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Stochastic Newton Equation with Reflecting Boundary Condition

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§1. Introduction

Let D be a bounded domain in $\mathbf{R}^d$ with a smooth boundary and $n(x)$, $x \in \partial D$, be an outer normal vector. Let $a^{ij} : \mathbf{R}^d \to \mathbf{R}$, $i, j = 1, \dots d$, be smooth functions such that $a^{ij}(x) = a^{ji}(x)$, $x \in \mathbf{R}^d$. Also, let $b^i : \mathbf{R}^{2d} \to \mathbf{R}$, $i = 1, \dots d$, be bounded measurable functions. We assume that there are positive constants C_0, C_1 such that

$$C_0|\xi|^2 \le \sum_{i,j=1}^d a^{ij}(x)\xi_i\xi_j \le C_1|\xi|^2, \qquad x, \xi \in \mathbf{R}^d.$$

Let L_0 be a second order linear differential operator in $\mathbf{R}^{2d}$ given by

$$L_0 = \sum_{i=1}^d v^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial v^i \partial v^j} + \sum_{i=1}^d b^i(x, v) \frac{\partial}{\partial v^i}$$

Let $\tilde{W}^d = C([0, \infty); \mathbf{R}^d) \times D([0, \infty); \mathbf{R}^d)$. Now let $\Phi : \mathbf{R}^d \times \partial D \to \mathbf{R}^d$ be a smooth map satisfying the following .
(i) $\Phi(\cdot, x) : \mathbf{R}^d \to \mathbf{R}^d$ is linear for all $x \in \partial D$.
(ii) $\Phi(v, x) = v$ for any $x \in \partial D$ and $v \in T_x(\partial D)$, i.e., $\Phi(v, x) = v$ if $x \in \partial M$, $v \in \mathbf{R}^d$ and $v \cdot n(x) = 0$.
(iii) $\Phi(\Phi(v, x), x) = v$ for all $v \in \mathbf{R}^d$ and $x \in \partial D$.
(iv) $\Phi(n(x), x) \ne n(x)$ for any $x \in \partial D$.

The main theorem in the present paper is the following.

Theorem 1. *Let $(x_0, v_0) \in (\bar{D})^c \times \mathbf{R}^d$. Then there exists a unique probability measure μ over $\tilde{W}^d$ satisfying the following conditions.*
(1) $\mu(w(0) = (x_0, v_0)) = 1$.
(2) $\mu(w(t) \in D^c \times \mathbf{R}^d, t \in [0, \infty)) = 1$.

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(3) *For any* $f \in C_0^\infty((\bar{D})^c \times \mathbf{R}^d)$, $\{f(w(t)) - \int_0^t L_0 f(w(s))ds; t \geq 0\}$ *is a martingale under* $\mu(dw)$.
(4) $\mu(1_{\partial D}(x(t))(v(t) - \Phi(v(t-), x(t))) = 0$ *for all* $t \in [0, \infty)) = 1$.
Here $w(\cdot) = (x(\cdot), v(\cdot)) \in \tilde{W}^d$.

Now let us think of the following Stochastic Newton equation

$$\begin{aligned} dX_t^\lambda &= V_t^\lambda dt \\ dV_t^\lambda &= \sigma(X_t^\lambda)dB(t) + (b(X_t^\lambda, V_t^\lambda) - \lambda \nabla U(X_t^\lambda))dt \end{aligned}$$

$$X_0^\lambda = x_0, \qquad V_0^\lambda = v_0.$$

Here $B(t)$ is a d-dimensional Brownian motion, $\sigma \in C^\infty(\mathbf{R}^d; \mathbf{R}^d)$, $b : \mathbf{R}^{2d} \to \mathbf{R}^d$ is a bounded Lipschitz continuous function, and $U \in C_0^\infty(\mathbf{R}^d)$.

We assume the following also.
(A-1) There are positive constants C_0, C_1 such that

$$C_0|\xi|^2 \leq |\sigma(x)\xi|^2 \leq C_1|\xi|^2, \qquad x, \xi \in \mathbf{R}^d.$$

(A-2) Let $D = \{x \in \mathbf{R}^d;\ U(x) > 0\}$. Then there are $\varepsilon_0 > 0$, $U_0 \in C^\infty(\mathbf{R}^d; \mathbf{R})$ and a non-increasing C^1-function $\rho : \mathbf{R} \to \mathbf{R}$ satisfying the following.
(1) $x \in \partial D$, if and only if $U_0(x) = 0$ and $dis(x, \partial D) < \varepsilon_0$.
(2) $\nabla U_0(x) \neq 0$, $x \in \partial D$.
(3) $\rho(t) = 0$, $t \geq 0$, $\rho(t) > 0$, $t < 0$, and $U(x) = \rho(U_0(x))$ for $x \in \mathbf{R}^d$ with $dis(x, \partial D) < \varepsilon_0$.
(4) $\lim_{t \uparrow 0} \dfrac{\rho'(t)}{\rho(t)} = -\infty$.

Now let $\tilde{dis}$ be a metric function on $\tilde{W}^d$ given by

$$\begin{aligned} &\tilde{dis}(w_0, w_1) \\ &= \sum_{n=1}^\infty 2^{-n}(1 \wedge ((\max_{t \in [0,n]} |x_0(t) - x_1(t)|) + (\int_0^n |v_0(t) - v_1(t)|^n)^{1/n})), \end{aligned}$$

for $w_i(\cdot) = (x_i(\cdot), v_i(\cdot)) \in \tilde{W}^d$, $i = 0, 1$.
Then we will show the following.

Theorem 2. *Let* ν^λ, $\lambda \in [1, \infty)$, *be the probability law of* $(X_t^\lambda, V_t^\lambda)$, $t \in [0, \infty)$, *on* $\tilde{W}_0$, *and* μ *be the probability measure given in Theorem 1 in the case when* $\Phi(v, x) = v - 2(v \cdot n(x))n(x)$, $v \in \mathbf{R}^d$, $x \in \partial D$. *Then* ν^λ *conveges to* μ *weakly as* $\lambda \to \infty$ *as probability measures on* $(\tilde{W}_0, \tilde{dis})$.

§2. Basic lemmas

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,\infty)}, P)$ be a filtered probability space, and $B(t) = (B^1(t), \ldots, B^d(t))$ be a d-dimensional Brownian motion. Let $B^0(t) = t$, $t \in [0, \infty)$. Let $\sigma_i : \mathbf{R}^N \to \mathbf{R}^N$, $i = 0, 1, \ldots, d$, be Lipschitz continuous functions, and let $X : [0, \infty) \times \mathbf{R}^N \times \Omega \to \mathbf{R}^N$ be the solution to the following SDE

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t \sigma_i(X(s,x))dB^i(s), \qquad t \geq 0,\ x \in \mathbf{R}^N.$$

We may assume that $X(t,x)$ is continuous in (t,x) (cf. Kunita [2]).
Then we have the following.

Lemma 3. *For any $T > 0$ and $p_0, p_1, \ldots, p_m \in (1, \infty)$, $m \geq 1$, with $\sum_{k=0}^m p_k^{-1} = 1$, there is a constant $C > 0$ such that*

$$E[\int_{\mathbf{R}^N} \prod_{k=0}^{m} |f_k(X(t_k,x))|dx] \leq C \prod_{k=0}^{m} \| f_k \|_{L^{p_k}(\mathbf{R}^N,dx)}$$

for all $0 = t_0 < t_1 < \ldots < t_m \leq T$, and $f_k \in C_0^\infty(\mathbf{R}^N)$, $k = 0, 1, \ldots, m$.

Proof. From the assumption, there is a $C_0 > 0$ such that

$$|\sigma_i(x) - \sigma_i(y)| \leq C_0|x - y|, \qquad x, y \in \mathbf{R}^N.$$

Let $\varphi \in C_0^\infty(\mathbf{R}^N)$ such that $\int_{\mathbf{R}^N} \varphi(x)dx = 1$. Let $\varphi_n(x) = n^N\varphi(nx)$, $x \in \mathbf{R}^N$, for $n \geq 1$, and let $\sigma_i^{(n)} = \varphi_n * \sigma_i$, $i = 0, \ldots, d$. Then $\sigma_i^{(n)} \in C^\infty(\mathbf{R}^N; \mathbf{R}^N)$. Let

$$W_{i,k}^{(n),j}(x) = \frac{\partial}{\partial x^k}\sigma_i^{(n),j}(x),$$
$$x \in \mathbf{R}^N,\ j,k = 1\ldots,N,\ i = 0,1,\ldots,d,\ n \geq 1.$$

Then we see that $|W_{i,k}^{(n),j}(x)| \leq C_0$, $x \in \mathbf{R}^N$. Let $X^{(n)} : [0, \infty) \times \mathbf{R}^N \times \Omega \to \mathbf{R}^N$ be the solution to the following SDE

$$X^{(n)}(t,x) = x + \sum_{i=0}^{d} \int_0^t \sigma_i^{(n)}(X^{(n)}(s,x))dB^i(s), \qquad t \geq 0,\ x \in \mathbf{R}^N.$$

Then we may think that $X^{(n)}(t, \cdot) : \mathbf{R}^N \to \mathbf{R}^N$ is a diffeomorphism with probability one. Let $J_k^{(n),j}(t,x) = \frac{\partial}{\partial x^k}X^{(n),j}(t,x)$. Let $W_i^n(x) =$

$(W_{i,k}^{(n),j}(x))_{k,j=1,\dots,N}$ and $J^{(n)}(t,x) = (J_k^{(n),j}(t,x))_{k,j=1,\dots,N}$. Then the $N\times N$-matrix valued process $J^{(n)}(t,x)$ satisfies the following SDE

$$J^{(n)}(t,x) = I_N + \sum_{i=0}^{d}\int_0^t W_i^{(n)}(X^{(n)}(s,x))J^{(n)}(s,x)dB_i(s).$$

Also, we see that

$$\begin{aligned} &J^{(n)}(t,x)^{-1} \\ &= I_N - \sum_{i=0}^{d}\int_0^t J^{(n)}(s,x)^{-1}W_i^{(n)}(X^{(n)}(s,x))dB_i(s) \\ &+\frac{1}{2}\sum_{i=1}^{d}\int_0^t J^{(n)}(s,x)^{-1}W_i^{(n)}(X^{(n)}(s,x))^2 ds. \end{aligned}$$

Then we see that

$$C_T = \sup\{E[\det J^{(n)}(t,x)^{-p_0+1}];\ t\in[0,T], x\in \mathbf{R}^N, n\geq 1\} < \infty.$$

So we have

$$\begin{aligned} &E[\int_{\mathbf{R}^N}\prod_{k=0}^{n}|f_k(X^{(n)}(t_k,x))|dx] \\ &\leq E[\int_{\mathbf{R}^N}|f_0(x)|_0^p(\prod_{k=1}^{m}\det J^{(n)}(t_k,x)^{-p_0/p_k})dx]^{1/p_0} \\ &\times\prod_{k=1}^{m}E[\int_{\mathbf{R}^N}|f_k(X^{(n)}(t_k,x))|^{p_k}\det J^{(n)}(t_k,x)dx]^{1/p_k} \\ &\leq C_T(\int_{\mathbf{R}^N}|f(x)|_0^p dx)^{1/p_0}\prod_{k=1}^{m}(\int_{\mathbf{R}^N}|f_k(x))|^{p_k}dx)^{1/p_k} \end{aligned}$$

Letting $n\to\infty$, we have our assertion. ∎

Now let D be a bounded domain in $\mathbf{R}^N$ and $F^j : \mathbf{R}^N \to \mathbf{R}$, $j=1,2$, be C^2 functions satisfying the following assumptions (F1),(F2), furthermore.

(F1) For $x\in D$ and $i=1,\dots,d$,

$$\sum_{j=1}^{N}\sigma_i^j(x)\frac{\partial}{\partial x^j}F^1(x) = 0.$$

(F2) $\inf\{\det(\nabla F^i(x)\cdot\nabla F^j(x))_{i,j=1,2};\ x\in D\} > 0$.

Then we have the following

Lemma 4. *For a.e.x,*

$$P(X(t,x) \in D, F(X(t,x)) = 0 \textit{ for some } t > 0) = 0.$$

Here $F = (F^1, F^2) : \mathbf{R}^N \to \mathbf{R}^2$.

Proof. Let

$$\tau(s,x) = \inf\{t \geq s; X(t,x) \in D^c\} \wedge (s+1), \qquad x \in \mathbf{R}^N, s > 0.$$

Also, let

$$p(x,s) = P(F(X(t,x)) = 0 \text{ for some } t \in [s, \tau(s,x))), \qquad x \in \mathbf{R}^N, s > 0.$$

Then we see that

$$P(X(t,x) \in D, F(X(t,x)) = 0 \text{ for some } t > 0) \leq \sum_{r \in \mathbf{Q}_+} p(x,r),$$

where $\mathbf{Q}_+$ is the set of positive rational numbers. Let $V(m) = \{x \in \mathbf{R}^N;\ |x| \leq m\}$, $m \geq 1$. Let us define random variables $Z_{T,m}$, $T > 0$, $m \geq 1$, and constant C_1 by

$$Z_{T,m} = \sup\{|t-s|^{-1/3}|X(t,x) - X(s,x)|;\ 0 \leq s < t \leq T,\ x \in V(m)\},$$

and

$$C_1 = \sup\{|\sigma_0(x)||\nabla F^1(x)| + \frac{1}{2}\sum_{i=1}^{d} |\nabla^2 F^1(x)||\sigma_i(x)|^2 + |\nabla F^2(x)|;\ x \in \bar{D}\}.$$

Then we see that $P(Z_{T,m} < \infty) = 1$ (cf. Kunita[2]). By the assumtion (F1), we see that

$$F^1(X(t,x)) = F^1(x) + \int_0^t (\sigma_0(X(s,x))\nabla F^1(X(s,x))$$

$$+ \sum_{i=1}^{d} \frac{1}{2}\nabla^2 F^1(X(s,x))(\sigma_i(X(s,x)), \sigma_i(X(s,x)))ds.$$

So we see that

$$|F^1(X(t,x)) - F^1(X(s,x))| \leq C_1|t-s|, \qquad t \in [s, \tau(s,x)), s \geq 0, x \in \mathbf{R}^N,$$

and

$$|F^2(X(t,x)) - F^2(X(s,x))| \leq C_1 Z_{T,m}|t-s|^{1/3} \qquad t,s \in [0,T], x \in V(m).$$

Also, by the assumption (F2), we see that there is a constant $C_2 > 0$ such that

$$\int_D 1_A(F(x))dx \le C_2|A|$$

for any Borel set A in $\mathbf{R}^2$, where $|A|$ denotes the area of A.

Let $\Delta_{\ell,n,k} = [-C_1 n^{-1}, C_1 n^{-1}] \times [-\ell C_1 n^{-1/3}, \ell C_1 n^{-1/3}]$, $\ell, n \ge 1$, $k = 1, \dots, n$. Then we have for any $\ell \ge 1$,

$$\begin{aligned}
&\int_{V(m)} dx P(F(X(t,x)) = 0 \text{ for some } t \in [s, \tau(s,x)), Z_{s+1,m} \le \ell) \\
&\le \sum_{k=1}^{n} \int_{V(m)} dx\, P(X(s,x) \in D, X(s+(k-1)/n, x) \in D, \\
&\qquad\qquad F(X(s+(k-1)/n,x)) \in \Delta_{\ell,n,k}) \\
&= \sum_{k=1}^{n} E[\int_{\mathbf{R}^N} dx 1_{V(m)}(x) 1_D(X(s,x)) \\
&\qquad 1_D(X(s+(k-1)/n,x)) 1_{\Delta_{\ell,n,k}}(F(X(s+(k-1)/n,x)))] \\
&\le C \sum_{k=1}^{n} |V(m)|^{1/10} |D|^{1/10} (\int_D 1_{\Delta_{\ell,n,k}}(F(x))dx)^{4/5} \\
&\le C C_2 n |V(m)|^{1/10} |D|^{1/10} (4C_1^2 \ell n^{-4/3})^{4/5}.
\end{aligned}$$

Here C is the constant in Lemma 3 for $T = s+1$, $p_0 = p_1 = 10$ and $p_3 = 5/4$. Since $n \ge 1$ is arbitrary, we see that

$$\int_{V(m)} dx P(F(X(t,x)) = 0 \text{ for some } t \in [s,, \tau(s,x)), Z_{s+1,m} \le \ell) = 0, \qquad \ell \ge 1.$$

This implies that $\int_{\mathbf{R}^N} p(x,s) = 0$, $s > 0$.

Therefore we have our assertion. ∎

Corollary 5. *Suppose moreover that* $x_0 \in (\bar{D})^c$, σ_i, $i = 0, \dots, d$, *are smooth around* x_0 *and that* $\dim Lie[\frac{\partial}{\partial t} - V_0, V_1, \dots, V_d](0, x_0) = N+1$. *Here*

$$V_i(x) = \sum_{j=1}^{d} \sigma_i^j(x) \frac{\partial}{\partial x^j}, \qquad i = 1, \dots, d,$$

and

$$V_0(x) = \sum_{j=1}^{d} (\sigma_0^j(x) - \frac{1}{2} \sum_{i=1}^{d} \sum_{k=1}^{N} \sigma_i^k(x) \frac{\partial \sigma_i^j}{\partial x^k}(x)) \frac{\partial}{\partial x^j}.$$

Then

$$P(X(t,x_0) \in D, F(X(t,x_0)) = 0 \text{ for some } t > 0) = 0.$$

Proof. Let U be an open neighborhood of x_0 such that σ_i, $i = 0,\dots,d$, are smooth around $\bar{U}$ and that $\bar{U} \cap \bar{D} = \emptyset$. Let $\tau = \inf\{t > 0; X(t,x_0) \in U^c\}$. Then we see that

$$\begin{aligned}
&P(X(t,x_0) \in D, F(X(t,x_0)) = 0 \text{ for some } t > 0) \\
&\leq \sum_{n=1}^{\infty} P(X(t,x_0) \in D, F(X(t,x_0)) = 0 \text{ for some } t > \frac{1}{n}, \tau > \frac{1}{n}) \\
&\leq \sum_{n=1}^{\infty} \int_U P(X(\frac{1}{n},x_0) \in dx, \tau > \frac{1}{n}) P(X(t,x) \in D, F(X(t,x)) = 0 \\
&\qquad\qquad\qquad\qquad \text{for some } t > 0).
\end{aligned}$$

However, by [3], we see that $P(X(\frac{1}{n},x_0) \in dx, \tau > \frac{1}{n})$ is absolutely continuous. So by Lemma 4, we have our assertion. ∎

§3. Proof of Theorem 1

Since the proof is similar, we prove Theorem 1 in the case that $D = \{x = (x^1,\dots,x^d) \in \mathbf{R}^d;\ x^1 < 0\} \subset \mathbf{R}^d$, and $\Phi(v,x) = (-v^1, v^2,\dots,v^d)$ for $v = (v^1, v^2, \dots, v^d)$ and $x \in \partial D$. In general, if we take a double cover of D^c and change the coordinate functions, we can apply a similar proof. Let $a^{ij} : \mathbf{R}^d \to \mathbf{R}$, $i,j = 1,\dots d$, be bounded Lipschitz continuous function such that $a^{ij}(x) = a^{ji}(x)$, $x \in \mathbf{R}^d$ and that there are positive constants C_0, C_1 such that

$$C_0|\xi|^2 \leq \sum_{i,j}^{d} a^{ij}(x)\xi_i\xi_j \leq C_1|\xi|^2, \qquad x,\xi \in \mathbf{R}^d.$$

Let $b : \mathbf{R}^{2d} \to \mathbf{R}^d$ be a bounded measurable function.

Let L_0 be a second order linear differential operator in $\mathbf{R}^{2d}$ given by

$$L_0 = \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2}{\partial v^i \partial v^j} + \sum_{i=1}^{d} b^i(x,v)\frac{\partial}{\partial v^i}$$

Then Theorem 1 is somehow equivalent to the following Theorem. So we prove this Theorem.

Theorem 6. *Let* $(x_0, v_0) \in (\bar{D})^c \times \mathbf{R}^d$, *and suppose that* a^{ij}, $i, j = 1, \dots, d$, *are smooth around* x_0. *Then there exists a unique probability measure* μ *over* $\tilde{W}^d$ *satisfying the following conditions.*
(1) $\mu(w(0) = (x_0, v_0)) = 1$.
(2) $\mu(w(t) \in D^c \times \mathbf{R}^d, t \in [0, \infty)) = 1$.
(3) *For any* $f \in C_0^\infty((\bar{D})^c \times \mathbf{R}^d)$, $\{f(w(t)) - \int_0^t L_0 f(w(s))ds; t \geq 0\}$ *is a martingale under* $\mu(dw)$.
(4) $\mu(1_{\{0\}}(x^1(t))(v^1(t) + v^1(t-)) = 0,\ t \in [0, \infty)) = 1$ *and*

$$\mu(v^i(t) \text{ is continuous in } t \in [0, \infty),\ i = 2, \dots, d) = 1.$$

Proof. Let $\tilde{a}^{ij} : \mathbf{R}^d \to \mathbf{R}$, $i, j = 1, \dots d$, be given by

$$\tilde{a}^{ij}(x) = a^{ij}(|x^1|, x^2, \dots, x^d), \qquad x = (x^1, x^2, \dots, x^d) \in \mathbf{R}^d.$$

Let $\tilde{b}^i : \mathbf{R}^{2d} \to \mathbf{R}$, $i = 1, \dots d$, be given by

$$\tilde{b}^1(x) = sgn(x^1) b^1(|x^1|, x^2, \dots, x^d),$$

and

$$\tilde{b}^i(x) = b^i(|x^1|, x^2, \dots, x^d),\ i = 2, \dots, d$$

for $x = (x^1, x^2, \dots, x^d) \in \mathbf{R}^d$. Let $\tilde{L}_0$ be second order linear differential operators in $\mathbf{R}^{2d}$ given by

$$\tilde{L}_0 = \sum_{i=1}^d v^i \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^d \tilde{a}^{ij}(x) \frac{\partial^2}{\partial v^i \partial v^j} + \sum_{i=1}^d \tilde{b}^i(x, v) \frac{\partial}{\partial v^i}.$$

Then by transformation of drift (cf. Ikeda-Watanabe[1]), we see that there is a unique probability measure ν on $C([0, \infty); \mathbf{R}^{2d})$ such that $\nu(w(0) = (x_0, v_0)) = 1$ and that $\{f(w(t)) - \int_0^t \tilde{L}_0 f(w(s))ds; t \geq 0\}$ is a martingale under $\nu(dw)$ for any $f \in C_0^\infty(\mathbf{R}^{2d})$.

Let $\tilde{\xi}(w) = \inf\{t > 0;\ x^1(t) = 0, v^1(t-) = 0\}$. Then by Corollary 5 and Girsanov's transformation, we see that $\nu(\tilde{\xi}(w) = \infty) = 1$. Let

$$X(t, w) = (|x^1(t)|, x^2(t), \dots, x^d(t)), \qquad t \in [0, \infty),$$

and

$$V(t, w) = \frac{d^+}{dt} X(t, w), \qquad t \in [0, \infty).$$

Let μ is the probability law of $(X(\cdot, w), V(\cdot, w))$ under ν. Then we see that μ satisfies the conditions (1)-(4). So we see the existence.

Now let us prove the uniqueness. Let μ be a probability measure as in Theorem. Let $\xi(w) = \inf\{t > 0;\ x^1(t) = 0, v^1(t-) = 0\}$. Also, let us define stopping times $\tau_k : \tilde{W}_0 \to [0,\infty]$, $k = 0, 1, 2, \ldots$, inductively by $\tau_0(w) = 0$ and

$$\tau_{k+1}(w) = \inf\{t > \tau_k(w);\ x^1(t) = 0\}, \qquad w \in \tilde{W}^d,\ k = 0, 1, \ldots.$$

Then we see from the assumption (4) that if $\tau_k(w) < \xi(w)$, then $\tau_k(w) < \tau_{k+1}(w)$ for μ-a.s.w. Also, it is easy to see that $\xi(w) \le \sup_k \tau_k(w)$, $w \in \tilde{W}^d$.

For any $\varepsilon > 0$ and $k = 0, 1, 2, \ldots$, let

$$\sigma_k^0(w) = \inf\{t > \tau_k(w);\ x^1(t) > \varepsilon\},$$

and

$$\sigma_k^1(w) = \inf\{t > \sigma_k^0(w);\ x^1(t) < \varepsilon/2\}, \qquad w \in \tilde{W}^d,\ k = 0, 1, \ldots.$$

Then we see from the assumption (3) that

$$f(x(t\wedge\sigma_k^1), v(t\wedge\sigma_k^1)) - f(x(t\wedge\sigma_k^0), v(t\wedge\sigma_k^0)) - \int_{t\wedge\sigma_k^0}^{t\wedge\sigma_k^1} L_0 f(x(s), v(s))ds$$

is a bounded continuous martingale for any $f \in C_0^\infty(\mathbf{R}^{2d})$.

Now let

$$\tilde{X}(t,w) = \begin{cases} x(t), & t \in [\tau_k(w), \tau_{k+1}(w)),\ \text{if } k \text{ is even}, \\ (-x^1(t), x^2(t), \ldots, x^d(t)), & t \in [\tau_k(w), \tau_{k+1}(w)),\ \text{if } k \text{ is odd}, \end{cases}$$

$$\tilde{V}(t,w) = \begin{cases} v(t), & t \in [\tau_k(w), \tau_{k+1}(w)),\ \text{if } k \text{ is even}, \\ (-v^1(t), v^2(t), \ldots, v^d(t)), & t \in [\tau_k(w), \tau_{k+1}(w)),\ \text{if } k \text{ is odd}. \end{cases}$$

Then we can see that $(\tilde{X}(t\wedge\xi), \tilde{V}(t\wedge\xi))$ is continuous in t for μ-a.s.w. Also, we see that

$$f(\tilde{X}(t\wedge\sigma_k^1), \tilde{V}(t\wedge\sigma_k^1)) - f(\tilde{X}(t\wedge\sigma_k^0), \tilde{V}(t\wedge\sigma_k^0)) - \int_{t\wedge\sigma_k^0}^{t\wedge\sigma_k^1} \tilde{L}_0 f(\tilde{X}(s), \tilde{V}(s))ds$$

is a continuous martingale for any $f \in C_0^\infty(\mathbf{R}^{2d})$.

Therefore we see that

$$f(\tilde{X}(t\wedge\tau_{k+1}), \tilde{V}(t\wedge\tau_{k+1})) - f(\tilde{X}(t\wedge\tau_k), \tilde{V}(t\wedge\tau_k)) - \int_{t\wedge\tau_k}^{t\wedge\tau_{k+1}} \tilde{L}_0 f(\tilde{X}(s), \tilde{V}(s))ds$$

is a continuous martingale for any $f \in C_0^\infty(\mathbf{R}^{2d})$. So we can conclude that

$$f(\tilde{X}(t\wedge\xi),\tilde{V}(t\wedge\xi)) - \int_0^{t\wedge\xi} \tilde{L}_0 f(\tilde{X}(s),\tilde{V}(s))ds$$

is a continuous martingale for any $f \in C_0^\infty(\mathbf{R}^{2d})$.

Therefore we see that the probability law of $(\tilde{X}(\cdot\wedge\xi),\tilde{V}(\cdot\wedge\xi))$ under μ is the same of $w(\cdot\wedge\tilde{\xi})$ under ν, by the argument of shift of drift and the fact that a strong solution of stochastic differential equation with Lipschitz continuous coefficients is unique. So we see that $\mu(\xi(w) = \infty) = 1$. Since we see that

$$x(t) = (|\tilde{X}^1(t)|,\tilde{X}^2(t),\ldots,\tilde{X}^d(t)), \qquad t \in [0,\xi),$$

and

$$v(t) = (\frac{d^+}{dt}|\tilde{X}^1(t)|,\tilde{V}^2(t),\ldots,\tilde{V}^d(t)), \qquad t \in [0,\xi),$$

we see the uniqueness.

This completes the proof.

§4. Proof of Theorem 2

We will make some preparations to prove Theorem 2.

Proposition 7. *Let $T > 0$. Let A_0 be the set of $w \in D([0,T);\mathbf{R})$ for which $w(0) = 0$, $w(T-) \leq 1$, and $w(t)$ is non-decreasing in t. Then A_0 is compact in $L^p((0,T),dt)$, $p \in (1,\infty)$, and its cluster points are in $D([0,T);\mathbf{R})$.*

Proof. Suppose that $w_n \in A_0$, $n = 1,2,\ldots$. Then we see that $w_n(t) \in [0,1]$, $t \in [0,T)$, $n \geq 1$. So taking subsequence if necessary, we may assume that $\{w_n(r)\}_{n=1}^\infty$ is convergent for any $r \in [0,T) \cap \mathbf{Q}$. Let $\tilde{w}(r) = \lim_{n\to\infty} w_n(r)$, $r \in \mathbf{Q}$, and let $w(t) = \lim_{r\downarrow t} \tilde{w}(r)$, $t \in [0,T)$, and $w(T)$ be arbitrary such that $\sup_{t\in[0,T)} w(t) \leq w(T) \leq 1$. Then we see that $w \in D([0,T);\mathbf{R})$ and w is non-decreasing, and that $w_n(t) \to w(t)$, $t \in [0,T)$, if t is a continuous point of w. So we see that $w_n \to w$, $n \to \infty$, in $L^p((0,T),dt)$.

This completes the proof. ∎

We have the following as an easy consequence of Proposition 7.

Corollary 8. *Let $T > 0$. Let A be the set of $w \in D([0,T);\mathbf{R}^d)$ for which $w(0) = 0$ and the total variation of w is less than 1. Then A is compact in $L^p((0,T);\mathbf{R}^d,dt)$, $p \in (1,\infty)$, and its cluster points are in $D([0,T);\mathbf{R}^d)$.*

Now let us prove Theorem 2. Let

$$H_t^\lambda = \lambda U(X_t^\lambda) + \frac{1}{2}|V_t^\lambda|^2, \qquad t \geq 0.$$

Then we have

$$H_t^\lambda = \frac{1}{2}|v_0|^2 + \int_0^t V_s^\lambda \cdot \sigma(X_s^\lambda) dB_s + \int_0^t V_s^\lambda \cdot b(X_s^\lambda, V_s^\lambda) ds + \frac{1}{2}\int_0^t trace(\sigma(X_s^\lambda)^* \sigma(X_s^\lambda)) ds.$$

So we see that for any $p \in [2,\infty)$ there is a constant C independent of λ such that

$$E[\sup_{t\in[0,T]} (H_t^\lambda)^p] \leq C(|v_0|^{2p} + 1 + E[\int_0^T |V_t^\lambda|^p dt])$$

$$\leq C(|v_0|^{2p} + 1 + 2^{p/2} T E[\sup_{t\in[0,T]} (H_t^\lambda)^p]^{1/2}).$$

So we see that

(1) $$\sup_{\lambda>0} E[\sup_{t\in[0,T]} (H_t^\lambda)^p] < \infty, \qquad p \in [1,\infty).$$

Therefore we see that

$$\sup_{\lambda>0} E[\sup_{t\in[0,T]} |V_t^\lambda|^p] < \infty, \qquad p \in [1,\infty).$$

So we see that $\{H_t^\lambda\}_{t\in[0,\infty)}$, and $\{X_t^\lambda\}_{t\in[0,\infty)}$, $\lambda \geq 0$, are tight in C. Moreover, we see that

(2) $$E[\sup_{t\in[0,T]} U(X_t^\lambda)^p] \to 0, \quad \lambda \to \infty, \qquad p \in [1,\infty).$$

Let us take an $\varepsilon \in (0, \varepsilon_0)$ such that

$$C_0 = \sup\{|\nabla U_0(x)|^{-1};\ dis(x, \partial D) \leq \varepsilon\} < \infty.$$

Let $\varphi \in C_0^\infty(\mathbf{R}^d)$, such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$, if $dis(x, \partial D) < \varepsilon/3$, and $\varphi(x) = 0$, if $dis(x; \partial D) > \varepsilon/2$. Let $D_0 = \{x \in D;\ dis(x, \partial D) > \varepsilon/4\}$, and let $\tau = \tau^\lambda = \inf\{t > 0;\ X_t^\lambda \in D_0\}$. Then we see by Equation (2) that

$$P(\tau^\lambda < T) \to 0, \quad \lambda \to \infty,$$

for any $T > 0$. Let A_t^λ, $t \geq 0$ be a non-decreasing continuous process given by

$$A_t^\lambda = -\lambda \int_0^{t\wedge\tau^\lambda} \varphi(X_s^\lambda)\rho'(U_0(X_s^\lambda))|\nabla U_0(X_s^\lambda)|^2 ds, \qquad t \geq 0.$$

Note that $A_0^\lambda = 0$. Since we have

$$\begin{aligned}
&\varphi(X_{t\wedge\tau^\lambda}^\lambda)(\nabla U_0(X_{t\wedge\tau^\lambda}^\lambda)\cdot V_{t\wedge\tau^\lambda}^\lambda) - \varphi(X_0^\lambda)(\nabla U_0(X_0^\lambda)\cdot V_0^\lambda)\\
&= A_t^\lambda + \int_0^{t\wedge\tau^\lambda} \varphi(X_s^\lambda)\nabla^2 U_0(X_s^\lambda)(V_s^\lambda, V_s^\lambda)ds\\
&\quad + \int_0^{t\wedge\tau^\lambda} \varphi(X_s^\lambda)(\nabla U_0(X_s^\lambda)\cdot b(X_s^\lambda, V_s^\lambda))ds\\
&\quad + \int_0^{t\wedge\tau^\lambda} \varphi(X_s^\lambda)(\nabla U_0(X_s^\lambda))^*\sigma(X_s^\lambda)dB_s\\
&\quad + \int_0^{t\wedge\tau^\lambda} (\nabla\varphi(X_s^\lambda)\cdot V_s^\lambda)(\nabla U_0(X_s^\lambda)\cdot V_s^\lambda)ds,
\end{aligned}$$

we see that

$$\sup_{\lambda>0} E[(A_T^\lambda)^p] < \infty, \qquad p \in [1,\infty).$$

Since we have

$$\int_0^{T\wedge\tau^\lambda} \lambda U(X_t^\lambda)dt = \int_0^{T\wedge\tau^\lambda} \frac{\rho(U_0(X_t^\lambda))}{|\rho'(U_0(X_t^\lambda))|}|\nabla U_0(X_t^\lambda)|^{-2}dA_t^\lambda,$$

we see that

$$\begin{aligned}
&P(\int_0^{T\wedge\tau^\lambda} \lambda U(X_t^\lambda)dt > \delta)\\
&\leq P(\sup_{t\in[0,T]} U(X_t^\lambda) > \eta) + P(C_0^2 A_T^\lambda \sup_{\rho^{-1}(\eta)\leq s<0} \frac{\rho(s)}{|\rho'(s)|} > \delta)
\end{aligned}$$

for any $\delta, \eta > 0$. So we see that

$$P(\int_0^{T\wedge\tau^\lambda} |H_t^\lambda - \frac{1}{2}|V_t^\lambda|^2|dt > \delta) \to 0, \quad \lambda\to\infty \tag{3}$$

for any $\delta > 0$.

Also, we see that

$$V_{t\wedge\tau^\lambda}^\lambda = v_0 + V_t^{\lambda,0} + V_t^{\lambda,1},$$

where

$$V_t^{\lambda,0} = +\int_0^{t\wedge\tau^\lambda} |\nabla U_0(X_s^\lambda))|^{-2}\nabla U_0(X_s^\lambda)dA_s^\lambda,$$

and

$$V_t^{\lambda,1} = \int_0^{t\wedge\tau^\lambda} \sigma(X_s^\lambda)dB_s + \int_0^{t\wedge\tau} b(X_s^\lambda, V_s^\lambda)ds.$$

So we see that the total variation of $V_t^{\lambda,0}$, $t \in [0,T]$, is dominated by $C_0 A_T^\lambda$. Also, $\{V_t^{\lambda,0}\}_{t\in[0,\infty)}$ is tight in C.

Then by Corollary 8 it is easy to see that $\{V_t^\lambda\}_{t\in[0,T)}$ is tight in $L^p((0,T);\mathbf{R}^d)$ and its limit process is in $D([0,T);\mathbf{R}^d)$ with probability one for any $T > 0$ and $p \in (1,\infty)$.

Let $F \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d;\mathbf{R}^d)$ be given by

$$F(x,v) = \varphi(x)(v - |\nabla U_0(x)|^{-2}(\nabla U_0(x)\cdot v)\nabla U_0(x)), \qquad (x,v) \in \mathbf{R}^d \times \mathbf{R}^d.$$

Then by Itô's lemma it is easy to see that $\{F(X_t^\lambda, V_t^\lambda)\}_{t\in[0,\infty)}$, $\lambda \in (0,\infty)$, is tight in C, and that $\{f(X_t^\lambda, V_t^\lambda) - \int_0^t L_0 f(X_s^\lambda, V_s^\lambda)ds\}$ is a continuous martingale for any $\lambda \in (0,\infty)$ and $f \in C_0^\infty((\bar{D})^c \times \mathbf{R}^d)$.

So we see that there are stochastic processes $\{(X_t, V_t)\}_{t\in[0,\infty)}$ and $\{H_t\}_{t\in[0,\infty)}$ and a subsequence $\{\lambda_n\}_{n=1}^\infty$, $\lambda_n \to \infty$, $n \to \infty$, such that $\{((X_t^{\lambda_n}, V_t^{\lambda_n}), H_t^{\lambda_n})\}_{t\in[0,\infty)}$ converges in law to $\{((X_t, V_t), H_t)\}_{t\in[0,\infty)}$ in $\tilde{W}^d \times C$ with respect the metric function $\tilde{dis} + dis_C$.

Then we see that $\{f(X_t, V_t) - \int_0^t L_0 f(X_s, V_s)ds\}_{t\in[0,\infty)}$ is a continuous martingale for any $f \in C_0^\infty((\bar{D})^c \times \mathbf{R}^d)$, and that $\{F(X_t, V_t)\}_{t\in[0,\infty)}$ is a continuous process. Also, we see by Equation (3) that

$$\int_0^T |H_t - \frac{1}{2}|V_t|^2|dt = 0 \qquad a.s.$$

for any $T > 0$. So we see that $\{|V_t|^2\}_{t\in[0,\infty)}$ is a continuous process. Therefore we have

$$P(1_{\partial D}(X_t)(V_t - V_{t-} - 2(n(X_t)\cdot V_{t-})n(X_t)) = 0,\ t \in [0,\infty)) = 1.$$

So we see that the probability law of $\{(X_t, V_t)\}_{t\in[0,\infty)}$ in $\tilde{W}$ is μ in Theorem 1.

This completes the proof of Theorem 2

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