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Associativity Breaks Down in Deformation Quantization

Hideki Omori

§1. Introduction

The Weyl algebra $W_\hbar$ is the associative algebra generated over $\mathbb{C}$ by u, v with the fundamental relation $u * v - v * u = -\hbar i$ where $\hbar$ is a positive constant. (u, v) is called a *canonical conjugate pair*. This is one of the simplest algebra which appears in the theory of deformation quantization [BFLS].

In such a noncommutative algebra, the *ordering problem* may be viewed as the problem of expressing elements of the algebra in a unique way. In the Weyl algebra, three kind of orderings; normal ordering, anti-normal ordering, and Weyl ordering, are mainly used. The normal ordering expression is the way of writing elements in the form $\sum a_{m,n} u^m * v^n$ by arranging u to the left hand side in each term. The anti-normal ordering is in the form $\sum a_{m,n} v^m * u^n$. The Weyl ordering is in the form $\sum a_{m,n} u^m \odot v^n$ by using the symmetric product $\odot$ defined by $u \odot v = \frac{1}{2}(u * v + v * u)$ etc. (See [OMY] for the detail of symmetric product.)

Through such an ordering, one can linearly identify the algebra with the space of all polynomials.

In other words, the Weyl algebra can be viewed, through each ordering mentioned above, as a non commutative associative product structure defined on the space $\mathbb{C}[u, v]$ of all polynomials with the ordinary commutative product. Product formulas are given respectively as follows: (We denote the ordinary commutative product by $\circ$, $\bullet$, $\cdot$ in order to distinguish what ordering expression is used.)

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- In the normal ordering expression: the product $*$ of the Weyl algebra is given by the ΨDO-*product formula* as follows:

$$f(u,v) * g(u,v) = f \exp\{\hbar i(\overleftarrow{\partial_v} \circ \overrightarrow{\partial_u})\}g. \tag{1.1}$$

- In the anti-normal ordering expression: the product $*$ of the Weyl algebra is given by the $\overline{\Psi}$DO-*product formula* as follows:

$$f(u,v) * g(u,v) = f \exp\{-\hbar i(\overleftarrow{\partial_u} \bullet \overrightarrow{\partial_v})\}g. \tag{1.2}$$

- In the Weyl ordering expression: the product $*$ of the Weyl algebra is given by the *Moyal product formula* as follows:

$$f(u,v) * g(u,v) = f \exp \frac{\hbar i}{2}\{\overleftarrow{\partial_v} \dot{\wedge} \overrightarrow{\partial_u}\}g \tag{1.3}$$

where $\overleftarrow{\partial_v} \dot{\wedge} \overrightarrow{\partial_u} = \overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}$. Every product formula yields $u*v - v*u = -\hbar i$, and hence defines the Weyl algebra. Here, commutative products $\circ$, $\bullet$, $\cdot$ play only a supplementary role to express elements in the unique way. We distinguish these to indicate what ordering expression is used.

Remark that we can change generators. For every $A \in SL(2,\mathbb{C})$, let

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix}, \quad A \in SL(2,\mathbb{C}).$$

Then, it is obvious that $[u', v']_* = -\hbar i$, and hence u', v' may be viewed as generators. The replacement (pull-back) A^* of u, v by u', v' gives an algebra isomorphism of $W_\hbar$. Thus, we may consider the ordering problem by using u', v' instead of u, v.

Moreover, using a suitable canonical conjugate pair u, v, we can extend the algebra by using one of the above product formulas.

Let $Hol(\mathbb{C}^2)$ be the space of all entire functions on $\mathbb{C}^2$ with the compact open topology. In the case that the parameter $\hbar$ is treated as a formal parameter, which has been the usual attitude in the theory of deformation quantization (cf. [O,el.2]), the product $*$ extends associatively in any ordering expression to the space $Hol(\mathbb{C}^2)[[\hbar]]$ of all formal power series of $\hbar$ with coefficients in $Hol(\mathbb{C}^2)$. This is because product formulas mentioned above are bidifferential operators of total order $2k$ at the level of the coefficients of $\hbar^k$. (See [Om], §13 for more general treatment.)

However, it is obvious that $\hbar$ should be a positive parameter in a true quantum theory.

In this paper, we treat $\hbar$ is a positive parameter. Since all product formulas are given by concrete forms, these extend to the following:

- $f * g$ is defined if one of f, g is a polynomial.
- For every polynomial $p = p(u, v)$, the left-(resp. right-) multiplication $p*$ (resp. $*p$) is a continuous linear mapping of $Hol(\mathbb{C}^2)$ into itself under the compact open topology.

We call such a system a $(\mathbb{C}[u, v]; *)$-*bimodule.*

Proposition 1. *In every product formula mentioned above,* $(Hol(\mathbb{C}^2), \mathbb{C}[u, v], *)$ *is a* $(\mathbb{C}[u, v]; *)$-*bimodule.*

By the polynomial approximation theorem, the associativity $f * (g * h) = (f * g) * h$ holds if two of f, g, h are polynomials. We refer this as 2-*p*-*associativity.*

On the other hand, it is easy to see that the set of all quadratic forms in $W_\hbar$ is closed under the commutator bracket $[\ ,\ ]_*$, hence it forms a Lie algebra. $X = \frac{1}{\hbar}u^2$, $Y = \frac{1}{\hbar}v^2$, $H = \frac{i}{\hbar}uv$, where $uv = u * v + \frac{\hbar i}{2}$, form a basis of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$: We see

$$\left[\frac{i}{2\hbar}uv, \frac{1}{\hbar\sqrt{8}}u^2\right] = -\frac{1}{\hbar\sqrt{8}}u^2, \quad \left[\frac{i}{2\hbar}uv, \frac{1}{\hbar\sqrt{8}}v^2\right] = \frac{1}{\hbar\sqrt{8}}v^2,$$
$$\left[\frac{1}{\hbar\sqrt{8}}u^2, \frac{1}{\hbar\sqrt{8}}v^2\right] = -\frac{i}{2\hbar}uv.$$

X, Y, H generate an associative algebra in the space $\mathbb{C}[u, v]$ of all polynomials. This is an enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$.

The Casimir element $C = H^2 + (X * Y + Y * X)$, that is

$$C = \left(\frac{i}{2\hbar}uv\right)^2_* + \frac{1}{\hbar\sqrt{8}}u^2 * \frac{1}{\hbar\sqrt{8}}v^2 + \frac{1}{\hbar\sqrt{8}}v^2 * \frac{1}{\hbar\sqrt{8}}u^2$$

is given by

$$\begin{aligned} 8\hbar^2 C &= u^2 * v^2 + v^2 * u^2 - 2\left(u * v + \frac{\hbar i}{2}\right)^2 \\ &= u^2 * v^2 + v^2 * u^2 - 2u * v * u * v - 2\hbar i u * v + \frac{\hbar^2}{2}. \end{aligned}$$

Hence, $C = -\frac{3}{16}$. This means that our enveloping algebra is constrained in the space $C = -\frac{3}{16}$.

In a $(\mathbb{C}[u, v]; *)$-*bimodule* with an ordering expression mentioned above, we can consider the differential equation

$$\frac{d}{dt}f_t(u, v) = p(u, v) * f_t(u, v), \quad f_0(u, v) = f(u, v)$$

for every polynomial $p(u,v)$. If $p(u,v) = u^2 + (\frac{i}{\hbar}v)^2$, this equation is viewed as that of standard harmonic oscillator. If the complex variable t is considered, the existence of the solution for arbitrary initial function does not hold, but a real analytic solution in t is unique, if exists. If the real analytic solution exists, then we denote this by $e_*^{tp(u,v)} * f(u,v)$, where $e_*^{tp(u,v)}$ is the solution with initial condition 1.

The purpose of this paper is to investigate the group generated by $e_*^{aH+bX+cY}$. It is obvious that the obtained group should be $SL(2,\mathbb{C})$ or $SL(2,\mathbb{C})/\mathbb{Z}_2$.

However, we have to use several ordering expressions to define $e_*^{aH+bX+cY}$ for all a, b, $c \in \mathbb{C}$. This is just like a 2-sphere can not be covered by one coordinate sheet. We need at least three ordering expressions to cover $SL(2,\mathbb{C})$. The precise meaning of the "union" will become clear in the proof.

Moreover, we see that the $*$-product $e_*^{aH+bX+cY} * e_*^{a'H+b'X+c'Y}$ is defined in general with an ambiguity of $\pm$-sign of $\sqrt{\cdot}$, and the ambiguity can not be eliminated. Since the group structure is considered by using $*$-multiplication and the addition is not used, we can calculate the group operation with $\pm$ ambiguity. We show the following in this paper:

Theorem 2. *There is no $(\mathbb{C}[u,v];*)$-bimodule with an ordering expression containing $e_*^{aH+bX+cY}$ for all a, b, $c \in \mathbb{C}$.*

However, if we use several $(\mathbb{C}[u,v];*)$-bimodules with ordering expressions and forget about the ambiguity of $\sqrt{\cdot}$, then the group generated by $\{e_*^{aH+bX+cY}; a,\ b,\ c \in \mathbb{C}\}$ is embedded in the union of such bimodules, and the image is $SL(2,\mathbb{C})$.

Several anomalous phenomena relating this theorem will be also discussed in this paper. Especially, we discuss how the associativity breaks down in the calculation of extended $*$-product.

§2. Extensions of product formula

In this section, we mainly use the Weyl ordering expression. The following is the most useful property of Moyal product formula (1.3):

Proposition 3. *For every $A \in SL(2,\mathbb{C})$, let Φ^* be the replacement (pull-back) of u, v into u', v' by the combination of the linear transformation by the matrix A and the parallel displacement:*

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A \in SL(2,\mathbb{C}), \quad (\alpha,\beta) \in \mathbb{C}^2.$$

Then, Φ^* is an isomorphism in both $*$-product and $\cdot$-product.

Remark that other expressions do not have such a property. It is easily seen that

$$(au+bv)^m_* = (au+bv)^m_{\cdot}, \quad \text{but} \quad (au+bv)^m_* \neq (au+bv)^m_\circ \quad \text{for} \quad ab \neq 0.$$

For the proof of Proposition 3, we have only to remark the following identity:

$$\overleftarrow{\partial_v} \wedge \overrightarrow{\partial_u} = \overleftarrow{\partial_{v'}} \wedge \overrightarrow{\partial_{u'}}.$$

It is clear that if $A = \operatorname{diag}\{\lambda, \lambda^{-1}\}$, then the replacement Φ^* of (u, v) by (u', v') which is given by

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \lambda \in \mathbb{C}_*, \ (\alpha, \beta) \in \mathbb{C}^2,$$

gives an isomorphism in both $*$-product and $\circ$-product or in both $*$-product and $\bullet$-product.

Starting from a $(\mathbb{C}[u, v]; *)$-bimodule, $*$-product extends to a wider class of functions. For every positive real number p, we set

$$(2.1) \quad \mathcal{E}_p(\mathbb{C}^2) = \{f \in Hol(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f| e^{-s|\xi|^p} < \infty, \ \forall s > 0\}$$

where $|\xi| = (|u|^2 + |v|^2)^{1/2}$. The family $\{\| \ \|_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^2)$ and $(\mathcal{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dot $\cdot$ is the ordinary multiplication for functions in $\mathcal{E}_p(\mathbb{C}^2)$. Thus, $\cdot$ may be replaced by $\circ$ or $\bullet$ to indicate ordering of expression. It is easily seen that for $0 < p < p'$, there is a continuous embedding

$$(2.2) \qquad \mathcal{E}_p(\mathbb{C}^2) \subset \mathcal{E}_{p'}(\mathbb{C}^2)$$

as commutative Fréchet algebras (cf. [GS]), and that $\mathcal{E}_p(\mathbb{C}^2)$ is $SL(2, \mathbb{C})$-invariant.

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^2)$ and $\mathbb{C}[u, v]$ is dense in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 0$ in the Fréchet topology defined by the family $\{\| \ \|_{p,s}\}_{s>0}$.

Every exponential function $e_{\cdot}^{\alpha u+\beta v}$ is contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathcal{E}_1(\mathbb{C}^2)$, and functions such as $e_{\cdot}^{au^2+bv^2+2cuv}$ are contained in $\mathcal{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathcal{E}_2(\mathbb{C}^2)$. Functions such as $\sum \frac{1}{(n!)^{1/p}} u^k$ is contained in $\mathcal{E}_q(\mathbb{C}^2)$ for any $q > p$, but not in $\mathcal{E}_p(\mathbb{C}^2)$.

$Hol(\mathbb{C}^2)$ is a complete topological linear space under the compact open topology.

The following theorem is the main result of [OMMY]: [1]

[1] In [OMMY], the proof is given in the case of Weyl ordering expression, but the same proof works for other orderings.

Theorem 4. *Any product formula* (1.1), (1.2), (1.3) *extend to give the following*:

- **(i):** *For* $0 < p \leq 2$, *the space* $(\mathcal{E}_p(\mathbb{C}^2), *)$ *forms a topological associative algebra.*
- **(ii):** *For* $p > 2$, *every product formula gives a continuous bi-linear mapping of*

$$\mathcal{E}_p(\mathbb{C}^2) \times \mathcal{E}_{p'}(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \quad \mathcal{E}_{p'}(\mathbb{C}^2) \times \mathcal{E}_p(\mathbb{C}^2) \to \mathcal{E}_p(\mathbb{C}^2), \tag{2.3}$$

for every p' *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$.

We remark here about the statement (ii). Since $p > 2$, p' must be $p' < 2$, hence the statement (i) gives that $(\mathcal{E}_{p'}(\mathbb{C}^2); *)$ is a Fréchet algebra. So the statement (ii) means that every $\mathcal{E}_p(\mathbb{C}^2)$, $p > 2$, is a topological $\mathcal{E}_{p'}(\mathbb{C}^2)$-bimodule.

We remark also that if $\hbar > 0$, then $e_{\cdot}^{\pm(1/\hbar)(au^2+bv^2+2cuv)} \in \mathcal{E}_p(\mathbb{C}^2)$ for every $p > 2$. Remark also that such an element does not appear in the theory of *formal* deformation quantization.

Let $\mathcal{E}_{2+}(\mathbb{C}^2) = \bigcap_{p>2} \mathcal{E}_p(\mathbb{C}^2)$. $\mathcal{E}_{2+}(\mathbb{C}^2)$ is a Fréchet space under the natural intersection topology, $e^{\pm(1/\hbar)(au^2+bv^2+2cuv)}$ is continuous in $\mathcal{E}_{2+}(\mathbb{C}^2)$ with respect to $(a, b, c) \in \mathbb{C}^3$.

The following are examples of elements of $\mathcal{E}_{2+}(\mathbb{C}^2)$ which play important role in the later sections:

$$\int_{-\infty}^{\infty} \frac{1}{\cosh t} e^{(\tanh t)uv}\, dt, \quad \frac{1}{u}(1 - e^{(2i/\hbar)uv}), \quad \frac{1}{v}(1 - e^{-(2i/\hbar)uv}).$$

2.1. Intertwiner, or coordinate transformations

We have three kind of $(\mathbb{C}[u, v]; *)$-bimodules according to normal ordering expressions, the anti-normal ordering expression and the Weyl ordering expression.

Let e_*^{su}, e_*^{tv} be $*$-exponential functions defined by $e_*^{su} = \sum \frac{1}{k!}(su)^k$ or equivalently by the solution of $\frac{d}{dt} f_t(u) = u * f_t(u)$ with $f_0(u) = 1$. By each product formula, $e_*^{su} * e_*^{tv}$ is computed as follows:

- $e_*^{su} * e_*^{tv} = e_\circ^{su+tv}$ in the ΨDO-product formula,
- $e_*^{su} * e_*^{tv} = e^{-\hbar ist} e_\bullet^{su+tv}$ in the $\bar{\Psi}$DO-product formula,
- $e_*^{su} * e_*^{tv} = e^{-(\hbar ist/2)} e_{\cdot}^{su+tv}$ in the Moyal product formula,

where $\circ$, $\bullet$, $\cdot$ indicate the commutative product used in each expression. We have also

- $e_*^{\alpha u+\beta v} = e_{\cdot}^{\alpha u+\beta v}$ in the Weyl ordering expression, but
- $e_*^{\alpha u+\beta v} = e^{(\hbar i/2)\alpha\beta} e_\circ^{\alpha u+\beta v}$ in the normal ordering expression with respect to (u, v).

Thus, we must identify $e_{\circ}^{su+tv}$, $e^{-\hbar ist}e_{\bullet}^{su+tv}$, $e^{-(\hbar ist/2)}e_{\cdot}^{su+tv}$ through linear transformations. These are obtained by the following

$$e^{\hbar i\partial_u\partial_v}e_{\circ}^{su+tv} \longleftrightarrow e^{-\hbar ist}e_{\bullet}^{su+tv}, \quad e^{(\hbar i/2)\partial_u\partial_v}e_{\circ}^{su+tv} \longleftrightarrow e^{-(\hbar ist/2)}e_{\cdot}^{su+tv}.$$

Thus we define intertwiners as follows:

$$I_{\circ}^{\bullet}(f) = e^{-\hbar i\partial_u\partial_v}f, \quad I_{\circ}^{\cdot}(f) = e^{-(\hbar i/2)\partial_u\partial_v}f.$$

We consider also the intertwiner between normal ordering expression with respect to (u,v) and the normal ordering expression with respect to (u',v') when (u,v) and (u',v') are related by $u' = au+bv$, $v' = cu+dv$ such that $ad-bc=1$.

The principle of making the intertwiner is that the $*$-exponential functions $e_*^{\alpha u+\beta v}$ and $e_*^{\alpha' u'+\beta' v'}$ coincide if (u,v) and (u',v') are canonical conjugate pairs related linearly by each other and $\alpha u+\beta v = \alpha' u' + \beta' v'$.

Lemma 5. *If $u' = \alpha u+\beta v$, and $v' = \gamma u+\delta v$ is a canonical conjugate pair, then $e_*^{tu'} = e_{\circ'}^{tu'}$ in the normal ordering expression with respect to (u',v').*

Applying Lemma 5 to a canonical conjugate pair (u',v'), we take the normal ordering expression with respect to (u',v'):

$$e_*^{\alpha' u'+\beta' v'} = e^{(\hbar i/2)\alpha'\beta'}e_{\diamond}^{\alpha' u'+\beta' v'}$$

Suppose $\alpha u+\beta v = \alpha' u'+\beta' v'$ and $u' = au+bv$, $v' = cu+dv$, $ad-bc=1$. Then, we must identify $e^{(\hbar i/2)\alpha\beta}e_{\circ}^{\alpha u+\beta v}$ with $e^{(\hbar i/2)\alpha'\beta'}e_{\diamond}^{\alpha' u'+\beta' v'}$.

Hence, we have to define the intertwiner $I_{\circ}^{\diamond}$ as a linear mappings:

$$I_{\circ}^{\diamond}f = e^{(\hbar i/2)\partial_{u'}\partial_{v'}-(\hbar i/2)\partial_u\partial_v}f. \tag{2.4}$$

Precisely speaking, if (u,v) and (u',v') relate by

$$u' = au+bv, \quad v' = cu+dv, \quad ad-bc=1,$$

we first consider the exponential of the operator

$$\partial_{u'}\partial_{v'} - \partial_u\partial_v = -bd\partial_u^2 + (ad+bc-1)\partial_u\partial_v - ac\partial_v^2$$

and then we replace the variable (u,v) by $(du'-bv', -cv'+av')$ to obtain $I_{\circ}^{\diamond}f(u,v)$. That is, if

$$e^{-bd\partial_u^2+(ad+bc-1)\partial_u\partial_v-ac\partial_v^2}f(u,v) = g(u,v),$$

then we set $I_{\circ}^{\diamond}f(u,v) = g(du'-bv', -cv'+av')$.

These are first defined on the space $\mathbb{C}[u,v]$, and these give different expressions to a *same element* written by using $*$-product via different commutative algebras.

Theorem 6. *The intertwiners defined above extend to continuous linear isomorphisms of $\mathcal{E}_p(\mathbb{C}^2)$ onto itself for every $0 < p \leq 2$, and to give algebra isomorphisms of $(\mathcal{E}_p(\mathbb{C}^2); *)$ onto $(\mathcal{E}_p(\mathbb{C}^2); *)$.*
However, these do not extend to the space $\mathcal{E}_{2+}(\mathbb{C}^2)$.

Just like a coordinate transformation, the intertwiner is defined only on a part of $\mathcal{E}_{2+}(\mathbb{C}^2)$ onto a part of another $\mathcal{E}_{2+}(\mathbb{C}^2)$.

In spite of this, it is remarkable that the patching property, that is, $I_{\diamond}^{\diamond'} I_{\circ}^{\diamond}(f) = I_{\circ}^{\diamond'}(f)$ holds for $f \in \mathcal{E}_2(\mathbb{C}^2)$, and this hold also for $f \in \mathcal{E}_{2+}(\mathbb{C}^2)$ if both sides are defined. This is proved by the approximation by elements of $\mathcal{E}_2(\mathbb{C}^2)$. *Intertwiners have the property of gluing maps of bimodules.*

By the above observation we see in particular:

Lemma 7. *The anti-normal ordering expression with respect to (u,v), and the normal ordering expression with respect to $(-v,u)$ coincides.*

By the observation as above, we have to consider the differential equations

$$\frac{\partial}{\partial t} f = \hbar i \partial_u \partial_v f, \quad \frac{d}{d\tau} f = \hbar i \partial_u^2 f. \tag{2.5}$$

The solution with initial function e^{au+bv} is given by $e^{\hbar iabt} e^{au+bv}$, $e^{\hbar a^2 t} e^{au+bv}$. To obtain the solution with the initial function $e^{\alpha u^2+\beta v^2+2\gamma uv}$, we set $f = s(t)e^{\phi_1(t)u^2+\phi_2(t)v^2+\phi_3(t)2uv}$. Then, the equations in (2.5) are rewritten respectively as systems of ordinary differential equations:

$$\begin{aligned} s'(t) &= 2\hbar i s(t)\phi_3(t), \quad \phi_1'(t) = 4\hbar i \phi_1(t)\phi_3(t), \\ \phi_2'(t) &= 4\hbar i \phi_2(t)\phi_3(t), \quad \phi_3'(t) = 2\hbar i(\phi_1(t)\phi_2(t) + \phi_3(t)^2). \end{aligned} \tag{2.6}$$

$$\begin{aligned} s'(\tau) &= 2\hbar i \phi_1(\tau)s(\tau), \quad \phi_1'(\tau) = 4\hbar i \phi_1(\tau)^2, \\ \phi_2'(\tau) &= 4\hbar i \phi_3(\tau)^2, \quad \phi_3'(\tau) = 4\hbar i \phi_1(\tau)\phi_3(\tau). \end{aligned} \tag{2.7}$$

Through the solutions, we can patch exponential functions of quadratic forms together, and although the domain and the region are not clearly stated, intertwiners give patching identities of $\mathcal{E}_p(\mathbb{C}^2)$-bimodules for $p < 2$, to define a certain $\mathcal{E}_p(\mathbb{C}^2)$-bimodule as a patched object.

§3. Vacuums, half-inverses and the break down of the associativity

A direct calculation using the Moyal product formula (1.3) shows that the coordinate function v has a right inverse $v^\circ = \frac{1}{v}(1 - e^{(2i/\hbar)uv})$, and a left inverse $v^\bullet = \frac{1}{v}(1 - e^{-(2i/\hbar)uv})$ in $\mathcal{E}_{2+}(\mathbb{C}^2)$, i.e,

$$v * v^\circ = 1 = v^\bullet * v, \quad v^\circ * v = 1 - 2e^{(2i/\hbar)uv}, \quad v * v^\bullet = 1 - 2e^{-(2i/\hbar)uv}.$$

If the associativity holds, then these should be the same genuine inverse. Hence we must set $\frac{1}{v}\sin\frac{2}{\hbar}uv = 0$. Since this is impossible (cf. [O,el.1]), we loose the associativity in $\mathcal{E}_{2+}(\mathbb{C}^2)$. This is one of the most basic phenomenon which breaks the associativity. That is, *coordinate functions have both left- and right-inverses.*

3.1. Star-exponentials of quadratic forms in the Weyl ordering expression

These strange phenomena are deeply related to the $*$-exponential function such as $e_*^{(t/\hbar)u\cdot v}$ defined by the equation $\frac{d}{dt}f_t(u,v) = \frac{1}{\hbar}(u\cdot v) * f_t(u,v)$, $\quad f_0(u,v) = 1$. Recall again that such an element can not appear in the formal deformation theory.

For every point $(a,b,c;s)$ in $\mathbb{C}^4$, consider a curve $s(t)\exp\{\frac{1}{\hbar}(a(t)u^2 + b(t)v^2 + 2c(t)uv)\}$ starting at the point $s\exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ then the tangent vector of this curve is given as

$$\left(\frac{1}{\hbar}\big(a'u^2 + b'v^2 + 2c'uv\big)s + s'\right) e^{(1/\hbar)(au^2+bv^2+2cuv)}.$$

On the other hand, consider the $*$-product

$$\frac{d}{dt}\Big|_{t=0} e^{(t/\hbar)(a'u^2+b'v^2+2c'uv)} * se^{(1/\hbar)(au^2+bv^2+2cuv)}.$$

This is computed as follows:

$$\begin{aligned}
&\frac{1}{\hbar}(a'u^2 + b'v^2 + 2c'uv) * se^{(1/\hbar)(au^2+bv^2+2cuv)}\\
&\quad = \frac{1}{\hbar}(a'u^2 + b'v^2 + 2c'uv)se^{(1/\hbar)(au^2+bv^2+2cuv)}\\
&\qquad + \frac{2i}{\hbar}\{(b'v + c'u)(au+cv) - (a'u + c'v)(bv+cu)\}\\
&\qquad\qquad \times se^{(1/\hbar)(au^2+bv^2+2cuv)}\\
&\qquad - \frac{1}{2\hbar}\{b'(\hbar a + 2(au+cv)^2) - 2c'(\hbar c + 2(au+cv)(bv+cu))\\
&\qquad\qquad + a'(\hbar b + 2(bv+cu)^2)\}se^{(1/\hbar)(au^2+bv^2+2cuv)}
\end{aligned}$$

This may be written as

(3.1)

$$\frac{1}{\hbar}(a',b',c')\begin{bmatrix} -(c+i)^2, & -b^2, & -b(c+i), & -\frac{b}{2} \\ -a^2, & -(c-i)^2, & -a(c-i), & -\frac{a}{2} \\ 2a(c+i), & 2b(c-i), & 1+ab+c^2, & c \end{bmatrix}\begin{bmatrix} u^2 \\ v^2 \\ 2uv \\ \hbar \end{bmatrix}$$

$$\times\, se^{(1/\hbar)(au^2+bv^2+2cuv)}.$$

We denote this matrix by $M(a,b,c;s)$, and by $M(a,b,c)$ the submatrix of first three columns.

Remark that

$$\det M(a,b,c) = (c^2 - ab + 1)^3. \tag{3.2}$$

The feature of this matrix is that the radial direction is the direction of eigen vector:

$$(a,b,c)M(\tau a,\tau b,\tau c) = (1+(c^2-ab)\tau^2)(a,b,c), \tag{3.3}$$

holds for every (a,b,c).

If $c^2 - ab + 1 = 0$, then we can write

$$au^2 + bv^2 + 2cuv = 2i(\alpha u + \beta v)(\gamma u + \delta v), \quad \alpha\delta - \beta\gamma = 1.$$

Clearly, $[\alpha u + \beta v, \gamma u + \delta v] = -\hbar i$. Hence, setting $u' = \alpha u + \beta v$, $v' = \gamma u + \delta v$, (u',v') is a canonical conjugate pair, and hence by Proposition 3, we easily see by (1.3) that

$$(\gamma u + \delta v) * e^{(2i/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)} = 0, \quad \text{for} \quad \alpha\delta - \beta\gamma = 1. \tag{3.4}$$

It follows that

$$(\gamma u + \delta v)^2_* * e^{(1/\hbar)(au^2+bv^2+2cuv)} = 0,$$
$$(\alpha u + \beta v) * (\gamma u + \delta v) * e^{(1/\hbar)(au^2+bv^2+2cuv)} = 0.$$

The second identity yields $(a,b,c)M(a,b,c) = 0$, if $c^2 - ab + 1 = 0$, which corresponds to (3.3), and the first one yields

$$(\gamma^2, \delta^2, \gamma\delta)M(a,b,c) = 0, \quad c^2 - ab + 1 = 0.$$

Hence we see that $M(a,b,c)$ is rank 1 at the point $c^2 - ab + 1 = 0$, but the rank of $M(a,b,c;s)$ is 2 at such a point. $2e^{(2i/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)}$ and $2e^{-(2i/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)}$ are called *vacuums*. Remark that $(\alpha u + \beta v) \times (\gamma u + \delta v)$ and $(\gamma u + \delta v)(\alpha u + \beta v)$ are distinguished in the expression of vacuums.

3.2. Horizontal distributions

Using (3.1), we consider a holomorphic singular distribution D given by

$$D(a,b,c;s) = \{(a',b',c')M(a,b,c;s) \mid (a',b',c') \in \mathbb{C}^3\}$$

on the space $\mathbb{C}^3 \times \mathbb{C}_*$. Let $\pi\colon \mathbb{C}^3 \times \mathbb{C}_* \to \mathbb{C}^3$ be the natural projection.

Let $\Sigma = \{(a,b,c); c^2 - ab + 1 = 0\}$. $\Sigma \times \mathbb{C}_*$ is a 3-dimensional complex submanifold of $\mathbb{C}^3 \times \mathbb{C}_*$.

Though $\{D\}$ is singular on $\Sigma \times \mathbb{C}_*$, $\{D\}$ is a strongly involutive distribution in the sense of [Om] p. 51, for $\{D\}$ is given as an infinitesimal action of a Lie group. This gives an ordinary involutive distribution on $(\mathbb{C}^3 - \Sigma) \times \mathbb{C}_*$ and hence there is the 3-dimensional maximal integral holomorphic submanifold M^3 through the origin $(0,0,0;1)$.

A curve $\boldsymbol{g}(t) = (a(t), b(t), c(t); s(t))$ is an *integral curve* of $\{D\}$, if $\frac{d}{dt}\boldsymbol{g}(t) \in D(\boldsymbol{g}(t))$ for every t. For every curve $\boldsymbol{c}(t)$ in $\mathbb{C}^3 - \Sigma$, we have an integral curve $\boldsymbol{g}(t)$ such that $\pi(\boldsymbol{g}(t)) = \boldsymbol{c}(t)$. $\boldsymbol{g}(t)$ is a *lift* of $\boldsymbol{c}(t)$. Remark that $\boldsymbol{g}(1)$ depends only on the homotopy class of curves joining $(0,0,0)$ and $\boldsymbol{c}(1)$.

Points of M^3 is given as the homotopy equivalence class of lift of curves in $\mathbb{C}^3 - \Sigma$ starting at the origin $(0,0,0)$.

Every integral curve $\boldsymbol{g}(t)$ staring at a point of $\Sigma \times \mathbb{C}_*$ remains in this space. The maximal integral submanifold through a point of $\Sigma \times \mathbb{C}_*$ is a 2-dimensional complex submanifold M^2 such that $\pi(M^2)$ is a one dimensional submanifold of Σ. Hence, $\Sigma \times \mathbb{C}_*$ is foliated by maximal integral submanifolds.

3.3. $*$-exponentials and vacuums

In this subsection we define the exponential function $e_*^{t(au^2+bv^2+2cuv)}$. Set $e_*^{t(au^2+bv^2+2cuv)} = F(t,u,v)$, and consider the evolution equation

$$\frac{\partial}{\partial t}F(t,u,v) = (au^2 + bv^2 + 2cuv) * F(t,u,v), \quad F(0,u,v) = 1. \tag{3.5}$$

The right hand side of (3.5) is computed by the Moyal product formula (1.3) as follows:

$$\begin{aligned}
&(au^2 + bv^2 + 2cuv) * F(t,u,v) \\
&\quad = (au^2 + bv^2 + 2cuv)F + \hbar i\{(bv + cu)\partial_u F - (au + cv)\partial_v F\} \\
&\qquad - \frac{\hbar^2}{4}\{b\partial_u^2 F - 2c\partial_v\partial_u F + a\partial_v^2 F\}
\end{aligned}$$

This is a partial differential equation. If $ab - c^2 > 0$, then this is the heat equation and the existence of solutions is not ensured in general.

This implies that the mapping $f(u,v) \to e_*^{(t/\hbar)(au^2+bv^2+2cuv)} * f(u,v)$ is not always defined for C^∞-functions.

However, we see that real analytic solution in t is unique, if it exists. Hence we assume that $e_*^{t(au^2+bv^2+2cuv)}$ is a function of au^2+bv^2+2cuv; that is $e_*^{t(au^2+bv^2+2cuv)} = f_t(au^2+bv^2+2cuv)$. Then, setting $x = au^2+bv^2+2cuv$, we have

$$\frac{d}{dt} f_t(x) = x f_t(x) - \hbar^2(ab-c^2)\big(f_t'(x) + x f_t''(x)\big). \tag{3.6}$$

The right hand side is the Bessel operator.

However, there is another method to treat this differential equation. We assume that

$$e_*^{t(au^2+bv^2+2cuv)} = s(t) e^{a(t)u^2+b(t)v^2+2c(t)uv},$$

then we have only to solve the system of ordinary differential equations

$$\begin{aligned} \frac{d}{dt}(a(t), b(t), c(t); s(t)) &= (a,b,c) M(a(t), b(t), c(t); s(t)), \\ (a(0), b(0), c(0); s(0)) &= (0,0,0;1). \end{aligned} \tag{3.7}$$

Lemma 8. *The solution of* (3.6) *with the initial function* 1 *is given by*

$$f_t(x) = \frac{1}{\cosh(\hbar\sqrt{ab-c^2}\,t)} \exp \frac{x}{\hbar\sqrt{ab-c^2}} \tanh\left(\hbar\sqrt{ab-c^2}\,t\right).$$

If $ab-c^2 = 0$, *then we set*

$$\frac{1}{\hbar\sqrt{ab-c^2}} \tanh\left(\hbar\sqrt{ab-c^2}\,t\right) = t.$$

This shows that e_*^{tuv} cannot be defined for all $t \in \mathbb{C}$, if the equation $\frac{d}{dt} f_t(uv) = (uv) * f_t(uv)$ is considered in the Weyl ordering expression.

We shall show that such singularities appear also in the other ordering expressions. Such an observation gives the first half of Theorem 2.

By Lemma 8, we have

(3.8)

$$\begin{aligned}
&\exp_*\left\{\frac{t}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&=\frac{1}{\cosh(\sqrt{ab-c^2}\,t)}\\
&\quad\times\exp\left\{(au^2+bv^2+2cuv)\left(\frac{1}{\hbar\sqrt{ab-c^2}}\tanh\left(\sqrt{ab-c^2}\,t\right)\right)\right\}\\
&=\frac{1}{\cos(\sqrt{c^2-ab}\,t)}\\
&\quad\times\exp\left\{(au^2+bv^2+2cuv)\left(\frac{1}{\hbar\sqrt{c^2-ab}}\tan\left(\sqrt{c^2-ab}\,t\right)\right)\right\}
\end{aligned}$$

(cf. same formula is seen also in [MS].) Remark here that $e_*^{t(au^2+bv^2+2cuv)}\in M^3$. Though the ambiguity of $\pm\sqrt{ab-c^2}$ makes no difference for the result, the difference of the periodicity of cos and tan gives that if $c^2-ab\neq 0$, then

$$\pi^{-1}\pi\{e_*^{t(au^2+bv^2+2cuv)};t\in\mathbb{C}\}=\{\pm e_*^{t(au^2+bv^2+2cuv)};t\in\mathbb{C}\}. \tag{3.9}$$

Since $\tan\theta=\sqrt{c^2-ab}$ gives $\frac{1}{\cos^2\theta}=c^2-ab+1$, (3.8) is equivalent with

$$\begin{aligned}
&\sqrt{c^2-ab+1}\,\exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&\quad=\exp_*\left\{\frac{1}{\hbar\sqrt{c^2-ab}}\left(\arctan\sqrt{c^2-ab}\right)(au^2+bv^2+2cuv)\right\}.
\end{aligned}\tag{3.10}$$

Using this, we have the following:

Proposition 9. *If $c^2-ab+1\neq 0$, then $\pm\sqrt{c^2-ab+1}\times\exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ are elements of M^3. Conversely, if $\pi(Q)=\exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ with $c^2-ab+1\neq 0$ for some $Q\in M^3$, then*

$$\begin{aligned}
Q=\sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)}\quad &or\\
-\sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)}&.
\end{aligned}$$

These are written as $*$-exponential functions written in the form

$$e_*^{(t/\hbar)(au^2+bv^2+2cuv)};\quad a,\ b,\ c\in\mathbb{C}$$

except the case $Q=-e^{(t/\hbar)(au^2+bv^2+2cuv)}$, $c^2-ab=0$.

By (3.8), we have in particular, if $c^2 \neq ab$, then $\exp_*\{\frac{\pi}{\hbar\sqrt{c^2-ab}} \times (au^2+bv^2+2cuv)\} = -1$, but $\exp_*\{\frac{\pi}{2\hbar\sqrt{c^2-ab}}(au^2+bv^2+2cuv)\}$ diverges in the Weyl ordering expression.

Let Π_0 be the subset defined as follows:

$$\Pi_0 = \{(a,b,c) \in \mathbb{C}^3; e_*^{(1/\hbar)(au^2+bv^2+2cuv)} \quad \text{does not defined}\}.$$

Remark that Proposition 9 shows that $\pi\colon M^3 \to \mathbb{C}^3 - \Sigma$ is surjective, but the difference of period of cos and tan, and the ambiguity of the sign of $\sqrt{c^2-ab+1}$ of (3.10) shows that π gives a double cover. Hence we have the following result:

Proposition 10. $\exp_*\colon \mathbb{C}^3 - \Pi_0 \to \mathbb{C}^3 - \Sigma$ *is a holomorphic mapping such that*

$$\begin{aligned}&\exp_*(\mathbb{C}^3 - \Pi_0)\\ &\quad = M^3 - \{-e^{(1/\hbar)(au^2+bv^2+2cuv)}; \quad c^2-ab=0, \quad (a,b,c) \neq (0,0,0)\}.\end{aligned}$$

The element -1 *is on a* $*$*-exponential function as* $\exp_*(\frac{\pi}{\hbar}2uv) = -1$.

By the uniqueness of analytic solutions, the exponential law

$$e_*^{isx} * e_*^{itx} = e_*^{i(s+t)x}$$

holds where both sides are defined.

Lemma 11. *For* $s, \sigma \in \mathbb{C}$ *such that* $1 + s\sigma(ab-c^2) \neq 0$, *we have*

$$\begin{aligned}&\exp\left\{\frac{s}{\hbar}(au^2+bv^2+2cuv)\right\} * \exp\left\{\frac{\sigma}{\hbar}(au^2+bv^2+2cuv)\right\}\\ &\quad = \frac{1}{1+s\sigma(ab-c^2)}\exp\left\{\frac{s+\sigma}{\hbar(1+s\sigma(ab-c^2))}(au^2+bv^2+2cuv)\right\}.\end{aligned}$$

Thus, we have idempotent elements

$$\begin{aligned}2\exp&\left\{\pm\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}\\ &* 2\exp\left\{\pm\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}\\ &= 2\exp\left\{\pm\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}.\end{aligned}$$

Recall $2\exp\{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\}$ is a vacuum.

Corollary 12. *Vacuums are obtained as the limit point of $*$-exponential functions*:

$$
\begin{aligned}
&2\exp\left\{\frac{1}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}\\
&= \lim_{t\to\infty}\exp\left\{it\sqrt{ab-c^2}\right\}\exp_*\left\{\frac{t}{\hbar\sqrt{ab-c^2}}(au^2+bv^2+2cuv)\right\}
\end{aligned}
$$

is a vacuum.

This shows that vacuums may be regarded as certain equilibrium states (cf. [BL]).

The following lemma is useful in the computation, and is proved by that both quantities satisfy the same partial differential equation with the same initial condition:

Lemma 13 (Bumping lemma).

$$v * e_*^{itu*v} = e_*^{itv*u} * v, \quad e_*^{itu*v} * u = u * e_*^{itv*u}$$

3.4. Anomarous phenomena

We easily see by the Moyal product formula (1.3) that

$$v * e^{(2i/\hbar)uv} = 0 = e^{(2i/\hbar)uv} * u, \quad u * e^{-(2i/\hbar)uv} = 0 = e^{-(2i/\hbar)uv} * v.$$

We call $2e^{(2i/\hbar)uv}$ a *vacuum* and $2e^{-(2i/\hbar)uv}$ a *bar-vacuum* and denote these by $\varpi_{0,0}$, and $\overline{\varpi}_{0,0}$ respectively. By the Moyal product formula and the 2-p-associativity, we see easily

$$\left(uv-\frac{\hbar i}{2}\right) * e^{(2i/\hbar)uv} = u * v * e^{(2i/\hbar)uv} = 0.$$

However, $uv-\hbar i/2 = u*v$ has the inverse $i\int_0^\infty e_*^{-(it/\hbar)u*v}\,dt$ in $\mathcal{E}_{2+}(\mathbb{C}^2)$. Thus, the associativity fails in $\mathcal{E}_{2+}(\mathbb{C}^2)$:

$$
\begin{aligned}
&\left(\left(uv-\frac{\hbar i}{2}\right)^{-1} * \left(uv-\frac{\hbar i}{2}\right)\right) * e^{(2i/\hbar)uv}\\
&\neq \left(uv-\frac{\hbar i}{2}\right)^{-1} * \left(\left(uv-\frac{\hbar i}{2}\right) * e^{(2i/\hbar)uv}\right).
\end{aligned}
\tag{3.11}
$$

Furthermore, we see that

$$
\begin{aligned}
&\int_0^\infty \frac{1}{\cosh(t/2)}\exp\left\{\frac{i}{\hbar}\left(\tanh\frac{t}{2}\right)2u\cdot v\right\}dt,\\
&\int_{-\infty}^0 \frac{1}{\cosh(t/2)}\exp\left\{\frac{i}{\hbar}\left(\tanh\frac{t}{2}\right)2u\cdot v\right\}dt
\end{aligned}
$$

exist in the space $\mathcal{E}_{2+}(\mathbb{C}^2)$. It follows that $u \cdot v$ has *two different* inverses as follows:

$$(u\cdot v)^{-1}_{+i0} = -i\int_0^{\infty} e_*^{(it/\hbar)u\cdot v}\,dt, \quad (u\cdot v)^{-1}_{-i0} = i\int_{-\infty}^{0} e_*^{(it/\hbar)u\cdot v}\,dt.$$

The difference is given as

$$(u\cdot v)^{-1}_{+i0} - (u\cdot v)^{-1}_{-i0} = -i\int_{-\infty}^{\infty} e_*^{(it/\hbar)u\cdot v}\,dt. \tag{3.12}$$

Since the right hand side of (3.12) can be viewed as the $*$-Fourier transform of 1, this may be written as the $*$-*delta function* $-i\delta_*(u\cdot v)$ (cf. [OMMY]). Hence the associativity must break down again, and it holds $(u\cdot v)*\delta_*(u\cdot v) = \delta_*(u\cdot v)*(u\cdot v) = 0$.

Thus, it is impossible to treat $(u\cdot v)^{-1}_{+i0}$ and $(u\cdot v)^{-1}_{-i0}$ in the same associative algebra. In spite of this, the right hand side of (3.12) has the expression as follows by using Hansen-Bessel formula:

$$\begin{aligned}\int_{-\infty}^{\infty} e_*^{(it/\hbar)u\cdot v}\,dt &= \int_{-\infty}^{\infty} \frac{1}{\cosh(t/2)} \exp\left\{\frac{i}{\hbar}\left(\tanh\frac{t}{2}\right)2u\cdot v\right\}dt \\ &= \frac{\pi}{2}J_0\left(\frac{2}{\hbar}u\cdot v\right).\end{aligned}$$

Hence, $-i\delta_*(u\cdot v)$ is expressed as an entire function by the Weyl ordering expression.

Several fancy relations to Sato's hyper functions [M] can be seen, since $(u\cdot v \pm z)^{-1}_{\pm i0}$ is defined as a holomorphic function with respect to z on the upper half plane, and $-i\delta_*(u\cdot v)$ is viewed as the difference $(u\cdot v + z)^{-1}_{+i0} - (u\cdot v - z)^{-1}_{-i0}$. These will be discussed in another paper.

3.5. Several product formulas

Every quadratic form $Q(u,v)$ is written in the form

- $(\alpha u + \beta v)^2$, if $ab - c^2 = 0$,
- $\lambda(\alpha u + \beta v)(\gamma u + \delta v)$ with $\alpha\delta - \beta\gamma = 1$, if $ab - c^2 \neq 0$.

By Proposition 3, the general product formula for quadratic exponential functions can be obtained from only the two cases as follows:

$$e^{tu^2} * e^{au^2+bv^2+2cuv}, \quad e^{\tau uv} * e^{au^2+bv^2+2cuv}.$$

By solving the system of ordinary equations (3.7) with the general initial condition

$$(a(0), b(0), c(0); s(0)) = (a, b, c; 1),$$

we see that the first one is written as

(3.13)
$$
\begin{aligned}
&\exp_*\left\{\frac{t}{\hbar}u^2\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&=\frac{1}{\sqrt{1+bt}}\\
&\quad\times\exp\left\{\frac{1}{\hbar(1+bt)}\{(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv\}\right\}.
\end{aligned}
$$

The ambiguity of $\pm\sqrt{1+bt}$ can not be eliminated for all t, b.

The formula (3.13) yields several results for the $*$-product. Remark first that $e_*^{(t/\hbar)u^2}=e^{(t/\hbar)u^2}$.

Lemma 14. *For* $\exp\{\frac{t}{\hbar}u^2\}$, $Q\in M^3$ *such that* $\pi(Q)=\exp\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\}$ *and* $bt\neq -1$, *the product* $\exp\{\frac{t}{\hbar}u^2\}*Q$ *is defined as an element of* M^3 *written as*

$$
\begin{aligned}
&\sqrt{\frac{c^2-ab+1}{1+bt}}\\
&\quad\times\exp\left\{\frac{1}{\hbar(1+bt)}\{(a+(ab-c^2-2ci+1)t)u^2+bv^2+2(c-ibt)uv\}\right\}.
\end{aligned}
$$

Similar to (3.13), we have

(3.14)
$$
\begin{aligned}
&\exp_*\left\{\frac{t}{\hbar}v^2\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\
&=\frac{1}{\sqrt{1+at}}\\
&\quad\times\exp\left\{\frac{1}{\hbar(1+at)}\{au^2+(b+(ab-c^2+2ci+1)t)v^2+2(c+iat)uv\}\right\},
\end{aligned}
$$

and hence we have the similar result as Lemma 14.

Remarking $e_*^{(t/\hbar)2uv}=\sqrt{1+s^2}\,e^{(s/\hbar)2uv}$, and solving carefully the system of ordinary equations (3.7) with the general initial condition, we have

(3.15)

$$\begin{aligned}&\exp\left\{\frac{s}{\hbar}2uv\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\&=\frac{1}{\sqrt{1-2cs+(c^2-ab)s^2}}\\&\quad\times\exp\left\{\frac{1}{\hbar(1-2cs+(c^2-ab)s^2)}\right.\\&\qquad\left.\times\ (a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)\right\}.\end{aligned}$$

The following identity is useful for the computation of discriminant D:

$$\begin{aligned}&\big(1-2cs+(c^2-ab)s^2\big)^2+\big(c-(c^2-ab-1)s-cs^2\big)^2\\&\qquad -ab(1+is)^2(1-is)^2\\&\quad=(c^2-ab+1)(1+s^2)\big((c^2-ab)s^2-2cs+1\big),\end{aligned}\tag{3.16}$$

but the ambiguity of $\pm\sqrt{1-2cs+(c^2-ab)s^2}$ can not be eliminated.

Using (3.15) and (3.10), we have several results as follows:

Lemma 15. *If $Q_1,\ Q_2\in M^3$ such that $\pi(Q_1)=e^{(s/\hbar)2uv}$, $\pi(Q_2)=e^{(1/\hbar)(au^2+bv^2+2cuv)}$, then*

$$Q_1=\pm\sqrt{1+s^2}\,e^{(s/\hbar)2uv},\quad Q_2=\pm\sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)}$$

with $1+s^2\neq 0$, $c^2-ab+1\neq 0$.

If $1-2cs+(c^2-ab)s^2\neq 0$, then the $*$-product $Q_1 * Q_2$ is defined as an element of M^3 by

$$\begin{aligned}&\left(\sqrt{1+s^2}\exp\left\{\frac{s}{\hbar}2uv\right\}\right) * \left(\sqrt{c^2-ab+1}\right)\exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\&=\sqrt{1+D}\exp\left\{\frac{1}{\hbar(1-2cs+(c^2-ab)s^2)}\right.\\&\qquad\left.\times\ (a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv)\right\}\end{aligned}$$

where D is the discriminant of the quadratic form

$$\begin{aligned}&\frac{1}{1-2cs+(c^2-ab)s^2}\\&\quad\times\big(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv\big).\end{aligned}$$

Hence the right hand side is also an element of $*$-exponential function.

By Lemmas 15, 14, we have the following:

Theorem 16. *M^3 forms a local group, which is locally isomorphic to $SL(2,\mathbb{C})$ and M^3 is embedded in $\mathcal{E}_{2+}(\mathbb{C}^2)$ as*

$$M^3 = \left\{\pm\sqrt{c^2-ab+1}\,e^{(1/\hbar)(au^2+bv^2+2cuv)};\quad c^2-ab+1\neq 0\right\}.$$

The open dense subset

$$M^3 - \left\{-e^{(1/\hbar)(au^2+bv^2+2cuv)};\quad c^2-ab=0,\ (a,b,c)\neq(0,0,0)\right\}$$

is covered by $*$-exponential functions $\{e_*^{(1/\hbar)(au^2+bv^2+2cuv)}\}$.

§4. Star exponential functions in the normal ordering expression

Although $e_*^{\pm(\pi/\hbar)uv}$ diverge in the Weyl ordering expression, we prove in this section that such elements make sense in the normal ordering expression.

Since $uv = u\circ v + (\hbar i/2)$, we have $au^2+2cuv+bv^2 = au^2+2cu\circ v+bv^2+\hbar ci$. In this section, we compute $e^{-cit}e_*^{(t/\hbar)(au^2+bv^2+2cuv)} = e_*^{(t/\hbar)(au^2+bv^2+2cu\circ v)}$ by ΨDO-product formula. Thus, we set

$$e_*^{(t/\hbar)(au^2+bv^2+2cu*v)} = s(t)e_\circ^{(1/\hbar)(a(t)u^2+b(t)v^2+2c(t)u\circ v)}.$$

We first compute

$$\begin{aligned}&\frac{1}{\hbar}(a'u^2+b'v^2+2c'u\circ v)*e_\circ^{(1/\hbar)(au^2+bv^2+2cu\circ v)}\\ &\quad=\left\{\frac{1}{\hbar}(a'u^2+b'v^2+2c'u\circ v)+\frac{i}{\hbar}(2b'v+2c'u)\circ(2au+2cv)\right.\\ &\qquad\left.+\frac{-1}{\hbar}\frac{1}{2}(2b')((2au+2bv)^2+2a\hbar)\right\}\circ e_\circ^{(1/\hbar)(au^2+bv^2+2cu\circ v)}.\end{aligned}$$

This is

(4.1)

$$\frac{1}{\hbar}(a',b',c')\begin{bmatrix}1, & 0, & 0, & 0\\ -4a^2, & 1+4ci-4c^2, & 2ai-2ac, & -2\hbar a\\ 4ai, & 0, & 1+2ci, & 0\end{bmatrix}\begin{bmatrix}u^2\\ v^2\\ 2u\circ v\\ \hbar\end{bmatrix}$$

$$\circ\, se_\circ^{(1/\hbar)(au^2+bv^2+2cu\circ v)}$$

Submatrix of first 3-columns is singular only at $1+2ci=0$, i.e. at $e_\circ^{(1/\hbar)iu\circ v}$. This is in fact a vacuum computed by ΨDO-product formula (cf. Corollary 12 and (4.3) below).

Hence, setting $e_*^{(t/\hbar)(au^2+bv^2+2cu\circ v)} = \psi(t)e_\circ^{\phi_1(t)u^2+\phi_2(t)v^2+2\phi_3(t)u\circ v}$, we have only to solve the system of ordinary differential equations

$$
(4.2)\qquad
\begin{aligned}
\phi_1'(t) &= \frac{1}{\hbar}a + 4ic\phi_1(t) - 4\hbar b\phi_1(t)^2\\
\phi_2'(t) &= \frac{1}{\hbar}b + 4ib\phi_3(t) - 4\hbar b\phi_3(t)^2\\
\phi_3'(t) &= \frac{1}{\hbar}c + 2ic\phi_3(t) + 2ib\phi_1(t) - 4\hbar b\phi_1(t)\phi_3(t)\\
\psi'(t) &= -2\hbar b\phi_1(t)\psi(t)
\end{aligned}
$$

with the initial condition $\phi_i(0)=0$ and $\psi(0)=1$.

4.1. The case $b=0$ as the simplest case

(4.2) is easily solved if $b=0$, and we have

$$
(4.3)\qquad e_*^{(t/\hbar)(au^2+2cu\circ v)} = e_\circ^{(a/4ci\hbar)(e^{4cit}-1)u^2+(1/2i\hbar)(e^{2cit}-1)2u\circ v}.
$$

In particular, $\lim_{t\to\infty} e_*^{(it/\hbar)u\circ v} = e_\circ^{(1/\hbar)iu\circ v}$. By Corollary 12, the limit is the vacuum ϖ_{00}.

Note also that the case $1+2ci=0$ in (4.1) is written as follows:

$$
\begin{aligned}
e_\circ^{(1/\hbar)(au^2+bv^2+iu\circ v)} &= \left(e_\circ^{(1/\hbar)au^2} * \varpi_{00}\right) * e_\circ^{(1/\hbar)bv^2}\\
&= e_\circ^{(1/\hbar)au^2} * \left(\varpi_{00} * e_\circ^{(1/\hbar)bv^2}\right).
\end{aligned}
$$

We have also the following remarkable fact:

$$
(4.4)\qquad e_*^{(\pi/2\hbar)(au^2+2u*v)} = e_\circ^{-(1/\hbar i)2u\circ v} = e_*^{(\pi/\hbar)u*v}
$$

that is, $e_*^{(\pi/\hbar)(au^2+u\circ v)}$ does not depend on a.

Using the exponential law we have the following:

Proposition 17. *In the normal ordering expression with respect to (u,v), the exponential law holds*

$$
\begin{aligned}
&\left(\exp_\circ\left\{\frac{a}{4ci\hbar}(e^{4cis}-1)u^2 + \frac{1}{2i\hbar}(e^{2cis}-1)2u\circ v\right\}\right)\\
&\quad * \left(\exp_\circ\left\{\frac{a}{4ci\hbar}(e^{4cit}-1)u^2 + \frac{1}{2i\hbar}(e^{2cit}-1)2u\circ v\right\}\right)\\
&= \exp_\circ\left\{\frac{a}{4ci\hbar}(e^{4ci(s+t)}-1)u^2 + \frac{1}{2i\hbar}(e^{2ci(s+t)}-1)2u\circ v\right\}
\end{aligned}
$$

In particular, we have the exponential law:

$$e_\circ^{(1/\hbar i)(e^{is}-1)u\circ v} * e_\circ^{(1/\hbar i)(e^{it}-1)u\circ v} = e_\circ^{(1/\hbar i)(e^{i(s+t)}-1)u\circ v}$$

If we set $\sigma = e^{is} - 1$, $\tau = e^{it} - 1$, then the exponential law gives the following product formula:

$$e_\circ^{(1/\hbar i)\sigma u\circ v} * e_\circ^{(1/\hbar i)\tau u\circ v} = e_\circ^{(1/\hbar i)(\sigma\tau+\sigma+\tau)u\circ v} \tag{4.5}$$

Though the product has no singularity, the inverse has a singular point:

$$(e_\circ^{(1/\hbar i)\sigma u\circ v})_*^{-1} = e_\circ^{-(1/\hbar i)(\sigma/1+\sigma)u\circ v}. \tag{4.6}$$

The singular point $e_\circ^{-(1/\hbar i)u\circ v}$ is in fact the normal ordering expression of the vacuum $\varpi_{0,0}$.

4.2. Several facts, concluded from the case $a = 0$

If $a = 0$ in (4.2), then we have

$$e_*^{(t/\hbar)(bv^2+2cu\circ v)} = e_\circ^{(b/4ci\hbar)(e^{4cit}-1)v^2+(1/2\hbar i)(e^{2cit}-1)2u\circ v} \tag{4.7}$$

The same exponential law as in Proposition 17 holds.

In particular, we see that

$$e_*^{(\pi/2\hbar)(bv^2+2u*v)} = e_\circ^{-(2/\hbar i)u\circ v} = e_*^{(\pi/\hbar)u*v} \tag{4.8}$$

and this quantity does not depend on b.

By (4.4), (4.8), we have the following remarkable fact:

Lemma 18. *In the normal ordering expression with respect to the canonical conjugate pair (u, v), the identities*

$$e_*^{(\pi/\hbar)u*(v+au)} = e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)(u+bv)*v} = e_\circ^{(2i/\hbar)u\circ v}$$

hold for any $a, b \in \mathbb{C}$.

An element $e_*^{(\pi/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)} = ie_*^{(\pi/\hbar)(\alpha u+\beta v)*(\gamma u+\delta v)}$ with $\alpha\delta - \beta\gamma = 1$ is called a *polar element.* This element is computed in the normal ordering expression with respect to $u' = \alpha u + \beta v$, $v' = \gamma u + \delta v$. We denote the set of all polar elements by ϵ_{00}. Obviously,

$$\begin{aligned}\epsilon_{00} &= \left\{e_*^{(\pi/\hbar)(\alpha u+\beta v)(\gamma u+\delta v)};\ \alpha\delta - \beta\gamma = 1\right\}\\ &= \left\{e_*^{(\pi/\hbar)(au^2+bv^2+2cuv)};\ c^2 - ab = \frac{1}{4}\right\}\end{aligned}$$

$e_*^{(\pi/\hbar)uv} = ie_*^{(\pi/\hbar)u*v}$ is a polar element. Though this is not computed in the Moyal product formula, this is computed in the ΨDO-product formula (1.1) as $e_\circ^{(2i/\hbar)u\circ v}$.

Note that $u' = u$, $v' = au + v$ gives a canonical conjugate pair. Hence, by Lemma 18 applied for u', v', we have

$$e_*^{(\pi/\hbar)u'*v'} = e_*^{(\pi/\hbar)(u'+bv')*v'} = e_{\circ'}^{(2i/\hbar)u'\circ v'}$$

in the normal ordering expression with respect to (u', v').

Note that for every $c \neq 0$,

$$(u'+bv')*v' = (u+b(v+au))*(v+au) = \left(\frac{1+ab}{c}u + \frac{b}{c}v\right)*(cau+cv).$$

Thus, in a first glance, it looks very natural to set as

$$e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)u*(v+au)} = e_*^{(\pi/\hbar)u'*v'} = e_*^{(\pi/\hbar)(u'+bv')*v'},$$

hence we have

$$\exp_*\left\{\frac{\pi}{\hbar}u*v\right\} = \exp_*\left\{\frac{\pi}{\hbar}\left(\frac{1+ab}{c}u + \frac{b}{c}v\right)*(cau+cv)\right\}.$$

However, such equalities are dangerous, because quantities of left and right members are computed separately by using different canonical conjugate pairs. Such two elements should be compared through intertwiners mentioned in § 2.1.

Although $e_*^{\pm(\pi/\hbar)uv}$ is defined only by normal ordering expression, the equality (3.15) gives also the following:

Lemma 19. *If $c^2 - ab \neq 0$, then*

$$\exp_*\left\{\pm\frac{\pi}{\hbar}uv\right\} * \exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}$$
$$= \frac{1}{\sqrt{c^2-ab}}\exp\left\{\frac{1}{\hbar(ab-c^2)}(au^2+bv^2+2cuv)\right\}.$$

Proof. Remark $e_*^{\pm(t/\hbar)2uv} = \sqrt{1+s^2}\, e^{(s/\hbar)2uv}$ and if $t \to \pm\frac{\pi}{2}$, then $s \to \infty$. Multiplying $\sqrt{1+s^2}$ to the both sides of (3.15), and take $s \to \infty$. We have the lemma. Q.E.D.

Since linearly related canonical conjugate pairs form an arcwise connected subset, polar elements look like forming connected complex 2-dimensional manifold. In fact, however, we have the following:

Proposition 20. *For every $a,\ b \in \epsilon_{00}$, we have $a * b = \pm 1$, $a * a = -1$. Hence $a = -a^{-1}$ and hence $a = \pm b$ by applying a^{-1}. Consequently, ϵ_{00} forms a single point.*

Proof. By Lemma 8, we see easily that $a * a = -1$. To prove $a * b = \pm 1$, we have only to compute

$$\lim_{c^2 - ab \to \frac{\pi^2}{4}} \Bigg[e_*^{(\pi/\hbar) u*v} * \frac{1}{\cos\sqrt{c^2 - ab}} \times \exp\left\{ \left(\frac{1}{\hbar\sqrt{c^2 - ab}} \tan\sqrt{c^2 - ab} \right) (au^2 + bv^2 + 2cuv) \right\} \Bigg]$$

in the Weyl ordering expression. By Lemma 19, this is rewritten as

$$\lim_{c^2 - ab \to \frac{\pi^2}{4}} \frac{1}{\sin\sqrt{c^2 - ab}} \exp\left\{ - \left(\frac{1}{\hbar} \cot^2 \sqrt{c^2 - ab} \right) (au^2 + bv^2 + 2cuv) \right\}.$$

Since $\cos\theta = 0$ implies $\sin\theta = \pm 1$, the above quantity tends to ± 1. This shows $a * b = \pm 1$.

Since the set $\alpha\delta - \beta\gamma = 1$ is connected, we see that $\{\exp_*\{(\pi/\hbar) \times (\alpha u + \beta v)(\gamma u + \delta v)\}\}$ forms a single element and $a * b = 1$ in fact. Q.E.D.

We denote the polar element by the same notation ϵ_{00}.

Remark. This is a little tricky, because $(-v, u)$ is also a canonical conjugate pair. Hence at the first glance the above result looks like insisting $e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)(-v)*u}$. If this were true, then since $-v * u = -u * v - \hbar i$, we must have

$$e_*^{(\pi/\hbar)(-v)*u} = -e_*^{-(\pi/\hbar)u*v}.$$

However, we have already seen that $\epsilon_{00} * \epsilon_{00} = -1$. This gives $e_*^{-(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)u*v}$, and hence we have $\epsilon_{00} = -\epsilon_{00}$. This looks like a contradiction. Remark however, that $\epsilon_{00} = -\epsilon_{00}$ does not necessarily imply $2\epsilon_{00} = 0$.

In Lemma 23, we will see that ϵ_{00} is expressed as $e_\circ^{(2i/\hbar)u \circ v}$ and $e_\bullet^{(2i/\hbar)(-v) \bullet u}$ by normal ordering expressions with respect to (u, v) and $(-v, u)$ respectively. Thus, we have to use the intertwiner between canonical conjugate pairs (u, v) and $(-v, u)$ to compare $e_*^{(\pi/\hbar)u*v}$ and $e_*^{(\pi/\hbar)(-v)*u}$. Consequently, we have to set $e_*^{(\pi/\hbar)u*v} = -e_*^{(\pi/\hbar)(-v)*u}$.

Although ϵ_{00} forms a single element and $\epsilon_{00} * \epsilon_{00} = -1$, this does not imply that $\epsilon_{00} = i$, because the following holds by the bumping Lemma 13:

Proposition 21. $u * \epsilon_{00} + \epsilon_{00} * u = 0$, $v * \epsilon_{00} + \epsilon_{00} * v = 0$. *In particular,* ϵ_{00} *commutes with every even element.*

This suggests that ϵ_{00} has some super theoretic character [W]. There are several *odd variables* in our system, but a systematic treatment of these will be given some other paper [O,el.4].

On the contrary, the normal ordering expressions of ϵ_{00} with respect to the canonical conjugate pair (u, v) is $e_\circ^{(2i/\hbar)u\circ v}$. Hence the normal ordering expression of $e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)u*(v+a'u)}$, with respect to $(u, v + a'u)$ is $e_\diamond^{(2i/\hbar)u\diamond(v+a'u)}$. Similarly, the normal ordering expressions of $e_*^{(\pi/\hbar)u*v} = e_*^{(\pi/\hbar)(u+b'v)*v}$ with respect to $(u + b'v, v)$ is $e_{\diamond'}^{(2i/\hbar)(u+b'v)\diamond' v}$.

In the Weyl ordering expression, we have had

$$\begin{aligned}\exp_*\left\{\frac{t}{\hbar}(bv^2 + 2cu\circ v)\right\} &= \exp_*\left\{\frac{t}{\hbar}(bv^2 + 2cu * v)\right\}\\ &= \exp\{-c\hbar it\}\exp_*\left\{\frac{t}{\hbar}(bv^2 + 2cu\cdot v)\right\}\\ &= \frac{\exp\{-c\hbar it\}}{\cos ct}\exp\left\{\left(\frac{1}{\hbar c}\tan ct\right)(bv^2 + 2cuv)\right\}.\end{aligned}$$

Thus, we have

Proposition 22. *In the normal ordering expression with respect to* (u, v), *the product*

$$e_*^{(t/\hbar)u^2} * e_*^{(1/\hbar)(bv^2+2cu\circ v)} = e_\circ^{(t/\hbar)u^2} * e_\circ^{(b/4ci\hbar)(e^{4ci}-1)v^2+(1/2\hbar i)(e^{2ci}-1)2u\circ v}$$

is welldefined for every t *as* $e_\circ^{(t/\hbar)u^2+(b/4ci\hbar)(e^{4ci}-1)v^2+(1/2\hbar i)(e^{2ci}-1)2u\circ v}$.

Recall this is defined only for $1 + bt \neq 0$ in the Weyl ordering expression (cf. Lemma 14).

4.3. The case $ab \neq 0$, Proof of the first half of Theorem 2

If $ab \neq 0$ in (4.2), there appear singularities in the $*$-exponential functions, and this gives the first half of Theorem 2. This is because that the exponential function $e_*^{aH+bX+cY}$ is not defined for all $(a, b, c) \in \mathbb{C}^3$ under any ordering expression.

The first equation of (4.2) is

$$\left(\phi_1(t) - \frac{ci}{2\hbar b}\right)' = \frac{ab - c^2}{\hbar b} - 4\hbar b\left(\phi_1(t) - \frac{ci}{2\hbar b}\right)^2.$$

It follows

$$\phi_1(t) = \frac{ic}{2\hbar b} - \frac{\sqrt{c^2 - ab}}{2\hbar b}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0)\right),$$

where t_0 is fixed by the initial condition $\phi_1(0) = 0$, i.e.

$$\sqrt{c^2 - ab}\left(\tan 2\sqrt{c^2 - ab}\,t_0\right) = ic.$$

The forth equation gives that

$$\frac{d}{dt}\psi(t) = \left(\sqrt{c^2 - ab}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0)\right) - ic\right)\psi(t).$$

It follows

$$\psi(t) = e^{-ict}\left(\frac{\cos(2\sqrt{c^2 - ab}\,t_0)}{\cos 2\sqrt{c^2 - ab}\,(t + t_0)}\right)^{1/2}.$$

The third equation

$$\left(\phi_3(t) + \frac{1}{2\hbar i}\right)' = 2\sqrt{c^2 - ab}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0)\right)\left(\phi_3(t) + \frac{1}{2\hbar i}\right)$$

gives

$$\phi_3(t) + \frac{1}{2\hbar i} = \frac{A}{\cos 2\sqrt{c^2 - ab}\,(t + t_0)}$$

where A is fixed by the initial condition $\phi_3(0) = 0$, i.e. $A = \frac{1}{2\hbar i} \times \cos 2\sqrt{c^2 - ab}\,t_0$.

The second equation is

$$\phi_2'(t) = -4\hbar b\left(\phi_3(t) + \frac{1}{2\hbar i}\right)^2 = -4\hbar b A^2\left(\cos 2\sqrt{c^2 - ab}\,(t + t_0)\right)^{-2}.$$

Hence

$$\phi_2(t) = -\frac{2\hbar b A^2}{\sqrt{c^2 - ab}}\left(\tan 2\sqrt{c^2 - ab}\,(t + t_0) - \frac{ic}{\sqrt{c^2 - ab}}\right).$$

If $c^2 - ab = 0$, then the first equation of (4.2) is

$$\left(\phi_1(t) - \frac{ci}{2\hbar b}\right)' = -4\hbar b\left(\phi_1(t) - \frac{ci}{2\hbar b}\right)^2.$$

It follows

$$\phi_1(t) - \frac{ci}{2\hbar b} = \frac{1}{1 + 4\hbar b t}.$$

$$\psi(t) = e^{2\hbar t}(1 + 4\hbar bt)^{-1/(4b)}.$$

Hence, $e_*^{(t/\hbar)(au^2+bv^2+2cuv)}$ with $ab - c^2 = 0$, $ab \neq 0$, is singular at $1 + 4\hbar bt = 0$ in the ΨDO-expression, while this is computed as $e_\cdot^{(t/\hbar)(au^2+bv^2+2cuv)}$ in the Weyl ordering expression.

Some of $*$-products are easy to compute in the normal ordering expression. Note that $e_*^{(t/\hbar)u^2} = e_\cdot^{(t/\hbar)u^2} = e_\circ^{(t/\hbar)u^2}$ and

$$e_\circ^{(t/\hbar)u^2} * e_\circ^{(1/\hbar)(au^2+bv^2+2cuv)} = e_\circ^{(1/\hbar)((a+t)u^2+bv^2+2cuv)},$$
$$e_\circ^{(t/\hbar)u\circ v} * e_\circ^{(1/\hbar)(bv^2+2cuv)} = e_\circ^{(1/\hbar)(t(1+2ic)u^2+bv^2+2cuv)}$$

in the ΨDO-product formula under the normal ordering expression with respect to the canonical conjugate pair (u, v). Remark these are defined for all t.

By these computations, we see also the following:

Lemma 23. *In the normal ordering expression with respect to* (u, v), $e_*^{(\pi/2\hbar)(au^2+bv^2+cu*v)}$ *with* $c^2 - ab = 1$ *is given identically as* $e_\circ^{(2i/\hbar)u\circ v}$.

§5. Proof of Theorem 2

We have already seen the first half of Theorem 2. To prove the second half, we consider the set obtained by gluing M^3 and $\epsilon_{00} * M^3$ by the mapping $\epsilon_{00}*$. We set

$$M_0^3 = \{e_*^{(1/\hbar)(au^2+bv^2+2cuv)};\ c^2 - ab = 0\}.$$

Since ϵ_{00} commutes with every $e^{(1/\hbar)(au^2+bv^2+2cuv)}$, and $\epsilon_{00}{}_*^2 = -1$, Lemma 19 gives that $\epsilon_{00}*$ gives a diffeomorphism of $M^3 - M_0^3$ onto itself, but this can not extend to the whole space:

For a point P of M_0^3, the computation is represented by setting $P = e^{au^2}$. Since

$$e_*^{(t/\hbar)2uv} * e^{au^2} = \sqrt{1+s^2}\, e^{(1/\hbar)(a(1+is)^2u^2+2suv)}, \quad \tan t = s,$$

this is written in the form of $*$-exponential function and hence this is a member of M^3, if $t \neq \pm\frac{\pi}{2}$. However, if $t \to \pm\frac{\pi}{2}$, then $s \to \infty$. Hence, we see that $e_*^{(\pi/\hbar)uv} * e^{au^2}$ can not be a member of M^3, but of $\epsilon_{00} * M^3$.

We show the following in this section:

For Q_1, $Q_2 \in M^3$, *if* $Q_1 * Q_2$ *is not defined in the Weyl ordering expression, then*

$$Q_1 * (\epsilon_{00} * Q_2) = (Q_1 * \epsilon_{00}) * Q_2$$

is defined in the Weyl ordering expression as an element of M^3.

If $Q_1 * Q_2$ is defined, then the product $Q_1 * (\epsilon_{00} * Q_2)$, $(\epsilon_{00} * Q_1) * (\epsilon_{00} * Q_2)$ are defined by $\epsilon_{00} * (Q_1 * Q_2)$, $-Q_1 * Q_2$ respectively. If $Q_1 * Q_2$ is not defined in the Weyl ordering expression, then $-\epsilon_{00} * (Q_1 * (\epsilon_{00} * Q_2))$ is defined.

This shows that $M^3 \cup (\epsilon_{00} * M^3)$ forms a group. We already know that by Theorem 16, M^3 forms a local group, which is locally isomorphic to $SL(2, \mathbb{C})$ and M^3 is embedded in $\mathcal{E}_{2+}(\mathbb{C}^2)$ as

$$M^3 = \left\{ \pm\sqrt{c^2 - ab + 1} \exp\left\{ \frac{1}{\hbar}(au^2 + bv^2 + 2cuv);\ c^2 - ab + 1 \neq 0 \right\} \right\}.$$

It is well known that $SL(2; \mathbb{C})$ is simply connected with the non-trivial discrete center $\{\pm 1\}$.

Since $\pm 1 \in M^3 \cup (\epsilon_{00} * M^3)$, we see that $M^3 \cup (\epsilon_{00} * M^3)$ is isomorphic to $SL(2, \mathbb{C})$.

By the argument in the first paragraph of §3.5, the case that $Q_1 * Q_2$ is not defined in the Weyl ordering expression is represented by the following two cases: Namely,

$$e^{(t/\hbar)u^2} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)},$$
$$\sqrt{1 + s^2}\, e^{(s/\hbar)2uv} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)}$$

are not defined only for $1 + bt = 0$, and $1 - 2cs + (c^2 - ab)s^2 = 0$ respectively.

However using the polar element combined with Lemma 19, we show these are defined by Weyl ordering expressions.

By the computation in Lemma14, we remark first the following:

Lemma 24. *Under the condition* $1 + bt \neq 0$, $\exp_*\{\frac{t}{\hbar}u^2\} * \exp\{\frac{1}{\hbar}(au^2 + bv^2 + 2cuv)\}$ *is a vacuum, if and only if* $\exp\{\frac{1}{\hbar}(au^2 + bv^2 + 2cuv)\}$ *is a vacuum, i.e.* $c^2 - ab + 1 = 0$.

Lemma 19 is used for the computation of

$$e^{(t/\hbar)u^2} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)},$$
$$\sqrt{1 + s^2}\, e^{(s/\hbar)2uv} * \sqrt{c^2 - ab + 1}\, e^{(1/\hbar)(au^2 + bv^2 + 2cuv)}$$

for $1 + bt = 0$, and $1 - 2cs + (c^2 - ab)s^2 = 0$ respectively.

Corollary 25. *If* $1 + bt = 0$, *then*

$$\exp_*\left\{ \pm\frac{\pi}{\hbar}uv \right\} * \exp\left\{ \frac{t}{\hbar}u^2 \right\} * \sqrt{c^2 - ab + 1} \exp\left\{ \frac{1}{\hbar}(au^2 + bv^2 + 2cuv) \right\}$$
$$= \exp\left\{ \frac{1}{\hbar(c^2 - ab + 1)}((ci - 1)^2 tu^2 + bv^2 + 2(c - ibt)uv) \right\}$$

and the right hand side is written in the form $e^{(1/\hbar)(\alpha u+\beta v)^2}$.

If $1-2cs+(c^2-ab)s^2=0$, *then remarking*

$$\begin{aligned}&\big(c-(c^2-ab-1)s-cs^2\big)^2-ab(1+is)^2(1-is)^2\\&\quad=\big((c^2-ab+1)(1+s^2)-((c^2-ab)s^2-2cs+1)\big)\\&\qquad\times\big((c^2-ab)s^2-2cs+1\big)\end{aligned}$$

we have

$$\begin{aligned}&\exp_*\left\{\pm\frac{\pi}{\hbar}uv\right\}*\sqrt{1+s^2}\,\exp\left\{\frac{s}{\hbar}2uv\right\}\\&\quad*\sqrt{c^2-ab+1}\,\exp\left\{\frac{1}{\hbar}(au^2+bv^2+2cuv)\right\}\\&=\exp\left\{\frac{1}{\hbar(c^2-ab+1)(1+s^2)}\right\}\\&\quad\times\left(a(1+is)^2u^2+b(1-is)^2v^2+(c-(c^2-ab-1)s-cs^2)2uv\right).\end{aligned}$$

The discriminant of the right hand side vanishes, and hence it is written in the form $e^{(1/\hbar)(\alpha u+\beta v)^2}$.

This completes the proof of Theorem 2.

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Department of Mathematics
Faculty of Science and Technology
Science University of Tokyo
Noda, Chiba, 278
Japan
omori@ma.noda.sut.ac.jp