of $f(T)$ in $W\langle\text{cusp}\rangle$. Every member C of T' contains $\{x_+, x_-\}$, where (x_+, x_-) is the image of $y \in Y$ in $X \times X$. The condition ($\clubsuit$) above says that a general member of T' has no cusp at $x_\pm$ but some member does; thus $T' \subset W\langle\text{cusp}\rangle$ defines a nontrivial one-parameter family of cuspidal curves passing through x_+, x_-. However, (2.6) asserts that the cupidal locus cannot pass through one of the $x_\pm$, which contradicts our construction.

Corollary 3.11. *Let $\mathfrak{M}_{x_+} \subset \mathcal{O}_X$ be the maximal ideal which defines x_+. Then*

(1) $\mathfrak{M}_{x_+}\mathcal{O}_{\overline{V}\langle x_+, x_-\rangle} = \mathfrak{J}(-\sigma_+)$, *where $\mathfrak{J} \subset \mathcal{O}_{\overline{V}\langle x_+, x_-\rangle}$ is an ideal*

sheaf of a closed subscheme away from σ_+.

(2) *If* C *is a general member of* $W\langle x_+, x_-\rangle$, *then*

$$\overline{\mathrm{pr}}_X^{-1} C = \sigma_+ + \sigma_- + \overline{V}_{[C]} + B + E$$

where B *is a union of finitely many curves away from* $\sigma_+ \cup \sigma_-$ *and* E *is a finite set* $\subset \overline{V}\langle x_+, x_-\rangle \setminus (\sigma_+ \cup \sigma_-)$ *such that* $\overline{\mathrm{pr}}_X(E) \subset \{x_+, x_-\}$.

Proof. (1) The statement is a direct consequence of (3.10). In particular, on an open neighbourhood of the Cartier divisor σ_+, the projection $\overline{\mathrm{pr}}_X : \overline{V}\langle x_+, x_-\rangle \to X$ lifts to a morphism $\tilde{\mathrm{pr}}_X$ to $\mathrm{Bl}_{x_+}(X)$. The scheme theoretic inverse image of C in $\mathrm{Bl}_{x_+} X$ is $\mathcal{I}_{\tilde{C}}(-E_+)$, where $\tilde{C}$ the strict transform and E_+ the exceptional divisor.

(2) Since $\overline{\mathrm{pr}}_X$ is finite over $X \setminus \{x_+, x_-\}$ and $C \subset X$ is a locally complete intersection of codimension $n-1$, it is clear that there is a decomposition of the above type and we have only to show that B is away from $\sigma_+ \cup \sigma_-$. If it meets $\sigma_+ \cup \sigma_-$ for general C, then the same should hold for any specialisation of C, which is not the case for $C = L_+ + L_-$ by (3.8) and (3.9).

Corollary 3.12. *Take a small open analytic neighbourhood* U^* *of* x_+ *in* X. *Then* $\overline{\mathrm{pr}}_X^{-1}(U^*) \subset \overline{V}\langle x_+, x_-\rangle$ *is a disjoint union of a small open neighbourhood* U *of* σ_+ *and an extra open subset* U'.

The X*-projection* $\overline{\mathrm{pr}}_X$ *induces proper bimeromorphic morphisms between a small analytic neighbourhood* $U \to U^*$ *finite over* $U^* \setminus \{x_+\}$ *and* $U \to \tilde{U}^*$, *where* $\tilde{U}^* \subset \mathrm{Bl}_{x_+}(X)$ *is the inverse image of* U^*. *In particular, the ramification locus of* $\overline{\mathrm{pr}}_X$ *has codimension* ≥ 2 *on* $\overline{V}\langle x_1, x_2\rangle \setminus (\sigma_+ \cup \sigma_-)$.

Proof. Let C be a general member of $W\langle x_+, x_-\rangle$ and $x \in C$ a closed point sufficiently close to x_+ but not equal to $x_\pm$. Then (3.11) asserts that $\overline{\mathrm{pr}}_X^{-1}(x) \cap U = \{([C], x)\}$, a single point. Hence $\overline{\mathrm{pr}}_X$ is a bimeromorphism on U, finite over $U^* \setminus \sigma_+$.

If ramification locus of $\overline{\mathrm{pr}}_X$ contains an $(n-1)$-dimensional irreducible component $R_0 \neq \sigma_\pm$, it must be a pull-back $\overline{\mathrm{pr}}_{\overline{W}}^* B_0$ of a divisor B_0 on $\overline{W}\langle x_+, x_-\rangle$. However, this contradicts the fact that $\overline{\mathrm{pr}}_X$ is unramified in codimension one on $U \setminus \sigma_+$.

Corollary 3.13. (1) $\overline{\mathrm{pr}}_X$ *is birational.*

(2) $\overline{V}\langle x_+, x_-\rangle \setminus (\sigma_+ \cup \sigma_-) \simeq X \setminus \{x_+, x_-\}$.

(3) $\overline{W}\langle x_+, x_-\rangle$ *is nonsingular.*

(4) *There are isomorphisms*

$$\begin{aligned}\overline{V}\langle x_+, x_-\rangle &\simeq V\langle x_+, x_-\rangle \simeq \mathrm{Bl}_{\{x_+,x_-\}}(X)\\ \overline{W}\langle x_+, x_-\rangle &\simeq W\langle x_+, x_-\rangle \simeq E_\pm \simeq \mathbb{P}^{n-1}.\end{aligned}$$

Proof. Because the Fano manifold X is smooth and simply connected, a generically finite morphism $f\colon Y \to X$ with branch locus of dimension $\leq n-2$ is necessarily birational, whence (1) follows. By Grothendieck's version of Zariski's Main Theorem, the inverse image of x_+ in $\overline{V}\langle x_+, x_-\rangle$ is connected, so that $\mathfrak{M}_{x_\pm}\mathcal{O}_{\overline{V}\langle x_+,x_-\rangle} = \mathcal{O}(-\sigma_\pm)$. This shows that $\overline{F}\langle x_+, x_-\rangle \to X \setminus \{x_+, x_-\}$ is a well defined, proper, finite, birational morphism, and hence an isomorphism.

Since $\overline{\mathrm{pr}}_{\overline{W}}|_{\overline{V}\langle x_+,x_-\rangle\setminus(\sigma_+\cup\sigma_-)}$ has reduced fibres and hence admits an analytic local section over any closed point $w \in \overline{W}\langle x_+, x_-\rangle$, the smoothness of the total space implies that of the base space, *i.e.,* the assertion (3).

The fibre space $V\langle x_+, x_-\rangle$ has fibres smooth near the section $\sigma_\pm$, and hence the smoothness of the base $\overline{W}\langle x_+, x_-\rangle$ is inherited by the total space near $\sigma_\pm$, thereby showing the global smoothness of $\overline{F}\langle x_+, x_-\rangle$. Once the smoothness is established, the purity of ramification loci tells us that the naturally induced morphism $\overline{\mathrm{pr}}_{\tilde{X}}\colon \overline{V}\langle x_+, x_-\rangle \to \mathrm{Bl}_{\{x_+,x_-\}}(X)$, which has ramification of codimension ≥ 2, is an isomorphism, thereby inducing $\sigma_\pm \simeq E_\pm$.

We now arrive at the conclusion:

Corollary 3.14. *The pullback* pr_W^*L *of the hyperplane divisor* L *on* $W\langle x_+, x_-\rangle \simeq \mathbb{P}^{n-1}$ *is linearly equivalent to* $\mathrm{pr}_X^*H_0 - \sigma_+ - \sigma_-$*, where* H_0 *is an ample divisor on* X *with* $H_0^n = 2$*. The linear system* $|H_0|$ *is free from base points, defining an isomorphism* $X \to Q_n \subset \mathbb{P}^{n+1}$.

Proof. Since pr_W^*L cuts out a hyperplane from the section $\sigma_\pm = \tilde{\mathrm{pr}}_X^*E_\pm$, it is linearly equivalent to $\mathrm{pr}_X^*H_0 - \sigma_+ - \sigma_-$. It follows that $H_0^n = L^n + 2 = 2$. Noting that $|L|$ is free from base point, we see that $|H_0|$ has no base point outside $\{x_+, x_-\}$. On the other hand, since $\mathrm{Pic}(X) \simeq \mathbb{Z}$ is discrete, the linear system $|H_0|$ does not depend on the choice of the general base points $x_\pm \in X$, meaning that it is free from base points and has dimension $\dim|L| + 2 = n + 2$.

The semiample divisor H_0 is ample or, equivalently, $(\Gamma, H_0) > 0$ for every irreducible curve $\Gamma \subset X$. Indeed, for every irreducible curve $\Gamma \not\subset \sigma_+ \cup \sigma_-$, we have $(\Gamma, \tilde{\mathrm{pr}}_X^*H_0) \geq (\Gamma, \mathrm{pr}_W^*L)$, the equality holding if and only if Γ is away from $\sigma \cup \sigma_2$. By construction $(\Gamma, \mathrm{pr}_W^* L) \geq 0$, the equality holding if and only if Γ is an irreducible component of the fibre of pr_W. Hence $(\Gamma, \tilde{\mathrm{pr}}_X^* H_0) \geq (\Gamma, \mathrm{pr}_W^* L) \geq 0$ and at least one of the inequalities is strict.

Thus $|H_0|$ defines a finite morphims $X \to \mathbb{P}^{n+1}$ onto a non-degenerate hypersurface of degree ≤ 2, which is necessarily an isomorphism onto a hyperquadric.

§4. Proof of main theorems and concluding remarks

Let us complete the proof of (0.1) and (0.2).

In Theorem 0.1, the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial, and it suffices to show that $\rho(X) = 1$ when the global length $l(X) = n$.

Lemma 4.1. *Let X be a Fano n-fold of dimension $n \geq 3$. If $l(X) = n$, then the Picard number $\rho(X)$ is one.*

Proof. Suppose $\rho(X) \geq 2$. Fix an extremal ray, and we have a non-trivial extremal contraction $\pi\colon X \to Y$ (see, for instance, [7, §3]). The fibre of a closed point $y \in Y$ is uniruled.

When π is birational, take the exceptional locus E of π. Let $C \subset E$ be a rational curve which is contracted to a point in Y and suppose that $(C, -K_X)$ attains the minimum among such curves. Then any deformation of the normalisation morphism $f\colon \mathbb{P}^1 \to C$ belongs to $\mathrm{Hom}(\mathbb{P}^1, E)$, and thanks to [2, Theorem 2.8] we have

$$(C,-K_X)+n \leq \dim_{[f]}\mathrm{Hom}(\mathbb{P}^1,X) = \dim_{[f]}\mathrm{Hom}(\mathbb{P}^1,E) \leq 2\dim E+1 \leq 2n-1,$$

contradicting the inequality $(C, -K_X) \geq n$.

In case X is a fibre space over Y, take a rational curve C contained in a smooth fibre X_y, and assume that $(C, -K_X)$ attains the minimum among such. Then we have $\dim X_y + 1 \geq (C, -K_{X_y}) = (C, -K_X) \geq n$, so that $\dim X_y = n-1$ and $X_y \simeq \mathbb{P}^{n-1}$. Choose another extremal ray inducing a sectond morphism $\varphi\colon X \to Z$. By what we have seen before, φ defines another fibre space structure on X.

A fibre X_y of π is $\mathbb{P}^{n-1} \subset X$ which is non-trivially mapped to Z, a projective variety. The pullback of an ample divisor H on Z is non-trivial on $X_y \simeq \mathbb{P}^{n-1}$ and hence ample, so that H^{n-1} cannot be numerically trivial on Z. In particular $\dim Z \geq n-1$, and a general fibre X_z of φ must be $\mathbb{P}^1$ with $(X_z, -K_X) = 2 < n$, another contradiction.

Remark 4.2. In Theorem 0.1, we cannot drop the condition $\rho(X) = 1$ in (3). For instance, let A be a smooth hypersurface of degree $d \leq n$

of the linear subspace $\mathbb{P}^{n-1} = H = \{x_n = 0\} \subset \mathbb{P}^n$ and let $\mu\colon X \to \mathbb{P}^n$ be the blowup along A. X is a smooth Fano manifold with $\rho(X) = 2$, $-K_X = (n+1)\mu^*H - E$, where E stands for the exceptional divisor. If $x_0 \in X \setminus H$, then the local length $l(X, x_0)$ is n, which is attained by the strict transforms of the lines connecting x_0 and A.[5] In this case, any curve C with $(C, -K_X) = -2n$, $C \ni x_+, x_-$ is a disjoint union of two components provided $x_\pm$ are general.

In Theorem 0.2, the following implication relations are trivial:

— (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2),
— (1) $\Rightarrow$ (6),
— (1) $\Rightarrow$ (7),

while the equivalence between (1)(2)(3) were established by (4.1). Thus it suffices to check the implications (6) $\Rightarrow$ (2) and (7) $\Rightarrow$ (3) to complete the proof of (0.2).

The implication (6) $\Rightarrow$ (3) follows from

Lemma 4.3. *Let X be a smooth Fano n-fold, $n \geq 3$. If $\wedge^2\Theta_X$ is ample, then $l(X) \geq n$.*

Proof. Let C be an arbitrary rational curve on X and let $\nu\colon \mathbb{P}^1 \to C \subset X$ denote a birational map induced by the normalisation of C. Put $\nu^*\Theta_X \simeq \bigoplus_{i=1}^n \mathcal{O}(d_i)$, $d_1 \leq d_2 \leq \cdots \leq d_2$. Then the condition on $\wedge^2\Theta_X$ implies that $2d_2 \geq d_2 + d_1 \geq 1$. If $d_1 \geq 1$, then $(C, -K_X) = \sum_i d_i \geq n$. If $d_1 = 0$, then $d_1 + 0 \geq 1$, while $d_n \geq 2$ thanks to the inclusion $\Theta_{\mathbb{P}^1} \simeq \mathcal{O}(2) \subset \nu^*\Theta_X$. Suppose that $d_1 < 0$. Then $d_2 \geq -d_1 + 1$ so that

$$(C, -K_X) = \sum_{i=1}^n d_i = d_1 + \sum_{i=2}^n d_i \geq d_1 + (n-1)(-d_1+1) = n-1+(n-2)(-d_1).$$

Since $n \geq 3$, we have $(C, -K_X) \geq n$ whenever $d_1 < 0$.

Finally we have

Lemma 4.4. *Assume that $n \geq 3$. Let $f\colon Q_m \to X$ be a surjective morphism from an smooth hyperquadric in $\mathbb{P}^{m+1}$ to a smooth projective vaiety of dimension n. Then X is a Fano n-fold with Picard number one and the local length satisfies $l(X, x_0) \geq n$ if x_0 is away from the branch locus of f.*

Proof. Because $\rho(Q_m) = 1$, $m \geq n \geq 3$, the pullback f^*H of the hyperplane bundle H on X is ample, implying $m = n$ and the equality

[5]The global length $l(X)$ is of course 1 attained by the fibres of the $\mathbb{P}^1$-bundle $E \to A$.

$\rho(X) = 1$ as well. Because $\mathcal{O}(-f^*K_X)$ contains the ample line bundle $\mathcal{O}(-f^*K_{Q_n})$, X with Picard number one must be Fano.

Let $C \subset X$ be a rational curve passing through x_0 and $\nu \colon \mathbb{P}^1 \to C \subset X$ the normalisation morphism. Then $\nu^*\Theta_X \simeq \sum_i \mathcal{O}(d_i)$, $d_1 \leq d_2 \leq \cdots \leq d_n$, $d_n \geq 2$. Hence $(C, K_X) \geq n$ follows if we show that $d_1 \geq 0$, $d_2 \geq 1$.

Let $\Gamma \subset Q_n$ be an irreducible curve which surjects onto C, with normalisation $\nu' \colon \tilde{\Gamma} \to \Gamma \subset Q_n$. Consider the commutative diagram

$$\begin{array}{ccccc}
\tilde{\Gamma} & \xrightarrow[\nu']{} & \Gamma & \longrightarrow & Q_n \\
\downarrow \tilde{f}_\Gamma & & \downarrow f_\Gamma & & \downarrow f \\
\mathbb{P}^1 & \xrightarrow[\nu]{} & C & \longrightarrow & X.
\end{array}$$

Noticing that the ramification locus of f does not contain Γ (which meets $f^{-1}(x_0)$), we have a natural inclusion

$$\nu'^*\Theta_{Q_n} \subset \nu'^* f^* \Theta_X = \tilde{f}_\Gamma^* \Big(\bigoplus_{i=1}^n \mathcal{O}(d_i) \Big).$$

Then the semipositivity of Θ_{Q_n} gives $d_1 \geq 0$, while the ampleness of $\wedge^2 \Theta_{Q_n}$ yields $2d_2 \geq d_1 + d_2 > 0$.

Remark 4.5. In the proof of Theorem 0.1, we used the condition that X is nonsingular in order to establish the dimension estimates for $S\langle x_\pm \rangle$, $W\langle x_+, x_- \rangle$ and the birationality (generic one-to-one property) of pr_X. If we relax the smoothness condition into normality, we obtain the following

THEOREM. *Let X be a normal, projective, $\mathbb{Q}$-factorial, $\mathbb{Q}$-Fano n-fold with Picard number one defined over the complex numbers. Let x_0 be a sufficiently general closed point of X and assume that any rational curve passing through x_0 deforms in $n-2$ independent parameters. Then X is a finite quotient of a normal hyperquadric $\subset \mathbb{P}^{n+1}$ (possibly with irreducible singular locus of dimension $\leq n-2$) by a finite group action without divisorial fixed point set. In particular, X is isomorphic to a normal hyperpquadric if and only if the open subset $X \setminus \mathrm{Sing}(X)$ is simply connected.*

The proof of Theorem 0.1 carries over into this situation without essential change. The variety $\overline{V}\langle x_+, x_- \rangle$ is now a two-point blowup of a normal hyperquadric, while pr_X is unramified over $X \setminus \big(\{x_+, x_-\} \cup \mathrm{Sing}(X)\big)$.

Remark 4.6. The author does not know if Theorem 0.1 stays true in positive characteristics. Almost all of our arguments work well regardless of the characteristic. The exceptions are those related to Sard's theorem, which, unfortunately, permeate throughout the paper. The most serious question to be checked is the separability of the projection $\mathrm{pr}_X \colon V\langle x_+, x_-\rangle \to X$.

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