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Lifting of Holomorphic Actions on Complex Supermanifolds

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Abstract.

We study the problem of lifting analytic actions of a Lie group G to a non-split complex analytic supermanifold $(M, \mathcal{O})$ from its retract $(M, \mathcal{O}_{\mathrm{gr}})$. In the case when G is compact (or complex reductive), two criteria for lifting a Lie group action are found. The first one is invariance of the Čech 1-cocycle with values in a special automorphism sheaf of $(M, \mathcal{O}_{\mathrm{gr}})$ determining the non-split supermanifold $(M, \mathcal{O})$, while the second one is invariance of a certain differential form of a special kind which can also be viewed as a global derivation of a sheaf of differential forms on M. If the action is transitive on M, then the second criterion allows to reduce the lifting problem to the study of invariants of a finite dimensional linear representation.

Introduction

The paper is devoted to the problem of lifting analytic actions on complex analytic supermanifolds. If a supermanifold $(M, \mathcal{O})$ is split, i.e., is constructed by means of a holomorphic vector bundle $\mathbf{E} \to M$, then an analytic action of a Lie group G on $(M, \mathcal{O})$ preserving the $\mathbb{Z}$-grading in $\mathcal{O}$ is the same as an analytic action of G on $\mathbf{E}$. In the non-split case, any G-action on $(M, \mathcal{O})$ naturally induces a G-action on the corresponding split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ (the retract of $(M, \mathcal{O})$) preserving the $\mathbb{Z}$-grading, but the converse is false. The important question here is to settle, whether a given G-action on $\mathbf{E}$ is induced by an action on $(M, \mathcal{O})$ or, as we say, lifts to an action of $(M, \mathcal{O})$. The infinitesimal version of this question is the question about lifting vector fields

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or actions of Lie algebras (and Lie superalgebras), see [8], [10] for some results in this direction. After discussing the general problem, we find two criteria for lifting a Lie group action in the case when G is compact or complex reductive. The first one is expressed as invariance of a Čech 1-cocycle with values in a special automorphism sheaf $\mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}$ of $(M, \mathcal{O}_{\mathrm{gr}})$ lying in the cohomology class $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ that determines the non-split supermanifold $(M, \mathcal{O})$ (see Theorem 2.1). The second one is invariance of a certain differential form of a special kind which can also be viewed as a global derivation of a sheaf of differential forms on M. This main result is formulated in Theorem 3.1. The proof is based on triviality of the 1-cohomology of a compact Lie group G with values in some non-abelian topological G-groups.

If the given G-action is transitive on M, then we deal with a homogeneous vector bundle $\mathbf{E}$, and the action on $\mathbf{E}$ is determined by the corresponding linear representation of the isotropy subgroup G_x of G in the fibre E_x of $\mathbf{E}$ over a point $x \in M$. Then the second criterion can be expressed in terms of invariants of this representation. In another place we will study this situation and apply the results to the case when M is a flag manifold.

The preliminary study of the lifting problem (see Proposition 3.1) shows that if a lifting of a certain action of an arbitrary group G exists, then ζ is G-invariant. The natural question arises: is the converse true if G is compact or complex reductive? Equivalently, does any G-invariant class from $H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ contain a G-invariant cocycle? The answer seems to be positive.

§1. Preliminaries on supermanifolds

1.1. Split and non-split supermanifolds

In this paper, only complex analytic supermanifolds $(M, \mathcal{O})$ are considered; here M is an ordinary complex manifold, called the *reduction* of M, and $\mathcal{O}$ is the *structure sheaf* of $(M, \mathcal{O})$. The simplest way to get a supermanifold is as follows. Let $(M, \mathcal{F})$ be a complex manifold of dimension n and $\mathcal{E}$ a locally free analytic sheaf of rank m on it. Defining $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{E}$, we get a supermanifold $(M, \mathcal{O})$ of dimension $n|m$. A supermanifold is called *split* if it is isomorphic to a supermanifold of this form. The structure sheaf $\mathcal{O}$ of a split supermanifold admits the $\mathbb{Z}$-grading $\mathcal{O} = \bigoplus_{p \geq 0} \mathcal{O}_p$, where $\mathcal{O}_p \simeq \bigwedge^p_{\mathcal{F}} \mathcal{E}$; the $\mathbb{Z}_2$-grading on it is derived from the $\mathbb{Z}$-grading by reducing $\bmod 2$. In what follows, we often omit the subscript $\mathcal{F}$ while denoting the exterior powers, the tensor products etc. of the sheaves of $\mathcal{F}$-modules. Instead of the locally free sheaf $\mathcal{E}$, one may consider the corresponding holomorphic vector bundle $\mathbf{E} \to M$.

On the other hand, there is a construction associating with any supermanifold $(M, \mathcal{O})$ a split one. Let $\mathcal{J} \subset \mathcal{O}$ be the subsheaf of ideals generated by odd elements. Consider the filtration

$$\mathcal{O} = \mathcal{J}^0 \supset \mathcal{J}^1 \supset \mathcal{J}^2 \supset \cdots \tag{1}$$

of $\mathcal{O}$. The associated graded sheaf

$$\operatorname{gr} \mathcal{O} = \bigoplus_{p \geq 0} \operatorname{gr}_p \mathcal{O},$$

where $\operatorname{gr}_p \mathcal{O} = \mathcal{J}^p / \mathcal{J}^{p+1}$, gives rise to the split supermanifold $(M, \operatorname{gr} \mathcal{O})$. In fact, $\operatorname{gr} \mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{E}$, where $\mathcal{F} = \operatorname{gr}_0 \mathcal{O}$ and $\mathcal{E} = \operatorname{gr}_1 \mathcal{O}$. Clearly, $(M, \mathcal{O})$ and $(M, \operatorname{gr} \mathcal{O})$ have the same dimension. We call $(M, \operatorname{gr} \mathcal{O})$ the *retract* of $(M, \mathcal{O})$.

Let $\pi_p \colon \mathcal{J}^p \to \operatorname{gr}_p \mathcal{O}$ denote the natural projection. Then we have the exact sequences of sheaves

$$0 \to \mathcal{J}^{p+1} \to \mathcal{J}^p \xrightarrow{\pi_p} \operatorname{gr}_p \mathcal{O} \to 0. \tag{2}$$

The supermanifold $(M, \mathcal{O})$ is split if and only if there exists an isomorphism of superalgebra sheaves $h \colon \operatorname{gr} \mathcal{O} \to \mathcal{O}$ such that its restriction $h_p \colon \operatorname{gr}_p \mathcal{O} \to \mathcal{J}^p$ splits (2), i.e., satisfies $\pi_p \circ h_p = \mathrm{id}$. In the general case, such an isomorphism exists over a neighbourhood of any point of M.

A classical example of a split supermanifold is (M, Ω), where Ω is the sheaf of holomorphic forms on M; its dimension is $n|n$. The corresponding vector bundle is the cotangent bundle $\mathbf{T}(M)^*$.

The first example of a non-split supermanifold was published in [2]; this is the quadric in the projective superplane $\mathbb{CP}^{2|2}$. In [5] four series of supermanifolds of flags were constructed, corresponding to four series of classical linear Lie superalgebras. These supermanifolds are mostly non-split.

1.2. The tangent sheaf

Let $(M, \mathcal{O})$ be an arbitrary supermanifold. Denote by $\mathcal{T} = \mathcal{D}er\, \mathcal{O}$ the sheaf of derivations of the structure sheaf $\mathcal{O}$. Its stalk at $x \in M$ is the Lie superalgebra $\mathfrak{der}\, \mathcal{O}_x = \mathfrak{der}_{\bar{0}}\, \mathcal{O}_x \oplus \mathfrak{der}_{\bar{1}}\, \mathcal{O}_x$ of derivations of the superalgebra $\mathcal{O}_x$ (the summands with indices $\bar{0}$ and $\bar{1}$ consist of even and odd derivations respectively). The sheaf $\mathcal{T}$ is called the *tangent sheaf* of M. The tangent sheaf is in a natural way a sheaf of $\mathbb{Z}_2$-graded left

$\mathcal{O}$-modules. On the other hand, it can be regarded as a sheaf of complex Lie superalgebras with the graded Lie bracket

$$[u, v] = uv + (-1)^{p(u)p(v)+1} vu. \tag{3}$$

Sections of $\mathcal{T}$ (*holomorphic vector fields* on $(M, \mathcal{O})$) form the Lie superalgebra $\mathfrak{v}(M, \mathcal{O}) = \Gamma(M, \mathcal{T})$; it is finite-dimensional whenever M is compact.

We endow the tangent sheaf $\mathcal{T}$ with the following filtration:

$$\begin{gathered} \mathcal{T} = \mathcal{T}_{(-1)} \supset \mathcal{T}_{(0)} \supset \cdots \supset \mathcal{T}_{(m)} \supset \mathcal{T}_{(m+1)} = 0, \\ \mathcal{T}_{(p)} = \{\delta \in \mathcal{T} \mid \delta(\mathcal{O}) \subset \mathcal{J}^p,\ \delta(\mathcal{J}) \subset \mathcal{J}^{p+1}\} \quad \text{for } p \geq 0. \end{gathered} \tag{4}$$

Thus we obtain a filtered sheaf of Lie superalgebras. One sees that the associated graded sheaf of Lie superalgebras $\operatorname{gr} \mathcal{T}$ is naturally isomorphic to $\mathcal{T}_{\mathrm{gr}} = \mathcal{D}er(\operatorname{gr} \mathcal{T})$. In fact, any $v \in \mathcal{T}_{(p)}$ maps $\mathcal{J}^q$ into $\mathcal{J}^{q+p}$ inducing a derivation from $(\mathcal{T}_{\mathrm{gr}})_p$. As a result, we get a homomorphism $\sigma_p \colon \mathcal{T}_{(p)} \to (\mathcal{T}_{\mathrm{gr}})_p$. One checks easily that $\operatorname{Ker} \sigma_p = \mathcal{T}_{(p+1)}$, and hence we have the exact sequence of sheaves

$$0 \to \mathcal{T}_{(p+1)} \to \mathcal{T}_{(p)} \xrightarrow{\sigma_p} (\mathcal{T}_{\mathrm{gr}})_p \to 0. \tag{5}$$

Now we make some remarks concerning vector fields on the split supermanifolds. If $(M, \mathcal{O})$ is split, then $\mathcal{T}$ is a $\mathbb{Z}$-graded sheaf of Lie superalgebras, the grading being given by

$$\mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p,$$

where

$$\mathcal{T}_p = (\mathcal{D}er\, \mathcal{O})_p = \{v \in \mathcal{T} \mid v(\mathcal{O}_q) \subset \mathcal{O}_{q+p} \text{ for all } q \in \mathbb{Z}\}.$$

One sees easily that $\mathcal{T}_0$ is the sheaf of infinitesimal automorphisms of the corresponding holomorphic vector bundle $\mathbf{E}$. The filtration (4) has the form

$$\mathcal{T}_{(p)} = \bigoplus_{q \geq p} \mathcal{T}_q.$$

The $\mathbb{Z}$-grading implies that $\mathfrak{v}(M, \mathcal{O}) = \bigoplus_{p \geq -1} \mathfrak{v}(M, \mathcal{O})_p$ is a $\mathbb{Z}$-graded Lie superalgebra. Moreover, we get a $\mathbb{Z}$-grading in any cohomology group $H^p(M, \mathcal{T})$ turning $H^*(M, \mathcal{T})$ into a bigraded Lie superalgebra.

In the split case, $\mathcal{T}$ can be regarded as a locally free analytic sheaf on the complex manifold M (see [10]).

1.3. Sheaves of automorphisms

Let $(M, \mathcal{O})$ be a supermanifold. Let us denote by $\mathrm{Aut}(M, \mathcal{O})$ the group of all automorphisms of the $\mathbb{Z}_2$-graded ringed space $(M, \mathcal{O})$. By definition, $F \in \mathrm{Aut}(M, \mathcal{O})$ is a pair (f, φ), where $f\colon M \to M$ belongs to the group $\mathrm{Bih}\, M$ of all biholomorphic transformations of M and φ is an isomorphism of superalgebra sheaves $\mathcal{O} \to \mathcal{O}$ over f. On the other hand, denote by $\mathcal{A}ut\, \mathcal{O}$ the sheaf of automorphisms of the sheaf $\mathcal{O}$ (mapping each stalk $\mathcal{O}_x$, $x \in M$, onto itself). This is a sheaf of groups. Its sections form the normal subgroup $\mathrm{Aut}\, \mathcal{O}$ of $\mathrm{Aut}(M, \mathcal{O})$ consisting of automorphisms that are identical on M. Moreover, for any $F = (f, \varphi) \in \mathrm{Aut}(M, \mathcal{O})$, the mapping $\mathrm{Int}\, F\colon a \mapsto \varphi \circ a \circ \varphi^{-1}$ is an automorphism of the sheaf of groups $\mathcal{A}ut\, \mathcal{O}$ which gives an action Int of the group $\mathrm{Aut}(M, \mathcal{O})$ on $\mathcal{A}ut\, \mathcal{O}$ by automorphisms of this sheaf.

Clearly, any $a \in \mathcal{A}ut\, \mathcal{O}$ maps $\mathcal{J}$ onto itself and hence preserves the filtration (1) and induces a germ of an automorphism of $\mathrm{gr}\, \mathcal{O}$. By definition, a induces the identity mapping on $\mathcal{F} = \mathcal{O}/\mathcal{J}$. Consider

$$(6)\qquad \mathcal{A}ut_{(2p)}\, \mathcal{O} = \{a \in \mathcal{A}ut\, \mathcal{O} \mid a(f) - f \in \mathcal{J}^{2p} \text{ for any } f \in \mathcal{O}\}.$$

Then we get the following finite filtration:

$$(7)\qquad \mathcal{A}ut\, \mathcal{O} = \mathcal{A}ut_{(0)}\, \mathcal{O} \supset \mathcal{A}ut_{(2)}\, \mathcal{O} \supset \cdots .$$

Any automorphism preserves the filtration (1), and hence any $\mathcal{A}ut_{(2p)}\, \mathcal{O}$ is a subsheaf of normal subgroups. It also is invariant under the action Int of $\mathrm{Aut}(M, \mathcal{O})$ defined above.

To study the sheaves $\mathcal{A}ut_{(2p)}\, \mathcal{O}$, one can use the linearization method proposed in [15] (see also [8], [13]). As in the classical Lie theory, there exists a natural relationship between automorphisms and even derivations of the sheaf $\mathcal{O}$. In particular, the filtration (4) gives rise to a filtration of $\mathcal{T}_{\bar{0}}$ corresponding to (7); it is given by

$$(8)\qquad \mathcal{T}_{\bar{0}} \supset \mathcal{T}_{\bar{0}(2)} \supset \mathcal{T}_{\bar{0}(4)} \supset \cdots ,$$

where

$$\mathcal{T}_{\bar{0}(2p)} = \mathcal{T}_{\bar{0}} \cap \mathcal{T}_{(2p)}.$$

Now, we have the exponential mapping

$$\exp\colon \mathcal{T}_{\bar{0}(2)} \to \mathcal{A}ut_{(2)}\, \mathcal{O}.$$

It is expressed by the usual exponential series which is actually a polynomial, since any $v \in \mathcal{T}_{\bar{0}(2)}$ satisfies $v^k = 0$ for ally $k > [m/2]$. One proves that exp is an isomorphism of sheaves of sets (but in general not of groups). We denote $\log = \exp^{-1}$. Consider the mappings

$$\lambda_{2p}\colon \mathcal{A}ut_{(2p)}\,\mathcal{O} \xrightarrow{\log} \mathcal{T}_{\bar{0}(2p)} \xrightarrow{\sigma_p} (\mathcal{T}_{\mathrm{gr}})_{2p},\ p \geq 1. \tag{9}$$

One verifies that these are homomorphisms of sheaves of groups. Clearly, $\operatorname{Ker}\lambda_{2p} = \mathcal{A}ut_{(2p+2)}\,\mathcal{O}$, and hence for any $p \geq 1$ we have the exact sequence of sheaves of groups

$$e \to \mathcal{A}ut_{(2p+2)}\,\mathcal{O} \to \mathcal{A}ut_{(2p)}\,\mathcal{O} \xrightarrow{\lambda_{2p}} (\mathcal{T}_{\mathrm{gr}})_{2p} \to 0. \tag{10}$$

If $(M,\mathcal{O})$ is split, then $\lambda_{2p}(a)$ is the $2p$-component of $\log a \in \mathcal{T}_{2p} = (\mathcal{T}_{gr})_{2p}$.

§2. Cochain complexes and classification theorems

2.1. The Čech complex

We remind the well-known definition of the 1-cohomology set $H^1(M,\mathcal{F})$, where $\mathcal{F}$ is a sheaf of groups on M. For an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of M, denote by $C^p(\mathfrak{U},\mathcal{F})$, $p \geq 0$, the group of $\mathcal{F}$-valued p-cochains on $\mathfrak{U}$. By definition, $c \in C^p(\mathfrak{U},\mathcal{F})$ is a family $c = (c_{i_0\cdots i_p})$, where $c_{i_0\cdots i_p} \in \Gamma(U_{i_0}\cap\cdots\cap U_{i_p},\mathcal{F})$. One defines the coboundary operators $\partial_p\colon C^p(\mathfrak{U},\mathcal{F}) \to C^{p+1}(\mathfrak{U},\mathcal{F})$, $p = 0,\,1$, by

$$\begin{aligned}(\partial_0(a))_{ij} &= a_i a_j^{-1},\\ (\partial_1(b))_{ijk} &= b_{ij}b_{jk}b_{ik}^{-1}.\end{aligned} \tag{11}$$

One also defines the action ρ of the group $C^0(\mathfrak{U},\mathcal{F})$ on $C^1(\mathfrak{U},\mathcal{F})$ by

$$(\rho(a)(b))_{ij} = a_i b_{ij} a_j^{-1}.$$

Then $\partial_0(a) = \rho(a)(e)$, where $e_{ij} = e$, i.e., e is the unit of the group $C^1(\mathfrak{U},\mathcal{F})$.

Now, we have the sets of cocycles $Z^p(\mathfrak{U},\mathcal{F}) = \operatorname{Ker}\partial_p$, the cohomology group $H^0(\mathfrak{U},\mathcal{F}) = Z^0(\mathfrak{U},\mathcal{F})$ that coincides with the group of global sections $H^0(M,\mathcal{F}) = \Gamma(M,\mathcal{F})$, and the cohomology set $H^1(\mathfrak{U},\mathcal{F}) = Z^1(\mathfrak{U},\mathcal{F})/\rho$. Passing to the limit over all open covers $\mathfrak{U}$ of M, we get the *1-cohomology set of M* with values in $\mathcal{F}$ denoted by $H^1(M,\mathcal{F})$.

If $\mathcal{F}$ is a sheaf of not necessarily abelian groups, then its 1-cohomology set possesses, in general, no natural group structure, but has the distinguished element ε determined by the unit 1-cocycles e.

Suppose now that an action of a group G on M is given. Then we may consider open *G-covers*, i.e., open covers $\mathfrak{U} = (U_i)_{i\in I}$ of M such that G acts on I and $gU_i = U_{gi}$ for any $g \in G$, $i \in I$. Suppose that $\mathcal{F}$

is a *G-sheaf*, i.e., that G acts on $\mathcal{F}$ by sheaf automorphisms inducing the given action on M. Then we get the natural action of G by on any group $C^p(\mathfrak{U}, \mathcal{F})$, $p \geq 0$, by automorphisms, given by

$$(gc)_{i_0 \cdots i_p} = gc_{g^{-1}i_0 \cdots g^{-1}i_p}, \; g \in G.$$

Clearly, this action commutes with ∂_p, and we have $g\rho(a)(b) = \rho(ga)(gb)$ for any 0-cochain a, 1-cochain b and $g \in G$. This gives rise to natural actions of G on $H^0(M, \mathcal{F})$ (by group automorphisms) and on $H^1(\mathfrak{U}, \mathcal{F})$ and $H^1(M, \mathcal{F})$ (leaving the unit element invariant).

As an example, consider the sheaves of automorphisms of a supermanifold. As we saw in Subsection 1.3, the sheaves $\mathcal{A}ut\, \mathcal{O}$ and $\mathcal{A}ut_{(2p)}\, \mathcal{O}$ are G-sheaves under the action Int of the group $G = \mathrm{Aut}(M, \mathcal{O})$. Note also that G acts on the sheaf $\mathcal{T}$ by the automorphisms of Lie superalgebra sheaf $u \mapsto \mathrm{Int}\, F(u) = \varphi \circ u \circ \varphi^{-1}$, where $F = (f, \varphi) \in G$.

Let $\mathbf{E}$ be a holomorphic vector bundle over a complex manifold M, and $\mathcal{E}$ the sheaf of holomorphic sections of $\mathbf{E}$. Then we have the split supermanifold $(M, \mathcal{O})$, where $\mathcal{O} = \bigwedge \mathcal{E}$. Clearly, any automorphism of the vector bundle $\mathbf{E}$ gives rise to an automorphism of $(M, \mathcal{O})$, and thus we may identify $\mathrm{Aut}\, \mathbf{E}$ with a subgroup of $\mathrm{Aut}(M, \mathcal{O})$ (this is just the subgroup preserving the $\mathbb{Z}$-grading of $\mathcal{O}$). It follows that $\mathcal{A}ut\, \mathcal{O}$ and $\mathcal{A}ut_{(2p)}\, \mathcal{O}$ may be considered as $(\mathrm{Aut}\, \mathbf{E})$-sheaves. Further, $\mathrm{Aut}\, \mathbf{E}$ preserves the $\mathbb{Z}$-grading of $\mathcal{T}$, and therefore $\mathcal{T}_p$ are $(\mathrm{Aut}\, \mathbf{E})$-sheaves.

Let again $(M, \mathcal{O})$ be an arbitrary complex supermanifold and let $(M, \mathcal{O}_{\mathrm{gr}})$ denote its retract. Clearly, any $F = (f, \varphi) \in \mathrm{Aut}(M, \mathcal{O})$ leaves the ideal sheaves $\mathcal{J}^p \subset \mathcal{O}$ invariant and hence induces an automorphism $\bar{F} = (f, \bar{\varphi}) \in \mathrm{Aut}(M, \mathcal{O}_{\mathrm{gr}})$ preserving the $\mathbb{Z}$-grading of $\mathcal{O}_{\mathrm{gr}}$. Clearly, $\bar{\varphi}$ is uniquely determined by the relation $\pi_p \circ \varphi = \bar{\varphi} \circ \pi_p$ on $\mathcal{J}^p$. We say that the automorphism $\bar{F}$ *lifts to* F. The correspondence $F \mapsto \bar{F}$ is a homomorphism $\mathrm{Aut}(M, \mathcal{O}) \to \mathrm{Aut}(M, \mathcal{O}_{\mathrm{gr}})$ which is in general neither injective nor surjective. It follows that any $(\mathrm{Aut}\, \mathbf{E})$-sheaf, where $\mathbf{E}$ is the vector bundle corresponding to $(\mathcal{O}_{\mathrm{gr}})_1$, may be regarded as a $\mathrm{Aut}(M, \mathcal{O})$-sheaf. In particular, (10) is a sequence of $\mathrm{Aut}(M, \mathcal{O})$-sheaves (one sees that (9) is $\mathrm{Aut}(M, \mathcal{O})$-equivariant).

The set $H^1(M, \mathcal{A}ut_{(2)}\, \mathcal{O})$ can be used for describing the family of all supermanifolds having a given split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ as their retract. The following statement is proved in [2].

Theorem 2.1. *Let $(M, \mathcal{O}_{\mathrm{gr}})$ be the split supermanifold determined by a holomorphic vector bundle $\mathbf{E}$. Then to any supermanifold $(M, \mathcal{O})$ having $(M, \mathcal{O}_{\mathrm{gr}})$ as its retract there corresponds an element of the set $H^1(M, \mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}})$. This correspondence gives rise to a bijection between the isomorphy classes of supermanifolds satisfying the above condition* *and the orbits of the group* $\operatorname{Aut}\mathbf{E}$ *on* $H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ *under the action* Int. *The given split supermanifold* $(M, \mathcal{O}_{\mathrm{gr}})$ *corresponds to the unit element* $e \in H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$.

Let us describe the correspondence mentioned in Theorem 2.1. Let $(M, \mathcal{O})$ be a supermanifold such that $\operatorname{gr}\mathcal{O} = \mathcal{O}_{\mathrm{gr}}$. We can choose an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of M such that there exist isomorphisms $h_i\colon \mathcal{O}_{\mathrm{gr}}|U_i \to \mathcal{O}|U_i$, $i \in I$, satisfying $\pi_p \circ (h_i)_p = \mathrm{id}$ on $(\mathcal{O}_{\mathrm{gr}})_p|U_i$. Setting $z_{ij} = h_i^{-1} \circ h_j$, we obtain a 1-cocycle $z = (z_{ij}) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$. Its cohomology class $\zeta \in H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ does not depend of the choice of h_i; this is the desired class.

2.2. A resolution of the automorphism sheaf

We retain the notation of the preceeding sections. Following [12], [14], we are now going to express the cohomology set $H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ in terms of a non-linear complex similar to the non-linear de Rham and Dolbeault complexes studied, e.g., in [3, 7, 14]. Actually, a general complex of this sort was considered in [6], but it was used there only in the finite-dimensional situation. This complex will be constructed as the set of global sections of a "resolution" of the sheaf $\mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}$ having trivial 1-cohomology.

First we construct a fine resolution of the sheaf $\mathcal{T}_{\mathrm{gr}} = \mathcal{D}er\,\mathcal{O}_{\mathrm{gr}}$ endowed with a bracket operation that extends the operation (3) given in $\mathcal{T}_{\mathrm{gr}}$. Let us denote by $\Phi^{p,q}$ the sheaf of smooth complex-valued forms of type (p,q) on M. We form the standard Dolbeault-Serre resolution $\hat{\Phi}$ of $\mathcal{O}$, setting

$$\hat{\Phi}^{p,q} = \Phi^{0,q} \otimes (\mathcal{O}_{\mathrm{gr}})_p,$$
$$\hat{\Phi} = \bigoplus_{p,q\geq 0} \hat{\Phi}^{p,q},$$
$$\bar{\partial}(\varphi \otimes u) = (\bar{\partial}\varphi) \otimes u, \quad \varphi \in \Phi^{0,q},\ u \in \mathcal{O}_p.$$

Then, regarding $\hat{\Phi}$ as a sheaf of graded algebras with respect to the total degree, consider the sheaf of bigraded Lie superalgebras $\hat{\mathcal{T}} = \mathcal{D}er\,\hat{\Phi}$. Clearly, $\bar{D} = \operatorname{ad}\bar{\partial}$ is a derivation of bidegree $(0,1)$ of $\hat{\mathcal{T}}$ satisfying $\bar{D}^2 = 0$. Set

$$\mathcal{S} = \{u \in \hat{\mathcal{T}} \mid u(\bar{f}) = u(d\bar{f}) = 0 \quad \text{for any } f \in \mathcal{F}\}.$$

One sees readily that $\mathcal{S}$ is a subsheaf of bigraded subalgebras of $\hat{\mathcal{T}}$ that is invariant under $\bar{D}$. Moreover, $\mathcal{T}_{\mathrm{gr}}$ is identified with the kernel of the mapping $\bar{D}\colon \mathcal{S}_{*,0} \to \mathcal{S}_{*,1}$. Thus, we get the sequence

$$0 \to \mathcal{T}_{\mathrm{gr}} \xrightarrow{i} \mathcal{S}_{*,0} \xrightarrow{\bar{D}} \mathcal{S}_{*,1} \xrightarrow{\bar{D}} \cdots .$$

By [10], this is a fine resolution of $\mathcal{T}_{\mathrm{gr}}$ isomorphic to the standard Dolbeault-Serre resolution of the locally free analytic sheaf $\mathcal{T}_{\mathrm{gr}}$. Moreover, i is a homomorphism of graded Lie algebra sheaves, and hence the natural bracket in $\mathcal{S}$ may be used to calculate the bracket in $H^*(M, \mathcal{T}_{\mathrm{gr}})$ induced by the Lie bracket defined in $\mathcal{T}_{\mathrm{gr}}$.

We also need some sheaves associated with the smooth supermanifold corresponding to $(M, \mathcal{O}_{\mathrm{gr}})$. Denote by $\mathcal{F}^\infty$ the sheaf of differentiable complex-valued functions on M and by $\mathcal{E}^\infty$ the sheaf of differentiable sections of $\mathbf{E}$. Then the sheaf of algebras

$$\mathcal{O}_{\mathrm{gr}}^\infty = \bigwedge_{\mathcal{F}^\infty} \mathcal{E}^\infty$$

is the sheaf of differentiable sections of $\bigwedge \mathbf{E}$. We have

$$\mathcal{O}_{\mathrm{gr}}^\infty = \mathcal{F}^\infty \otimes \mathcal{O}_{\mathrm{gr}}.$$

Let $\mathcal{T}^\infty = \mathcal{D}er\, \mathcal{O}_{\mathrm{gr}}^\infty$ denote the tangent sheaf of $(M, \mathcal{O}_{\mathrm{gr}}^\infty)$. Consider the sheaf

$$\mathcal{P}\mathcal{A}ut\, \mathcal{O}_{\mathrm{gr}}^\infty = \{a \in \mathcal{A}ut\, \mathcal{O}_{\mathrm{gr}}^\infty \mid a(\bar{f}) = \bar{f} \text{ for all } f \in \mathcal{F}\}$$

and its subsheaf

$$\mathcal{P}\mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}}^\infty = \left\{ a \in \mathcal{A}ut\, \mathcal{O}_{\mathrm{gr}}^\infty \;\middle|\; a(u) - u \in \bigoplus_{p \geq 2} (\mathcal{O}_{\mathrm{gr}}^\infty)^p,\ u \in \mathcal{O}^\infty \right\}.$$

The sheaf of groups $\mathcal{P}\mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}}^\infty$ acts on $\mathcal{S}$ by the automorphisms $\operatorname{Int} a(u) = aua^{-1}$.

Now we can construct the resolution of $\mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}}$ mentioned above. It has the form

$$e \to \mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}} \to \mathcal{P}\mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}}^\infty \xrightarrow{\delta_0} \bigoplus_{k \geq 2} \mathcal{S}_{2k,1} \xrightarrow{\delta_1} \bigoplus_{k \geq 2} \mathcal{S}_{2k,2}, \tag{12}$$

where δ_0 and δ_1 are given by

$$\begin{aligned} \delta_0(a) &= \bar{\partial} - a\bar{\partial}a^{-1}, \\ \delta_1(u) &= \bar{D}u - \frac{1}{2}[u,u] = -\frac{1}{2}[u - \bar{\partial}, u - \bar{\partial}], \end{aligned} \tag{13}$$

The corresponding cochain complex is the triple $K = (K^0, K^1, K^2)$, where

$$K^0 = \Gamma(M, \mathcal{P}\mathcal{A}ut_{(2)}\, \mathcal{O}_{\mathrm{gr}}^\infty), \quad K^p = \bigoplus_{k \geq 2} \Gamma(M, \mathcal{S}_{2k,p}), \quad p = 1,\ 2,$$

together with the mappings $\delta_0\colon K^0 \to K^1$, $\delta_1\colon K^1 \to K^2$ and the action ρ of K^0 on K^1 defined by

$$\rho(a)(u) = a(u - \bar{\partial})a^{-1} + \bar{\partial}.$$

One verifies that K is a non-abelian cochain complex in the sense of [7, 11]. We define the group $H^0(K) = Z^0(K) = \operatorname{Ker}\delta_0$, the set $Z^1(K) = \operatorname{Ker}\delta_1$ of 1-cocycles of K and the 1-cohomology set

$$H^1(K) = Z^1(K)/\rho$$

with the distinguished point 0. Using the machinery of non-abelian complexes, we get the following result (see [12], [14]).

Theorem 2.2. *We have an isomorphism of sets with distinguished points*

$$\mu\colon H^1(K) \to H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}).$$

The mapping μ can be expressed quite explicitly. Take $\alpha \in Z^1(K)$. There exists an open cover $\mathfrak{U} = (U_i)_{i\in I}$ of M such that $\alpha = \delta_0(a_i)$, where $a_i \in \Gamma(U_i, \mathcal{P}\mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}})$ for any i. Define $z \in Z^1(\mathfrak{U}, \mathcal{P}\mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}})$ by $z_{ij} = a_i^{-1}a_j$. One sees that z_{ij} leaves invariant the subsheaf $\mathcal{O}_{\mathrm{gr}}|U_i \cap U_j$, and hence we may regard z as a cocycle from $Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$. Then μ maps the cohomology class of α in the complex K onto the Čech cohomology class of z.

We note that the group $\operatorname{Aut}\mathbf{E}$ acts in a natural way on the resolution (12) and on the complex (K^0, K^1, K^2). In fact, first we get the natural action $F \mapsto F_*$ of $\operatorname{Aut}\mathbf{E}$ on the bigraded sheaf $\hat{\Phi}$ such that $F_*\circ\bar{\partial} = \bar{\partial}\circ F_*$. This action induces the action $F \mapsto \operatorname{Int}F_*$ of $\operatorname{Aut}\mathbf{E}$ on $\hat{\mathcal{T}}$ given by $\operatorname{Int}F_*(u) = F_*\circ u\circ F_*^{-1}$. Clearly, this latter action preserves the bracket in $\hat{\mathcal{T}}$ and commutes with $\bar{D}$. Moreover, the subsheaf $\mathcal{S}$ is invariant under $\operatorname{Int}F_*$, $F \in \operatorname{Aut}\mathbf{E}$. Consider the corresponding action $F \mapsto \operatorname{Int}F_*$ of $\operatorname{Aut}\mathbf{E}$ on K^p, $p = 1, 2$, and the action $F \mapsto \operatorname{Int}F$ of this group on $K^0 = \Gamma(M, \mathcal{P}\mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}})$ by inner automorphisms. One sees readily that $\rho(\operatorname{Int}F(a))(\operatorname{Int}F_*)(u) = (\operatorname{Int}F_*)(\rho(a)(u))$, $F \in \operatorname{Aut}\mathbf{E}$. Hence our action leaves $Z^1(K)$ invariant and induces an action of $\operatorname{Aut}\mathbf{E}$ on $H^1(K)$. One verifies that the isomorphism μ of Theorem 2.2 is equivariant if you consider the action Int of $\operatorname{Aut}\mathbf{E}$ on $H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$.

2.3. Non-abelian cohomology of groups

We remind the definition of the non-abelian 1-cohomology of a group (see [16], [7]), considering the topological version of this theory. Let G and A be two topological groups. Suppose that A is a G-group, i.e., let a homomorphism $h\colon G \to \operatorname{Aut} A$ be given, such that the corresponding action of G on A is continuous. We will write $ga = h(g)(a)$ for $g \in G$, $a \in A$. The group of A-valued p-cochains $C^p(G, A)$ of G is defined as the group of all continuous mappings $G^p \to A$, $p \geq 1$; we also set $C^0(G, A) = A$. Define the coboundary operators $\delta_p\colon C^p(G, A) \to C^{p+1}(G, A)$, $p = 0, 1$, by

$$
\begin{aligned}
&(\delta_0(a))(g) = a(ga)^{-1}, \ g \in G,\\
&(\delta_1(c))(g_1, g_2) = c(g_1)^{-1}c(g_1g_2)(g_1c(g_2)^{-1}), \ g_1, \ g_2 \in G.
\end{aligned} \tag{14}
$$

Also, define the action ρ of the group $C^0(G, A) = A$ on $C^1(G, A)$ by

$$(\rho(a)(c))(g) = ac(g)(ga)^{-1}, \ g \in G.$$

Clearly, $\delta_0(a) = \rho(a)(e)$. One denotes $Z^p(G, A) = \operatorname{Ker} \delta_p$, $p = 0, 1$, and

$$
\begin{aligned}
H^0(G, A) &= Z^0(G, A) = A^G,\\
H^1(G, A) &= Z^1(G, A)/\rho.
\end{aligned}
$$

The 1-cocycles are the so-called *crossed homomorphisms* of G to A, i.e., the (continuous) mappings $c\colon A \to G$ satisfying

$$c(g_1g_2) = c(g_1)(g_1c(g_2)).$$

Note that the coboundary operators ∂_1, δ_1 of the non-abelian complexes considered in Subsections 2.1 and 2.2 (see (11) and (13)) are crossed homomorphisms, if one defines an action σ of the group of 0-cochains on that of 1-cochains in an appropriate way. Namely, for the Čech complex, we set

$$(\sigma(a)(b))_{ij} = a_jb_{ij}a_j^{-1},$$

while $\sigma(a) = \operatorname{Int} a$ for the complex K. For any crossed homomorphism $c\colon A \to G$, the subset $\operatorname{Ker} c = c^{-1}(e)$ is a subgroup of G, and we have $f(a) = f(b)$ if and only if $b = an$, where $n \in \operatorname{Ker} c$.

In the next section, we will use the following simple lemma exploiting the 1-cohomology of groups.

Lemma 2.1. *Let G be a topological group, A and B two topological G-groups, and let a continuous crossed homomorphism $f\colon A \to B$ satisfying $f(ga) = gf(a)$, $g \in G$, $a \in A$, be given. Denote $N = \operatorname{Ker} f$. Then $f(A^G) = f(A) \cap B^G$, whenever $H^1(G, N) = \{\varepsilon\}$.*

Proof. Clearly, N is a G-invariant subgroup of A and hence a G-group. Suppose that $H^1(G, N) = \{\varepsilon\}$. Since $f(A^G) \subset f(A) \cap B^G$, we only have to prove the converse inclusion. Let us fix $b_0 \in f(A) \cap B^G$ and

$a_0 \in A$ such that $f(a_0) = b_0$ and consider $c(g) = a_0^{-1}(ga_0)$. The relation $f(ga_0) = gf(a_0) = f(a_0)$ implies $c(g) \in N$. It follows that $c \in Z^1(G, N)$. Therefore $c = \delta_0(n)$ for a certain $n \in N$. Thus $c(g) = n(gn)^{-1}$, whence $g(a_0n) = a_0n$. Setting $a = a_0n \in A^G$, we get $f(a) = b_0$. □

§3. Actions of Lie groups on supermanifolds

3.1. Actions and the lifting problem

Let $(M, \mathcal{O})$ be a supermanifold. An *action* of a group G on $(M, \mathcal{O})$ is, by definition, a homomorphism $\Phi\colon G \to \mathrm{Aut}(M, \mathcal{O})$. If G is a real or complex Lie group, then we suppose that the action is real (respectively complex) analytic in a natural way. For any $g \in G$, we have $\Phi(g) = (f(g), \varphi(g))$, where $f\colon g \mapsto f(g) \in \mathrm{Bih}\, M$ is an (analytic) action of G on the complex manifold M and $\varphi(g)$ is an automorphism of $\mathcal{O}$ over $f(g)$.

Let $\mathbf{E}$ be a holomorphic vector bundle over a complex manifold M and G a (real or complex) Lie group. Suppose that $\mathbf{E}$ has a structure of the G-bundle, i.e., a homomorphism $\Phi\colon G \to \mathrm{Aut}\,\mathbf{E}$ satisfying the natural analyticity conditions is given. Using the natural inclusion $\mathrm{Aut}\,\mathbf{E} \subset \mathrm{Aut}(M, \mathcal{O})$ (see Subsection 2.1), we may consider Φ as an action on the split supermanifold $(M, \mathcal{O})$ corresponding to $\mathbf{E}$. This action is *$\mathbb{Z}$-graded*, i.e., all $\varphi(g)$, $g \in G$, preserve the $\mathbb{Z}$-grading of the structure sheaf. Conversely, any $\mathbb{Z}$-graded action of G on $(M, \mathcal{O})$ extends an action on the vector bundle $\mathbf{E}$. We will usually consider the $\mathbb{Z}$-graded actions whenever the supermanifold is split.

Let again $(M, \mathcal{O})$ be an arbitrary complex supermanifold and let $(M, \mathcal{O}_{\mathrm{gr}})$ denote its retract and $\mathbf{E}$ the corresponding vector bundle. As we saw in Subsection 2.1, there exists a homomorphism $\mathrm{Aut}(M, \mathcal{O}) \to \mathrm{Aut}(M, \mathbf{E})$ attaching to any $F = (f, \varphi) \in \mathrm{Aut}(M, \mathcal{O})$ the automorphism $\bar{F} = (f, \bar{\varphi}) \in \mathrm{Aut}(M, \mathcal{O}_{\mathrm{gr}})$ preserving the $\mathbb{Z}$-grading of $\mathcal{O}_{\mathrm{gr}}$. Here $\bar{\varphi}$ is uniquely determined by the relation $\pi_p \circ \varphi = \bar{\varphi} \circ \pi_p$ on $\mathcal{J}^p$. It follows that any action $\Phi\colon G \to \mathrm{Aut}(M, \mathcal{O})$ induces a $\mathbb{Z}$-graded action $\bar{\Phi}\colon G \to \mathrm{Aut}(M, \mathcal{O}_{\mathrm{gr}})$, where $\bar{\Phi}(g) = \overline{\Phi(g)}$, $g \in G$. In this situation, we also say that the action $\bar{\Phi}$ *lifts to* Φ. An important problem is to describe those $\mathbb{Z}$-graded actions on $(M, \mathcal{O}_{\mathrm{gr}})$ which lift to $(M, \mathcal{O})$.

Let us also formulate the corresponding infinitesimal problem (see also [8], [10]). Namely, any even vector field $v \in \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$ preserves all $\mathcal{J}^p$ and hence induces a vector field $\bar{v} \in \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_0$ uniquely determined by the relation $\sigma_p \circ v = \bar{v} \circ \sigma_p$ on $\mathcal{J}^p$. We say that $\bar{v}$ *lifts to* v. The correspondence $v \mapsto \bar{v}$ is a Lie algebra homomorphism $\mathfrak{v}(M, \mathcal{O})_{\bar{0}} \to \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_0$ which is in general neither injective nor surjective. It follows that any Lie algebra homomorphism $\alpha\colon \mathfrak{g} \to \mathfrak{v}(M, \mathcal{O})_{\bar{0}}$ induces a Lie algebra homomorphism $\bar{\alpha}\colon \mathfrak{g} \to \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_0$, and we say that $\bar{\alpha}$ *lifts* *to* α. Thus, the lifting problem for Lie algebra homomorphisms arises which is naturally related to that for Lie group actions. In fact, to any analytic action $\Phi\colon G \underset{\imath}{\rightarrow} \mathrm{Aut}(M,\mathcal{O})$ there corresponds a homomorphism $d\Phi\colon \mathfrak{g} \to \mathfrak{v}(M,\mathcal{O})_{\bar{0}}$, where $\mathfrak{g}$ is the Lie algebra of G, and one easily verifies that $\overline{d\Phi} = d\bar{\Phi}$.

Suppose now that a $\mathbb{Z}$-graded action $g \mapsto \Psi(g) = (f(g), \psi(g))$ of a group G on a split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ is given. Let us fix an open G-cover $\mathfrak{U} = (U_i)_{i\in I}$ on M and isomorphisms $h_i\colon \mathcal{O}_{\mathrm{gr}}|U_i \to \mathcal{O}|U_i$, $i \in I$, satisfying $\pi_p \circ (h_i)_p = \mathrm{id}$. Using the action Int of Aut $\mathbf{E}$ on the sheaf $\mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}$, we will consider it as a G-sheaf.

Proposition 3.1. *Let $(M,\mathcal{O})$ be the supermanifold with retract $(M,\mathcal{O}_{\mathrm{gr}})$ corresponding to a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$. The action Ψ lifts to an action on $(M,\mathcal{O})$ if and only if there exists a cocycle $a \in Z^1(G, C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}))$ analytically dependent on $g \in G$ and such that*

$$gz = \rho(a(g)^{-1})(z), \;\; g \in G. \tag{15}$$

In particular, in this situation $g\zeta = \zeta$, $g \in G$.

If the action Ψ satisfies $gz = z$, $g \in G$, then it lifts to an action on $(M,\mathcal{O})$.

Proof. Let $\Phi(g) = (f(g), \varphi(g))$, $g \in G$, be a lift of Ψ. Then

$$\pi_p \circ \varphi(g) = \psi(g) \circ \pi_p \tag{16}$$

on $\mathcal{J}^p$. It follows that

$$\pi_p \circ \varphi(g) \circ h_i = \psi(g) \text{ on } \mathcal{O}^p_{\mathrm{gr}}|U_i, \;\; i \in I. \tag{17}$$

Define the sheaf isomorphisms $\bar{\varphi}_i(g)\colon \mathcal{O}_{\mathrm{gr}}|U_i \to \mathcal{O}_{\mathrm{gr}}|U_{gi}$ over $f\colon U_i \to U_{gi}$ by

$$\bar{\varphi}_i(g) = h_{gi}^{-1} \circ \varphi(g) \circ h_i, \;\; g \in G, \;\; i \in I. \tag{18}$$

Then, clearly,

$$z_{ij} = \bar{\varphi}_i(g)^{-1} \circ z_{gi,gj} \circ \bar{\varphi}_j(g), \;\; g \in G, \tag{19}$$

over any $U_i \cap U_j \neq \emptyset$. Now, (17) and (18) imply that $\bar{\varphi}_i(g)(\xi) - \psi(g)(\xi) \in \mathcal{J}^{p+2}_{\mathrm{gr}}$ for any $\xi \in \mathcal{O}^p_{\mathrm{gr}}|U_i$, where $\mathcal{J}_{\mathrm{gr}}$ is the ideal subsheaf of $\mathcal{O}_{\mathrm{gr}}$ generated by odd elements. Therefore, we may write $\bar{\varphi}_i(g)$ in the form

$$\bar{\varphi}_i(g) = a(g)_{gi} \circ \psi(g), \tag{20}$$

where $a(g) \in C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$. The automorphisms $a(g)_i$ depend analytically on $g \in G$, since $\psi(g)$ and $\varphi_i(g)$ do. Clearly, (18) also implies the following relations:

$$(21) \qquad \bar{\varphi}_i(gg') = \bar{\varphi}_{g'i}(g) \circ \bar{\varphi}_i(g'), \ g, \ g' \in G, \ i \in I.$$

Since $\psi(gg') = \psi(g) \circ \psi(g')$, it follows from (21) that

$$a(gg')_i = a(g)_i \circ \psi(g) \circ a(g')_i \circ \psi(g)^{-1}, \ i \in I,$$

which means that $a \in Z^1(G, C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}))$, where the G-group structure on $C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ is defined by $\mathrm{Int}\,\Psi$.

Moreover, (20) yields that over $U_{gi} \cap U_{gj}$ we have

$$(22) \qquad \psi(g) \circ z_{ij} \circ \psi(g)^{-1} = a(g)_{gi}^{-1} \circ z_{gi,gj} \circ a(g)_{gj}.$$

Clearly, this is the same as

$$(gz)_{gi,gj} = a(g)_{gi}^{-1} \circ z_{gi,gj} \circ a(g)_{gj},$$

and thus (15) is proved.

Conversely, suppose that there exists a cocycle $a \in Z^1(G, C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}))$ analytically dependent on $g \in G$ and satisfying (15). Then we may define sheaf isomorphisms $\bar{\varphi}_i(g)\colon \mathcal{O}_{\mathrm{gr}}|U_i \to \mathcal{O}_{\mathrm{gr}}|U_{gi}$ over $f\colon U_i \to U_{gi}$, $i \in I$, by the formula (20); these isomorphisms depend on $g \in G$ analytically. Now, define the isomorphisms $\varphi_i(g)\colon \mathcal{O}|U_i \to \mathcal{O}|U_{gi}$ over $f\colon U_i \to U_{gi}$ by

$$\varphi_i(g) = h_{gi} \circ \bar{\varphi}(g) \circ h_i^{-1}, \ g \in G, \ i \in I.$$

Clearly, (15) implies the coincidence $\varphi_i(g) = \varphi_j(g)$ on $U_i \cap U_j$. Setting $\varphi(g) = \varphi_i(g)$ over U_i, we get the automorphisms $\Phi(g) = (f(g), \varphi(g))$, $g \in G$, of $(M, \mathcal{O})$. Moreover, (16) is satisfied. In fact, we should prove that

$$(23) \qquad \pi_p(\varphi_i(g)(\xi)) = \psi(g)(\pi_p(\xi))$$

for all $\xi \in \mathcal{J}^p|U_i$. But we have $\mathcal{J}^p|U_i = h_i(\mathcal{O}_{\mathrm{gr}}^p|U_i) \oplus \mathcal{J}^{p+1}|U_i$. If $\xi \in \mathcal{J}^{p+1}$, then both sides of (23) vanish. If $\xi = h_i(\eta)$, where $\eta \in \mathcal{O}_{\mathrm{gr}}^p$, then $\psi(g)(\pi_p(\xi)) = \psi(g)(\eta)$. On the other hand, $\pi_p(\varphi_i(g)(\xi)) = (\pi_p \circ h_{gi})\big(a(g)_{gi}(\psi(g)(\eta))\big) = (\pi_p \circ h_{gi})(\psi(g)(\eta) + \eta_1) = \psi(g)(\eta)$, where $\eta_1 \in \mathcal{J}_{\mathrm{gr}}^{p+2}$. Thus, (23) is proved, and $\Phi(g)$ is a lift of $\Psi(g)$ for any $g \in G$. To prove that Φ is an action, we note that (21) is satisfied, since $a(g)$ are crossed homomorphisms. This readily implies that $\varphi(gg') = \varphi(g) \circ \varphi(g')$, $g, \ g' \in G$.

Suppose now that the cocycle z is invariant under Ψ. Then we may take $a(g) = e$ for all $g \in G$, and (15) will be satisfied. Therefore Ψ lifts to an action on $(M, \mathcal{O})$. $\square$

3.2. The 1-cohomology of compact Lie groups

Let G be a compact topological group. This is a classical fact that the continuous cohomology groups $H^p(G, E)$ vanish for all $p > 0$ for a wide class of continuous G-modules E (see [4], Ch. 3, Corollary 2.1, where this is proved for any quasi-complete G-module). We will prove a vanishing theorem for the non-abelian 1-cohomology of a compact Lie group arising from its action on a complex supermanifold.

Let $(M, \mathcal{O})$ be a complex supermanifold such that M has countable topology. Then the group $\operatorname{Aut}(M, \mathcal{O})$ can be endowed with a natural Hausdorff countable topology turning it into a topological group. The convergence of sequences in this topology can be described as the compact convergence of sequences of usual holomorphic functions entering into the expressions of automorphisms in local charts. We omit the details, referring to Ch. 2 of [1], where the similar theory for non necessarily reduced complex spaces is developed. For any $p \geq 1$, the group $\operatorname{Aut}_{(2p)} \mathcal{O} = \Gamma(M, \mathcal{A}ut_{(2p)} \mathcal{O})$ is a closed normal subgroup of $\operatorname{Aut}(M, \mathcal{O})$. Moreover, for any analytic action Φ of a Lie group G on $(M, \mathcal{O})$, the correspondence $g \mapsto \operatorname{Int} \Phi(g)$ gives natural structures of the G-group on $\operatorname{Aut}(M, \mathcal{O})$ and $\operatorname{Aut}_{(2p)} \mathcal{O}$. On the other hand, $\mathfrak{v}(M, \mathcal{O})$ is a Fréchet space endowed with a natural structure of the continuous G-module.

Proposition 3.2. *Let an analytic action of a compact Lie group G on a supermanifold $(M, \mathcal{O})$ be given. Then $H^1(G, \operatorname{Aut}_{(2p)} \mathcal{O}) = \{\varepsilon\}$ for all $p \geq 1$.*

Proof. For any $p \geq 1$, the exact sequence of sheaves (10) gives rise to the exact sequence of G-groups

$$e \to \operatorname{Aut}_{(2p+2)} \mathcal{O} \to \operatorname{Aut}_{(2p)} \mathcal{O} \stackrel{\lambda_{2p}}{\to} \tilde{\mathfrak{v}}(M, \mathcal{O}_{\mathrm{gr}})_{2p} \to 0,$$

where $\tilde{\mathfrak{v}}(M, \mathcal{O}_{\mathrm{gr}})_{2p} = \operatorname{Im} \lambda_{2p} \subset \mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_{2p}$ is actually the subspace of vector fields that lift to $(M, \mathcal{O})$ (in the split case it coincides with $\mathfrak{v}(M, \mathcal{O}_{\mathrm{gr}})_{2p}$). This sequence gives rise to the exact sequence of sets with distinguished points (see [16], Proposition 38):

$$\begin{aligned} H^1(G, \operatorname{Aut}_{(2p+2)} \mathcal{O}_{\mathrm{gr}}) &\longrightarrow H^1(G, \operatorname{Aut}_{(2p)} \mathcal{O}_{\mathrm{gr}}) \\ &\xrightarrow{\lambda_{2p}^*} H^1(G, \tilde{\mathfrak{v}}(M, \mathcal{O}_{\mathrm{gr}})_{2p}). \end{aligned} \tag{24}$$

Note that $H^1(G, \tilde{\mathfrak{v}}(M, \mathcal{O}_{\mathrm{gr}})_{2p}) = \{0\}$ for any $p \geq 1$ by the classical result cited above. Therefore, (24) allows to prove the triviality of

$H^1(G, \mathrm{Aut}_{(2p)} \mathcal{O}_{\mathrm{gr}})$ by induction on $-p$. Namely, $\mathrm{Aut}_{(2q)} \mathcal{O}_{\mathrm{gr}}) = \{e\}$ for sufficiently large q, and if $H^1(G, \mathrm{Aut}_{(2p+2)} \mathcal{O}_{\mathrm{gr}}) = \{\varepsilon\}$, then the exactness of (24) implies that $H^1(G, \mathrm{Aut}_{(2p)} \mathcal{O}_{\mathrm{gr}}) = \{\varepsilon\}$. □

A similar argument applies in some other situations. First, the groups of sections can be replaced by the groups of 0-cochains of an open G-cover.

Proposition 3.3. *Under the assumptions of Proposition* 3.2, *choose an open G-cover $\mathfrak{U} = (U_i)_{i\in I}$ and consider the topological group $C^0(\mathfrak{U}, \mathrm{Aut}_{(2p)} \mathcal{O})$ endowed with a natural G-group structure. Then $H^1(G, C^0(\mathfrak{U}, \mathcal{A}ut_{(2p)} \mathcal{O})) = \{\varepsilon\}$ for all $p \geq 1$.*

Further, we may consider the smooth supermanifold $(M, \mathcal{O}^\infty_{\mathrm{gr}})$ corresponding to a split complex analytic supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$. Using the notation of Subsection 2.2, define the groups $\mathrm{PAut}_{(2p)} O^\infty_{\mathrm{gr}} = \Gamma(M, \mathcal{P}\mathcal{A}ut_{(2p)} O^\infty_{\mathrm{gr}})$, $p \geq 1$. They can be regarded as topological groups, using the topology determined by the compact convergence of functions and their derivatives of arbitrary orders. For any $p \geq 1$, we have the following exact sequence of sheaves of groups similar to (10):

$$e \to \mathcal{P}\mathcal{A}ut_{(2p+2)} \mathcal{O}^\infty_{\mathrm{gr}} \to \mathcal{P}\mathcal{A}ut_{(2p)} \mathcal{O}^\infty_{\mathrm{gr}} \overset{\lambda_{2p}}{\to} \mathcal{P}\mathcal{T}_{2p} \to 0.$$

Here we denote by $\mathcal{P}\mathcal{T}$ the subsheaf of $\mathcal{T}^\infty$ formed by derivations v of $\mathcal{O}^\infty_{\mathrm{gr}}$ satisfying $v(\bar{f}) = 0$ for all $f \in \mathcal{F}$. Applying the same argument as above, we come to the following result.

Proposition 3.4. *Let a $\mathbb{Z}$-graded action of a compact Lie group G on a split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ be given. Then $H^1(G, \mathrm{PAut}_{(2p)} \mathcal{O}^\infty_{\mathrm{gr}}) = \{\varepsilon\}$ for all $p \geq 1$. If we choose an open G-cover $\mathfrak{U} = (U_i)_{i\in I}$ of M, then $H^1(G, C^0(\mathfrak{U}, \mathcal{P}\mathcal{A}ut_{(2p)} \mathcal{O}^\infty_{\mathrm{gr}})) = \{\varepsilon\}$ for all $p \geq 1$.*

3.3. Lifting and invariant cocycles

Our aim is to prove the following result.

Theorem 3.1. *Let G be a compact or a complex reductive Lie group, and suppose an analytic action Ψ of G on a split supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ be given. Let $(M, \mathcal{O})$ be the supermanifold corresponding to a given class $\zeta \in H^1(M, \mathcal{A}ut_{(2)} \mathcal{O}_{\mathrm{gr}})$ by Theorem* 2.1. *Then the following conditions are equivalent:*

(i) *the action Ψ lifts to $(M, \mathcal{O})$;*
(ii) *the class ζ contains a G-invariant cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)} \mathcal{O}_{\mathrm{gr}})$, where $\mathfrak{U}$ is an open G-cover of M.*

If G is compact, then these conditions are equivalent to the following one:

(iii) *the class* $\mu^{-1}(\zeta) \in H^1(K)$ *(see Theorem 2.2) contains a* G*-invariant cocycle.*

Proof. First we consider the case, when G is compact. Suppose that Ψ lifts to $(M, \mathcal{O})$. Choose an open G-cover $\mathfrak{U}$ of M, such that there exists a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$ representing ζ. Due to Proposition 3.1, there exists $a \in Z^1(G, C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}}))$ such that (15) holds. Applying Proposition 3.3, we see that $a(g) = a_0(ga_0)^{-1}$, $g \in G$, for a certain $a_0 \in C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$. Then we get $gz = \rho((ga_0)a_0^{-1})(z)$, $g \in G$, whence $z_0 = \rho(a_0^{-1})(z) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})^G$, and (ii) is satisfied. Conversely, (ii) implies (i) due to Proposition 3.1.

Now suppose that ζ is represented by a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})^G$ for an open G-cover $\mathfrak{U}$. Consider the crossed homomorphism of G-groups $\partial_0 \colon C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}}) \to C^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}})$. Then $\operatorname{Ker}\partial_0 = \operatorname{Aut}_{(2)}\mathcal{O}^{\infty}_{\mathrm{gr}}$. Further, $\operatorname{Im}\partial = Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}})$. In fact, it is proved in [12], [14] that $H^1(M, \mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}}) = \{\varepsilon\}$, but the proof is actually valid for the Čech cohomology of any open cover $\mathfrak{U}$, since only the exact sequences (10) and triviality of the cohomology of the fine sheaf $\mathcal{PT}_{2p}$ are used. Applying Proposition 3.4 and Lemma 2.1, we see that there exists a cochain $c \in C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}})^G$ such that $\partial_0(c) = z$. Then $\delta_0(c_i) = \delta_0(c_j)$ in $U_i \cap U_j$, which determines the cocycle $\alpha = \delta_0(c_i) \in Z^1(K)^G$, whose cohomology class corresponds to ζ.

Conversely, suppose that $\mu^{-1}(\zeta)$ contains a cocycle $\alpha \in Z^1(K)^G$. Then there exists an open G-cover $\mathfrak{U}$ of M such that $\alpha = \delta_0(c_i)$ for a certain $c \in C^0(\mathfrak{U}, K^0)$. Consider the crossed homomorphism of G-groups $\delta_0 \colon C^0(\mathfrak{U}, K^0) = C^0(\mathfrak{U}, \mathcal{PA}ut_{(2)}\,\mathcal{O}^{\infty}_{\mathrm{gr}}) \to C^0(\mathfrak{U}, K^1)$ given by (13). Clearly, $\operatorname{Ker}\delta_0 = C^0(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})$. Applying Proposition 3.3 and Lemma 2.1, we see that the cochain c can be replaced by a G-invariant one. But then $z = \partial_0(c) \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})^G$ will be a G-invariant cocycle representing ζ.

If G is complex reductive, then $G = K(\mathbb{C})$ is the complexification of its maximal compact subgroup K. If (i) for G is satisfied, then, as we have proved above, ζ is represented by a cocycle $z \in Z^1(\mathfrak{U}, \mathcal{A}ut_{(2)}\,\mathcal{O}_{\mathrm{gr}})^K$, where $\mathfrak{U}$ may be supposed to be a G-cover. Consider the action $d\Psi$ of the Lie algebra $\mathfrak{g}$ of G (see Subsection 3.1). Then the relations

$$z_{gi,gj} \circ \psi(g) = \psi(g) \circ z_{i,j}, \ g \in K, \tag{25}$$

imply

$$z_{i,j} \circ d\psi(u) = d\psi(u) \circ z_{i,j}, \ u \in \mathfrak{k}, \tag{26}$$

where $\mathfrak{k} \subset \mathfrak{g}$ is the subalgebra corresponding to K. Since $\mathfrak{g} = \mathfrak{k}(\mathbb{C})$, (26) is true for all $u \in \mathfrak{g}$, whence one deduces that (25) is true for all g

from the identity component G° of G. But $G = G^\circ K$, and hence (25) is true for all $g \in G$. Thus, (ii) is proved for G. The converse follows from Proposition 3.1. $\square$

References

[1] D. N. Akhiezer, Lie Group Actions in Complex Analysis, Vieweg & Sohn, Braunschweig/Wiesbaden, 1995.

[2] P. Green, *On holomorphic graded manifolds*, Proc. Amer. Math. Soc., **85** (1982), 587–590.

[3] W. M. Goldman and J. J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Publ. Math. IHES, **67** (1988), 43–96.

[4] A. Guichardet, Cohomologie des groupes topologiques et des algèbres de Lie, Cedic/Fernand Nathan, Paris, 1980.

[5] Yu. I. Manin, Gauge Field Theory and Complex Geometry, Springer-Verlag, Berlin e.a. 1988.

[6] A. Nijenhuis and R. W. Richardson, Jr., *Cohomology and deformations in graded Lie algebras*, Bull. Amer. Math. Soc., **72** (1966), 1–29.

[7] A. L. Onishchik, *Some concepts and applications of non-abelian cohomology theory*, Trudy Mosk. Mat. Obshch., **17** (1967), 45–88, (in Russian), English transl. in Transact. Moscow Math. Soc., **17** (1967), Amer. Math. Soc. 1969, 49–98.

[8] A. L. Onishchik, Non-split supermanifolds associated with the cotangent bundle, Université de Poitiers, Département de Math., No. 109, Poitiers 1997.

[9] A. L. Onishchik, *About derivations and vector-valued differential forms*, J. Math. Sci., **90** (1988), 2274–2286.

[10] A. L. Onishchik, *A construction of non-split supermanifolds*, Ann. Global Analysis and Geometry, **16** (1998), 309–333.

[11] A. L. Onishchik, *On non-abelian cochain complexes*, Voprosy Teorii Grupp i Gomologicheskoi Algebry, Yaroslavl State University, Yaroslavl, 1998, 171–197.

[12] A. L. Onishchik, Non-abelian cohomology and supermanifolds, SFB 288, Preprint No. 360, Berlin, 1998.

[13] A. L. Onishchik, *A moduli problem related to complex supermanifolds*, Algebra and Operator Theory. Proc. of the Colloquium in Tashkent, 1997, Kluwer Ac. Publ. Dordrecht e.a. 1998, 13–24.

[14] A. L. Onishchik, *On the classification of complex analytic supermanifolds*, Lobachevskii J. Math., **4** (1999), 47–70.

[15] M. J. Rothstein, *Deformations of complex supermanifolds*, Proc. Amer. Math. Soc., **95** (1985), 255–260.

[16] J.-P. Serre, Cohomologie Galoisienne, Springer-Verlag, Berlin e.a. 1964.

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