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Single generation and rank of C*-algebras

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§1. Introduction

We mainly treat a separable C*-algebra A in this article. Let S be a subset of A_{sa}. We call S a generator of A when any C*-subalgebra B of A containing S is equal to A, and we denote $A = C^*(S)$. If S is finite, then we call A finitely generated and we define the number of generators $\mathfrak{gen}(A)$ by the minimum cardinality of S which generates A. We denote $\mathfrak{gen}(A) = \infty$ unless A is finitely generated. We call a C*-algebra A singly generated if $\mathfrak{gen}(A) \leq 2$. Indeed, if $A = C^*(x,y)$ for $x, y \in A_{sa}$, then any C*-subalgebra B of A containing the element $x + \sqrt{-1}y$ is equal to A.

There are many works on single generation of operator algebras. Many of them concern to von Neumann algebras ([2],[6],[17], [19], [20], [24]). Concerning to C*-algebras, there are interesting works of D. Topping([22]), C. L. Olsen and W. R. Zame([15]). With related to them, we introduce the recent work ([11],[12]) of singly generated C*-algebras in the next section and mention the relation between singly generated C*-algebras and their ranks in the last section.

§2. Single generation of C*-algebras

Let S be a subset of a C*-algebra A satisfying $A = C^*(S)$. If A is unital, then $\{s + 2\|s\| \mid s \in S\}$ also generates A. So we may assume that an element of S is invertible. We mention about the fundamental property of $\mathfrak{gen}(\cdot)$ without the proof.

Lemma 1. [12] *Let A and B be C^*-algebras.*

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(1) $\mathfrak{gen}(A) = \mathfrak{gen}(\tilde{A})$, *where* $\tilde{A}$ *is the* C^**-algebraic unitization of* A.
(2) *If* A *and* B *are subalgebras of a* C^**-algebra* C, *then we have*

$$\mathfrak{gen}(C^*(A,B)) \le \mathfrak{gen}(A) + \mathfrak{gen}(B).$$

(3) *If one of* A *and* B *has a unit, then we have*

$$\mathfrak{gen}(A \oplus B) = \max\{\mathfrak{gen}(A), \mathfrak{gen}(B)\}.$$

For a commutative C*-algebra A, we can make clear the meaning of $\mathfrak{gen}(A)$ as follows:

Proposition 2. [12] *Let* A *be a unital commutative* C^**-algebra and* Ω *the spectrum of* A. *Then we have*

$$\mathfrak{gen}(A) = \min\{m \in \mathbb{N} \mid \text{ there is an embedding of } \Omega \text{ into } \mathbb{R}^m\}.$$

Thanks to this statement, we can consider $\mathfrak{gen}(A)$ as a sort of non-commutative topological dimension of a C*-algebra A. So we investigate the relation of $\mathfrak{gen}(A)$ and $\mathfrak{gen}(M_n(A))$, where $M_n(A) \cong M_n(\mathbb{C}) \otimes A$.

Theorem 3. [12] *Let* A *be a unital* C^**-algebra with* $\mathfrak{gen}(A) \le n^2 + 1$ $(n \in \mathbb{N})$. *Then we have* $\mathfrak{gen}(M_n(A)) \le 2$.

Outline of Proof. Let $a_1, a_2, \ldots, a_{(n-1)^2}, b, c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_{n-1}$ be self-adjoint elements of A. We assume that they generate A and satisfy the following condition:

$$b \ge 1 \text{ and } d_1, d_2, \ldots, d_{n-1} \ge \delta \text{ for some } \delta > 0.$$

We define two self-adjoint elements x, y in $M_n(A)$ as follows:

$$x = \begin{pmatrix} a_1 & a_2 + \sqrt{-1}a_3 & \cdots & a_{2n-4} + \sqrt{-1}a_{2n-3} & 0 \\ a_2 - \sqrt{-1}a_3 & a_{2n-2} & \cdots & a_{4n-9} + \sqrt{-1}a_{4n-8} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{2n-4} - \sqrt{-1}a_{2n-3} & a_{4n-9} - \sqrt{-1}a_{4n-8} & \cdots & a_{(n-1)^2} & 0 \\ 0 & 0 & \cdots & 0 & b \end{pmatrix}$$

and

$$y = \begin{pmatrix} c_1 & d_1 & & & \\ d_1 & c_2 & d_2 & & \\ & d_2 & \ddots & \ddots & \\ & & \ddots & \ddots & d_{n-1} \\ & & & d_{n-1} & c_n \end{pmatrix}.$$

If we assume that

$$\varepsilon 1 \leq (x_{ij})_{i,j=1}^{n-1} \leq (1-\varepsilon)1 \quad \text{for some } \varepsilon > 0,$$

then x and y generate A. Q.E.D.

It is proved that $M_n(A)$ is singly generated if $\mathfrak{gen}(A) \leq (n^2+3n)/2$ ([15]), and if $\mathfrak{gen}(A) \leq (n-1)^2$ ([14]). The above result implies the following estimation for unital C*-algebra A:

$$\mathfrak{gen}(M_n(A)) \leq \lceil \frac{\mathfrak{gen}(A)-1}{n^2} + 1 \rceil,$$

where $\lceil \cdot \rceil$ means "the least integer greater than or equal to". We can see that the above estimation is best possible. C. L. Olsen and W. R. Zame [15] prove that $M_2(C([0,1]^n))$ is singly generated if and only if $n \leq 5$.

Theorem 4. [12] *Let n and m be positive integers. Then we have*

$$\mathfrak{gen}(M_m(C[0,1]^n)) = \lceil \frac{n-1}{m^2} + 1 \rceil.$$

Let Ω be an n-dimensional compact manifold. By Whitney's theorem, Ω is embeddable to $\mathbb{R}^{2n}$, so we have

$$\mathfrak{gen}(M_m(C(\Omega))) \leq \lceil \frac{2n-1}{m^2} + 1 \rceil.$$

Now we shall investigate generators for a simple C*-algebra or a C*-algebra which is tensored with a simple C*-algebra.

Theorem 5. [12] *Let A be a simple, infinitely dimensional C*-algebra. Then we have*

$$\mathfrak{gen}(A \otimes_{max} B) \leq \mathfrak{gen}(A) + 1$$

for any unital C-algebra B.*

Outline of Proof. We assume that A is unital and $x_1, x_2, \ldots, x_n \in A_{sa}$ generate A. We choose $\{y_k | k = 1, 2, \ldots\} \subset B_{sa}$ such that $\{y_k | k = 1, 2, \ldots\}$ generates B and $\|y_k\| = 1$. By the infinite dimensionality of A, we can choose a family of positive elements in A satisfying $p_i p_j = 0$ if $i \neq j$. We set

$$s_i = x_i \otimes 1, \; t = \sum_{k=1}^{\infty} \frac{1}{k} p_k \otimes y_k.$$

Then we have $p_k^2 \otimes y_k = k(p_k \otimes 1)t \in C^*(s_1, \dots, s_n, t)$. By the simplicity of A, we have

$$\sum_{i=1}^{m} a_i p_k^2 b_i = 1$$

for suitable elements a_i, b_i in A. This means that $\{s_1, \dots, s_n, t\}$ generates $A \otimes_{max} B$. Q.E.D.

Corollary 6. *Let A be a simple, singly generated, infinitely dimensional C^*-algebra. Then we have*

$$\mathfrak{gen}(A \otimes_{max} B) \leq 3$$

for any unital C^*-algebra B.

In particular, $M_k(\mathbb{C}) \otimes A \otimes_{max} B$ is singly generated for $k \geq 2$.

Examples. (1) Let $\mathbb{K}$ be a C*-algebra of all compact operators on a separable Hilbert space. Then $\mathbb{K}$ is singly generated, and

$$\begin{pmatrix} 1 & & & \\ & 1/2 & & \\ & & 1/3 & \\ & & & \ddots \end{pmatrix} + \sqrt{-1} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1/2 & \\ & 1/2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

is a generator of $\mathbb{K}$.

Every UHF C*-algebra is also singly generated([22]). So we have $A \otimes_{min} \mathbb{K}$ and $A \otimes_{min}(\mathrm{UHF})$ are singly generated for any unital C*-algebra A ([15]) by Corollary 6.

(2) Let A be a unital C*-algebra with a unitary $u \in A$ and $h \in A_{sa}$ satisfying $A = C^*(u, h)$. Then A is singly generated and $u(h + 2\|h\|)$ is a generator of A.

For any compact subspace Ω of $\mathbb{R}$, the C*-crossed product $C(\Omega) \rtimes_\alpha \mathbb{Z}$ is also singly generated.

Let $A_\theta = C^*(u, v)$ be an irrational rotation C*-algebra. Then A_θ is singly generated([10]) and $u(v + v^* + 3)$ is a generator of A_θ.

(3) Every simple AF C*-algebra is singly generated([9]).

(4) The Cuntz algebra $\mathcal{O}_n$ has the property $M_n(\mathcal{O}_n) \cong \mathcal{O}_n$. So we have $A \otimes_{min} \mathcal{O}_n$ is singly generated for any unital C*-algebras. In general, E. Kirchberg ([13],[12]) shows that a C*-algebras A is singly generated if A has two isometries with orthogonal ranges.

(5) By Proposition 2, $C(\mathbb{T} \times \mathbb{T})$ is not singly generated, so the enveloping group C*-algebra $C^*(F_2)$ of the free group F_2 with two generators is not singly generated. By Theorem 3, $M_2(C^*(F_2))$ is singly generated.

§3. Rank of C*-algebras

In this section, we assume that a C*-algebra A has a unit. The notion of real rank is defined by L. G. Brown and G. K. Pedersen [4], and that of stable rank is defined by M. A. Rieffel [18] as follows:

$$\begin{aligned} \mathrm{RR}(A) = \min\{ & n \in \mathbb{N} \cup \{0\} \mid \\ & \{(a_1, a_2, \dots, a_{n+1}) \in (A_{sa})^{n+1} \mid Aa_1 + Aa_2 + \cdots + Aa_n = A\} \\ & \text{is dense in } (A_{sa})^{n+1}\}, \\ & \text{or } \infty, \end{aligned}$$

$$\begin{aligned} \mathrm{sr}(A) = \min\{ & n \in \mathbb{N} \mid \\ & \{(a_1, a_2, \dots, a_n) \in A^n \mid Aa_1 + Aa_2 + \cdots + Aa_n = A\} \\ & \text{is dense in } A^n\}, \\ & \text{or } \infty. \end{aligned}$$

If A is commutative, then $\mathrm{RR}(A)$ is equal to the covering dimension $\dim(\Omega)$ of its spectrum Ω, and

$$\mathrm{sr}(A) = \lceil \frac{\dim(\Omega) + 1}{2} \rceil.$$

In the case that A is not commutative, we have

$$\mathrm{RR}(A) \le 2\mathrm{sr}(A) - 1.$$

E. J. Beggs and D. E. Evans [1] show that the following formula:

$$\mathrm{RR}(M_m(C(\Omega))) = \lceil \frac{\dim(\Omega)}{2m - 1} \rceil.$$

We shall construct an example of C*-algebra whose rank is infinite using free products of C*-algebras([23]).

Theorem 7. [14] *If C^*-algebras A and B have surjective *-homomorphisms to $C[0,1]$, then we have*

$$\mathrm{RR}(A * B) = \infty,$$

where $A*B$ is the enveloping C^*-algebra of the free product of A and B.

Proof. For any $n \in \mathbb{N}$, by Theorem 4, we have

$$\mathfrak{gen}(M_n(C[0,1]^{n^2})) = 2.$$

Let a, b be invertible self-adjoint generators of $M_n(C[0,1]^{n^2})$. There are surjective C*-homomorphisms from A (resp. B) to $C^*(a)$ (resp.$C^*(b)$). This means that there exists a surjective C*-homomorphism from $A*B$ to $M_n(C[0,1]^{n^2})$. By Beggs-Evans' formula, we have

$$\mathrm{RR}(A * B) \geq \lceil \frac{n^2}{2n-1} \rceil$$

for any n, that is, $\mathrm{RR}(A * B) = \infty$. Q.E.D.

Both $C[0,1] * C[0,1]$ and $C^*(F_2)$ have their real rank ∞ (in particular, their stable rank ∞). The former is singly generated and the latter is not as we have shown. M. A. Rieffel [18] show that $\mathrm{sr}(A) = \infty$ when A contains two isometries with orthogonal ranges. But, for unital C*-algebras $A \subset B$, it is not necessarily true that $\mathrm{sr}(A) = \infty$ implies $\mathrm{sr}(B) = \infty$. We give here such an example.

Lemma 8. *Let A be a unital, separable, residually finite C^*-algebra and M a factor of type II_1. Then there exists a unital embedding of A to M.*

Proof. Since A is residually finite, there exists a countable family $\{\pi_n\}_{n=1}^{\infty}$ of finite-dimensional *-representation of A such that $\oplus_{n=1}^{\infty}\pi_n$ is a faithful representation of A. We can choose a family $\{p_n\}_{n=1}^{\infty}$ of orthogonal projections of M such that

$$\sum_{n=1}^{\infty} p_n = 1.$$

For each n, $p_n M p_n$ contains a unital *-subalgebra which isomorphic to $M_{\dim \pi_n}(\mathbb{C})$. Using these isomorphisms, we can construct an embedding of A to $\sum_{n=1}^{\infty} p_n M p_n \subset M$. Q.E.D.

M.-D. Choi [5] prove that $C^*(F_2)$ is residually finite. So $C^*(F_2)$ can be embedded in a factor M of type II_1. Every finite factor M is simple and has $\mathrm{RR}(M) = 0$ and $\mathrm{sr}(M) = 1$. More precisely, using N. C. Phillips' argument [16], we can choose a unital, separable, simple C*-algebra A which contains $C^*(F_2)$ and $\mathrm{RR}(A) = 0$ and $\mathrm{sr}(A) = 1$.

Indeed, there exists a simple, separable C*-algebra A_1 such that $C^*(F_2) \subset A_1 \subset M$ [3]. Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a positive decreasing sequence tending to 0. We can choose a countable sequence $\{a_n\}_{n=1}^{\infty} \subset (A_1)_{sa}$ and $\{b_n\}_{n=1}^{\infty} \subset A_1$ such that $\{a_n\}_{n=1}^{\infty}$ (resp.$\{b_n\}_{n=1}^{\infty}$) is dense in the unit ball of $(A_1)_{sa}$ (resp. A_1). By the fact $\mathrm{RR}(M) = 0$ and $\mathrm{sr}(M) = 1$, we can choose invertible elements $a'_n \in M_{sa}$, $b'_n \in M$ such that

$$\|a'_n\|, \|b'_n\| \leq 1, \ \|a_n - a'_n\| < \epsilon_1, \ \|b_n - b'_n\| < \epsilon_1.$$

We put A_2 the C*-algebra generated by A_1, a'_n and b'_n. Then there exists a simple, separable C*-algebra A_3 such that $A_2 \subset A_3 \subset M$. We also choose a countable sequence $\{a''_n\}_{n=1}^{\infty} \subset (A_3)_{sa}$ and $\{b''_n\}_{n=1}^{\infty} \subset A_3$ such that $\{a''_n\}_{n=1}^{\infty}$ (resp.$\{b''_n\}_{n=1}^{\infty}$) is dense in the unit ball of $(A_3)_{sa}$ (resp. A_3), and invertible elements $a'''_n \in M_{sa}$, $b'''_n \in M$ such that

$$\|a'''_n\|, \|b'''_n\| \leq 1, \ \|a''_n - a'''_n\| < \epsilon_2, \ \|b''_n - b'''_n\| < \epsilon_2.$$

We put A_4 the C*-algebra generated by A_3, a'''_n and b'''_n. Repeating this argument, we can construct

$$C^*(F_2) \subset A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset M.$$

Then the inductive limit C*-algebra $\lim_{n\to\infty} A_n$ is the desired one.

We have no example that a separable C*-algebra A is simple and is not singly generated. Many researcher consider the reduced group C*-algebra $C^*_{red}(F_2)$ as a candidate of such a C*-algebra. K. Dykema, U. Haagerup and M. Rørdam [7] prove that $\mathrm{sr}(C^*_{red}(F_2)) = 1$. Since $C^*_{red}(F_2)$ does not have non-trivial projections, its real rank is one. We do not know whether a separable, simple, C*-algebra of real rank zero is singly generated. This fact is related to the problem of a singly generated factor of type II_1 .

We remark that, if any separable, simple C*-algebra of real rank zero is singly generated, then every factor of type II_1 with the separable predual is singly generated as a von Neumann algebra. Indeed, we choose elements $a_1, a_2, \ldots \in M$ such that $\{a_1, a_2, \ldots\}$ generates M as a von Neumann algebra. For the C*-algebra A generated by $\{a_1, a_2, \ldots\}$, by the above argument, we can choose a separable, simple C*-algebra B of real rank zero such that

$$A \subset B \subset M.$$

By the assumption, there exists an element $x \in B$ such that x generates B. Then x generates M as a von Neumann algebras.

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