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Differential Algebra and Differential Geometry

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§1. Introduction

There are now two theories devoted to partial differential equations in the algebraic or analytic domain:

On one side, the theory of involutive differential systems, based on the Cartan-Kähler theorem, and developed namely by Matsushima, Kuranishi, Guillemin-Singer-Sternberg, Quillen, Goldschmidt. This theory is of constant use in differential geometry, f.i. in the study of the "equivalence problems" in the sense of E. Cartan.

On the other side, the "differential algebra" of Ritt, Kolchin, and others, which studies the differential ideals and their properties of finiteness, dimension, etc. cf. [Ri 1], [Ko]. There is a nice application by Buium [Bu] to some problems of algebraic geometry on functions fields. A part from this application, this theory seems to have had practically no contact with geometry, especially with differential geometry; compare f.i. the bibliographies of [Ko] and [B-C-G 3]: their intersection is empty; see however [Po].

It seems to me that a mutual interaction should be useful for both theories. For instance, with the help of the ideas of Ritt, one can prove rather easily the "generic involutiveness" of analytic systems of p.d.e.'s; see a precise statement in §3. Hopefully, the result could be useful in several contexts, namely in the theory of Lie groupoids and in differential Galois theory; I will develop this point elsewhere.

On the opposite side, I mention only the following fact: differential algebraists use classically Riquier-Janet theory of "passive orthonormic systems" rather than Cartan involutiveness. But it seems that they are now becoming aware of this last theory; see f.i. [Se].

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§2. *D*-varieties

(2.1). — For a general theory of analytic p.d.e.'s, one needs a context generalizing both analytic spaces and D-modules (which correspond to the linear case). Here, I will describe the formalism adopted in [Ma 1]; later, I discuss briefly some other possibilities.

A few words of informal explanations: we are interested to systems of equations of the form $f_k(x_i, \partial^\alpha y_j) = 0$, $1 \leq i \leq n$, $1 \leq j \leq p$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| \leq l$ with, as usual $\partial_i = \frac{\partial}{\partial x_i}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$; the f_k are supposed analytic in all the variables, which we denote x_i, y_j^α.

In the course of the study, one has to differentiate the equations, with the usual derivations: $D_i f = \frac{\partial f}{\partial x_i} + \sum \frac{\partial f}{\partial y_j^\alpha} y_j^{\alpha+\varepsilon_i}$, $\varepsilon_i = (0, \dots, 1, \dots, 0)$. We note the following fact: suppose f of order $\leq k$, i.e. the y_j^α occuring in f verify $|\alpha| \leq l$. Then, when we differentiate f as many times as we want, the y_j^α, $|\alpha| \geq l+1$ occur only *in polynomial form.* Now, by the usual trick of adding some derivatives as new functions, we can suppose that all our equations are polynomial in the y_j^α, $|\alpha| \geq 1$.

This is the point of view adopted by Ritt himself in [Ri 2]. However, he considers only local situations, and we want global objects; this explains the definitions below.

In this pages, as in [Ma 1], I will call (analytic) *variety* what is called in the literature $\mathbb{C}$-analytic space in the sense of Grothendieck [Gr]; a priori, I will not suppose a variety smooth, and not even reduced (= without nilpotent elements). However, in the applications to differential equations, only the reduced case will be really interesting.

Let Y be a variety; I note $|Y|$ the underlying topological space, and $\mathcal{O}_Y$ the structural sheaf. By definition, an *affine variety* Z *over* Y is defined by the ringed space $(|Y|, \mathcal{A})$, with $\mathcal{A}$ and $\mathcal{O}_Y$-algebra of locally finite presentation, i.e. verifying the following property: over a small open set $U \subset |Y|$, one has

$$\mathcal{A} = \mathcal{O}_Y[t_1, \dots, t_n]/(f_1, \dots, f_m), \quad \text{with} \quad f_i \in \Gamma(U, \mathcal{O}_Y[t_1, \dots, t_n]).$$

If we have two such varieties over Y; $Z = (|Y|, \mathcal{A})$ and $Z' = (|Y|, \mathcal{A}')$, a morphism $Z \to Z'$ is, of course, a morphism of $\mathcal{O}_Y$-algebras: $\mathcal{A}' \to \mathcal{A}$; if Z' is affine over Y', one defines in the same way a morphism $Z \to Z'$ over a morphism on $Y \to Y'$. I will write often $\mathcal{O}_Z$ instead of $\mathcal{A}$, although this is a little bit confusing ($\mathcal{O}_Z$ is a sheaf on $|Y|$); on the other hand, I denote Z^{an} the analytic space specan Z [Ho]. If π denotes the projection $|Z^{\text{an}}| \to |Y|$, one has a natural map $\pi^{-1}\mathcal{O}_Z \to \mathcal{O}_{Z^{\text{an}}}$. If we have a morphism $Z \to Z'$ of affine varieties over Y, we say that "Z is an affine variety over Z'". As usual in algebraic geometry, we say that "$Z \to Z'$

is dominant" if the corresponding morphism of (sheaves of) rings is *injective.* Note that this does not imply that $Z^{\mathrm{an}} \to Z'^{\,\mathrm{an}}$ is surjective: this is only true generically, in an obvious sense.

(2.2). — Now, let Y_0 be a variety: a "projective system of affine varieties over Y_0" is a collection $Y_i \xrightarrow{\psi_i} Y_0$ of affine varieties over Y_0 $(i = 1,\ 2, \ldots)$, with a family of morphisms $Y_i \xrightarrow{\pi_i} Y_{i-1}$; one has $\psi_{i-1}\pi_i = \psi_i$ $(i \geq 1)$, and $\psi_1 = \pi_1$; we say that $\{Y_i\}$ is an *affine provariety over* Y_0 if the morphisms φ_i are dominant. If we have two affine provarieties $\{Y_i\}$, $\{Z_i\}$, a strict morphism is defined by an analytic morphism $u_0 \colon Y_0 \to Z_0$ and morphisms $u_i \colon Y_i \to Z_i$ over u_0, with the condition of commutativity of the obvious diagram

$$\begin{array}{ccc} Y_i & \longrightarrow & Y_{i-1} \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & Z_{i-1} \end{array}$$

(I use the word "strict", since there is a weaker notion of morphisms; cf. loc. cit., or below).

(2.3). — Now, I can define a D-variety, as the object naturally associated to a system of analytic p.d.e's (including all its prolongations); it is defined by the following datas

i) A variety X, which is supposed non singular (and, in particular, reduced)
ii) A variety Y_0, may be singular, provided with a morphism $p \colon Y_0 \to X$
iii) An affine provariety $Y = (Y_i, \pi_i)$ over Y_0; we note $\mathcal{O}_Y = \varinjlim \mathcal{O}_{Y_i}$
iv) A derivation (or "connection") $D \colon \mathcal{O}_Y \to p^{-1}\Omega^1_X \otimes_{p^{-1}\mathcal{O}_X} \mathcal{O}_Y$, where Ω^1_X denotes, the differential 1-forms over X.

These datas are submitted to conditions which will be described below. Before to do it, we need a definition: if we have two such datas (X, Y, D) and (X, Z, D), with same X, a strict D-morphism is defined by a morphism $u_0 \colon Y_0 \to Z_0$ of analytic varieties, commuting with the projections over X, and a strict morphism $u \colon Y \to Z$ over u_0; these datas should commute with the derivation D.

Suppose now (X, Y, D) given; let X' (resp. Y_0') an open subvariety of X (resp. Y_0), with $p|Y_0'| \subset |X'|$; one defines in an obvious way the restriction (X', Y', D) of (X, Y, D) to (X', Y_0').

With these definitions, a D-variety is a system (X, Y, D) with the following property: for every $y \in |Y_0|$, one can find a pair (X', Y_0'), with $y \in |Y'_0|$ such that the restriction of (X, Y, D) à (X', Y'_0) is strictly isomorphic to a model which we will now describe.

To do that, we give

i) An open set $U \subset \mathbb{C}^n$, of coordinates $(x_1, \dots, x_n)$
ii) An open set $V \subset \mathbb{C}^n$, of coordinates $(y_1, \dots, y_p)$.

We put $\mathcal{A}_0 = \mathcal{O}_{U\times V}$, the sheaf of holomorphic functions on $U \times V$. For $l \geq 1$, we put $\mathcal{A}_l = \mathcal{O}_{U\times V}[y_j^\alpha]$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $1 \leq |\alpha| \leq l$ and $\mathcal{A} = \varinjlim \mathcal{A}_l$.

On $\mathcal{A}$, one has a natural derivation $Df = \sum dx_i \otimes D_i f$, D_i as in (2.1).

Now, let $\mathcal{J}$ be a sheaf of ideals of $\mathcal{A}$, which is *differential*, e.g. stable by the D_i's and *pseudocoherent*, e.g. the $\mathcal{J}_l = \mathcal{J} \cap \mathcal{A}_l$ are coherent.

Now the model is as follows: one takes $X = U$, $Y_0 =$ the closed analytic subspace of $U \times V$ defined by $\mathcal{J}_0$; p is induced by the projection $U \times V \to U$; for $i \geq 1$, Y_i is the affine variety over Y_0 defined by $\mathcal{A}_l/\mathcal{J}_l$. Finally, D is defined in the obvious way by the "D" given on $\mathcal{A}$.

A D-variety is reduced if all the Y_i are reduced $=$ the corresponding sheaves have no nilpotent element. If we have a D-variety (X, Y, D), $Y = \{Y_i\}$, one defines naturally its "reduction" (X, Y^{red}, D), with $Y^{\text{red}} = \{Y_i^{\text{red}}\}$. Using the local model, the reader will find the following interpretation of a reduced D-variety, in terms of differential equations: the points of $|Y_l^{\text{an}}|$ are the jets of order l of solutions, and the points of $|Y^{\text{an}}| = \varprojlim_l |Y_l^{\text{an}}|$ are the formal solutions. Note that the maps $|Y_l^{\text{an}}| \to |Y_{l-1}^{\text{an}}|$ are only generically surjective; this explain the interest of the results of the next section.

§3. Formal integrability and generic involutiveness

To express these properties, it is simpler to work in a local model: so, let U, V, $\mathcal{A}$ and $\mathcal{J}$ as before, and suppose that $\mathcal{J}$ is reduced, i.e. the $\mathcal{J}_l$ are equal to their radical. One has the following theorems

Theorem 3.1. *Let $U' \subset U$ and $V' \subset V$ be polycylinders relatively compacts in U and V; then there exists $l \geq 1$ and $f \in \Gamma(U' \times V', \mathcal{A}_l)$ with the following properties*

i) *On $U' \times V'$, f is injective on $\mathcal{A}/\mathcal{J}$.*
ii) *Outside of $f = 0$, $\mathcal{J}_l$ is involutive, and the $\mathcal{J}_k$, $(k \geq l+1)$ are the prolongation of $\mathcal{J}_l$.*

For the notion of involutiveness (smoothness+formal integrability+ acyclicity), I refer f.i. to Goldschmidt [Go]; I leave it to the reader to translate these notions in the present context; this translation can be made in terms of $Y_k^{\mathrm{an}} - \{f = 0\}$, or more precisely in terms of $\mathcal{A}[f^{-1}]$ and $\mathcal{J}[f^{-1}]$.

Theorem 3.2. *An increasing sequence of differential pseudocoherent and reduced ideals of $\mathcal{A}$ is stationary on every relatively compact $U' \times V' \subset U \times V$.*

The first theorem express the "generic involutiveness" of reduced differential ideals; the second theorem is the version in our context of the finiteness theorem of Ritt-Raudenbush. We note that (3.2) has been already proved by Ritt [Ri 2] in the case where we take germs at a point $a \in U \times V$.

Theorems 3.1 and 3.2 are proved simultaneously; the main lines of the proof can be found in [Ma 2]; complete proofs will be given later. Roughly speaking, the idea is the following: we take for U' and V' *closed* polydiscs, instead of open ones; according to Cartan-Oka theorems, $\mathcal{J}$ is determined on $U' \times V'$ by its global sections; and, according to Frisch theorem, the $\Gamma(U' \times V', \mathcal{A}_l)$ are *noetherian rings*; so, we have only to study reduced differential ideals $\mathfrak{p}$ of $\Gamma(U' \times V', \mathcal{A})$; one proves successively the following results, which imply easily (3.1) and (3.2):

i) Theorem 3.1 is true when $\mathfrak{p}$ is prime.
ii) Any increasing sequence of reduced differential ideals $\mathfrak{p}$ is stationary.
iii) Any such $\mathfrak{p}$ is a finite intersection of primes.

The main point is i). Then ii) follows by an argument of differential algebra to be found, f.i. in [Ka]. Finally, ii) $\Rightarrow$ iii) is standard.

§4. General morphisms

In many problems, one has to consider two kinds of transformations which cannot be represented by the "strict morphisms" considered in §2.

A) Transformations of the type $z_k = f_k(x, y_j^\alpha)$ [and, of course $z_k^\alpha = D^\alpha f_k(x, y_j^\beta)$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$]. These transformations are called classically "Lie Bäcklund transformations".

B) Change of independent variables, f.i. Legendre transformation where y' is taken as the new independent variable; this is more generally the case when the system is given as an exterior differential system "with independence condition" in the sense of [B-C-G 3].

Concerning **A)**, let me first mention that these transformations are very simple to express in a more special context, the "affine D-varieties": they are given by families (X, Y, D), with X $\mathbb{C}$-analytic smooth, and the system $X \leftarrow Y_0 \leftarrow \cdots \leftarrow Y_l \leftarrow \cdots$ an affine provariety over X; the local models are given by pseudocoherent differential ideals $\mathcal{J}$ of $\mathcal{O}_X[y_j^\alpha]$, $|\alpha| \geq 0$. [In other words, these "varieties" represent differential systems which are polynomial in all the $\partial^\alpha y_j$. This context is sufficient for many applications; but, f.i. it would not contain the equation $y' = e^y$.]

If we have another affine D-variety (X, Z, D), with the same basis X, a Lie-Bäcklund transformation or "morphism" $(X, Y, D) \to (X, Z, D)$ is simply given by a morphism of $\mathcal{O}_X$-algebras $u\colon \mathcal{O}_Z \to \mathcal{O}_Y$, commuting with D. In the interesting cases there will be an $l \geq 0$ such that $u(\mathcal{O}_{Z_0}) \subset \mathcal{O}_{Y_l}$, and therefore $u(\mathcal{O}_{Z_k}) \subset \mathcal{O}_{Y_{l+k}}$, $k \geq 0$ (use commutation with D); we will say that "u is of order $\leq l$". One can express this in another way; call $Y(l)$ the affine system over X defined by $Y(l)_k = Y_{k+l}$; then a morphism of order $\leq l$, $(X, Y, D) \to (X, Z, D)$ is simply defined by a strict morphism $(X, Y(l), D) \to (X, Z, D)$; if $m \geq l$, a morphism of order on $(X, Y(m), D) \to (X, Z, D)$ is identified with the preceding one if it is obtained by composition with the obvious morphism $(X, Y(m), D) \to X, Y(l), D)$ given by the identity on the structure sheaf (note that both spaces have the same structure sheaf).

In the context of D-varieties, I will copy the last procedure: I define $Y(l)$ by $Y(l)_0 = Y_l^{\mathrm{an}}$, the "analytic spectrum" of Y_l, and $Y(l)_k = Y_l^{\mathrm{an}} \times_{Y_l} Y_{l+k}$, the "analytisation up to order l" (see [Ma 1] for more details). Then the morphisms are defined as the previous morphisms of finite order. This analytisation procedure is a little bit unpleasant; but, due to the good properties of analytisation, things behave in a reasonable good way. For instance, one can prove that the characteristic variety is invariant, outside of the zero section, by general isomorphisms cf. [Ma 1]; this generalizes the well-known result of "independence of the filtration" of the characteristic variety in the linear case (= in the theory of D-modules).

Concerning **B)**, the point of view of affine D-varieties is obviously irrelevant, since any change of the independent variables would destroy the affine structure. On the other hand, to include such changes, the category of D-varieties is "suitable" in the following sense: it has to be enlarged, but no new local model is needed; cf. [Ma 1], §4; this is the main reason for which I have adopted this point of view.

Another point of view which is also adapted to **A)** and **B)** consists in a separation of "analytic" and "algebraic" variables in Y_0. More precisely, one takes Y_0' analytic over X, and Y_0 affine over Y_0', then $Y_l \to \cdots \to Y_1 \to Y_0 \to Y_0'$ an affine provariety; the local models are made with such Y's, like in the preceding cases (I omit the details); here, $Y(l)$ is defined in the following way: $Y(l)_0' = Y_0'$; $Y(l)_k = Y_{l+k}$, $k \geq 0$; no analytisation is required.

One could improve also this model by taking Y_0 algebraic over Y_0', e.g. a relative schema in the sense of [Ha] (this is made by gluing affine models, as schemes are defined by gluing affine varieties or schemes; but here the "gluing" is more sophisticated, and require 2-categories; therefore the simplicity of morphism is "compensated" by a greater difficulty of definition). Of course, as in local models, $Y_0 \to Y_0'$ is affine, and there is here nothing new locally; f.i. the results of §3 are still true.

Generally speaking, it seems to me that the "good" definition of D-varieties one should take depends on the problem to be studied. I will give an example in a forth coming paper about "Lie groupoids", i.e. p.d.e's in the space of *invertible* jets $X \to X$, whose solutions form a groupoid (the invertibility of jets forces to change slightly the previous definitions). Here, the generic involutiveness shows that, at the general points, they coincide with the "infinite groups" of Lie and Cartan. But the consideration of singular points seems to me very important, and essentially overlooked in the literature.

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