Advanced Studies in Pure Mathematics 35, 2002
Higher Dimensional Birational Geometry
pp. 1–88

Characterizations of Projective Space and Applications to Complex Symplectic Manifolds

Koji Cho, Yoichi Miyaoka[1] and N. I. Shepherd-Barron

Dedicated to Professor Tetsuji Shioda on his 60th birthday

Abstract.

We obtain new criteria for a normal projective variety to be projective n-space. Our main result asserts that a normal projective variety which carries a closed, doubly-dominant, unsplitting family of rational curves is isomorphic to projective space. An immediate consequence of this is the solution of a long standing conjecture of Mori and Mukai that a smooth projective n-fold X is isomorphic to $\mathbb{P}^n$ if and only if $(C, -\mathrm{K}_X) \geq n+1$ for every curve C on X. As applications of the criteria, we study fibre space structures and birational contractions of compact complex symplectic manifolds.

Contents

Received December 15, 2000.

[1] Partially supported by Grant-in-Aid #01540066, #06452007, #08640029 (Japanese Ministry of Education, Science, Sports and Culture) and #12440006 (Japan Society for Promotion of Sciences).

Introduction

Projective n-space $\mathbb{P}^n$ is the simplest n-dimensional algebraic variety and can accordingly be characterized in various ways. The main objectives of the present paper are:

A. To establish new characterizations of projective n-space in such a way that all the known characterizations are thereby systematically explained;

B. To apply our characterizations to morphisms from complex symplectic manifolds;

and, as prerequisites to the above two,

C. To provide a self-contained exposition of basic theory of families of rational curves, which is important for understanding detailed structure of rationally connected varieties.

Let X be a projective variety and $\mathrm{Chow}(X)$ the Chow scheme (see Section 1 below). Let $S \subset \mathrm{Chow}(X)$ be an irreducible subvariety and $\mathrm{pr}_S : F \to S$ the associated universal family.[2]

We say that F is a *closed family of rational curves* if S is proper and the fibre $F_s = \mathrm{pr}^{-1}(s) \subset \{s\} \times X \simeq X$ over a general point $s \in S$ is an irreducible, reduced rational curve as an effective 1-cycle.[3] Any special fibre of a family of rational curves is a 1-cycle supported by a union of rational curves. A closed family of rational curves $F \to S$ is called *maximal* if F is a union of irreducible components of F' for any family of rational curves $F' \supset F$. When every fibre F_s is irreducible and reduced (as 1-cycles), we say that F is *unsplitting.* A family of rational curves F is *dominant* if the natural projection $\mathrm{pr}_X : F \to X$ is surjective. F is *doubly dominant* if $\mathrm{pr}^{(2)}_{X\times X} : F \times_S F \to X \times X$ is surjective.

---

[2] $F \subset S \times X$ is a Zariski closed subset with equidimensional fibres over S. The family F is proper over S, but not necessarily flat. Although we give a brief overview on Chow schemes and the associated universal families in Section 1, we refer the reader to [Kol] for full exposition and discussion.

[3] However, the scheme theoretic fibre F_s can contain 0-dimensional embedded components.

Main Theorem 0.1. *Let X be a normal projective variety defined over the complex number field $\mathbb{C}$ (or over an algebraically closed field of characteristic zero). If X carries a closed, maximal, unsplitting, doubly dominant family $\mathrm{pr}_S : F \to S$ of rational curves, then X is isomorphic to projective n-space $\mathbb{P}^n$, and F is the family of the lines on X parameterized by the Grassmann variety $S = \mathrm{Grass}(\mathbb{P}^n, 1)$.*

Roughly speaking, this theorem means that X is a projective space if and only if its two general points can be joined by a single rational curve of minimum degree (i.e., a line) with respect to a polarization of X. If we impose a slightly weaker condition than in Theorem 0.1, we have the following result.

Theorem 0.2. *Let X be a normal projective variety of dimension n over $\mathbb{C}$ and x a prescribed general point on it. Let $\mathrm{pr}_S : F \to S$ be a closed, maximal, doubly-dominant family of rational curves on X, and write $F\langle x\rangle \to S\langle x\rangle$ for the closed subfamily consisting of curves passing through x. If $F\langle x\rangle$ is unsplitting, then X is a quotient of $\mathbb{P}^n$ by a finite group action without fixed point locus of codimension one. In particular, X is $\mathbb{P}^n$ if it is smooth.*

A smooth projective variety X is said to be a *Fano manifold* if its anticanonical divisor $-\mathrm{K}_X$ is ample. Our Main Theorem 0.1 yields a simple numerical criterion for a Fano manifold to be projective space in terms of the *length* $l(\cdot)$ of rational curves:

Corollary 0.3 (Conjecture of Mori and Mukai). *Let X be a smooth complex Fano n-fold. Put*

$$l(X) = \min\{(C, -\mathrm{K}_X); C \subset X \textit{is a rational curve}\}.$$

Then X is isomorphic to $\mathbb{P}^n$ if and only if $l(X) \geq n+1$.

Our criterion (Theorem 0.1), stated in terms of geometry of rational curves, is strong enough to yield a whole series of characterizations of projective n-space expressed in very different languages:

Corollary 0.4. *Let X be a complex projective manifold of dimension n and $x_0 \in X$ a general point. Then the following fourteen conditions are equivalent:*

1. $X \simeq \mathbb{P}^n$;
2. Hirzebruch-Kodaira-Yau condition [HK]: *X is homotopic to $\mathbb{P}^n$*;
3. Kobayashi-Ochiai condition [KO]: *X is Fano and $c_1(X)$ is divisible by $n+1$ in $H^2(X, \mathbb{Z})$*;
4. Frankel-Siu-Yau condition [SY]: *X carries a Kähler metric of positive holomorphic bisectional curvature*;

5. Hartshorne-Mori condition [Mo1]: *The tangent bundle* Θ_X *of* X *is ample*;
6. Mori condition [Mo1]: X *is uniruled and* $\Theta_X|_C$ *is ample for an arbitrary rational curve* C *on* X;
7. Doubly transitive group action: *The action of* Aut(X) *on* X *is doubly transitive*;
8. Remmert-Van de Ven-Lazarsfeld condition [La]: *There exists a surjective morphism from a suitable projective space onto* X;
9. Length condition: $(C, -\mathrm{K}_X) \geq n+1$ *for every curve* C *on* X;
10. Length condition on rational curves: X *is uniruled and* $(C, -\mathrm{K}_X) \geq n+1$ *for every rational curve* C *on* X;
11. Length condition on rational curves with base point: X *is uniruled and* $(C, -\mathrm{K}_X) \geq n+1$ *for every rational curve* C *containing the prescribed general point* x_0;
12. Double dominance condition on rational curves: X *is uniruled and every reduced irreducible rational curve on* X *is a member of a doubly dominant family of rational curves*;
13. Double dominance condition on rational curves of minimum degree: X *is uniruled and a rational curve of minimum degree (with respect to an arbitrary fixed polarization) on* X *is a member of a doubly dominant family of rational curves*;
14. Dominance condition on rational curves with base point: *Every rational curve* C *passing through* x_0 *is a member of a dominant family* $F = \{C_t\}$ *of rational curves* $\{C_t\}$ *passing through the base point* x_0.

Although our result (Theorem 0.1) is far stronger than the results known before, we are not completely independent of the preceding works. Our basic strategy is in fact very similar to the argument used in [Mo1]. Given a closed, unsplitting, doubly dominant family $F \to S$ of rational curves, consider the subfamily $F\langle x\rangle \to S\langle x\rangle$. We prove that the projection $\mathrm{pr}_X : F\langle x\rangle \to X$ is actually the blow-up $\mathrm{Bl}_x(X)$ of X at x, the base variety $S\langle x\rangle$ being isomorphic to the associated exceptional divisor $E_x \simeq \mathbb{P}^{n-1}$.

If one knows that *every* point of $S\langle x\rangle$ represents a curve smooth at x, then the birationality of pr_X follows from an elementary argument (see Proposition 2.7 below). But it is by no means obvious that this smoothness condition is always satisfied. On the contrary, when $S\langle x\rangle$ happens to contain a point which represents a curve singular at x, then $F\langle x\rangle$ is never birational to X. Thus we need to rule out the existence of such bad points in S, which is done with the aid of a theorem of Kebekus

(Theorem 3.10) saying that no point of S can represent a curve which has a cuspidal singularity at the base point x.

Our characterization of projective n-space (Theorem 0.1) provides intriguing information on complex symplectic manifolds. Given a compact complex symplectic manifold Y of dimension $2n$ and an arbitrary non-constant morphism $f : \mathbb{P}^1 \to Y$, one can show that $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) \geq 2n+1$. If one knows that $f_t(\mathbb{P}^1)$ stays in a fixed n-dimensional subvariety $X \subset Y$ for any (small) deformation f_t of f, then the Main Theorem 0.1 implies that the normalization of X is necessarily $\mathbb{P}^n$. This is indeed the case in some important situations, imposing very restrictive constraints on fibre space structure of, or birational contractions from, complex symplectic manifolds. Specifically, we completely understand the symplectic resolutions of a normal projective variety with only isolated singularities. For precise statements, see Theorems 7.2, 8.3 and 9.1 below.

This paper is organized as follows:

Part I, consisting of three sections, is a review of general theory concerning families of rational curves on projective varieties. This theory is expected to be a useful tool to analyse the structure of uniruled varieties. We need here nothing very special; almost every result derives from well known geometry of ruled surfaces modulo general theory of Chow schemes and deformation.

In Section 1, we recall basic concepts and facts necessary for, and/or closely related to, the family of rational curves. Most results there are more or less known to experts, yet they are included for the coherence of the account and for the convenience of the reader.

Section 2 discusses unsplitting families of rational curves. The unsplitting condition is a very strict constraint on the family, and we obtain various estimates of the dimension of the parameter space S.

Section 3 is the survey of a recent result by Kebekus [Ke1] and [Ke2] on unsplitting families of singular rational curves. It asserts among other things that, if $F \to S$ is an unsplitting family of rational curves on a projective variety X, then no member C of S has a cuspidal singularity at a general fixed point $x \in X$.

Part II (Sections 4 and 5) treats characterizations of projective n-space. The Main Theorem 0.1 as well as Theorem 0.2 is proved in Section 4. Given a closed, doubly-dominant family of rational curves $F \to S$ which is unsplitting at a general point x (i.e., we assume that the subfamily $F\langle x\rangle \to S\langle x\rangle$ is unsplitting), we argue that the normalization of $F\langle x\rangle$ is isomorphic to a one-point blow up of $\mathbb{P}^n$.

The relationship between the various conditions in Corollary 0.4 is discussed in Section 5.

Part III (Section 6 through 9) contains applications of the Main Theorem to compact complex symplectic manifolds.

In Section 6, we review generalities on compact complex symplectic manifolds. One of the key observations is that any holomorphic map from a rational curve to a $2n$-dimensional symplectic manifold moves in a family with at least $2n+1$ independent parameters.

The first application of our Main Theorem is to fibre space structure of primitive complex symplectic manifolds. Matsushita [Mats] showed that, if such a manifold has a nontrivial fibre space structure, it must be a Lagrangian torus fibration over a $\mathbb{Q}$-Fano variety. In Section 7, we see that the base space is necessarily a projective space, provided the fibration admits a global cross section.

The second application is to birational contractions. Let Z be a compact complex symplectic manifold of dimension $2n$ and $f : Z \to \hat{Z}$ a birational contraction to a normal variety. Let $E_i \subset Z$ be an irreducible component of the exceptional locus and $B_i \subset \hat{Z}$ its image. Then we verify that the base variety B_i is again a symplectic variety of dimension $2(n-a_i)$ (possibly with singularities) and a general fibre X of the projection $E_i \to B_i$ is a union of copies of projective a_i-space. In case $a_i > 1$, X is indeed a single smooth $\mathbb{P}^{a_i}$, and the local analytic structure of $f : Z \to \hat{Z}$ is uniquely determined on a small open neighbourhood of X in Z. In order to simplify the argument, we first deal with isolated singularities (Section 8) and then general singularities (Section 9). One of the key results (unramifiedness of the normalization) is proved in Section 10.

Throughout the article, all schemes are assumed to be separated. Schemes and varieties are usually defined over $\mathbb{C}$, or, more generally, over an algebraically closed field k of characteristic zero. The assumption on the characteristic is made because we use Sard's theorem in an essential way. As far as the authors know, it is still an open problem if our results (Theorems 0.1 and 0.2) stay true in positive characteristics.

Acknowledgement

This belated joint work was originally intended as the second author's contribution for the proceedings of a conference on algebraic geometry organized by Laura Livorni and others at University of L'Aquilla, Italy, in May, 1992, but could not be duly completed. Details were gradually and painfully worked out at several places: University of Warwick, Coventry; University of Utah, Salt Lake City; MSRI, Berkeley;

Kyushu University, Fukuoka; Rikkyo University, Tokyo; Mathematisches Forschungsinstitut, Oberwolfach; and RIMS, Kyoto. The authors express their gratitude to these institutions for hospitality. They are indebted to Shigefumi Mori and Shigeru Mukai for the motivation. They sincerely appreciate generous advice and comments given by Mori, János Kollár, Rob Lazarsfeld, Eichi Sato, Yujiro Kawamata, Stefan Kebekus and Frédéric Campana, some of whom also pointed out several serious gaps in older versions. They are thankful to Miles Reid, Daisuke Matsushita and Noboru Nakayama for careful reading and frank criticism.

Notation

In the present work, standard notation in algebraic geometry is freely used. For instance, $\mathrm{Hom}(Y, X)$, $\mathrm{Chow}(X)$ and $\mathrm{Hilb}(X)$ stand for the Hom scheme, the Chow scheme and the Hilbert scheme parameterizing the morphisms, the effective cycles and the subschemes, respectively. A curve $C \subset X$ or a morphism $f : Y \to X$ is denoted by $[C]$ or $[f]$ when viewed as an element of $\mathrm{Chow}(X)$ or $\mathrm{Hom}(Y, X)$.

We list below some of the *ad hoc* symbols which frequently appear in the article.

- $\overline{\bullet}$ and $\bullet^c$ generally stand for the normalization and the closure of $\bullet$.
- $\nu_\bullet : \overline{\bullet} \to \bullet$ will denote the normalization map.
- $S :$ a closed subset of $\mathrm{Chow}(X)$ consisting of (unions of) rational curves.
- $S\langle x \rangle$: the closed subset of S consisting of members which pass through a prescribed closed point $x \in X$.
- F, $F\langle x \rangle$: the family of rational curves on X parameterized by S and $S\langle x \rangle$.
- $\mathrm{pr}_S : F \to S$, $\mathrm{pr}_X : F \to X$: the natural projections.
- $\overline{F}$, $\overline{S}$, $\overline{F}\langle x \rangle$, $\overline{S}\langle x \rangle$: the normalizations of F, S, $F\langle x \rangle$, $S\langle x \rangle$, with natural projections $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$, $\overline{\mathrm{pr}}_{\overline{S}\langle x \rangle} : \overline{F}\langle x \rangle \to \overline{S}\langle x \rangle$, $\overline{\mathrm{pr}}_X : \overline{F}, \overline{F}\langle x \rangle \to X$.
- $\mathrm{Bl}_Y(X)$: the blowup of X along a closed subscheme Y.
- E_Y: the exceptional divisor on $\mathrm{Bl}_Y(X)$.
- $\mathrm{pr}_{\tilde{X}} : F\langle x \rangle \dashrightarrow \tilde{X} = \mathrm{Bl}_x(X)$: the natural rational map induced by the X-projection $\mathrm{pr}_X : F\langle x \rangle \to X$.

PART I. Families of Rational Curves on Projective Varieties

1. Review of basic concepts and results

In this section, we recall basic concepts and results such as deformation theory, Chow schemes and Mori's bend and break technique. Nothing very special or new appears here and experts can skip the whole section.

A. Cotangent sheaves, Zariski tangent spaces and infinitesimal deformation of morphisms

Let A be a scheme and X an A-scheme (assumed to be separated as usual). The diagonal $\Delta = \Delta_{X/A} \subset X \times_A X$ is a closed subscheme defined by the ideal sheaf $\mathfrak{I}_\Delta$. We regard $\mathcal{O}_{X\times_A X} = \mathcal{O}_X \otimes_{\mathcal{O}_A} \mathcal{O}_X$ as a left $\mathcal{O}_X$-module via the multiplication to the first factor. $\mathfrak{I}_\Delta^k$, $k = 0, 1, 2, \dots$ are naturally $\mathcal{O}_X$-modules. We have a canonical direct sum decomposition of $\mathcal{O}_{X\times_A X} = \mathfrak{I}_\Delta^0$ into the direct sum $\mathcal{O}_X \oplus \mathfrak{I}_\Delta$ as a left $\mathcal{O}_X$-module by virtue of the two canonical homomorphisms $\mathrm{pr}_1^* : \mathcal{O}_X \to \mathcal{O}_{X\times_A X}$, $\mathrm{rest}_\Delta : \mathcal{O}_{X\times_A X} \to \mathcal{O}_\Delta \simeq \mathcal{O}_X$.[4]

The coherent sheaf $\mathfrak{I}_\Delta/\mathfrak{I}_\Delta^2$ is called the *sheaf of Kähler differentials* or the *sheaf of relative* 1*-forms*, and denoted by $\Omega^1_{X/A}$. On a flat A-scheme X, $\Omega^1_{X/A}$ is locally free if and only if X is smooth over A, and in this case $\Omega^1_{X/A}$ is often called the *relative cotangent bundle* over A. When A is the spectrum of an algebraically closed field, we usually abbreviate $\Omega^1_{X/A}$ to Ω^1_X.

Explicit local description of $\Omega^1_{X/A}$ is as follows. Let $t_1, \dots, t_N$ be generators of the $\mathcal{O}_A$-algebra $\mathcal{O}_X$, with relations (or defining ideal) J. As an $\mathcal{O}_A$-module, $\mathcal{O}_X \otimes_{\mathcal{O}_A} \mathcal{O}_X$ is generated by the $t_i \otimes 1$ and $1 \otimes t_i$, with obvious relations $u \otimes 1 = 1 \otimes u = 0$ for $u \in J$ and $a \otimes 1 = 1 \otimes a$ for $a \in \mathcal{O}_A$. The ideal $\mathfrak{I}_\Delta$ is generated by $v \otimes 1 - 1 \otimes v$, $v \in \mathcal{O}_X$, and so is $\mathfrak{I}_\Delta^2$ by

$$
\begin{aligned}
&(v\otimes 1 - 1\otimes v)(w \otimes 1 - 1 \otimes w)\\
&\qquad = vw \otimes 1 - v \otimes w - w \otimes v + 1 \otimes vw\\
&\qquad = v(w\otimes 1 - 1 \otimes w) + w(v \otimes 1 - 1 \otimes v) - (vw \otimes 1 - 1 \otimes vw).
\end{aligned}
$$

Given $v \in \mathcal{O}_X$, let $\mathrm{d}v$ denote the equivalence class of $v \otimes 1 - 1 \otimes v$ modulo $\mathfrak{I}_\Delta^2$. Then the $\mathcal{O}_X$-module $\Omega^1_{X/A}$ is generated by $\mathrm{d}v$, $v \in \mathcal{O}_X$ with relations $\mathrm{d}(vw) = v\mathrm{d}w + w\mathrm{d}v$, $v, w \in \mathcal{O}_X$ and $\mathrm{d}a = 0$ for $a \in \mathcal{O}_A$. Eventually we conclude that the $\mathcal{O}_X$-module $\Omega^1_{X/A}$ is generated

[4] The direct sum decomposition of $a \otimes b$ is given by $(ab, a \otimes b - ab \otimes 1)$.

by $\mathrm{d}t_1, \ldots, \mathrm{d}t_N$ as an $\mathcal{O}_X$-module, with two relations $\mathrm{d}(t_it_j) = t_i\mathrm{d}t_j + t_j\mathrm{d}t_i$, $t_i, t_j \in \mathcal{O}_X$ and $\mathrm{d}u = 0, u \in \mathcal{O}_A$.

Let $f : Y \to X$ be an A-morphism between A-schemes. Then we have a commutative diagram

$$\begin{array}{ccc} \Delta_{Y/A} & \longrightarrow & \Delta_{X/A} \\ \downarrow & & \downarrow \\ Y \times_A Y & \longrightarrow & X \times_A X, \end{array}$$

which induces a natural $\mathcal{O}_X$-homomorphism $\Omega^1_{X/A} \to f_*\Omega^1_{Y/A}$ or, equivalently, an $\mathcal{O}_Y$-homomorphism $\mathrm{d}f^* : f^*\Omega^1_{X/A} \to \Omega^1_{Y/A}$, called the *differential* of f.

For an arbitrary closed embedding $f : Y \hookrightarrow X$, the differential $\mathrm{d}f^*$ is a surjection by the above description of Ω^1.

When A is the spectrum of an algebraically closed field k of characteristic zero and f is a dominant morphism between smooth k-varieties, we have the following

Theorem 1.1 (Sard's theorem). *Let k be an algebraically closed field of characteristic zero. Let X and Y be smooth k-varieties and $f : Y \to X$ a dominant morphism. Then there exists a non-empty open subset $U \subset X$ such that $\mathrm{d}f^* : f^*\Omega^1_X \to \Omega^1_Y$ is everywhere injective on $f^{-1}(U)$. Put in another way, a general fibre of f is smooth.*

Let k be an algebraically closed field and let A be Spec k; thus X is a k-scheme. Let x be a k-valued point defined by an maximal ideal $\mathfrak{M}$. Choose a generator $t_1, \ldots, t_N$ of $\mathcal{O}_{X,x}$ such that they form a k-basis of $\mathfrak{M}/\mathfrak{M}^2$. Then $(\mathcal{O}_X/\mathfrak{M}) \otimes_{\mathcal{O}_X} \Omega^1_X$ is precisely $k\mathrm{d}t_1 \oplus \cdots \oplus k\mathrm{d}t_N \simeq \mathfrak{M}/\mathfrak{M}^2$ as a k-vector space.

Given a k-valued point x of the k-scheme X, the *Zariski tangent space* $\Theta_{X,x}$ of X at x is, by definition, the set of the k-morphisms

$$f : \left(\mathrm{Spec}\ k[\varepsilon]/(\varepsilon)^2, \mathrm{Spec}\ k[\varepsilon]/(\varepsilon)\right) \to (X, x)$$

between the pointed k-schemes or, equivalently, the set of the k-algebra homomorphisms $v : \mathcal{O}_X \to k[\varepsilon]/(\varepsilon^2)$ such that $v \mod (\varepsilon)$ is equal to the evaluation map $v_x : \mathcal{O}_X \to k$ at x. By the correspondence $v \mapsto (v - v_x) : \mathcal{O}_X \to \varepsilon k$, $\Theta_{X,x}$ is naturally identified with

$$\mathrm{Hom}_k(\mathfrak{M}_x/\mathfrak{M}_x^2, k) \simeq \mathrm{Hom}_k(k(x) \otimes \Omega^1_X, k(x)).$$

(In particular, if $Y \subset X$ is a closed subscheme, there is a natural injection $\Theta_{Y,x} \subset \Theta_{X,x}$ for a k-valued point x on Y.) Grothendieck [Gro] generalized this standard fact as follows.

Proposition 1.2. *Let X and Y be k-schemes and let $\tilde{f}$: $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \times Y \to X$ be a k-morphism. There is a natural correspondence $\tilde{f} \mapsto \partial_\varepsilon \tilde{f} \in \operatorname{Hom}_{\mathcal{O}_Y}(f^*\Omega^1_X, \mathcal{O}_Y)$, where $f : Y \to X$ is the restriction of $\tilde{f}$ to $Y = \operatorname{Spec} k[\varepsilon]/(\varepsilon) \times Y$. Given a k-morphism $f : Y \to X$, the above gives a one-to-one correspondence between the set of the liftings of f to morphisms $\tilde{f}$: $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \times Y \to X$ and the k-vector space $\operatorname{Hom}_{\mathcal{O}_Y}(f^*\Omega^1_X, \mathcal{O}_Y)$.*

Proof. The topological space $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \times Y$ is identical with Y, and hence a k-morphism $\tilde{f}$ is uniquely determined by the continuous map f and a k-algebra homomorphism $\tilde{f}^* : \mathcal{O}_X \to \big(k[\varepsilon]/(\varepsilon^2)\big) \otimes \mathcal{O}_Y$ such that $\tilde{f}^* \mod (\varepsilon) = f^*$. By the standard embedding $k \to k[\varepsilon]/(\varepsilon^2)$, we regard f^* as a ring homomorphism to $k[\varepsilon]/(\varepsilon^2) \otimes \mathcal{O}_Y$.

Consider the natural two k-morphisms

$$(f, \tilde{f}), (f, f) : \operatorname{Spec} k[\varepsilon]/(\varepsilon)^2 \times Y \to X \times X$$

and the associated k-algebra homomorphisms

$$\begin{aligned} f^* \cdot f^*,\ f^* \cdot \tilde{f}^* : \mathcal{O}_X \otimes_k \mathcal{O}_X \to \big(k[\varepsilon]/(\varepsilon^2)\big) \otimes_k \mathcal{O}_Y, \\ a \otimes b \mapsto f^*af^*b,\ f^*a\tilde{f}^*b. \end{aligned}$$

By construction, they satisfy

$$\begin{aligned} (f^* \cdot f^*)(\mathfrak{I}_\Delta) &= 0, \\ (f^* \cdot \tilde{f}^*)(\mathfrak{I}_\Delta) &\subset \varepsilon\mathcal{O}_Y, \\ (f^* \cdot f^* - f^* \cdot \tilde{f}^*)(\mathcal{O}_X \otimes 1) &= 0. \end{aligned}$$

Hence $f^* \cdot f^* - f^* \cdot \tilde{f}^*$ is an $\mathcal{O}_X$-linear map from $\Omega^1_X \subset (\mathcal{O}_X \otimes \mathcal{O}_X)/\mathfrak{I}^2_\Delta$ to $\varepsilon\mathcal{O}_Y$. Then $\partial_\varepsilon \tilde{f}$ is defined to be $\varepsilon^{-1}(f^* \cdot f^* - f^* \cdot \tilde{f}^*)$. Given $\mathrm{d}\alpha = \alpha \otimes 1 - 1 \otimes \alpha \in \Omega^1_X$, $\alpha \in \mathcal{O}_X$, we have the explicit formula

$$\begin{aligned} \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha) &= (f^* \cdot \tilde{f}^* - f^* \cdot f^*)(\alpha \otimes 1 - 1 \otimes \alpha) \\ &= f^*(\alpha)\tilde{f}^*(1) - f^*(1)f^*(\alpha) - f^*(\alpha)\tilde{f}^*(1) + f^*(1)\tilde{f}^*(\alpha) \\ &= \tilde{f}^*(\alpha) - f^*(\alpha). \end{aligned}$$

Conversely, given $\partial_\varepsilon \tilde{f} : \Omega^1_X \to \mathcal{O}_Y$ and $\alpha \in \mathcal{O}_X$, we define $\tilde{f}^*(\alpha) = f^*(\alpha) + \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha)$. $\tilde{f}^*$ is in fact a ring homomorphism because

$$\begin{aligned} \tilde{f}^*(\alpha\beta) &= f^*(\alpha)f^*(\beta) + \varepsilon f^*(\alpha)\partial_\varepsilon \tilde{f}(\mathrm{d}\beta) + \varepsilon f^*(\beta)\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha) \\ &\equiv \big(f^*(\alpha) + \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\alpha)\big)\big(f^*(\beta) + \varepsilon\partial_\varepsilon \tilde{f}(\mathrm{d}\beta)\big) \\ &= \tilde{f}^*(\alpha)\tilde{f}^*(\beta) \mod (\varepsilon^2). \end{aligned}$$

Thus $\partial_\epsilon \tilde{f}$ uniquely determines the morphism $\tilde{f}$. Q.E.D.

The k-vector space $\Theta_{X,x}$ can be viewed as an $\mathcal{O}_X$-module through the natural surjection $\mathcal{O}_X \to k(x) \simeq k$, and hence there is a natural $\mathcal{O}_X$-homomorphism $\mathcal{T}_X^0 := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \to \Theta_{X,x}$. This map is, however, not a surjection in general. When X is smooth, we write Θ_X instead of $\mathcal{T}_X^0$ and call it the *tangent sheaf* of X. In this special case, the Zariski tangent space $\Theta_{X,x}$ is naturally identified with $k(x) \otimes_{\mathcal{O}_X} \Theta_X$.

Corollary 1.3. *Let X, Y and T be k-schemes and $\tilde{f} : T \times Y \to X$ a k-morphism. Given a k-valued point $t \in T$, let $f_t : Y \to X$ denote the restriction of $\tilde{f}$ to $\{t\} \times Y$. Then there exists a natural k-linear map* (Kodaira-Spencer map)

$$\partial_{T,t}\tilde{f} : \Theta_{T,t} \to \mathrm{Hom}_{\mathcal{O}_Y}(f_t^*\Omega_X^1, \mathcal{O}_Y).$$

If T is smooth, then the Kodaira-Spencer maps define a natural $\mathcal{O}_T$-linear map

$$\partial_T \tilde{f} : \Theta_T \to (\mathrm{pr}_T)_* \mathcal{H}om(\tilde{f}^*\Omega_X^1, \mathcal{O}_{T\times Y}).$$

B. Hilbert schemes, Chow schemes and Hom schemes

Let A be a connected scheme. Let X be a flat, projective A-scheme with an A-ample line bundle L. Consider an arbitrary A-scheme T and a T-flat closed subscheme $Y \subset T \times_A X$. For each integer m and for each point $t \in T$, the Euler characteristic $\chi(Y_t, \mathcal{O}_{Y_t}(mL))$ of the fibre Y_t over t is well-defined. If m is fixed, then it is a locally constant function on T. If we fix a connected component T_0 of T, it is a polynomial function in m of degree $\dim Y/T_0$ with coefficients in the rational numbers, and is called the *Hilbert polynomial* (of Y over the connected component).

Take a polynomial $h = h(m)$ with rational coefficients. The correspondence

$$\begin{aligned} T \mapsto\ & \mathcal{H}ilb_{X/A}^h(T) \\ & := \big\{T\text{-flat closed subscheme} \subset T \times_A X \text{ with Hilbert function } h(m)\big\} \end{aligned}$$

defines a contravariant functor from the category of A-schemes to the category of sets. Indeed, if $T_1 \to T_2$ is an A-morphism and $Y \subset T_2 \times_A X$ is a T_2-flat closed subscheme with Hilbert polynomial h, then the pull-back of the family or, equivalently, the base change, $T_1 \times_{T_2} Y$ is a T_1-flat closed subscheme of $T_1 \times_A X$ with the same Hilbert polynomial. A fundamental result of Grothendieck [FGA] is that this contravariant functor is representable by the *Hilbert scheme* $\mathrm{Hilb}^h(X/A)$:

Theorem 1.4. *Let the notation be as above. For a given polynomial $h = h(m) \in \mathbb{Q}[m]$, there exist a projective A-scheme* $\mathrm{Hilb}^h(X/A)$

and a $\mathrm{Hilb}^h(X/A)$*-flat closed subscheme* $\mathcal{U}^h(X/A) \subset \mathrm{Hilb}^h(X/A) \times_A X$ *which has the following universal property:*

(Univ) *Given an arbitrary* A*-scheme* T *and arbitrary* $Y \in \mathcal{H}ilb^h_{X/A}(T)$, *there exists a unique* A*-morphism* $T \to \mathrm{Hilb}^h(X/A)$ *such that* $Y = T \times_{\mathrm{Hilb}^h(X/A)} \mathcal{U}^h(X/A)$.

In particular, $\mathrm{Hilb}(X/A) = \coprod_h \mathrm{Hilb}^h(X/A)$ is a countable union of projective A-schemes parameterizing the closed A-subschemes of X.

In the following, we take A to be Spec k, k being an algebraically closed field. In this specific case, we simply write $\mathrm{Hilb}(X)$ instead of $\mathrm{Hilb}(X/\mathrm{Spec}\, k)$. $\mathrm{Hilb}^h(X)(k)$ is the set of closed k-subschemes of X with Hilbert polynomial h.

If Y is a closed subscheme of (pure) dimension m, we can view Y as an effective m-cycle by forgetting the scheme structure of Y. As for the space of effective cycles, we have also the universal family parameterized by the *Chow scheme.*

Theorem 1.5. *Let* (X, L) *be a polarized projective variety (i.e., a pair of a projective variety and an ample line bundle on it). The set of effective* m*-cycles of degree* d *on* X *is uniquely parameterized by the* k*-points of the Chow scheme* $\mathrm{Chow}^d_m(X)$, *a reduced, seminormal,*[5] *projective* k*-scheme. The product* $\mathrm{Chow}^d_m(X) \times X$ *carries the effective cycle* $\mathrm{Univ}^d_m(X)$ *which has the following universal property.*[6] *If there is a family of effective* m*-cycles of degree* d *parameterized by a seminormal* k*-scheme* S *(i.e., if there is a closed subset* $F \subset S \times X$ *such that every closed fibre* F_s *is an* m*-cycle of degree* d *in* X*), then there exists a unique* k*-morphism* $S \to \mathrm{Chow}^d_m(X)$ *such that the family is the pullback of* $\mathrm{Univ}^d_m(X)$.

For the proof, see Kollár [Kol, Theorem 3.21].

The relationship between $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ is messy in general. However, if a closed subscheme $Y \subset X$ is integral, we have a natural set theoretic one-to-one correspondence between the two schemes near $[Y]$ (which can be viewed also as an effective cycle). Hence, if in addition $\mathrm{Hilb}(X)$ is smooth at $[Y]$, $\mathrm{Hilb}(X)$ and $\mathrm{Chow}(X)$ are canonically isomorphic near $[Y]$ by seminormality of $\mathrm{Chow}(X)$.[7]

[5] A k-scheme S is *seminormal* iff any birational, bijective morphism $S' \to S$ is an isomorphism. For instance, an ordinary node is seminormal, while a cusp is not.

[6] N.B.: $\mathrm{Univ}^d_m(X)$ is usually not flat over $\mathrm{Chow}^d_m(X)$.

[7] Notice that there is a uniquely determined morphism $\mathrm{Hilb}(X) \to \mathrm{Chow}(X)$ near $[Y]$ thanks to the universal property of Chow.

A useful sufficient condition for $\mathrm{Hilb}(X)$ to be smooth at $[Y]$ is the following

Theorem 1.6 (Grothendieck [FGA]). *Let X be a k-variety and Y a closed subscheme defined by an ideal $\mathfrak{I}_Y \subset \mathcal{O}_X$. Assume that the conormal sheaf $\mathfrak{I}_Y/\mathfrak{I}_Y^2$ is a locally free $\mathcal{O}_Y$-module*[8] *(i.e., Y is locally complete intersection in X). Then the Zariski tangent space of* $\mathrm{Hilb}(X)$ *at the point $[Y]$ is canonically isomorphic to* $\mathrm{Hom}_{\mathcal{O}_Y}(\mathfrak{I}_Y/\mathfrak{I}_Y^2, \mathcal{O}_Y) \simeq \mathrm{H}^0(Y, (\mathfrak{I}_Y/\mathfrak{I}_Y^2)^*)$. $\mathrm{Hilb}(X)$ *is smooth at $[Y]$ if the obstruction space* $\mathrm{H}^1(Y, (\mathfrak{I}_Y/\mathfrak{I}_Y^2)^*)$ *vanishes.*

While the subschemes and the effective cycles are parametrized by the Hilbert scheme and the Chow scheme, the morphisms are parametrized by the *Hom scheme.* Let X and Y be projective k-schemes. The graph of a k-morphism $f : Y \to X$ determines a closed subscheme $\Gamma_f \subset Y \times X$ isomorphic to Y via the first projection, and *vice versa.* Thus the set of morphisms $\mathrm{Hom}(Y, X) = \{f : Y \to X\}$ is a locally closed subset of $\mathrm{Hilb}(Y \times X)$ in an obvious way, and as such a countable union of quasiprojective schemes. Given specified base points $x \in X, y \in Y$, we denote by $\mathrm{Hom}(Y, X\,; y \mapsto x)$ the closed subset $\{[f]; f(y) = x\} \subset \mathrm{Hom}(X, Y)$. The basic properties of the Hom scheme $\mathrm{Hom}(Y, X)$ are summarized as follows:

Theorem 1.7. *Let X and Y be projective k-schemes.*

(1) *Let (S, o) be a k-scheme with a specified k-valued point and $f : Y \to X$ a k-morphism. Let $\tilde{f} : S \times Y \to X$ be a morphism such that $\tilde{f}|_{\{o\}\times Y}$ is identical with f (i.e., $\tilde{f}$ is a deformation of f parameterized by S). Then there is a natural linear map $\kappa : \Theta_{S,o} \to \mathrm{Hom}_{\mathcal{O}_Y}(f^*\Omega^1_X, \mathcal{O}_Y)$, called the* Kodaira-Spencer map. *If $\tilde{f}$ is a deformation with a base point y (i.e., if $\tilde{f}(S \times \{y\})$ is a single point $x \in X$), then $\kappa(\Theta_{S,o}) \subset \mathrm{Hom}_{\mathcal{O}_Y}(f^*\Omega^1_X, \mathfrak{I}_y)$, where $\mathfrak{I}_y$ is the ideal sheaf defining $y \in Y$. When S is smooth, then there is a natural $\mathcal{O}_S$-homomorphism $\kappa : \Theta_S \to \mathrm{pr}_{S*}\mathcal{H}om_{\mathcal{O}_{S\times Y}}(\tilde{f}^*\Omega^1_X, \mathcal{O}_{S\times Y})$.*

(2) *When X is smooth, the Kodaira-Spencer map κ gives natural identifications*

$$\Theta_{\mathrm{Hom}(Y,X),[f]} \simeq \mathrm{H}^0(Y, f^*\Theta_X),$$
$$\Theta_{\mathrm{Hom}(Y,X;y\mapsto x),[f]} \simeq \mathrm{H}^0(Y, \mathfrak{I}_y f^*\Theta_X).$$

[8]When the conormal sheaf $\mathfrak{I}_Y/\mathfrak{I}_Y^2$ is a locally free $\mathcal{O}_Y$-module, the dual locally free sheaf $(\mathfrak{I}_Y/\mathfrak{I}_Y^2)^*$ is called the *normal bundle* of Y in X and is denoted by $\mathcal{N}_{Y/X}$.

If $\mathrm{H}^1(Y, f^*\Theta_X)$ *[resp.* $\mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X)$*] vanishes, then the k-scheme* $\mathrm{Hom}(Y,X)$ *[resp.* $\mathrm{Hom}(Y,X; y \mapsto x)$*] is smooth at* $[f]$.

(3) *The formal neighbourhood of* $[f]$ *in* $\mathrm{Hom}(Y,X)$ *[resp.* $\mathrm{Hom}(Y,X; y \mapsto x)$*] is isomorphic to a subvariety in the vector space* $\mathrm{H}^0(Y, f^*\Theta_X)$ *[resp.* $\mathrm{H}^0(Y, \mathfrak{I}_y f^*\Theta_X)$*] defined by at most* $\dim \mathrm{H}^1(Y, f^*\Theta_X)$ *[resp.* $\dim \mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X)$*] equations. In particular, we have the following estimate of the local dimensions of Hom schemes:*

$$\dim_{[f]} \mathrm{Hom}(Y,X) \geq \dim \mathrm{H}^0(Y, f^*\Theta_X) - \dim \mathrm{H}^1(Y, f^*\Theta_X),$$
$$\dim_{[f]} \mathrm{Hom}(Y,X; y \mapsto x) \geq \dim \mathrm{H}^0(Y, \mathfrak{I}_y f^*\Theta_X) - \dim \mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X).$$

More precisely, the dimension of each irreducible component has the estimate as above at $[f]$.

(4) *When* Y *is a complete curve, we have*

$$\dim_{[f]} \mathrm{Hom}(Y,X) \geq -\deg f^*\mathrm{K}_X + \big(\dim X\big)\chi(Y, \mathcal{O}_Y),$$
$$\dim_{[f]} \mathrm{Hom}(Y,X; y \mapsto x) \geq -\deg f^*\mathrm{K}_X + \big(\dim X\big)\big(\chi(Y, \mathcal{O}_Y) - 1\big).$$

The statements (1) and (2) essentially follow from Corollary 1.3 together with the definition of the Hom schemes. For details of the proof, see [Gro, Exposé III, (5.6)]. The statement (3) is derived from the analysis of the obstruction spaces $\mathrm{H}^1(Y, f^*\Theta_X)$, $\mathrm{H}^1(Y, \mathfrak{I}_y f^*\Theta_X)$ (see [Mo1, Section 1]). Once (3) is established, Riemann-Roch for vector bundles on curves yields (4).

C. Bend and Break

In this subsection, everything is defined over an algebraically closed field k of arbitrary characteristic.

Let C be a smooth projective curve, X a projective variety, and Δ a smooth curve with a smooth compactification Δ^c. Let $p_1, p_2 \in C$ be two distinct k-valued points.

Let $\tilde{f} : \Delta \times C \to X$ be a morphism and let $f_s : C \to X$ stand for the restriction of $\tilde{f}$ to $\{s\} \times C$, where s is a k-valued point of Δ. Given $s \in \Delta(k)$, let $(f_s)_*(C)$ denote the naturally defined 1-cycle on X.

Theorem 1.8 (Mori's Bend and Break [Mo1]). *In the notation as above, assume that* $\dim \tilde{f}(\Delta \times C) = 2$ *and that* $\tilde{f}(\Delta \times \{p_1\})$ *is a single point. Then we can find a boundary point* $s_\infty \in \Delta^c \setminus \Delta$ *such that the limiting cycle*

$$\lim_{s \to s_\infty} (f_s)_*(C) \in \mathrm{Chow}(X)$$

contains a rational curve as an irreducible component. Assume in addition that C *is* $\mathbb{P}^1$ *and that* $\tilde{f}(\Delta \times \{p_2\})$ *is also a point. Then some limit cycle is either reducible or nonreduced.*

Proof. The morphism $\tilde{f} : \Delta \times C \to X$ does not lift to a morphism from $\Delta^c \times C$ to X. Indeed, if it did, the extended morphism $\tilde{f}^c : \Delta^c \times C \to X$ would map the curve $D_1 = \Delta^c \times \{p_1\}$ to a single point. Let H be an ample divisor on X with support away from $\tilde{f}^c(D_1)$. Since $\tilde{f}^c$ has two-dimensional image, $(\tilde{f}^{c*}H)^2 > 0$, while $(\tilde{f}^{c*}H, D_1) = 0$. Then the Hodge index theorem says that D_1^2 is negative, which is obviously not the case (D_1 is the product $\Delta^c \times \{p_1\}$).

Thus we have to replace $\Delta^c \times C$ by a suitable blown up surface Y in order to extend $\tilde{f}$ to a morphism $\tilde{f}^c : Y \to X$. Pick up $(s_\infty, p) \in \Delta^c \times C$ at which the surface Y is blown up. By construction, Y has a fibration $\pi : Y \to \Delta^c$, the fibre over s_∞ is a union of C and several copies of $\mathbb{P}^1$, and at least one of the copies is mapped onto a rational curve on X. This proves the first statement.

Let us prove the second statement. In the above notation, put $\pi^*(s) = \sum m_{s,i}Y_{s,i}$, which is an effective Cartier divisor on the smooth blown up surface Y. If t is general in Δ^c, then $\pi^{-1}(t) = C \simeq \mathbb{P}^1$, so that $\sum m_{s,i}(f_s)_*Y_{s,i}$ is a limit cycle of $(f_t)_*(C)$. Suppose that $\sum m_{s,i}(f_s)_*Y_{s,i}$ is an irreducible reduced cycle for every $s \in \Delta^c$. Then, for each $s \in \Delta^c$, there is a unique irreducible component $Y_{s,i}$ with non-constant map to X, and its coefficient $m_{s,i}$ must be one. Renumber the indices so that this unique component is $Y_{s,0}$. Since the extra components $Y_{s,i}$, $i > 0$ are mapped to single points, we may blow down these components to keep $\tilde{f}^c$ still well defined on the blown down surface Y'.

The condition $m_{s,0} = 1$ guarantees that the resulting Y' is smooth. Indeed, for the reducible fibre $\sum m_{s,i}Y_{m,i}$, we see:

(1) $K_Y Y_{s,i} < 0$ if and only if $Y_{s,i}$ is (-1)-curve; and
(2) $K_Y(\sum m_{s,i}Y_{s,i}) = -2$.

From this observation it follows that $\bigcup Y_{s,m}$ contains a (-1)-curve $\neq Y_{s,0}$, and we can smoothly contract this extra curve. Repeating this process, we eventually arrive at the smooth geometric ruled surface Y' which preserves the distinguished component $Y_{s,0}$.

On the surface Y' thus obtained, the closure of $\Delta \times \{p_i\}$ gives a section D_i', which is projected to a single point on X. Similarly as before, the pullback of an ample divisor $\tilde{f}^{c*}H$, viewed as a divisor on Y', has positive self-intersection and is disjoint from the D_i'. Thus the two distinct sections D_i' on the complete geometric ruled surface $Y' \to \Delta^c$ have negative self-intersection. It is well known, however, that a geometric ruled surface carries at most one effective curve of negative self-intersection (see e.g. [BaPeVV]). This contradiction shows the second statement. Q.E.D.

2. Unsplitting family of rational curves

Definition 2.1. Let X be a projective variety.

A *rational curve* on X is by definition the image $C = f(\mathbb{P}^1)$ of a generically one-to-one morphism $f : \mathbb{P}^1 \to X$. Since the morphism f is naturally recovered from the image C by identifying the normalization of C with $\mathbb{P}^1$, the morphism f itself is sometimes called a rational curve.

Let S be an irreducible closed subvariety of $\mathrm{Chow}(X)$ and F the associated universal cycle parameterized by S. The family $F \subset S \times X$ (or the parameter space S when there is no danger of confusion) is said to be a *dominant family* if the natural projection $\mathrm{pr}_X : F \to X$ is surjective. F is a *family of rational curves* parameterized by S if a general point $s \in S$ represents an irreducible, reduced rational curve. Any closed point s of a family of rational curves represents a cycle supported by a union of finitely many rational curves. Let $S\langle x\rangle \subset S$ denote the closed subset parameterizing the cycles passing through a point $x \in X$. A family of rational curves $F \to S$ is said to be *unsplitting at* x if every point of $s \in S\langle x\rangle$ represents an irreducible, reduced rational curve. Given an open subset $U \subset X$, we say that F (or S by abuse of terminology) is *unsplitting on* U if F is unsplitting at every $x \in U$. When F is unsplitting at every point $x \in X$ or, equivalently, when every $s \in S$ represents an irreducible, reduced rational curve, F (or S) is simply called an *unsplitting family.*

If a rational curve C has the minimum degree (with respect to an arbitrary fixed polarization)

$$\min\deg = \min\{\deg\Gamma; \Gamma \subset X \text{ is a rational curve}\},$$

then an arbitrary family of rational curves $F \subset S \times X \to S$ which contains C as a closed fibre is necessarily unsplitting. If $C \ni x$ has the minimum degree among the rational curves passing through x, i.e., if

$$\deg C = \min\deg\langle x\rangle = \min\{\deg\Gamma\,;\ \Gamma \subset X \text{ is a rational curve through } x\},$$

then F is unsplitting at x.[9]

Theorem 1.8 asserts that any nontrivial family of irreducible rational curves splits at x_i whenever the family has two base points $x_1, x_2 \in X$.

An unsplitting family of rational curves has extremely simple structure after taking the normalization:

[9] Actually, if $\deg C < 2\min\deg$ or $\deg C < 2\min\deg\langle x\rangle$, then F is unsplitting globally or at x.

Theorem 2.2 (Kollár [Kol, Theorem II. 2.8]). *Let $\mathrm{pr}_S : F \to S$ be a locally projective family of irreducible, reduced rational curves. Let $\overline{F}$ and $\overline{S}$ denote the normalizations of F and S, respectively, with a naturally induced projection $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$. Assume that a general fibre of $\overline{\mathrm{pr}}_{\overline{S}}$ is smooth.*[10] *Then $\overline{\mathrm{pr}}_{\overline{S}}$ is a smooth morphism with every fibre isomorphic to $\mathbb{P}^1$. More precisely, $\overline{F}$ is an étale $\mathbb{P}^1$-bundle over $\overline{S}$.*

Proof. Let $s_0 \in \overline{S}$ be an arbitrary closed point and we check that $\overline{\mathrm{pr}}_{\overline{S}}$ is smooth over s_0. Since the statement is of local nature, we may replace $\overline{S}$ by an arbitrary étale neighbourhood of s_0. Let $s_0' \in S$ be the image of s_0. By our hypothesis that every fibre of pr_S is equidimensional and reduced, the projections pr_S and hence $\overline{\mathrm{pr}}_{\overline{S}}$ are smooth at the generic points of the closed fibres $\mathrm{pr}_S^{-1}(s_0')$, $\overline{\mathrm{pr}}_{\overline{S}}^{-1}(s_0)$ ([Ko, I.6.5]), and we may further assume that $\overline{\mathrm{pr}}_{\overline{S}}$ admits three disjoint sections σ_i over $\overline{S}$. Because a general fibre of $\overline{\mathrm{pr}}_{\overline{S}}$ is smooth $\mathbb{P}^1$, these three sections give rise to a (uniquely determined) rational map $f : \overline{F} \dashrightarrow \mathbb{P}^1$ which sends σ_i to $p_i \in \mathbb{P}^1$, $(p_1, p_2, p_3) = (0, 1, \infty)$. For a general point $s \in \overline{S}$, we have $\overline{F}_s \simeq \mathbb{P}^1$ and the graph Γ_s of $f|_{\overline{F}_s}$ is a well defined 1-cycle, thereby determining a rational map $\phi : \overline{S} \dashrightarrow \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$. By taking a suitable birational modification $S^\star \to \overline{S}$, we obtain a morphism $\phi^\star : S^\star \to \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$ and hence a family $\Gamma^\star \to S^\star$ of effective 1-cycles on $\overline{F} \times \mathbb{P}^1$. Let $\sigma_i^\star : S^\star \to F^\star = S^\star \times_{\overline{S}} \overline{F}$ be the disjoint three sections induced by σ_i. By construction, $\Gamma^\star$ contains the sections $\{(\sigma_i^\star(s^\star), p_i)\}_{s^\star \in S^\star}$.

The two projections $\Gamma^\star_{s^\star} \to F^\star_{s^\star}$ and $\Gamma^\star_{s^\star} \to \mathbb{P}^1$ are both of mapping degree one for each $s^\star \in S^\star$, and hence $\Gamma^\star_{s^\star} \subset F^\star_{s^\star} \times \mathbb{P}^1$ is either (a) a union of fibres of the two projections or (b) a graph of a birational morphism $g_{s^\star} : \mathbb{P}^1 \to F^\star_{s^\star} = \overline{F}_s$, where $s \in \overline{S}$ is the image of $s^\star \in S^\star$.

The subset $\Gamma^\star_{s^\star} \subset \overline{F}_s \times \mathbb{P}^1$ cannot contain all three sections $(\sigma_i^\star(s^\star), p_i)$ in the former case (a), and only the second case (b) occurs. Furthermore $g_{s^\star}$ depends only on $s \in \overline{S}$. Indeed, the source $\mathbb{P}^1$ and the target $\overline{F}_s$ depends only on s, while $g_{s^\star}(p_i) = \sigma_i^\star(s^\star) = \overline{\sigma}_i(s)$, $i = 1, 2, 3$. Hence $g_{s_1^\star} = g_{s_2^\star}$ if $s_1^\star$ and $s_2^\star \in S^\star$ lie over the same point $s \in S$. Thus the morphism $S^\star \to \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$ descends to a morphism $\overline{S} \to \mathrm{Chow}(\overline{F} \times \mathbb{P}^1)$ by Zariski's Main Theorem [Ha2, Corollary 11.4], and the associated relative 1-cycle Γ over $\overline{S}$ determines a birational morphism $\overline{S} \times \mathbb{P}^1 \to \overline{F}$, which is obviously finite. Since $\overline{F}$ is normal, we apply Zariski's Main Theorem once more to conclude that $\overline{F}$ and $\overline{S} \times \mathbb{P}^1$ are mutually isomorphic (over an étale neighbourhood of s_0). Q.E.D.

[10] This condition is automatic if the characteristic of the ground field is zero.

This in particular means the following

Corollary 2.3. *Assume that* char $k = 0$. *Let* $S \subset \mathrm{Chow}(X)$ *be a locally closed subset such that the associated family* $F \to S$ *consists of irreducible, reduced rational curves on* X. *Let* $\overline{F}$, $\overline{S}$ *and* $\overline{\mathrm{Hom}}(\mathbb{P}^1, X)$ *be the normalizations of* F, S *and* $\mathrm{Hom}(\mathbb{P}^1, X)$. *Then there exist a normal scheme* $\overline{M}$, *a finite surjective birational morphism* $\nu : \overline{M} \to M$ *onto a locally closed subset* $M \subset \mathrm{Hom}(\mathbb{P}^1, X)$ *and a commutative diagram*

$$\begin{array}{ccc} \overline{M} \times \mathbb{P}^1 & \xrightarrow[\Psi]{} & \overline{F} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_{\overline{M}}} & & \Big\downarrow{\scriptstyle \mathrm{pr}_{\overline{S}}} \\ \overline{M} & \xrightarrow[\psi]{} & \overline{S} \end{array}$$

which makes $\overline{M} \times \mathbb{P}^1 \to \overline{M}$ *equivariant* $\mathrm{Aut}(\mathbb{P}^1)$*-torsors (equivariant principal* $\mathrm{Aut}(\mathbb{P}^1)$*-bundles, in other words) over* $\overline{F} \to \overline{S}$.

Proof. In general, given a normal variety Z and a morphism $h : Z \to \mathrm{Hom}(\mathbb{P}^1, X)$, there is a natural morphism $\psi : Z \to \mathrm{Chow}(X)$ by the correspondence $z \mapsto f_*([\mathbb{P}^1])$, where $[f] = h(z)$.

The algebraic group $\mathrm{Aut}(\mathbb{P}^1)$ naturally acts on $\mathrm{Hom}(\mathbb{P}^1, X)$ from the left by $g([f]) = [f \cdot g^{-1}]$, $f \in \mathrm{Hom}(\mathbb{P}^1, X)$, $g \in \mathrm{Aut}(\mathbb{P}^1)$. If $M \subset \mathrm{Hom}(\mathbb{P}^1, X)$ is an $\mathrm{Aut}(\mathbb{P}^1)$-stable subset, then $\mathrm{Aut}(\mathbb{P}^1)$ acts also on $\overline{M}$, the normalization of M. It is clear that $\psi(g([f])) = \psi([f])$.

Given a family $F \to S$ as above, the normalization $\overline{F}$ is an étale $\mathbb{P}^1$-bundle over $\overline{S}$, so that the universal property of the Hom scheme tells us that there is an étale local morphism $\sigma_\iota : \overline{S} \to \mathrm{Hom}(\mathbb{P}^1, X)$ if we fix an étale local trivialization $\iota : \overline{F} \xrightarrow{\sim} \overline{S} \times \mathbb{P}^1$. If ι and ι' are two local trivializations, then there exists an $\overline{S}$-automorphism j of $\overline{S} \times \mathbb{P}^1$ such that $\iota' = \iota \circ j$, so that $\mathrm{Aut}_{\overline{S}}(\overline{S} \times \mathbb{P}^1)\overline{\sigma}_\iota$ is independent of the choice of local trivializations. Thus, by defining M to be the orbit $\mathrm{Aut}_{\overline{S}}(\overline{S} \times \mathbb{P}^1)(\sigma_\iota(\overline{S})) \subset \mathrm{Hom}(\mathbb{P}^1, X)$, we have a surjection $\psi : \overline{M} \to \overline{S}$, closed fibres being isomorphic to $\mathrm{Aut}(\mathbb{P}^1)$. Q.E.D.

From the observation above and Mori's Bend and Break (Theorem 1.8), we derive the following dimension estimate for unsplitting families:

Proposition 2.4. *We assume that* char $k = 0$. *Let* X *be a projective variety with a closed point* x *and a closed subset* $Z \subset X$ *off* x. *Let* $\mathrm{pr}_S : F \to S \subset \mathrm{Chow}(X)$ *be a closed family of rational curves parameterized by an irreducible variety* S, *and* pr_X *the natural second projection. Let* $F\langle x\rangle \to S\langle x\rangle$ *[resp.* $F\langle x, Z\rangle \to S\langle x, Z\rangle$*] denote the closed subfamily* *consisting of curves passing through x [resp. both x and Z]. Assume that $S\langle x\rangle$ is non-empty. Then:*

(1) *We have a general dimension estimate*

$$\dim S \leq \dim\ S\langle x\rangle + \dim\ \mathrm{pr}_X(F) - 1,$$
$$\dim S\langle x\rangle \leq \dim\ S\langle x, Z\rangle + \dim\ \mathrm{pr}_X(F\langle x\rangle) - \dim\big(Z \cap \mathrm{pr}_X(F\langle x\rangle)\big) - 1,$$

Moreover, there are open subsets $U \subset \mathrm{pr}_X(F)$, $U' \subset \mathrm{pr}_X(F\langle x\rangle)$ such that, if $x \in U$ and $Z \cap \mathrm{pr}_X F\langle x\rangle \subset U'$, the above inequalities become eualities.

(2) *If $F \to S$ is unsplitting at x, then we have*

$$\dim\ S\langle x, Z\rangle\ \leq\ \dim\big(Z \cap \mathrm{pr}_X(F\langle x\rangle)\big),$$
$$\dim S \leq \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x\rangle) - 2 \leq 2\dim X - 2.$$

Furthermore the projection $\mathrm{pr}_X : F\langle x\rangle \to X$ is finite over $X\setminus\{x\}$ (under the condition that F is unsplitting at x).

Proof. Take a closed point $x' \in Z$. By definition,

$$S\langle x\rangle = \mathrm{pr}_S(\mathrm{pr}_X^{-1}(x)),$$
$$S\langle x, x'\rangle = \mathrm{pr}_S\big((\mathrm{pr}_X|_{F\langle x\rangle})^{-1}(x')\big),$$

and the inequalities in (1) follow from standard dimension count. Under the unsplitting condition, the Bend-and-Break theorem (Theorem 1.8) shows that $\dim S\langle x, x'\rangle \leq 0$ for each $x' \in Z$, whence follows the first inequality in (2). When Z is a point x', substitute $\dim S\langle x, Z\rangle$ by 0 in the inequalities in (1), and we get the second inequality in (2). The fibre of $\mathrm{pr}_X|_{F\langle x\rangle}$ over x' is essentially $S\langle x, x'\rangle$ and hence finite. Q.E.D.

A family $F \to S$ of rational curves parameterized by an irreducible closed variety $S \subset \mathrm{Chow}(X)$ is said to be *maximal* if there is no family $F' \to S'$ of rational curves such that $S' \subset \mathrm{Chow}(X)$ is irreducible and that $S' \supsetneq S$. In view of Corollary 2.3, $F \to S$ is maximal if and only if there is an irreducible component $M \subset \mathrm{Hom}(\mathbb{P}^1, X)$ such that S is the natural image of $\overline{M}$.

Corollary 2.5 (Fujita, Ionescu [Io, Theorem 0.4], Wiśniewski [W, Theorem 1.1]). *Let C be an irreducible, reduced rational curve on an n-dimensional smooth projective variety X over a field of characteristic zero. Take a maximal family of rational curves $F \to S$ which contains C as a fibre. If $(C, -\mathrm{K}_X) > \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x\rangle) - n + 1$, then $F \to S$ is a splitting family.*

Proof. Let $f : \mathbb{P}^1 \to X$ be the composite of the normalization $\mathbb{P}^1 \to C$ and the embedding $C \hookrightarrow X$. The maximality condition on

S means that S is the image of an irreducible component $M \ni [f]$ of $\mathrm{Hom}(\mathbb{P}^1, X)$ and, in particular,

$$\dim S = \dim_{[f]} M - 3.$$

Then the estimate (Theorem 1.7) tells us that

$$\dim_{[f]} M \geq \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x \rangle) + 2,$$

so that

$$\dim_{[C]} S = \dim_{[f]} M - 3 \geq \dim \mathrm{pr}_X(F) + \dim \mathrm{pr}_X(F\langle x \rangle) - 1.$$

But Proposition 2.4(2) asserts that F must split at x under this condition. Q.E.D.

Example 2.6 (Dimension estimate of the exceptional loci of extremal contractions[11] of smooth varieties). Suppose that C is an extremal rational curve on a smooth projective variety X and that the associated extremal contraction $\mathrm{cont}_{[C]}$ is a birational morphism: $X \to Z$, with exceptional set $E \subset X$ mapped to $B \subset Z$. We may assume that $C \subset E_b = \mathrm{cont}_{[C]}^{-1}(b)$ for a general point $b \in B$ and C is a rational curve of minimum degree in E_b. Let $F \to S$ be the maximal family of rational curves on E which contains C as a closed fibre. Then, for an arbitrary closed point $x \in E_b$, the closed subset $S\langle x \rangle \subset S$ consists of rational curves on E_b, and hence $F\langle x \rangle$ is unsplitting (by the condition that $\deg C$ is minimum). Furthermore we have

$$\begin{gathered} \mathrm{pr}_X(F) \subset E, \\ \dim \mathrm{pr}(F\langle x \rangle) \leq \dim E_b = \dim E - \dim B \end{gathered}$$

for the general point x in E. Hence

$$\begin{gathered} 0 < (C, -\mathrm{K}_X) \leq 2 \dim E - \dim B - n + 1, \\ \dim E \geq \frac{n + \dim B}{2}, \end{gathered}$$

[11] A projective, surjective, birational morphism $X \to Z$ is called an *extremal contraction* if the relative Picard number $\rho(X/Y)$ is one with the anticanonical bundle $-K_X$ relatively ample over Z. Given such a contraction, we can find a curve $C \subset X$ such that an effective 1-cycle C' on X collapes to a point on Z if and only if $\mathbb{R}[C] = \mathbb{R}[C']$ in $N_1(X)$, the numerical equivalence classes of the 1-cycles. There is an one-to-one correspondence between the *extremal ray* $\mathbb{R}_{\geq 0}[C] \subset N_1(X)$ and the extremal contraction $\mathrm{cont}_{[C]}$. It is known that an extremal ray is generated by a rational curve (*extremal rational curve*). We refer the reader to [Mo2], [Mo3] and [KM] for the theory of extremal contractions.

thus ruling out birational extremal contractions with small exceptional loci.[12]

Thus the unsplitting condition is already a tight constraint for a closed family of rational curves $F \to S$. If we impose an additional condition, however, we can say much more about the projection $\mathrm{pr}_X : F\langle x\rangle \to X$. Indeed, we have the following proposition, which is a naive prototype of the more complex argument in Section 4 below.

Proposition 2.7 (char $k = 0$). *Let X be a projective variety and x a closed point on the smooth locus of X. Let $S \subset \mathrm{Chow}(X)$ be a closed irreducible subvariety and $F \to S$ the associated universal family of cycles. Assume that $F \to S$ satisfies the following three conditions*:

(a) (unsplitting) *Each fibre F_s is a reduced irreducible rational curve on X.*

(b) (base point) *Each F_s passes through the base point $x \in X$ (i.e., $S = S\langle x\rangle$).*

(c) (smoothness at the base point) *Each F_s is smooth at x.*

Then the natural projection $\mathrm{pr}_X : F \to X$ is birational onto the image.

Proof. Take the normalization $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$ of the fibre space $\mathrm{pr}_S : F \to S$. Let $\overline{\mathrm{pr}}_X : \overline{F} \to X$ be the natural projection. Then, by condition (c), the inverse image $\sigma_x = \overline{\mathrm{pr}}_X^{-1}(x)$ is a single nonsingular point on each fibre $\overline{F}_s$ over $s \in \overline{S}$. Thus σ_x is a well-defined section of the $\mathbb{P}^1$-bundle morphism $\overline{\mathrm{pr}}_{\overline{S}}$, and in particular defines a Cartier divisor on $\overline{F}$. By this property, $\overline{\mathrm{pr}}_X : \overline{F} \to X$ naturally lifts to a morphism $\overline{\mathrm{pr}}_{\tilde{X}} : \overline{F} \to \tilde{X} = \mathrm{Bl}_x(X)$. Let $\tilde{F}_s \subset \tilde{X}$ be the strict transform of $F_s \subset X$. It is clear that $\overline{\mathrm{pr}}_{\tilde{X}}(\overline{F}_s) = \tilde{F}_s$.

Put $Y = \overline{\mathrm{pr}}_{\tilde{X}}(\overline{F}) \subset \tilde{X}$. Let $Y^\circ \subset Y$ and $\tilde{F}_s^\circ \subset \tilde{F}_s$ be the smooth loci. Then we prove the inclusion relation

$$\overline{\mathrm{pr}}_{\tilde{X}}^{-1}\left(\tilde{F}_s^\circ \cap (Y^\circ \setminus \{x\})\right) \subset \{\overline{s}\} \times \overline{F}_s$$

where $\overline{s} \in \overline{S}$ lies over $s \in S$. Once this is proven, the assertion is more or less clear.

Note that $\tilde{F}_s^\circ \cap Y^\circ$ is locally complete intersection in Y°, so that its inverse image in $\overline{F}$ does not contain a zero-dimensional component. Thus it suffices to show that any complete curve Γ contained in $\overline{\mathrm{pr}}_{\tilde{X}}^{-1}(\tilde{F}_s)$ is of the form $\{\overline{s}\} \times \overline{F}_s$. If Γ is one of the fibre of $\overline{\mathrm{pr}}_{\overline{S}}$, then the universal

[12]In view of Theorem 0.1, we can show that the normalization of a general fibre E_b of the projection $E \to B$ is a disjoint union of finite quotients of projective $\frac{n-\dim B}{2}$-space if $\dim E$ attains the minimum possible value $\frac{n+\dim B}{2}$.

property of Chow schemes implies that the image of Γ on F must be $\{s\} \times F_s$, and the assertion readily follows.

We derive a contradiction from the hypothesis that Γ is not a fibre. Let B be the normalization of $\overline{\mathrm{pr}}_{\overline{S}}(\Gamma) \subset \overline{S}$. Consider the fibre product $\overline{F}_B = B \times_{\overline{S}} \overline{F}$, which is a geometric ruled surface over B with a distinguished section $\sigma_x : B \to \overline{F}_B$ and a projection $\phi : \overline{F}_B \to \tilde{X}$. Let $H \in \mathrm{Pic}(\tilde{X})$ be the total transform of an ample divisor on X and $E_x \subset \tilde{X}$ the exceptional divisor. Then $\phi^* E_x = \sigma_x$ by construction. It is well-known that the Néron-Severi group $\mathrm{NS}(\overline{F}_B)$ is freely generated by the section σ_x and a fibre $\mathfrak{f}$ of the ruling. If $d > 0$ denotes the mapping degree of the surjection $\Gamma \to \tilde{F}_s$, then we compute the intersection numbers on the ruled surface $\overline{F}_B$:

$$
\begin{aligned}
(\mathfrak{f}, \phi^* H) &= (\overline{F}_s, \overline{\mathrm{pr}}_{\tilde{X}}^* H) = (\tilde{F}_s, H) = \deg F_s > 0,\\
(\sigma_x, \phi^* H) &= \deg(\sigma_x / E_x)(E_x, H) = 0,\\
(\Gamma, \sigma_x) &= (\Gamma, \phi^* E_x) = d(\tilde{F}_s, E_x) = d = d(\mathfrak{f}, \sigma_x) > 0,\\
(\Gamma, \phi^* H) &= d(\tilde{F}_s, H) = d \deg(F_s) = d(\mathfrak{f}, \phi^* H) > 0.
\end{aligned}
$$

The first two equations show that the two divisors H and σ_x are independent in $\mathrm{NS}(\overline{F}_B) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{\oplus 2}$, and thus the last two equalities yield the numerical equivalence $\Gamma \approx d\mathfrak{f}$. However, an irreducible effective divisor numerically equivalent to a multiple of a fibre $\mathfrak{f}$ is necessarily a fibre, contradicting our assumption.[13] Q.E.D.

Corollary 2.5 bounds the intersection number $(C, -\mathrm{K}_X)$ from above under the condition that C cannot be deformed to split off extra components. The dominance condition on F gives a bound in the opposite direction.

Theorem 2.8 (char $k = 0$). *Let $\mathrm{pr}_S : F \to S$ be a family of rational curves on a projective variety X and C an irreducible, reduced member of the family. Fix a general closed point x on X.*

(1) *Assume that F is dominant and that $\mathrm{pr}_X(C)$ lies on the smooth locus of X. Then we have the inequality*

$$(C, -\mathrm{K}_X) \geq \dim \mathrm{pr}_X(F\langle x \rangle) + 1.$$

[13] We have actually proved something a little stronger than the statement of Proposition 2.7: $\overline{\mathrm{pr}}_{\tilde{X}} : \overline{F} \to \tilde{Y} = \overline{\mathrm{pr}}_{\tilde{X}}(\overline{F}) \subset \tilde{X} = \mathrm{Bl}_x(X)$ is an isomorphism over the smooth locus $\tilde{U}^\circ$ of an open neighbourhood $U \subset \tilde{Y}$ of the exceptional divisor $E_x \cap \tilde{Y}$.

If, in addition, $F \to S$ is closed, maximal and unsplitting at x, we have the equality

$$(C, -\mathrm{K}_X) = \dim \mathrm{pr}_X(F\langle x\rangle) + 1 = \dim F\langle x\rangle + 1.$$

(2) *Let $F \to S$ be a closed, maximal, dominant family of rational curves on X. Assume that*

a) *F is unsplitting at x, that*
b) *C is a general fibre of $F \to S$ and that*
c) *$\mathrm{pr}_X(C)$ lies on the smooth locus of X and passes through x.*

Let $f : \mathbb{P}^1 \to X$ be the composite of the normalization morphism $\mathbb{P}^1 \to C$ and the projection $\mathrm{pr}_X : C \to X$. Then

$$f^*\Theta_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus n-e},$$

where $e = \dim F\langle x\rangle = \dim S\langle x\rangle + 1$.

Proof. Since the intersection number is invariant under the flat deformation, we may assume that C is a general member of the family in order to compute $(C, -K_X)$. Consider the normalization $\overline{\mathrm{pr}}_{\overline{S}} : \overline{F} \to \overline{S}$ of the fibration with the second projection $\overline{\mathrm{pr}}_X : \overline{F} \to X$. Let $\overline{C} \simeq \mathbb{P}^1$ be the normalization of C. We view $\overline{C}$ as a general fibre over $\overline{S}$. Then $\overline{F}$ is étale locally a product $\overline{S} \times \mathbb{P}^1$. Hence we have a natural homomorphism between tangent spaces $\Theta_{\overline{C}\times\overline{S}} \to \overline{\mathrm{pr}}_X^*\Theta_X$, of maximal rank at a general point of $\overline{C} = \{[\overline{C}]\} \times \overline{C}$. The sheaf $\Theta_{\overline{C}\times\overline{S}}$ restricted to $\overline{C}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}^{\dim S}$. In particular, if F is dominant, then $\overline{\mathrm{pr}}_X^*\Theta_X$ is semipositive on $\overline{C}$ by Sard's theorem.

Next consider the subfamily $F\langle x\rangle \to S\langle x\rangle$ and its normalization $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$, a $\mathbb{P}^1$-bundle. The inverse image $\overline{\mathrm{pr}}_X^{-1}(x)$ in $\overline{F}\langle x\rangle$ contains a component σ which is finite and dominating over $\overline{S}\langle x\rangle$. Hence, after a suitable finite base change $W \to \overline{S}\langle x\rangle$, we get a section $\sigma_0 : W \to \overline{F}_W = W \times_{\overline{S}} \overline{F}$ which covers the multisection $\sigma \subset \overline{F}\langle x\rangle$. Then, étale locally, $\overline{F}_W = W \times \overline{C}_0$, $\sigma_0 = W \times \{\infty\}$, where $\overline{C}_0$ is the fibre of a general point of $\overline{S}\langle x\rangle$. Hence we have a natural morphism $W \to \mathrm{Hom}(\mathbb{P}^1, X; \infty \mapsto x)$. The differential of the associated morphism $W \times \mathbb{P}^1 \to X$ has rank equal to $\dim \mathrm{pr}_X(F\langle x\rangle)$ at a general point of C_0 and fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{pr}_W^*\Theta_W|_{C_0} & \longrightarrow & \overline{\mathrm{pr}}_X^*\Theta_X(-\sigma_0)|_{C_0} \\ \downarrow & & \downarrow \\ \Theta_{W\times\overline{C}_0}|_{C_0} \simeq \mathrm{pr}_W^*\Theta_W|_{C_0} \oplus \Theta_{C_0} & \longrightarrow & \overline{\mathrm{pr}}_X^*\Theta_X|_{C_0}, \end{array}$$

where the vertical arrows stand for natural inclusion maps. Since F is dominant over X, we may assume that x is general and that $C = C_0$. Thus we conclude that the vector bundle $\mathcal{E} = \overline{\mathrm{pr}}_X^* \Theta_X|_{\overline{C}}$ on $\overline{C} \simeq \mathbb{P}^1$ satisfies:

(1) $\mathcal{E}$ is a direct sum of line bundles $L_1, \ldots, L_n$;
(2) Each summand L_i has nonnegative degree;
(3) There are are at least $e = \dim \mathrm{pr}_X(F\langle x\rangle)$ summand of positive degree;
(4) There is a summand of degree ≥ 2.

Hence $(C, -\mathrm{K}_X) = \deg \Theta_X|_C = \deg \mathcal{E} \geq \dim \mathrm{pr}_X(F\langle x\rangle) + 1$. If F is unsplitting at x, we have the inequality of the converse direction by Corollary 2.5, so that $L_1 = \mathcal{O}(2)$, $L_2, \ldots, L_e = \mathcal{O}(1)$, $L_{e+1}, \ldots, L_n = \mathcal{O}$.
Q.E.D.

3. Unsplitting families of singular rational curves and a theorem of Kebekus

In this section, every scheme or morphism is defined over an algebraically closed field k of characteristic zero.

Definition 3.1. A singular rational curve C is said to be *nodal* if the normalization $\nu : \mathbb{P}^1 \to C$ is set theoretically not bijective.[14] Any nodal rational curve is (non-canonically) a birational image of the standard nodal curve obtained by identifying two distinct points (say 0 and ∞) of $\mathbb{P}^1$. The standard nodal curve is isomorphic to the plane cubic $y^2 = x^2(x-1)$.

Similarly, C is said to have a *cusp* at $x \in C$ if there is a point $p \in \overline{C} \simeq \mathbb{P}^1$ (say ∞) such that $\mathfrak{I}_x \mathcal{O}_{\mathbb{P}^1} \subset \mathfrak{I}_p^2$, or, equivalently, if a sufficiently small analytic (or formal) neighbourhood of x in C has an irreducible branch which has multiplicity ≥ 2 at x. (Here $\mathfrak{I}_\bullet$ denotes the defining ideal of $\bullet$.) Any cuspidal curve is an image of the standard cuspidal cubic $y^2 = x^3$ by a birational morphism.

Note that a singular curve C is nodal or cuspidal at some point $x \in C$, but these two properties are not mutually exclusive.

An unsplitting family of rational curves $F \to S$ is said to be a *family of singular* [resp. *nodal, cuspidal*] rational curves if a general member (= general fibre) is a singular [resp. nodal, cuspidal] rational curve. Every member of a family of singular [resp. cuspidal] rational curves is singular [resp. cuspidal], while special members of a family of nodal curves may not be nodal.

[14] Our definition of nodal curves is not quite standard. We do not require that the normalization map $\mathbb{P}^1 \to C$ is unramified. The curve C can be nodal and simultaneously cuspidal at a given point x.

Lemma 3.2. *Assume that* char $k = 0$.

(1) *Let $F \to S$ be an unsplitting family of nodal rational curves. Then, after a surjective finite base change $W \to S$ by a normal variety W, we can find two distinct sections $\sigma_1, \sigma_2 : W \to \overline{F}_W$, into the normalization of $F_W = W \times_S F$, such that the natural projection $\overline{F}_W \to F_W$ identifies the two sections σ_1 and σ_2.*

(2) *If $F \to S$ is an unsplitting family of cuspidal rational curves, then after a surjective finite base change $W \to S$, there exists a section $\sigma : W \to \overline{F}_W$ such that $\mathcal{O}_{F_W} \subset \mathcal{O}_W + \nu_* \mathcal{O}_{\overline{F}_W}(-2\sigma)$ via the projection $\nu : \overline{F}_W \to F_W$.*

Proof. Let $\mathrm{Sing}(F/S)$ denote the closed subset

$$\bigcup_{[C] \in S} \{[C]\} \times \mathrm{Sing}(C) \subset F.$$

When F is a family of singular rational curves, $\mathrm{Sing}(F/S)$ is finite and dominant over S. Choose an irreducible component $V \subset \mathrm{Sing}(F/S)$ which surjects onto S, and let W be the normal Galois closure of the projection $V \to S$. Then the inverse image $\Sigma \subset \overline{F}_W$ of V is a union of sections. If a general member of F is nodal, we can choose V so that Σ contains at least two distinct sections. If every member of F is cuspidal, we can choose V such that $\mathfrak{I}_V \mathcal{O}_{\overline{F}_W}$ is contained in $\mathfrak{I}_\sigma^2$ for a suitable irreducible component σ of Σ. Q.E.D.

Corollary 3.3 (Existence of singular cubic models for unsplitting family of singular rational curves [Ke1] and [Ke2]). *Let the notation be as in Lemma* 3.2. *If $F \to S$ is an unsplitting family of nodal* [resp. *cuspidal*] *rational curves, then after a finite surjective base change $W \to S$, there exists a family $\mathcal{C} \to W$ such that*

(1) *each fibre $\mathcal{C}_w$ is an irreducible singular plane cubic and a general fibre is a nodal plane cubic* [resp. *each fibre $\mathcal{C}_w$ is a cuspidal plane cubic*] *and that*

(2) *the natural projection $\overline{F}_W \to F_W$ factors through $\mathcal{C}$.*

The Bend and Break (Theorem 1.8) gives a constraint for an unsplitting family of nodal rational curves.

Proposition 3.4. *Let X be a projective variety and $S \subset$* $\mathrm{Chow}(X)$ *a closed subvariety which parameterizes an unsplitting family F of nodal rational curves on X (i.e., a general member of F is a nodal curve). Let* $\mathrm{Node}(F/S)$ *be the locally closed subset of nodal loci of the fibres (i.e.,* $\mathrm{Node}(F/S)$ *is* $\mathrm{Sing}(F/S)$ *minus the purely cuspidal locus). Then the natural projection* $\mathrm{Node}(F/S) \to X$ *via* $\mathrm{pr}_X : F \to X$ *is quasi-finite. In particular,* $\dim S = \dim \mathrm{Node}(F/S) \leq \dim X$.

Proof. By definition, $\mathrm{Node}(F/S)$ is dominant and quasi-finite over S. Assume that there is a (not necessarily complete) curve B in $\mathrm{Node}(F/S)$ which is contracted to a single point $x \in X$ via pr_X. Take the closure B^c of B in $F\langle x \rangle$. Its image $V \subset S$ via pr_S is a non-trivial complete curve. By the base change $V \to S$, we get a one-parameter family of nodal curves. Take a suitable finite, smooth base change $W \to V$ such that the inverse image of B in the normalization $\overline{F}_W$ of $F_W = W \times_S F$ is a union of two or more sections. Since $\overline{F}_W$ is a $\mathbb{P}^1$-bundle over the smooth curve W, it is a ruled surface with two distinct sections which are contracted to the point x. This means $\overline{F}_W$ contains two distinct curves with negative self-intersection number, which is impossible by elementary theory of ruled surfaces. Q.E.D.

Definition 3.5. Let Σ be a reduced scheme. By a *family of irreducible singular plane cubics* $\mathcal{C}$ over Σ, we mean a proper (not necessarily projective) flat morphism $\pi : \mathcal{C} \to \Sigma$ whose fibres are isomorphic to irreducible singular plane cubics. Note that, in our definition, the structure of plane cubics may change from fibre to fibre, and therefore π is not necessarily (globally) projective. A generic example of such families is indeed non-projective.

Corollary 3.3 asserts that an unsplitting family of singular curves $F \to S$ is dominated by $\mathcal{C} \to \Sigma$, a family of irreducible singular plane cubics.

In general, a one-parameter family of singular cubic curves has nodal generic fibre and several cuspidal special fibres. Only very special families have fibres of constant type. In such an exceptional case, the projectivity condition almost completely determines the global structure of the family.

Lemma 3.6 (Kebekus [Ke1] and [Ke2]). *Let $\pi : \mathcal{C} = \{(s, C_s)\}_{s\in\Sigma} \to \Sigma$ be a family of irreducible singular cubic curves over a smooth projective curve Σ. Suppose that one of the following mutually exclusive conditions is satisfied:*

(**C**) *A general member C_s is a cuspidal cubic, or*

(**N**) *Every member C_s is a nodal cubic.*

If $\mathcal{C}$ is furthermore projective[15] *over Σ, then after replacing the base curve Σ by a suitable finite étale cover, we can find a section $\sigma : \Sigma \to \mathcal{C}$ such that $\sigma(\Sigma)$ lies on the smooth locus of $\mathcal{C}$. In Case* (**N**) (*i.e., every*

[15]There are several ways to define projectivity. Our convention here is that an S-scheme Y is projective over S if there exists a line bundle L globally defined on Y which is relatively ample.

C_s is nodal), the normalization of $\mathcal{C}$ is a trivial bundle $\Sigma \times \mathbb{P}^1$ after an étale base change.

The proof of Lemma 3.6 relies on the theory of relative Picard schemes.

Given a family of irreducible singular cubic curves $\mathcal{C}$ parameterized by Σ, let

$$\mathrm{Pic}(\mathcal{C}/\Sigma) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d(\mathcal{C}/\Sigma)$$

denote the *relative Picard scheme*, which is a Σ-scheme-functor defined as follows. For an arbitrary Σ-scheme T, the set of the T-valued points $\mathrm{Pic}^d(\mathcal{C}/\Sigma)(T)$ consists of the equivalence classes of line bundles on $T \times_\Sigma \mathcal{C}$ of which restriction to each fibre over T has degree d. Here two line bundles L_1 and L_2 are said to be equivalent if and only if there exists a line bundle M on T such that L_2 and $\mathrm{pr}_T^* M \otimes L_1$ are mutually isomorphic as $\mathcal{O}_{T \times_\Sigma \mathcal{C}}$-modules.

The relative Picard scheme $\varpi : \mathrm{Pic}(\mathcal{C}/\Sigma) \to \Sigma$ and its open and closed subset (degree-zero part) $\varpi^0 : \mathrm{Pic}^0(\mathcal{C}/\Sigma) \to \Sigma$ are commutative group schemes over Σ, the multiplication being the tensor product. (In the following we adopt multiplicative notation for the group law of the relative Picard scheme.)

When $\mathcal{C} \to \Sigma$ is a family of irreducible singular plane cubics, the fibre $(\varpi^0)^{-1}(s)$ over a closed point $s \in \Sigma$ is either the algebraic torus $\mathbb{G}_{\mathrm{m}}$ ($= \mathbb{C}^\times$ if the ground field is $\mathbb{C}$) or the additive group scheme $\mathbb{G}_{\mathrm{a}}$ ($= \mathbb{C}$) according as C_s is nodal or cuspidal. $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$ has a canonical global section $[\mathcal{O}_\mathcal{C}]$, which is the unity section with respect to the group law.

Since each fibre C_s over $s \in \Sigma$ is reduced, the fibre space $\pi : \mathcal{C} \to \Sigma$ admits an analytic local section σ_s defined on a neighbourhood of s (alternatively, a local section in étale topology). An arbitrary local section σ_s determines

(1) a local identification (in analytic or étale topology)

$$\begin{aligned} \mathrm{Pic}^d(\mathcal{C}/\Sigma) &\xrightarrow{\sim} \mathrm{Pic}^0(\mathcal{C}/\Sigma) \\ [L] &\mapsto [L \otimes \mathcal{O}(-d\sigma_s)], \end{aligned}$$

and hence gives

(2) a natural $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor structure on $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$.

Let $\mathcal{U} = \{U_i\}$ be an open covering of Σ (in analytic or étale topology) and $\sigma_i : U_i \to \mathcal{C}$ a local section. It is straightforward to check that the cohomology class $\{\sigma_i^d \sigma_j^{-d}\} \in \mathrm{H}^1(\mathcal{U}, \mathrm{Pic}^0(\mathcal{C}/\Sigma))$ does not depend on the choice of the open covering $\mathcal{U}$ or on the choice of the σ_i, and thus determines an invariant (the characteristic class)

$$c(\mathrm{Pic}^d(\mathcal{C}/\Sigma)) = c(\mathrm{Pic}^1(\mathcal{C}/\Sigma))^d$$
$$\in \mathrm{H}^1_{\mathrm{ann}}(\Sigma, \mathrm{Pic}^0(\mathcal{C}/\Sigma)) \text{ , or } \in \mathrm{H}^1_{\text{ét}}(\Sigma, \mathrm{Pic}^0(\mathcal{C}/\Sigma))$$

of the torsor. The characteristic class vanishes if and only if $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ admits a global section over Σ.

General nonsense tells us that the structure of $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ as a $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor is completely determined by the characteristic class. In particular, $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ is isomorphic to $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$ (i.e., a trivial $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor) if and only if $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$ admits a global section over Σ.

These general remarks being said, we note the following

Lemma 3.7. *Let $\pi : \mathcal{C} \to \Sigma$ be a family of irreducible singular cubic curves as above. There is a natural Σ-isomorphism between $\mathrm{Pic}^1(\mathcal{C}/\Sigma)$ and the open subset $\mathcal{C}^\circ$ of $\mathcal{C}$ consisting of the smooth points of the fibres. Given integers d and m, the natural morphism $[m]$: $\mathrm{Pic}^d(\mathcal{C}/\Sigma) \to \mathrm{Pic}^{md}(\mathcal{C}/\Sigma)$ defined by $[L] \mapsto [L^{\otimes m}]$ is surjective. Let U be an open subset of Σ. If every geometric fibre is cuspidal or every fibre is nodal over U and if $\tau : U \to \mathrm{Pic}^{dm}(\mathcal{C}/\Sigma)$ is a section, then $[m]^{-1}(\tau(U)) \subset \mathrm{Pic}^d(\mathcal{C}/\Sigma)$ is étale and finite over U.*

Proof. For a given Σ-scheme T and a given T-valued point $\sigma : T \to \mathcal{C}^\circ_T = T \times_\Sigma \mathcal{C}^\circ$, the correspondence $\sigma \mapsto [\mathcal{O}_{\mathcal{C}_T}(\sigma(T))]$ defines a natural morphism $\mathcal{C}^\circ(T) \to \mathrm{Pic}^1(\mathcal{C}/\Sigma)(T)$.

Conversely, given an line bundle L of relative degree one on $\mathcal{C}_T$, Riemann-Roch for the curve C_t of arithmetic genus one tells us that the linear system $|L|_{C_t}|$ consists of a unique effective member $\sigma(t)$, a single smooth point. This correspondence induces the inverse morphism: $\mathrm{Pic}^1(\mathcal{C}/\Sigma)(T) \to \mathcal{C}^\circ(T)$, $[L] \mapsto \sigma(t)$. Thus we have a natural isomorphism $\mathcal{C}^\circ \simeq \mathrm{Pic}^1(\mathcal{C}/\Sigma)$.

The endomorphism $[m] : \mathrm{Pic}^0(\mathcal{C}/\Sigma) \to \mathrm{Pic}^0(\mathcal{C}/\Sigma)$ is surjective. Indeed, this map is fibrewise given by $z \mapsto z^m$ on $\mathbb{G}_\mathrm{m}$ and by $z \mapsto mz$ on $\mathbb{G}_\mathrm{a}$. Then the natural $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$-torsor structure on $\mathrm{Pic}(\mathcal{C}/\Sigma)$ yields the surjectivity of $[m] : \mathrm{Pic}^d(\mathcal{C}/\Sigma) \to \mathrm{Pic}^{md}(\mathcal{C}/\Sigma)$.

In order to check the final statements, we notice that there is a local isomorphism $\mathcal{C}|_V \simeq V \times C_{s_0}$ over a small analytic (or étale) open subset $V \subset U$. Then it is obvious that $[m] : \mathrm{Pic}^0(\mathcal{C}/\Sigma)|_V \to \mathrm{Pic}^0(\mathcal{C}/\Sigma)|_V$ is surjective with kernel isomorphic to μ_m (the group of the m-th roots of unity) or $\{0\}$ according as the fibres are multiplicative or additive.

Q.E.D.

Remark 3.8. If the type of the fibre of $\pi : \mathcal{C} \to \Sigma$ jumps at a closed point $s \in \Sigma$, then the kernel of $[m] : \mathrm{Pic}^0(\mathcal{C}/\Sigma) \to \mathrm{Pic}^0(\mathcal{C}/\Sigma)$ is not flat, not proper over Σ.

Assume that $\mathcal{C}_s$ is cuspidal with other fibres being nodal. Identify $\mathrm{Pic}^0(\mathcal{C}/\Sigma)$ locally with $\mathrm{Pic}^1(\mathcal{C}/\Sigma) = \mathcal{C}^\circ$. The normalization $\overline{\mathcal{C}}$ is a natural compactification of $\mathcal{C}^\circ$. The complement $D = \overline{\mathcal{C}} \setminus \mathcal{C}^\circ$ is a union of two sections meeting each other at $x \in \mathcal{C}$ over s. Then the closure of $\mathrm{Ker}[m]$ is a union of section $\sigma_0 = [\mathcal{O}_{\mathcal{C}}]$ and $m-1$ sections $\sigma_1, \ldots, \sigma_{m-1} : \Sigma \to \overline{\mathcal{C}}$ which meet D at x.

Proof of Lemma 3.6. Since $\pi : \mathcal{C} \to \Sigma$ is assumed to be projective, there exists a global ample line bundle L on $\mathcal{C}$. If $\deg L|_{\mathcal{C}_s} = d$, then $[L]$ is a global section of $\mathrm{Pic}^d(\mathcal{C}/\Sigma)$. Consider the surjective morphism $[d] : \mathrm{Pic}^1(\mathcal{C}/\Sigma) \to \mathrm{Pic}^d(\mathcal{C}/\Sigma)$. The inverse image $\tilde{\Sigma} = [d]^{-1}([L]) \subset \mathrm{Pic}^1(\mathcal{C}/\Sigma)$ is finite étale over Σ by Lemma 3.7. Hence by the étale base change $\mathcal{C}_{\tilde{\Sigma}} = \tilde{\Sigma} \times_\Sigma \mathcal{C}$, we get either a single section σ or d disjoint sections $\sigma_i : \tilde{\Sigma} \to \mathrm{Pic}^1(\mathcal{C}_{\tilde{\Sigma}}/\tilde{\Sigma})$ according to the case (**C**) or (**N**). Noting that $\mathrm{Pic}^1(\mathcal{C}_{\tilde{\Sigma}}/\tilde{\Sigma})$ is naturally identified with $\mathcal{C}^\circ_{\tilde{\Sigma}}$, we are done. Q.E.D.

Lemma 3.6 has intriguing applications to the geometry of rational curves on uniruled varieties.

Corollary 3.9. *Let X be a projective variety and $x \in X$ a closed point. Let $S \subset \mathrm{Chow}(X)$ be a closed subvariety such that $S = S\langle x \rangle$ (i.e., every $s \in S$ represents an effective cycle passing through x). Assume that S is a family of unsplitting rational curves (at x) and that each element s represents a singular rational curve.*

(1) *If every $s \in S$ represents a curve $C_s \subset X$ with at least one cuspidal singularity, then each irreducible component of $\mathrm{pr}_X(\mathrm{Cusp}(F/S))$, the locus of the cuspidal singularities of the C_s in X, is either identical with the one-point set $\{x\}$ or disjoint from x.*

(2) *If no $s \in S$ corresponds to a curve C_s with cuspidal singularities, then S is a finite set.*

Proof. Assume that S is a curve. Take a suitable smooth projective curve Σ which dominates S. In case (1) or (2), we can find a family of singular cubics $\mathcal{C} \to \Sigma$, with every fibre being accordingly cuspidal or nodal, such that $\mathcal{C} \to \Sigma$ dominates the universal family $F \to S$. Let $\tilde{\mathrm{pr}}_X : \mathcal{C} \to X$ be the projection naturally induced by $\mathrm{pr}_X : F \to S$. The normalization $\tilde{\mathcal{C}} \to \Sigma$ of the family $\mathcal{C} \to \Sigma$ is a $\mathbb{P}^1$-bundle. By the unsplitting property (at x) of $F \to S$, it follows that pr_X and $\tilde{\mathrm{pr}}_X$ are finite over $X \setminus \{x\}$. Hence the one-dimensional component of $\tilde{\mathrm{pr}}_X^{-1}(x)$ is a curve with negative self-intersection and hence the unique minimal section σ_x of the geometric ruled surface $\tilde{\mathcal{C}} \to \Sigma$.

When $\mathcal{C}$ is a family of nodal cubics, Lemma 3.6 says that $\mathcal{C}$ is essentially (namely after a base change) a trivial bundle without any negative section, meaning the assertion (2).

If $\mathcal{C}$ is a family of cuspidal cubics, let $\gamma \subset \tilde{\mathcal{C}}$ be the section obtained as the inverse image of the cuspidal locus. Then, by Lemma 3.6, there exists a section $\sigma : \Sigma \to \tilde{\mathcal{C}}$ which does not meet γ, implying that the unique negative section σ_x must coincide either with σ or with γ. Consequently, γ is either away from, or identical with, σ_x, yielding (1) when $\dim S = 1$. If S has dimension two or more, we can verify the assertion by taking all curves in it. Q.E.D.

Theorem 3.10 (Kebekus [Ke1] and [Ke2]). *Let X be a projective variety of dimension n and $S \subset \mathrm{Chow}(X)$ a closed, dominant family of rational curves on X. Assume that S is unsplitting on an open subset $U \subset X$ and fix a general point $x \in U$. Then*

(1) *there is no member of S which has a cuspidal singularity at x;*

(2) *there exist only finitely many members of S which have singularity at x (the singular point x of such a member C is necessarily nodal by (1)); and*

(3) *if C is a member of S which is singular at x, then there are an n-dimensional locally closed subset $\Sigma \subset S$ and a one-dimensional locally closed subset $\Sigma\langle x\rangle \subset S\langle x\rangle$ such that $[C] \in \Sigma\langle x\rangle \subset \Sigma$ and that Σ consists of nodal curves.*

Proof. We start with the proof of (1). Suppose that for every $x \in X$ there is a member of S with a cuspidal singularity at x. Then, by Corollary 3.3, it follows that there exists a family of cuspidal cubics $\pi : \mathcal{C} \to T$ with cuspidal locus $\mathrm{Cusp}(\mathcal{C}/T)$ and morphisms $\Phi : \mathcal{C} \to F$, $\varphi : T \to S$ such that

a) the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi"'] \arrow[d, "\pi"] & F \arrow[d, "\mathrm{pr}_S"] \\ T \arrow[r, "\varphi"'] & S \end{tikzcd}$$

is commutative, that

b) the closed fibre $\mathcal{C}_t$ over $t \in T$ is a partial normalization of $F_{\varphi(t)}$ and that

c) the restsriction of $\Psi = \mathrm{pr}_X \Phi : \mathcal{C} \to X$ to $\mathrm{Cusp}(\mathcal{C}/T)$ is a surjective morphism onto X.

In particular, by simple dimension count, we have

$$\begin{aligned}&\dim \pi(\Psi^{-1}(x)) = \dim \Psi^{-1}(x)\\ &= \dim(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)) + 1 = \dim \pi(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)) + 1,\end{aligned}$$

provided $x \in X$ is general. Hence we can find a pointed smooth complete curve (Σ, o) and a non-constant morphism $f : \Sigma \to \pi(\Psi^{-1}(x)) \subset T$ such that

$$\begin{aligned}f(o) &\in \pi(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)),\\ f(\Sigma) &\not\subset \pi(\Psi^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}/T)).\end{aligned}$$

Let $g : \mathcal{C}_\Sigma \to X$ be the naturally induced morphism from the one-parameter family of plane cubics $\mathcal{C}_\Sigma = \Sigma \times_T \mathcal{C}$ to X. By construction, $g^{-1}(x)$ contains a rational section over Σ but $\pi_\Sigma(g^{-1}(x) \cap \mathrm{Cusp}(\mathcal{C}_\Sigma/\Sigma))$ is a finite set containing o, contradicting Corollary 3.9(1). This completes the proof of (1).

The assertion (2) follows from Proposition 3.4.

In order to prove (3), let T be an irreducible component of the closed subset $\subset S$, which parameterizes the singular rational curves. If there is a member $C \in S$ which is singular at a general point x, then there exists a $T \subset S$ and the associated family $G \to T$, with $\mathrm{Sing}(G/T)$ dominating X. By (1), a general point of T represents a curve without cusps on an open subset $U \subset X$. Hence the dominant morphism $\mathrm{Sing}(G/T) \to X$ is generically finite over U, so that

$$\begin{aligned}\dim T &= \dim \mathrm{Sing}(G/T) = \dim U = \dim X = n,\\ \dim G &= \dim T + 1 = n + 1,\\ \dim T\langle x\rangle &= \dim G - \dim X = 1,\end{aligned}$$

whence follows the statement (3). Q.E.D.

PART II. Characterizations of Projective n-Space

4. A characterization of projective n-space by the existence of unsplitting, doubly-dominant family of rational curves

In this section, we prove that a normal variety which carries an unsplitting, doubly dominant family of rational curves F parametrized by an irreducible variety S is necessarily projective space.

Given a closed point $x \in X$ and a closed family $F \to S$ of rational curves in X, put

$$S\langle x\rangle = \mathrm{pr}_S(\mathrm{pr}_X^{-1}(x)),$$
$$F\langle x\rangle = S\langle x\rangle \times_S F.$$

The fibre space $F\langle x\rangle \to S\langle x\rangle$ is nothing but the closed subfamily consisting of curves passing through x.

Lemma 4.1. (1) *A family F of rational curves on X is doubly dominant if and only if the natural projection $\mathrm{pr}_X : F\langle x\rangle \to X$ is surjective for each closed point $x \in X$. If F is doubly dominant and x is general, then the restriction of pr_X to each irreducible component of $F\langle x\rangle$ is a morphism onto X.*

(2) *If F is doubly dominant and unsplitting at a general point $x \in X$, then $\mathrm{pr}_X : F\langle x\rangle \to X$ is finite over $X \setminus \{x\}$.*

Proof. The statement (1) is a verbal rephrasing of our definition, whereas (2) was proved by Proposition 2.4(2). Q.E.D.

Our goal in this section is the following

Theorem 4.2. *Let X be a normal projective variety defined over an algebraically closed field of characteristic zero. If X carries a closed, irreducible, maximal, doubly-dominant family of rational curves $F \to S$ which is unsplitting on an open subset U (i.e., every point of $S\langle x\rangle$ represents an irreducible and reduced curve for $x \in U$), then X is a finite quotient of $\mathbb{P}^n$ by $\pi_1(X \setminus \mathrm{Sing}(X))$. If $F \to S$ is everywhere unsplitting, then X is isomorphic to a projective space $\mathbb{P}^n$.*

The proof of Theorem 4.2 consists of ten steps. From Step 1 through Step 9, we require that the doubly-dominant family $F \to S$ is unsplitting on an open subset, while in Step 10 we assume that F is everywhere unsplitting. A rough plan of our proof is as follows:

Take the normalization $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$ of the subfamily $F\langle x\rangle \to S\langle x\rangle$ (more precisely, an irreducible component of this subfamily). This $\mathbb{P}^1$-bundle carries a distinguished section $\sigma : \overline{S}\langle x\rangle \to \overline{F}\langle x\rangle$, whose image in X is the base point x. Note that $\sigma = \sigma(\overline{S}\langle x\rangle)$ is a Cartier divisor on $\overline{F}\langle x\rangle$. The inverse image of x in $\overline{F}\langle x\rangle$ is the disjoint union of σ and a finite closed subscheme, say Δ. The monoidal transformation at Δ gives a morphism $\widetilde{\mathrm{pr}}_{\tilde{X}} : \tilde{F}\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X)$. In Step 3, we show that $\widetilde{\mathrm{pr}}_{\tilde{X}}$ maps $\sigma \simeq \overline{S}\langle x\rangle$ birationally onto the exceptional divisor $E_x \subset \tilde{X}$. In Step 4, the projection $\overline{F}\langle x\rangle \to X$ turns out to be unramified in codimension one. From Step 5 through 8, we observe that $\overline{F}\langle x\rangle$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, from which we conclude in Step 9 that $\overline{F}\langle x\rangle$ is a one-point blow up of $\mathbb{P}^n$, with X a quotient of $\mathbb{P}^n$ by a finite group. Finally in Step 10, we check that F cannot be globally unsplitting if X is a non-trivial quotient of $\mathbb{P}^n$.

Before going into the proof, we fix notation.

By $\overline{*}$, we denote the normalization of $*$. If F is unsplitting at x, then $\overline{F}\langle x\rangle$ is a $\mathbb{P}^1$-bundle over $\overline{S}\langle x\rangle$. $\overline{\mathrm{pr}}_X$ and $\overline{\mathrm{pr}}_{\overline{S}\langle x\rangle}$ stand for the natural projections from $\overline{F}\langle x\rangle$ to X and to $\overline{S}\langle x\rangle$.

Assume that F is unsplitting at a general point $x \in X$. Then by Kebekus' Theorem 3.10, almost every member C of $S\langle x\rangle$ is smooth at x, except for possibly finitely many C_i's which have nodal singularities at x. Thus the pullback of the closed subscheme $x \in X$ on the normalization of C_i is a disjoint union of reduced points. It follows that $\overline{\mathrm{pr}}_X^{-1}(x) \subset \overline{F}\langle x\rangle$ is a union of a divisor σ which is a (single-valued) section of the fibration $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$ plus a closed subscheme supported on finitely many points on $\overline{F}\langle x\rangle$ away from σ. In what follows, the section σ is referred as the *distinguished section* (with respect to the fibration $F\langle x\rangle \to S\langle x\rangle$).

In general, the closed subset $S\langle x\rangle$ of the irreducible projective variety S could be reducible. We denote by $S_0\langle x\rangle$ an (arbitrary) irreducible component of $S\langle x\rangle$.

When a smooth curve C lies in the smooth locus of X, the normal bundle of C is denoted by $\mathcal{N}_{C/X}$.

Given a closed smooth point z of a variety Z, let $\mu_z : \mathrm{Bl}_z(Z) \to Z$ be the blowing-up at z and let $E_z \subset \mathrm{Bl}_z(Z)$ stand for the associated exceptional divisor. $\mathrm{Bl}_x(X)$ is usually denoted by $\tilde{X}$ in order to simplify the notation.

Let us begin the proof of Theorem 4.2.

Step 1. *Let C be a general member of $S\langle x\rangle$. Then C is smooth and lies on the smooth locus of X with normal bundle $\mathcal{N}_{C/X}$ isomorphic to $\mathcal{O}(1)^{n-1}$. $S\langle x\rangle$ is smooth at $[C]$ and there exists an open (in Zariski topology) neighbourhood $V \subset S\langle x\rangle$ of $[C]$, such that the restriction of* $\mathrm{pr}_X : F \to X$ *to $V \times_S F$ naturally lifts to an étale morphism to* $\mathrm{Bl}_x(X)$.

Proof. By Theorem 3.10, almost every $C \in S\langle x\rangle$ is smooth at x. By Proposition 2.4(2), we have $\dim S\langle x, \mathrm{Sing}(X)\rangle \leq \dim \mathrm{Sing}(X)$. Since X is normal of dimension n, the singular locus of X has dimension $\leq n-2$. Hence $S\langle x, \mathrm{Sing}(X)\rangle$ has dimension $\leq n-2$, while the dominant family $S\langle x\rangle$ has dimension $n-1$. Thus a general member C is smooth at x and is off $\mathrm{Sing}(X)$. In particular, the inverse image of $x \in X$ via the projection $\overline{\mathrm{pr}}_X : \overline{F}\langle x\rangle \to X$ is, near the general fibre $\overline{C} \subset \overline{F}\langle x\rangle$, exactly the distinguished section σ, which is a Cartier divisor. Hence, around $\overline{C}$, we can naturally lift $\overline{\mathrm{pr}}_X$ to a morphism $\overline{\mathrm{pr}}_{\tilde{X}}$ to $\tilde{X} = \mathrm{Bl}_x(X)$.

Since $x \in X$ is general and $F\langle x\rangle$ is dominant, Theorem 2.8(2) applies to show that $\overline{\mathrm{pr}}_X^*\Theta_X|_{\overline{C}} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-1}$. Noticing that C is smooth at x and that $\overline{\mathrm{pr}}_X$ is of maximal rank at a general point of $\overline{C}$, we infer that

$$\overline{\mathrm{pr}}_{\tilde{X}}^*\Theta_{\tilde{X}}|_{\overline{C}} \simeq \mathcal{O}(2) \oplus \mathcal{O}^{\oplus n-1} = \Theta_{\overline{F}\langle x\rangle}|_{\overline{C}}.$$

This means that $\overline{\mathrm{pr}}_{\tilde{X}}$ is an analytic isomorphism near $\overline{C}$ and therefore that C is smooth outside x as well as at x. Q.E.D.

Step 2. *Let $T \to \overline{S}\langle x\rangle$ be a non-constant morphism from a smooth curve. Put $\overline{F}_T = T \times_{\overline{S}\langle x\rangle} \overline{F}\langle x\rangle$ and let $\Phi : \overline{F}_T \to X$ and $\overline{\mathrm{pr}}_T : \overline{F}_T \to T$ be the natural projections. Then we have*

$$J_x = \mathfrak{I}_x\mathcal{O}_{\overline{F}_T} = \mathfrak{I}_{\sigma T}\prod_i \mathfrak{I}_i,$$

where σ_T is a section $T \to \overline{F}_T$ and $\mathfrak{I}_i$ is the defining ideal of a zero-dimensional closed subscheme supported by a point p_i away from σ_T.

Furthermore, around p_i, there exists a local coordinate system (z_1, z_2) such that $\mathfrak{I}_i = (z_1, z_2^{m_i})$, $m_i \in \mathbb{Z}_{>0}$. In particular, the monoidal transformation $\nu : \tilde{F}_T \to \overline{F}_T$ with respect to the ideal $J_x \subset \mathcal{O}_{\overline{F}_T}$ gives a normal variety with at worst rational double points of type A_{m_i-1}. The ideal $\mathfrak{I}_i$ defines an exceptional Cartier divisor $m_i\tau_i$ over $p_i \in \overline{F}_T$, where $\tau_i \subset \tilde{F}_T$ is a Weil divisor isomorphic to $\mathbb{P}^1$.

Assume that a closed subscheme $Z \subsetneqq \Phi(\overline{F}_T)$ satisfies the following two conditions

(1) *$Z \ni x$,*

(2) *Z transversally meets all the smooth analytic branches of C_j at x whenever C_j in $S\langle x\rangle$ has a singularity at x.*

Then we have the inclusion relations

$$\mathcal{O}_{\tilde{F}_T}(-m_i\tau_i) \supset \mathfrak{I}_Z\mathcal{O}_{\tilde{F}_T} \not\subset \mathcal{O}_{\tilde{F}_T}((-m_i - 1)\tau_i).$$

Proof. Since each fibre of $F\langle x\rangle \to S\langle x\rangle$ passes through x and has no cuspidal singularity at x, the closed subset $\Phi^{-1}(x) \subset \overline{F}_T$ cuts out a non-empty, reduced closed subset from each fibre of $\overline{F}_T \to T$. Furthermore, there are only finitely many fibres $\subset F\langle x\rangle$ that have nodal singularities at x. This implies that the subset defined by J_x is a disjoint union of a section σ_T plus a 0-dimensional subscheme away from σ_T. Let p_i be a closed point which supports a zero-dimensional connected component, with $t_i = \overline{\mathrm{pr}}_T(p_i)$ its image in T. Let (z_1, z_2) be a local coordinate system of the smooth ruled surface $\overline{F}_T$ around p_i such that the fibre $\overline{\mathrm{pr}}_T^{-1}(t_i)$ is defined by $z_2 = 0$. Then we have

$$\mathfrak{I}_i/\mathfrak{I}_i \cap (z_2) = (z_1)/(z_1 z_2) \subset \mathcal{O}_{\overline{F}_T, p_i}/(z_2)$$

because $\mathfrak{I}_i$ defines the reduced point p_i on the fibre. Hence $\mathfrak{I}_{i,p_i}$ is of the form $(z_1, z_2^{m_i})$, an ideal generated by two generators. Near p_i, the monoidal transformation ν is thus a subvariety in $\overline{F}_T \times \mathbb{P}^1$ defined by

$$z_2^{m_i} = uz_1 \text{ or } z_1 = vz_2^{m_i}$$

on two affine subsets $\simeq \overline{F}_T \times \mathbb{A}^1$. If $m_i > 1$, then $\tilde{F}_T$ has a unique singularity $z_1 = z_2 = u = 0$ of type A_{m_i-1} over p_i. The ideal $\mathfrak{I}_i = (z_1, z_2^{m_i})$ is a principal ideal generated by z_1 or $z_2^{m_i}$ on the open subsets, while the Weil divisor τ_i is defined by $z_1 = z_2 = 0$.

Put $\overline{C}_i = \overline{\mathrm{pr}}_T^{-1}(\overline{\mathrm{pr}}_T(p_i)) \subset \overline{F}_T$. Since $\Phi(p_i) = \Phi(\sigma_T \cap \overline{C}_i) = x$, the image $\Phi(C_i)$ has a nodal singularity at x. Therefore the two conditions (1) and (2) on the closed subscheme Z mean that

(1*) $\mathfrak{I}_i \supset \mathfrak{I}_Z \mathcal{O}_{\overline{F}_T}$ and that

(2*) $\mathfrak{I}_Z \mathcal{O}_{\overline{C}_i} = (z_1)$ in terms of the local coordinate as above.

These two properties amount to saying that $\mathfrak{I}_Z \mathcal{O}_{\overline{F}_T, p_i}$ contains an element of the form $(\text{unit})z_1 + gz_2^{m_i}$, $g \in \mathcal{O}_{\overline{F}_T, p_i}$. The pullback of this element generates $z_1 \mathcal{O}_{\tilde{F}_T} = \mathcal{O}(-m_i \tau_i) \subset \mathcal{O}_{\tilde{F}_T}$ on the first affine open subset given by $z_2^{m_i} = uz_1$. Q.E.D.

Step 3. *Let $x \in X$ be a general closed point and $\tilde{X} = \mathrm{Bl}_x(X)$ the one-point blowup at x, with the exceptional divisor $E_x \subset \tilde{X}$. Fix an arbitrary irreducible component $F_0\langle x\rangle \to S_0\langle x\rangle$ of the fibre space $F\langle x\rangle \to S\langle x\rangle$. Let $\overline{\mathrm{pr}}_{\tilde{X}} : \overline{F}_0\langle x\rangle \dashrightarrow \tilde{X}$ be the dominant rational map induced by the projection $\overline{\mathrm{pr}}_X : \overline{F}_0\langle x\rangle \to X$. Blow up $\overline{F}\langle x\rangle$ along a zero-dimensional subscheme away from σ to eliminate the indeterminacy of $\overline{\mathrm{pr}}_{\tilde{X}}$ and we get a morphism $\tilde{\mathrm{pr}}_{\tilde{X}} : \tilde{F}\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X)$. Let $s \in \overline{S}_0\langle x\rangle$ be a general closed point such that*

(1) *$\overline{C}_s = \overline{\mathrm{pr}}_{\overline{S}}^{-1}(s) \subset \overline{F}_0\langle x\rangle$ is mapped onto a smooth curve C_s on $X \setminus \mathrm{Sing}(X)$ with normal bundle $\mathcal{O}(1)^{\oplus n-1}$ and that*

(2) *C_s is transversal with every member C' of $S\langle x\rangle$ which is singular at x.*

Let $\tilde{C}_s \subset \tilde{X}$ be the strict transform of $C_s \subset X$. Then $\tilde{\mathrm{pr}}_{\tilde{X}}^{-1}(\tilde{C}_s)$ is scheme-theoretically a union of $\overline{C}_s$ and a closed subscheme which does not meet σ. The restriction of $\tilde{\mathrm{pr}}_{\tilde{X}}$ to σ gives a birational morphism $\overline{S}_0\langle x\rangle \simeq \sigma \to E_x \simeq \mathbb{P}^{n-1}$. (Here $\overline{C}_s$ and σ are viewed as subschemes of

$\tilde{F}_0\langle x\rangle$ since the monoidal transformation $\tilde{F}_0\langle x\rangle \to \overline{F}_0\langle x\rangle$ does not affect the neighbourhoods of $\overline{C}_s$ and σ.)

Proof. For simplicity of the notation, assume that $S\langle x\rangle$ is irreducible. For general case, one has only to put the subscript $_0$ to everything relevant.

Fix a general point $s \in \overline{S}\langle x\rangle$ and let $\overline{C} \subset \overline{F}\langle x\rangle$ be the fibre over s. $C = \overline{\mathrm{pr}}_X(\overline{C}) \subset X$ is an everywhere smooth rational curve through x and off $\mathrm{Sing}(X)$. It is easy to check that C satisfies the conditions (1) and (2) above.

The smooth curve C is locally complete intersection on X, so that $\overline{\mathrm{pr}}_X^{-1}(C)$ is a union of the distinguished section σ and purely one-dimensional components (because $\overline{\mathrm{pr}}_X$ is finite over $X \setminus \{x\}$). One of the one-dimensional components is the trivial one $\overline{C} = B_0$. Let B_j, $j = 1, 2, \ldots$ be the extra one-dimensional components. Then we have the following claim

Claim. *The extra components $B_j \subset \overline{F}\langle x\rangle$ $(j \geq 1)$ do not meet the distinguished section σ.*

Assume for a while that this claim is true. Blow up $\overline{F}\langle x\rangle$ at the finitely many points p_i so that we have a well-defined morphism $\tilde{\mathrm{pr}}_{\tilde{X}} : \tilde{F}\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X)$, a process which does not affect open neighbourhoods of σ and of $\overline{C}$ in $\overline{F}\langle x\rangle$.

Let $\tilde{C} \subset \tilde{X}$ denote the strict transform of $C \subset X$. Then, by construction, $\tilde{\mathrm{pr}}_{\tilde{X}}^{-1}(\tilde{C})$ is the union of $\tilde{C}$ and the strict transforms $\tilde{B}_j$ of B_j, $j = 1, 2, \ldots$. In particular, the inverse image of $E_x \cap \tilde{C}$ in $\sigma \simeq \tilde{\sigma} \subset \tilde{F}\langle x\rangle$ is

$$\tilde{\sigma} \cap \left(\tilde{C} \cup \tilde{B}_1 \cup \tilde{B}_2 \cup \cdots\right) = \tilde{\sigma} \cap \overline{C},$$

a single point. Thus the projection $\tilde{\mathrm{pr}}_{\tilde{X}} : \sigma = \tilde{\sigma} \to E_x$ is generically one-to-one.

This shows that the assertion we want to prove follows from Claim above.

Proof of Claim. Fix an arbitrary component B_j and denote it by B for the sake of simplicity of notation. Let Γ be the normalization of $\overline{\mathrm{pr}}_{\overline{S}\langle x\rangle}(B) \subset \overline{S}\langle x\rangle$, and $\overline{F}_\Gamma$ the fibre product $\Gamma \times_{\overline{S}\langle x\rangle} \overline{F}\langle x\rangle$. $\overline{F}_\Gamma$ is a $\mathbb{P}^1$-bundle over the smooth curve Γ. The inverse image of $B \subset \overline{F}\langle x\rangle$ in $\overline{F}_\Gamma$ contains a unique one-dimensional irreducible component $\overline{B}_\Gamma$, dominating $C \subset X$ via the natural projection $\Phi : \overline{F}_\Gamma \to X$.

Let $\nu : \tilde{F}_\Gamma \to \overline{F}_\Gamma$ be the monoidal transformation with respect to the ideal $\mathfrak{I}_x\mathcal{O}_{\overline{F}_\Gamma}$. The naturally induced morphism $\tilde{\Phi} : \tilde{F}_\Gamma \to \tilde{X} = \mathrm{Bl}_x(X)$ is finite when restricted to $\tilde{F}_\Gamma \setminus \sigma_\Gamma$, and we have $\tilde{\Phi}^* E_x = \sigma_\Gamma + \sum m_i \tau_i$.

(Since ν is an isomorphism near the section σ_Γ, we refer by σ_Γ also its inverse image in $\tilde{F}_\Gamma$.)

Consider the commutative diagram

$$\begin{array}{ccccc} \tilde{F}_\Gamma & \xrightarrow[\tilde{\Phi}]{} & \tilde{X} = \mathrm{Bl}_x(X) & \longleftarrow & \tilde{C} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \simeq} \\ \overline{F}_\Gamma & \xrightarrow[\Phi]{} & X & \longleftarrow & C. \end{array}$$

By Step 2, we have

$$(*) \qquad \mathcal{O}_{\tilde{F}_\Gamma}(-\sigma_\Gamma - \sum m_i\tau_i) \supset \mathfrak{I}_C\mathcal{O}_{\tilde{F}_\Gamma} \not\subset \mathcal{O}_{\tilde{F}_\Gamma}(-\sigma_\Gamma - \sum m_i\tau_i - \tau_{i_0})$$

for any irreducible component τ_{i_0} of the exceptional divisor.

The ideal sheaf $\mathfrak{I}_C\mathcal{O}_{\overline{F}_\Gamma} \subset \mathcal{O}_{\overline{F}_\Gamma}$ is of the form $\mathfrak{q}(-D - \sigma_\Gamma)$, where D is an effective Cartier divisor on the smooth surface $\overline{F}_\Gamma$ and $\mathfrak{q} \subset \mathcal{O}_{\overline{F}_\Gamma}$ is an ideal defining a zero-dimensional closed subscheme. Thus the Cartier divisor ν^*D is necessarily of the form $\tilde{D} + \sum a_i\tau_i$, where a_i is a non-negative integer $\leq m_i$ (this is true because the Weil divisor τ_i is irreducible and reduced). By construction, $\overline{B}_\Gamma \subset \overline{F}_\Gamma$ is contained in D and so is its strict transform $\tilde{B}_\Gamma$ in $\tilde{D}$.

Let d be the mapping degree of the finite cover $\tilde{\Phi}|_{\tilde{D}} : \tilde{D} \to \tilde{C}$.

Since $(\tilde{C}, E_x) = 1$ by the smoothness of C at x, we have

$$d = (\tilde{D}, \tilde{\Phi}^*E_x) = (\tilde{D}, \sigma_\Gamma) + \sum(\tilde{D}, m_i\tau_i)$$

or, equivalently,

$$0 \leq (\tilde{D}, \sigma_\Gamma) = d - \delta, \quad \delta = \sum(\tilde{D}, m_i\tau_i) = -\sum a_i m_i \tau_i^2.$$

(The intersection number $\tau_i^2 \in \mathbb{Q}$ is well defined because $m_i\tau_i$ is Cartier.) Noting that $\tilde{D} = \nu^*D - \sum a_i\tau_i$ and that the exceptional divisors τ_i are disjoint from σ_Γ, we obtain

$$(D, \sigma_\Gamma) = (\tilde{D}, \sigma_\Gamma) = d - \delta,$$

which yields the numerical equivalence

$$(**) \qquad D \approx d\mathfrak{f} + \frac{\delta}{-\sigma_\Gamma^2}\sigma_\Gamma,$$

where $\mathfrak{f}$ denotes a fibre of the ruling $\mathrm{pr}_\Gamma : \overline{F}_\Gamma \to \Gamma$.

Indeed, this is a direct consequence of (a) the fact that the Néron-Severi group of $\overline{F}_\Gamma$ is freely generated by $\mathfrak{f}$ and σ_Γ, together with (b) the following table of intersection numbers:

$$\begin{aligned}
(\mathfrak{f}^2) = 0, \quad (\mathfrak{f}, \sigma_\Gamma) &= 1, \quad (\sigma_\Gamma^2) < 0, \\
(\mathfrak{f}, \Phi^* H) &= (C, H) > 0, \\
(\sigma_\Gamma, \Phi^* H) &= \deg(\sigma_\Gamma \to \Phi(\sigma_\Gamma))(\Phi(\sigma_\Gamma), H) = 0, \\
(D, \sigma_\Gamma) &= d - \delta = d(\mathfrak{f}, \sigma_\Gamma) - \delta, \\
(D, \Phi^* H) &= d(C, H) = d(\mathfrak{f}, \Phi^* H) > 0.
\end{aligned}$$

Here H denotes an ample divisor on X.

From the numerical equivalence (**) we deduce

$$\begin{aligned}
\frac{\delta}{-\sigma_\Gamma^2} &\geq 1 \in \mathbb{Z}, \\
D^2 &= \frac{2d\delta - \delta^2}{-\sigma_\Gamma^2}.
\end{aligned}$$

Then we get

$$\tilde{D}^2 = D^2 + (\sum a_i \tau_i)^2 \geq D^2 - \delta \geq D^2 - \delta \frac{\delta}{-\sigma_\Gamma^2} = 2(d - \delta) \frac{\delta}{-\sigma_\Gamma^2} \geq 0.$$

(Here we used the fact that $\tau_i^2 < 0, a_i^2 \leq a_i m_i$, $\tau_i \tau_j = \delta_{ij} \tau_i^2$.) The curve $B_\Gamma \subset \overline{F}_\Gamma$, which is an irreducible component of D, is away from σ_Γ if $d - \delta = (D, \sigma_\Gamma) = 0$. Hence Claim above reduces to the inequality $\tilde{D}^2 \leq 0$, which we derive from the following observation.[16]

Since C is general, the normal bundle $\mathcal{N}_{C/X}$ is of the form $\mathcal{O}(1)^{\oplus n-1}$ and so $\mathcal{N}_{\tilde{C}/\tilde{X}}$ is trivial. Let $\hat{X} \to \tilde{X}$ be the blow-up along $\tilde{C}$, with exceptional divisor $E_{\tilde{C}} \subset \hat{X}$. The triviality of the normal bundle means that the divisor $E_{\tilde{C}}|_{E_{\tilde{C}}}$ on $E_{\tilde{C}}$ is seminegative. Perform blowing-ups

---

[16]What we use below is the seminegativity of $\mathcal{N}_{\tilde{C}/\tilde{X}}$ rather than its triviality. If we start from a dominant family of rational curves unsplitting at a general point $x \in X$, then $\mathcal{N}_{\tilde{C}/\tilde{X}}$ is always seminegative by virtue of Theorem 2.8(2). Thus the above Claim as well as the statement of this step applies to a fairly wide class of families of rational curves.

$\hat{F}_\Gamma \to \tilde{F}_\Gamma$ to have a commutative diagram

$$\begin{array}{ccc} \hat{F}_\Gamma & \xrightarrow[\hat{\Phi}]{} & \hat{X} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \tilde{F}_\Gamma & \xrightarrow[\tilde{\Phi}]{} & \tilde{X}. \end{array}$$

Then $\hat{\Phi}^* E_{\tilde{C}} = \alpha^* \tilde{D} + A$, where A is an effective divisor lying over finitely many points on $\overline{F}_\Gamma$. Thus

$$\tilde{D}^2 = (\alpha^* \tilde{D})^2 \leq (\alpha^* \tilde{D}, \hat{\Phi}^* E_{\tilde{C}}) \leq 0,$$

because $\hat{\Phi}^* E_{\tilde{C}}$ restricted to $\alpha^* \tilde{D}$ is semi-negative.

This completes the proofs of Claim and of Step 3 as well. Q.E.D.

Step 4. *The projection $\overline{\mathrm{pr}}_X : \overline{F}_0\langle x\rangle \to X$ is unramified over $X \setminus \big(\mathrm{Sing}(X) \cup \{x\}\big)$. In particular, there exists a proper finite morphism $Y \to X$, étale over $X \setminus \mathrm{Sing}(X)$ and a birational morphism $\overline{F}_0\langle x\rangle \to Y$ which factor the morphism $\overline{\mathrm{pr}}_X$.*

Proof. In order to prove the assertion, take a (unique) normal birational modification $\lambda : F_0^\sharp\langle x\rangle \dashrightarrow \overline{F}_0\langle x\rangle$ such that $\mathrm{pr}_X : \overline{F}_0\langle x\rangle \to X$ lifts to a finite morphism

$$\mathrm{pr}^\sharp_{\tilde{X}} : F_0^\sharp\langle x\rangle \to \tilde{X} = \mathrm{Bl}_x(X).$$

(Such a modification is constructed by first blowing up $\overline{F}_0\langle x\rangle$ and then taking the Stein factorization with respect to the projection onto $\tilde{X}$.) Since $\overline{F}_0\langle x\rangle \to X$ is finite over $X \setminus \{x\}$, we have the identity $F_0^\sharp\langle x\rangle = \overline{F}_0\langle x\rangle$ outside the inverse images of $x \in X$.

The strict transform $\sigma^\sharp$ of the distinguished section σ is a subvariety in $F_0^\sharp\langle x\rangle$ which is finite and birational over E_x, and hence isomorphic to $E_x \simeq \mathbb{P}^{n-1}$ by Zariski's Main Theorem.

Recall that $\sigma^\sharp$ is a connected component of $(\mathrm{pr}^\sharp_{\tilde{X}})^{-1} E_x$. Furthermore, the normal variety $F_0^\sharp\langle x\rangle$ is smooth in codimension one and so is it at a general point of $\sigma^\sharp$. In particular, the Cartier divisor $\mathrm{pr}^{\sharp *}_{\tilde{X}} E_x$ is of the form $a\sigma^\sharp$ locally near a general point of $\sigma^\sharp$ and hence globally on a neighbourhood of $\sigma^\sharp$. On the other hand, the strict transform $C^\sharp \subset F_0^\sharp\langle x\rangle$ of a general fibre $\overline{C} \subset \overline{F}_0\langle x\rangle$ satisfies

$$(C^\sharp, \sigma^\sharp) = (\overline{C}, \sigma) = 1,$$
$$(C^\sharp, \mathrm{pr}^{\sharp *}_{\tilde{X}} E_x) = (\mathrm{pr}^\sharp_{\tilde{X}}(\tilde{C}), E_x) = 1.$$

Thus $a = 1$ and we have $\mathrm{pr}^{\sharp *}_{\tilde{X}} E_x = \sigma^\sharp$ near $\sigma^\sharp$, showing that $\mathrm{pr}^{\sharp *}_{\tilde{X}}$ is unramified near $\sigma^\sharp$.

By Step 1, $\overline{\mathrm{pr}}_X$ is unramified on $\left(U \cup \overline{\mathrm{pr}}^{-1}_{\overline{S}_0\langle x\rangle}(W)\right) \setminus \sigma$, where $W \subset \overline{S}_0\langle x\rangle$ is an open dense subset. This implies that $\overline{\mathrm{pr}}_X|_{(\overline{F}_0\langle x\rangle \setminus \sigma)}$ is unramified in codimension one, or equivalently, $\overline{\mathrm{pr}}_X$, finite over $X \setminus \{x\}$, is unramified over $X \setminus$ (subset of codimension ≥ 2), and we have the assertion by the purity of the branch loci. Q.E.D.

Step 5. *$\overline{S}_0\langle x\rangle$ is a smooth variety birational to $\mathbb{P}^{n-1}$. In particular, the $\mathbb{P}^1$-bundle $\overline{F}_0\langle y\rangle$ and Y defined in Step 4 are both smooth and simply connected.*

Proof. Let $U \subset \overline{F_0}\langle x\rangle$ be a small open neighbourhood of σ. $U \setminus \sigma$ is unramified over $X \setminus \{x\}$ and hence smooth. Furthermore, for each $s \in \overline{S}_0\langle x\rangle$, the fibre $\overline{C}_s = \overline{\mathrm{pr}}_S^{-1}(s)$ is a smooth $\mathbb{P}^1$. Therefore, for each point $p \in \overline{C}_s \cap (U \setminus \sigma_0)$, there is a smooth $(n-1)$-dimensional analytic (or étale) slice $\overline{S}^\dagger_p$ which cuts out p from $\overline{C}_s$, inducing an analytic local isomorphism $(\overline{S}_0\langle x\rangle, s) \simeq \overline{S}^\dagger_p$. Thus $\overline{S}_0\langle x\rangle$ is everywhere smooth.

In the proof of Step 3, we have checked that $\overline{S}_0\langle x\rangle$ was birational to $\mathbb{P}^{n-1}$, and hence $\pi_1(\overline{S}_0\langle x\rangle) \simeq \pi_1(\mathbb{P}^{n-1}) = (1)$ because of the birational invariance of the fundamental group of smooth projective varieties. Y is birational to the smooth variety $\overline{F}_0\langle x\rangle$. Furthermore, $Y -$ (the finitely many smooth points over $x \in X$) is isomorphic to an open subset of $\overline{F}_0\langle x\rangle$, so that Y is smooth and hence simply connected. Q.E.D.

Step 6. *Let Y be the smooth variety constructed above via $\overline{F}_0\langle x\rangle$. Then Y is a compactification of the universal cover of $X \setminus \mathrm{Sing}(X)$ by finitely many varieties and hence independent of the choice of the base point $x \in X$. Thus a point y lying over a general point x is again a general point of Y, and $\overline{F}\langle x\rangle \to \overline{S}\langle x\rangle$ defines a dominant unsplitting family of rational curves on Y through a general closed point $y \in Y$.*

Proof. For each fibre $\overline{C} \simeq \mathbb{P}^1$, the morphism $\overline{\mathrm{pr}}_Y|_{\overline{C}}$ is generically one-to-one because so is $\overline{\mathrm{pr}}_X|_{\overline{C}}$. Hence $\{C_Y\} = \{\overline{\mathrm{pr}}_Y(\overline{C})\}$ is a closed, dominant unsplitting family of rational curves (perhaps non-effectively) parameterized by $\overline{S}\langle x\rangle$. The image of σ in Y is a single point y because it is irreducible and finite over x. This implies the assertion. Q.E.D.

Step 7. *Put*

$$G_0 = \{(s, \overline{\mathrm{pr}}_Y(\overline{C}_s);\, s \in \overline{S}_0\langle x\rangle\} \subset \overline{S}_0\langle x\rangle \times Y.$$

Then we have $\overline{F}_0\langle x\rangle \simeq G_0$; i.e., every $\overline{\mathrm{pr}}_Y(\overline{C}_s)$ is smooth.

Proof. The projection $\overline{\mathrm{pr}}_Y : \overline{F}_0\langle x\rangle \to Y$ factors through G_0 by construction. Since $\overline{\mathrm{pr}}_Y$ is birational and finite over $Y \setminus \{y\}$, we have $\overline{F}_0\langle x\rangle \simeq G \simeq Y$ over $Y \setminus \{y\}$. Thus every $\overline{\mathrm{pr}}_Y(\overline{C}_s) \subset Y$ must be smooth off the base point y.

If, however, some $\overline{\mathrm{pr}}_Y(\overline{C}_s)$ has a singularity at a general point y, then Theorem 3.10(3) asserts that there is a one-parameter subfamily of nodal curves $\{\overline{\mathrm{pr}}_Y(\overline{C}_t)\}$ with moving nodal locus, which contradicts what we have just seen. Hence $\overline{\mathrm{pr}}_Y(\overline{C}_s)$ is smooth also at y. Q.E.D.

Step 8. *Let $\overline{\mathrm{pr}}_Y : \overline{F}_0\langle x\rangle \to Y$ be the birational morphism as above. Then $\overline{S}_0\langle x\rangle \simeq \overline{\mathrm{pr}}_Y^{-1}(y) \subset \overline{F}_0\langle x\rangle$ is isomorphic to $\mathbb{P}^{n-1}$ and $\overline{F}_0\langle x\rangle \simeq \mathrm{Bl}_y(Y)$.*

Proof. Since every fibre $\overline{C} = \overline{C}_s$ is isomorphically mapped onto a curve through $y = \overline{\mathrm{pr}}_Y(\sigma)$ in Y, we have a natural lift $\overline{\mathrm{pr}}_{\tilde{Y}} : \overline{F}\langle x\rangle \to \tilde{Y} = \mathrm{Bl}_y(Y)$. This birational morphism induces a birational morphism $\sigma \to E_y$ $(\simeq E_x)$ and the equality $\overline{\mathrm{pr}}_{\tilde{Y}}^* E_y = \sigma$. Hence $\overline{\mathrm{pr}}_{\tilde{Y}}$ is unramified at a general point of σ, and in the same time unramified on $U \setminus \sigma$, where U is an open neighbourhood of σ. This shows that $\overline{\mathrm{pr}}_{\tilde{Y}}$ is unramified in U by the purity of the ramification locus on smooth varieties. Thus we have a isomorphism between U and an open neighbourhood of E_y in Y, inducing $\sigma \simeq E_y$. In particular, $\overline{F}\langle x\rangle$ is birational, finite over $\mathrm{Bl}_y(Y)$, and hence isomorphic to $\mathrm{Bl}_y(Y)$ by Zariski's Main Theorem. Q.E.D.

Step 9. *$Y \simeq \mathbb{P}^n$ and $\overline{\mathrm{pr}}_Y(\overline{C})$ is a line. In other words, X is the finite quotient $\mathbb{P}^n/G$ and the unsplitting rational curve $C \subset X$ is the image of a line $\subset \mathbb{P}^n$, where $G = \pi_1(X \setminus \mathrm{Sing}(X))$.*

Proof. $\overline{F}_0\langle x\rangle \simeq \mathrm{Bl}_y(Y)$ is a $\mathbb{P}^1$-bundle over $\overline{S}_0\langle x\rangle \simeq \mathbb{P}^{n-1}$. Since the distinguished section σ is the exceptional divisor $E_y \simeq \mathbb{P}^{n-1}$ via the isomorphism $\overline{F}_0\langle x\rangle \simeq \mathrm{Bl}_y(Y)$, we have $\mathcal{O}_\sigma(\sigma) \simeq \mathcal{O}(-1)$. Thus the natural exact sequence

$$0 \to \mathcal{O}_{\overline{F}_0\langle x\rangle} \to \mathcal{O}_{\overline{F}_0\langle x\rangle}(\sigma) \to \mathcal{O}_\sigma(\sigma) \to 0$$

and the pushforward by $\mathrm{pr}_{\overline{S}_0\langle x\rangle}$ induce the exact sequence

$$0 \to \mathcal{O}_{\overline{S}_0\langle x\rangle} \to \mathrm{pr}_{\overline{S}_0\langle x\rangle\,*}\mathcal{O}_{\overline{F}_0\langle x\rangle}(\sigma) \to \mathcal{O}_{\overline{S}_0\langle x\rangle}(-1) \to 0.$$

This means that $\mathrm{Bl}_y(Y) \simeq \overline{F}_0\langle x\rangle$ is the projective bundle $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \mathcal{O}(-1))$ with σ being $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(-1))$, implying that Y is isomorphic to

$\mathbb{P}^n$.[17] Q.E.D.

We have thus completed the proof of Theorem 0.2, while Theorem 0.1 follows from

Step 10. *$X = Y \simeq \mathbb{P}^n$ if $F \to S$ is everywhere unsplitting.*

Proof. Since $Y \setminus$ (finite set) is the universal cover of $X \setminus \mathrm{Sing}(X)$, the fundamental group $G = \pi_1(X \setminus \mathrm{Sing}(X))$ naturally acts on $Y \simeq \mathbb{P}^n$ and the normal variety X is the finite quotient under this action.

Suppose that G is non-trivial. Take an arbitrary element $g \neq 1 \in G \subset \mathrm{Aut}(\mathbb{P}^n) = \mathbf{PGL}(n+1, k)$. Then $g \in \mathbf{PGL}(n+1, k)$, an element of finite order, can be diagonalized with at least two distinct eigenvalues λ_1, λ_2. Choose eigenvectors $v_1, v_2 \in \mathbb{C}^{n+1}$ corresponding to the eigenvalues. The two dimensional space $\mathbb{C}v_1 + \mathbb{C}v_2$ defines a $\langle g \rangle$-stable line C_0 in $\mathbb{P}^n \simeq Y$, on which g acts non-trivially.

Recall that $F\langle x \rangle$ was a family of lines in $Y = \mathbb{P}^n$ passing through y lying over $x \in X$. It follows, then, that each point of S is the image of a line $\subset Y$ in X. In particular, S can be viewed as a closed subset of the quotient of the Grassmann variety $\mathrm{Grass}(\mathbb{P}^n, 1)$ by the action of G. Since S is a doubly dominant family of curves, we have

[17] A more elementary and direct proof is the following: By pulling back a hyperplane h on $\overline{S}_0\langle x \rangle$, we get a base-point-free effective divisor $\tilde{H}_x$ on $\overline{F}_0\langle x \rangle \simeq \tilde{Y} = \mathrm{Bl}_y(Y)$. $\tilde{H}_x|_\sigma$ is a hyperplane in $E_x \simeq \sigma \simeq \overline{S}_0\langle x \rangle$, so that $\tilde{H}_x \simeq \mu_Y^* H_x - E_y$, where H_x is an effective divisor on Y. The obvious equality $\tilde{H}_x^n = h^n = 0$ on $\overline{F}_0\langle x \rangle = \mathrm{Bl}_y(Y)$ then gives $H_x^n = 1$. The free $(n-1)$-dimensional linear system $|\tilde{H}_x|$ on $\tilde{Y}$ can be viewed as a linear subsystem with a unique base point $y \in Y$ of the complete linear system $|H_x|$ on Y. Viewed as a linear system on Y, the base locus of the bigger linear system $|H_x|$ is contained in $y \in Y$ lying over $x \in X$. By moving around the prescribed base point $x \in X$, we get a new linear subsystem $|\tilde{H}_{x'}|$ of $|H_y|$ on Y with a single base point y' over $x' \neq x$. The two divisors H_x and $H_{x'}$ are obviously algebraically equivalent and hence linearly equivalent by the vanishing of the irregularity $q(Y)$ (it is birational to a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$). Thus $|H_x| = |H_{x'}| = |H|$. Take a general member $D \in |\tilde{H}_{x'}|$ and consider the linear system $\Lambda \subset |H|$ spanned by D and $|\tilde{H}_x|$. Λ is an n-dimensional linear system free from base points and hence gives rise to a morphism $\phi_\Lambda : Y \to \mathbb{P}^n$ of mapping degree $H^n = 1$. This birational morphism ϕ_Λ is finite and hence an isomorphism by Zariski's Main Theorem. To see this, it suffices to check that $(\Gamma, H) > 0$ for an arbitrary effective curve Γ on Y. The strict transform $\tilde{\Gamma} \subset \tilde{X} \simeq \overline{F}_0\langle x \rangle$ is a curve not contained in $E_y = \sigma$. Hence $(\Gamma, H) = (\tilde{\Gamma}, \tilde{H} + \sigma) \geq (\tilde{\Gamma}, \tilde{H})$, the third term being positive unless $\tilde{\Gamma}$ is a fibre $\overline{C}$ over a point in $\overline{S}_0\langle x \rangle$. For a fibre $\tilde{\Gamma} = \overline{C}$, we have $(\overline{\mathrm{pr}}_Y(\overline{C}), H) = (\overline{C}, \sigma) = 1$, so that the image of $\overline{\mathrm{pr}}_Y(\overline{C}) \subset Y$ in $\mathbb{P}^n$ is a line.

$\dim S \geq 2n-2 = \dim \mathrm{Grass}(1, \mathbb{P}^n)$, so that $S = \mathrm{Grass}(\mathbb{P}^n, 1)/G$, a $2(n-1)$-dimensional irreducible variety. Consequently the image of C_0 is represented by a point $\in S$, while $\mathrm{pr}_X(([C_0], C_0)) \subset X$ factors through the quotient $C_0/\langle g\rangle$. Hence $\mathrm{pr}_X|_{C_0} : C_0 \to X$ is not generically one-to-one, contradicting the global unsplitting property of the family. Q.E.D.

Example 4.3. The everywhere unsplitting condition on $F \to S$ is really necessary in order to characterize projective spaces. The easiest example is constructed in dimension two as follows.

The cyclic group $G = \mathbb{Z}/(p)$ of order p, an odd prime number, effectively acts on $Y = \mathbb{P}^2$ via the diagonal action $\mathrm{diag}(1, \exp \frac{2\pi\sqrt{-1}}{p}, \exp \frac{-2\pi\sqrt{-1}}{p})$ of a generator. If $y \in Y$ is general, then the orbit $G(y) \subset Y$ is not collinear, or, equivalently, no line passing through y is G-stable. This means that the images of the lines on Y in $X = Y/G$ form a closed, doubly-dominant family $F \to S$ of rational curves on X which is unsplitting at a general point. The line connecting y and $g(y)$ is mapped to a nodal curve on X, so that there are exactly $\frac{p-1}{2}$ points of S which represent curves with nodes at a fixed general point x.

Around the fixed point $(1:0:0) \in \mathbb{P}^2$, the quotient morphism $\mathbb{P}^2 \to X$ is given by

$$(s,t) \mapsto (st, s^p, t^p) \in X = \{(u,v,w); vw = u^p\}$$

in terms of affine coordinates. Hence a general line $\{(1:s:as)\}_{s\in\mathbb{C}\cup\infty}$ passing through $(1,0,0)$ is mapped to the curve $\{(as^2, s^p, a^p s^p)\}$ with a $(2,p)$-cusp. Thus there are finitely many cuspidal curves $\in S$ passing through a given general base point x.

This construction easily carries over to higher dimension (for instance, consider $X = \mathbb{P}^n/G$, $G = \mathbb{Z}/(p)$, where p is a prime number $\geq n+1$), showing that

a) There are lot of singular quotients of $\mathbb{P}^n$ which carry doubly dominant families of rational curves unsplitting at a general point, and also that
b) The dimension estimates (Theorem 3.10(1) – (3)) in Kebekus' theorem are optimal in general.

5. Various characterizations of projective spaces

In this section, we derive from Main Theorem 0.1 various characterizations of projective spaces given in Corollary 0.4.

For the proof of Corollary 0.4, let us begin with trivial implications:

(a) The condition $X \simeq \mathbb{P}^n$ implies all the other conditions;

(b) (Length condition) $\Rightarrow$ (Length condition for rational curves) $\Rightarrow$ (Length condition for rational curves with base point); and
(c) (double dominance condition for rational curves) $\Leftrightarrow$ (dominance condition for rational curves with base point) $\Rightarrow$ (double dominance condition for rational curves of minimum degree).

Furthermore we know the implication relations

(d) (Frankel-Siu-Yau condition) $\Rightarrow$ (Hartshorne-Mori condition) $\Rightarrow$ (Mori condition).

Indeed, as is well known, the positivity of the holomorphic bisectional curvature yields the ampleness of the tangent bundle, while from the ampleness of tangent bundle Θ_X (or from a weaker condition that $-\mathrm{K}_X$ is ample) follows the uniruledness of X.

Our Main Theorem asserts that the double dominance of an unsplitting family implies $X \simeq \mathbb{P}^n$, while

(e) a family of curves of minimum degree is always unsplitting,

and hence (double dominance condition of rational curves of minimum degree) implies $X \simeq \mathbb{P}^n$.

Thus only the following implications remain to be checked:

(f) (Hirzebruch-Kodaira-Yau condition) $\Rightarrow$ (Kobayashi-Ochiai condition) $\Rightarrow$ (Length condition);
(g) (Mori condition) $\Rightarrow$ (Length condition for rational curves);
(h) (doubly transitive group action) $\Rightarrow$ (Mori condition);
(i) (Remmert-Van de Ven-Lazarsfeld condition) $\Rightarrow$ (Length condition for rational curves with base point) $\Rightarrow$ (double dominance condition for rational curves of minimum degree).

In what follows, we check the implications above one by one. Almost everything is an easy exercise except for the proof of the implication (Hirzebruch-Kodaira-Yau) $\Rightarrow$ (Kobayashi-Ochiai), where we need the topological invariance of some Chern numbers plus the characterization of ball quotients due to S.-T. Yau.

Proof of (Hirzebruch-Kodaira-Yau) $\Rightarrow$ (Kobayashi-Ochiai): Assume that X is homotopic to $\mathbb{P}^n$. Noting that X is simply connected and complex projective, we have $\mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z}) \simeq H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$. The first Chern class $c_1(X)$ can be written as mh, where m is an integer and h is the positive generator of $\mathrm{Pic}(X)$. The Chern number $c_1^n(X) = m^n$ is a homotopy invariant up to sign [Hi], so that $m = \pm(n+1)$. The Kobayashi-Ochiai condition is thus satisfied modulo the positivity of m. Suppose that m were negative. Then K_X would be ample and hence X would carry a Kähler Einstein metric [Y1], [Y2] and [Au]. The Chern number $c_1^{n-2}(2(n+1)c_2 - nc_1^2)$ is again a homotopy invariant (up to sign) and hence zero because $X \approx \mathbb{P}^n$. By Chen-Ogiue-Yau's result [CO], [Y1] and [Y2], this would imply that the universal cover of X is the open unit ball $B_n = \mathbf{SU}(1,n)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(n))$, contradicting the assumption that the compact manifold X is simply connected. Q.E.D.

Proof of (Kobayashi-Ochiai) $\Rightarrow$ (Length): Since $-\mathrm{K}_X$ is ample and divisible by $n+1$, it follows that $(C, -\mathrm{K}_X) \geq n+1$ for any effective curve on X. Q.E.D.

Proof of (Doubly transitive group action) $\Rightarrow$ (Mori): A complex Lie group which holomorphically acts on X with a fixed point is necessarily a linear algebraic group, a rational variety (a non-trivial action of a complex torus cannot have fixed points). Hence the orbit space X is uniruled (actually unirational). Let $C \subset X$ be an irreducible curve and $x \in C$ a smooth point. A doubly transitive action of $\mathrm{Aut}(X)$ gives rise to vector fields on X with zero at a given point x. This shows that $\Theta_X \otimes \mathcal{O}_C(-x)$ is generated by global sections. Hence $\Theta_X|_C$ is ample. Q.E.D.

Proof of (Mori) $\Rightarrow$ (Length of rational curves): Let $f : \mathbb{P}^1 \to X$ be a morphism, which is birational onto its image. If $\Theta_X|_{f(\mathbb{P}^1)}$ is ample, then so is $f^*\Theta_X$. Thus $f^*\Theta_X = \mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n), d_i \geq 1$. Since there is a non-zero natural homomorphism $\mathcal{O}(2) \simeq \Theta_{\mathbb{P}^1} \to f^*\Theta_X$, some d_i must be at least 2. Hence $(C, -\mathrm{K}_X) = \deg f^*\Theta_X = \sum d_i \geq n+1$. Q.E.D.

Proof of (Remmert-Van de Ven-Lazarsfeld) $\Rightarrow$ (Length of rational curves with base point): Let $g : \mathbb{P}^N \to X$ be a surjective morphism. Clearly $N \geq n = \dim X$. Take an ample divisor H on X. If $n = 0$, then there is nothing to prove. If n is positive, then g^*H cannot be trivial so that $(g^*H)^N = g^*(H^N) > 0$. This shows that $N = n$. In particular by Sard's theorem, there is a non-empty open subset $U \subset X$ such that g is unramified over U. Pick up $x_0 \in X$ from U. Let $C \subset X$ be an irreducible curve through x_0 and $(\tilde{C}, \tilde{x_0}) \subset \mathbb{P}^n$ a pointed irreducible curve such that $g(\tilde{C}) = C, g(\tilde{x}_0) = x_0$. We have a natural homomorphism $\Theta_{\mathbb{P}^n}|_{\tilde{C}} \to (g|_{\tilde{C}})^*\Theta_X$, which is an isomorphism at $\tilde{x}_0$ and hence globally injective. Since $\Theta_{\mathbb{P}^n}$ is ample, so is $(g|_{\tilde{C}})^*\Theta_X$. Thus $\Theta_X|_C$ is ample and, if C is rational, $\deg \Theta_X|_C = \deg \nu^*\Theta_X \geq n+1$, where $\nu : \mathbb{P}^1 \to C \hookrightarrow X$ is the normalization. Q.E.D.

Proof of (Length of rational curves with base point) $\Rightarrow$ (double dominance of rational curves of minimum degree): Let C be a rational curve passing through x_0. Assume that C has minimal degree among such curves. Let $f : \mathbb{P}^1 \to X$ be the morphism induced by the normalization. Then, if $(C, -\mathrm{K}_X) \geq n+1$, then the

$\mathrm{Hom}(\mathbb{P}^1, X; \infty \mapsto x_0)$ has dimension at least $n+1$ at $[f]$, giving rise to a family of morphisms as desired. Q.E.D.

Remark 5.1. For a singular projective variety X, the conditions (Remmert-Van de Ven-Lazarsfeld), (Double dominance of rational curves of minimum degree) and (Dominance of rational curves with base point) in Corollary 0.4 still make sense. When X is normal, Theorem 0.2 asserts that

$$\begin{aligned}(X \simeq \mathbb{P}^n) \Leftrightarrow &(\text{Double dominance of rational curves of minimum degree})\\ \Rightarrow &(\text{Dominance of rational curves with base point})\\ \Rightarrow &(\text{Remmert-Van de Ven-Lazarsfeld}).\end{aligned}$$

If we drop the normality condition, the third still implies the fourth because they are both of birational nature. However, the first two conditions are not mutually equivalent any more. The most trivial example is this: let $x_1, x_2 \in \mathbb{P}^n$ be two distinct point and let X be $\mathbb{P}^n$ with these two points x_1, x_2 identified (it is easy to check that X is projective).

Remark 5.2. Let us have a quick glance at the history of characterizations of projective spaces.

The Hirzebruch-Kodaira characterization [HK] (1957) was obtained (under the condition that the first Chern class c_1 is positive) as a beautiful application of the two milestones of the age: Hirzebruch's Riemann-Roch [Hi] (1953) and Kodaira vanishing [Kod] (1953). The solution of Calabi's conjecture by S.-T. Yau [Y1], [Y2] and T. E. Aubin [Au] (1978) enabled us to drop the positivity assumption on c_1. S. Kobayashi and T. Ochiai [KO] (1973) found that one can relax the diffeomorphism condition to the divisibility condition on c_1; in fact, this condition together with Kodaira vanishing completely determines[18] the Hilbert polynomial $h(t) = \chi(X, \mathcal{O}(tH))$, where H is an ample divisor such that $[H]$ is a positive generator of $\mathrm{H}^2(X, \mathbb{Z}) \simeq \mathbb{Z}$. These two earlier characterizations are topological, or rather cohomological, in nature.

In 1960s, conjectures on more geometric characterizations were proposed by T. Frankel [Fr], R. Hartshorne [Ha1], R. Remmert and A. Van de Ven [RV]. The first two conjectures were formulated in terms

[18]Indeed, we have $h(-1) = \cdots = h(-n) = 0$, $h(0) = 1$ for the polynomial h of degree n, so that $h(t) = (1/n!)(t+1)\cdots(t+n)$. In particular, we have $H^n = 1$, $\dim \mathrm{H}^0(X, \mathcal{O}(H)) = h(1) = n+1$. Similarly, if X is an n-dimensional Fano with $-K_X = nH$, then $h(t) = (-1)^n h(-n-t)$ (Serre duality), $h(0) = 1$, $h(-1) = \cdots = h(-n+1) = 0$, and hence $h(t) = (2/n!)(t+(n/2))(t+1)\cdots(t+n-1)$, $H^n = 2$, $\dim \mathrm{H}^0(X, \mathcal{O}(H)) = h(1) = n+2$.

of the positivity of tangent bundles, while the third is concerned with holomorphic images of projective spaces.

It was perceived wisdom among experts that the key to the Frankel/Hartshorne conjecture was the existence of lines, i.e., rational curves C on the variety X with $(C, -\mathrm{K}_X) = n+1$. Y.-T. Siu and S.-T. Yau [SY] (1980) solved the Frankel conjecture by realizing lines as energy-minimizing C^∞ images of the Riemann sphere S^2 with the aid of an imposing work of Sacks-Uhlenbeck [SaU] on harmonic maps. In their proof, the positivity of the curvature tensor is essential to ensure global convergence of energy minimizing sequences, and so their method does not work under weaker conditions like the positivity of the Ricci tensor. Shortly before their work, however, S. Mori [Mo1] (1978) discovered a revolutionary technique (Bend and Break plus modulo p reduction) to produce rational curves under much milder conditions: if the canonical bundle is not nef, then we can always find rational curves [MiMo]. Once sufficiently many rational curves were found, the Hartshorne conjecture was not hard to prove any more.[19] R. Lazarsfeld [La] (1983) applied Mori's result to solve the conjecture of Remmert-Van de Ven.

The characterization in terms of the length was proposed by S. Mori and S. Mukai at Taniguchi Conference, Katata 1987, where the participants jointly compiled a list of open problems in algebraic geometry. The formulation of Theorem 0.1 in terms of the existence of a doubly-dominant, unsplitting family is presumably new. One of the cornerstones of our result is Theorem 3.10 due to S. Kebekus, who obtained the result in the spring of 2000 during his stay at RIMS, Kyoto University.

Remark 5.3. Various results on projective n-space have been extended to n-dimensional smooth hyperquadric $\mathrm{Q}_n \subset \mathbb{P}^{n+1}$, the second simplest n-fold. For instance, the following six conditions on a smooth complex Fano n-fold X $(n \geq 2)$ are known to be equivalent:

- $X \simeq \mathrm{Q}_n$;
- Brieskorn condition [Br]: X is diffeomorphic to Q_n;
- Kobayashi-Ochiai condition [KO]: $c_1(X)$ is divisible by n in $\mathrm{H}^2(X, \mathbb{Z})$;
- Siu condition [Si]: X carries a Kähler metric of semi-positive holomorphic bisectional curvature with certain non-degeneracy condition;
- Cho-Sato condition I [CS1]: X is a holomorphic image of Q_n and is not isomorphic to $\mathbb{P}^n$;

[19] Step 4 through Step 9 in the previous section essentially reproduce the proof.

- Cho-Sato condition II [CS2]: $\wedge^2\Theta_X$ is ample but Θ_X is not ample.

These are of course the counterparts of (Hirzebruch-Kodaira), (Kobayashi-Ochiai), (Frankel-Siu-Yau), (Remmert-Van de Ven-Lazarsfeld) and (Hartshorne-Mori) for projective spaces, respectively. In view of the apparent parallelism above, it is quite natural to ask if our method (possibly after some minor modifications) applies to hyperquadrics. To be more specific, we conjecture that the following two conditions on X are also equivalent to the above six:

- Length condition: $\min\{(C, -\mathrm{K}_X); C \subset X \text{ is a curve}\} = n$;
- Subdouble dominance of rational curves: Let $F \to S$ be an arbitrary maximal family of rational curves on X. Then, F is a dominant family and, for general $x \in X$, the projection $\mathrm{pr}_X : F\langle x\rangle \to X$ has image of dimension $\geq n-1$.[20]

Thanks to the classification of Del Pezzo surfaces and Fano threefolds [Is1], [Is2] and [MoMu], the conjecture is verified up to dimension three.

PART III. Applications to Complex Symplectic Manifolds

6. Complex symplectic manifolds: generalities

This section is a concise review of the theory of complex symplectic manifolds. For proofs and further discussion, we refer the reader for example to Beauville [Bea1] and Fujiki [Fu].

Let Y be a Kähler manifold and η a d-closed[21] holomorphic 2-form on it. η defines a skew-symmetric, $\mathcal{O}_X$-bilinear pairing: $\Theta_Y \times \Theta_Y \to \mathcal{O}_Y$. When this pairing is everywhere non-degenerate, we call the pair (Y, η), or simply Y itself, a *complex symplectic manifold*, and η is said to be a *complex symplectic form* or a *complex symplectic structure*[22] on Y.

A (complex) symplectic manifold is necessarily of even dimension $2n$. The non-degeneracy condition of η is equivalent to saying that $\wedge^n\eta$ is a nowhere vanishing $2n$-form. In particular, the canonical bundle K_Y of a (complex) symplectic manifold Y is trivial. The symplectic form η gives a standard isomorphism $\Theta_Y \simeq \Omega^1_Y$.

---

[20] A normal (singular) hyperquadric $\subset \mathbb{P}^{n+1}$ also satisfies this condition. One could ask if finite quotients of hyperquadrics are characterized as normal, projective, uniruled varieties which satisfy the above subdouble dominance condition.

[21] As is well known, the d-closedness of η is automatic if Y is compact.

[22] We often drop the adjective "complex" when there is no danger of confusion.

Let $Z \subset Y$ be an analytic subset (closed, open, locally closed, or whatever). A subbundle[23] $\mathcal{E} \subset \Theta_Y|_Z$ is said to be *isotropic* if η identically vanishes on $\mathcal{E} \times \mathcal{E}$. The rank of an isotropic subbundle $\mathcal{E}$ does not exceed $n = (1/2)\mathrm{rank}\, \Theta_Y$. An isotropic subbundle of rank n is called *Lagrangian.* If $\mathcal{E} \subset \Theta_Y|_Z$ is Lagrangian, η gives a non-degenerate pairing between $\mathcal{E}$ and $(\Theta_Y|_Z)/\mathcal{E}$, and thereby a natural isomorphism between $(\Theta_Y|_Z)/\mathcal{E}$ and the dual of $\mathcal{E}$.

In general, a symplectic structure η is not unique even modulo the equivalence via non-zero constant scalar multiples. For instance, if η' is any (possibly degenerate) d-closed holomorphic 2-form, $\eta + t\eta'$ is again a symplectic form if the parameter t is sufficiently close to zero. However, when $H^0(Y, \Omega_Y^2)$ is one-dimensional, there is essentially only one symplectic structure, and we call Y a *primitive symplectic manifold* in this case.

The importance of primitive complex symplectic manifolds in the framework of the classification theory of Kähler manifolds is illustrated by the following

Theorem 6.1 ("Bogomolov decomposition" due to Berger [Ber]-Yau [Y2]-Bogomolov [Bo]-Beauville [Bea1]). *Let W be a compact Kähler manifold whose canonical class $-c_1(W) \in \mathrm{H}^2(W, \mathbb{Q})$ is zero. Then there exist a finite étale cover $\tilde{W} \to W$, a Ricci-flat Kähler metric g on $\tilde{W}$ and a Riemannian decomposition $\tilde{W} \simeq A \times \prod Y_i \times \prod Z_j$ such that*

(a) *A is a flat complex torus (with trivial holonomy); that*
(b) *Y_i is a simply connected, primitive symplectic manifold of dimension $2d_i$ with holonomy group $\mathbf{Sp}(2d_i)$; and that*
(c) *Z_j is an n_j-dimensional simply connected manifold whose holonomy group is $\mathbf{SU}(n_j)$, $n_j \geq 3$ (i.e., Z_i is an $\mathbf{SU}$-manifold).*

Each class appearing in the above decomposition has trivial canonical class. The three classes are separated from each other by simple birational invariants:

Complex torus $\iff \mathrm{H}^0(\Omega^1) \neq 0$,
Symplectic manifold $\iff \mathrm{H}^0(\Omega^1) = 0,\ \mathrm{H}^0(\Omega^2) \neq 0$,
$\mathbf{SU}$*-manifold* $\iff \mathrm{H}^0(\Omega^1) = \mathrm{H}^0(\Omega^2) = 0$.

[23]A coherent subsheaf $\mathcal{E}$ of a vector bundle (= locally free sheaf) $\mathcal{F}$ is called a *subbundle* if $\mathcal{F}/\mathcal{E}$ is locally free. A subbundle is always locally free.

In essence, this theorem asserts that the classification of compact Kähler manifolds with trivial canonical classes reduces to that of compact primitive symplectic manifolds and **SU**-manifolds.[24]

The (analytic) local structure of a symplectic manifold is extremely rigid. Indeed we have the following

Theorem 6.2 (Darboux). *Let (Y, η) be a complex symplectic manifold. Then, around each point of Y, we can find a local analytic coordinate system, called a Darboux coordinate system, $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ such that*

$$\eta = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

Despite the rigidity of the local structure, the global structure of compact symplectic manifolds is rich and admits abundant non-trivial deformation of complex structure.

Theorem 6.3 (Bogomolov [Bo]-Beauville [Bea1]; for the generalization to **SU**-manifolds, see Tian [Ti] and Todorov [To]). *Let Y be a compact complex symplectic manifold. Then the deformation space* (the Kuranishi space) *of the complex structure of Y is smooth and its tangent space at $[M]$ is exactly $\mathrm{H}^1(Y, \Theta_Y) \simeq \mathrm{H}^1(Y, \Omega^1_Y)$. The Kuranishi space is a Kähler manifold via the Weil-Petersson metric defined by the canonical Hodge pairing on $\mathrm{H}^1(Y, \Theta_M)$ induced by a prescribed Ricci-flat Kähler metric on M. In particular, given a homology class $\alpha \in \mathrm{H}_2(Y, \mathbb{Q}) \cap (\mathrm{H}^{2,0}(Y))^\perp$ (which is represented by a rational algebraic 1-cycle by a theorem of Lefschetz), there is a one-parameter deformation $\mathcal{Y} = \{Y_t\}_{t \in T}$ such that the flat lifting $\alpha_t \in \mathrm{H}_2(Y_t, \mathbb{Q})$ of α is no more an algebraic cycle for general t.*

If we deform the complex structure of given compact Y in a general direction, then Y will not contain any compact analytic curve. In fact, given an algebraic cycle $\alpha \in Y$, the algebraicity of α is preserved in a $\mathbb{Q}$-rational hyperplane $\alpha^\perp$ in $\mathrm{H}^1(Y, \Omega^1) \subset \big(\mathrm{H}_2(Y, \mathbb{C})\big)^*$, which is analytically locally identified with the Kuranishi space. More generally, the set of the deformations with Picard number $\geq \rho$ locally forms a countable union of linear affine subspaces of codimension ρ in the Kuranishi space.

An immediate consequence of this observation is

Proposition 6.4. *Let Y be a compact complex symplectic manifold of dimension $2n$ and $f : \mathbb{P}^1 \to Y$ a non-constant morphism. Then $\mathrm{Hom}(\mathbb{P}^1, Y)$ is of dimension $\geq 2n + 1$ at $[f]$.*

---

[24] Manifolds within these two classes are often referred as "Calabi-Yau manifolds".

Proof. We can construct a one-parameter deformation $\mathcal{Y} = \{Y_t\}$, $Y = Y_0$ such that the algebraicity of the cycle $f(\mathbb{P}^1)$ is destroyed on general Y_t. Hence (the underlying reduced structure of) $\mathrm{Hom}(\mathbb{P}^1, \mathcal{Y})$ is locally identical with $\mathrm{Hom}(\mathbb{P}^1, Y)$ around $[f]$. On the other hand, since the canonical bundle $\mathrm{K}_{\mathcal{Y}}$ is trivial, we have

$$\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, \mathcal{Y}) \geq \chi(\mathbb{P}^1, f^*\Omega^1_{\mathcal{Y}}) = 2n + 1.$$

Q.E.D.

Definition 6.5. Let (Y, η) be a complex symplectic manifold of dimension $2n$. A subvariety $Z \subset Y$ is said to be *isotropic* if $\Theta_{Z^\circ} \subset \Theta_Y|_{Z^\circ}$ is isotropic (equivalently: if $\eta|_Z \in H^0(Z, \Omega^2_Z/(\text{torsion}))$ identically vanishes). Here the symbol Z° stands for the smooth locus of Z. Any isotropic subvariety Z has dimension $\leq n$. If $\dim Z$ attains the maximum n, Z is called a *Lagrangian* subvariety.

If a subvariety Z contains sufficiently many rational curves, then it is necessarily isotropic (Lagrangian provided $\dim Z = n$). Indeed, we have:

Proposition 6.6. *Let Z be a closed r-dimensional subvariety of a $2n$-dimensional compact complex symplectic manifold (Y, η). If there is a family of rational curves $\tilde{f} : T \times \mathbb{P}^1 \to Z \subset Y$, such that*

(1) *$\tilde{f}(T \times \{\infty\})$ is a single closed point and that*

(2) *$\tilde{f}(T \times \mathbb{P}^1)$ contains an open dense subset of Z,*

then Z is an isotropic (Lagrangian, if $r = n$) subvariety.

Proof. If necessary, by changing the parameter space T by a suitable resolution, we may assume that T is a complex manifold. We have a natural generically injective homomorphism

$$\mathrm{pr}_T^*\Theta_T \oplus \mathrm{pr}_{\mathbb{P}^1}^*\Theta_{\mathbb{P}^1}(-\infty) \to \mathcal{H}om(\tilde{f}^*\Omega^1_Z, \mathcal{O}_{T\times\mathbb{P}^1}(-\infty)) \subset (\tilde{f}^*\Theta_Y)(-\infty).$$

Hence $\mathcal{H}om(\tilde{f}^*\Omega^1_Z, \mathcal{O}_{T\times\mathbb{P}^1})$ is ample when restricted to general $\{t\} \times \mathbb{P}^1$. The restriction of η to Z induces a bilinear pairing on $\mathcal{H}om(\Omega^1_Z, \mathcal{O}_Z)$ with values in the trivial line bundle $\mathcal{O}_Z$. Since the former vector bundle is ample on general $f_t(\mathbb{P}^1)$, this pairing identically vanishes on $f_t(\mathbb{P}^1)$. In view of the condition (2), this proves the assertion. Q.E.D.

Remark 6.7. In Proposition 6.6, if the closed point $\tilde{f}(T \times \{\infty\})$ is a non-singlular point of Z, it follows that Z is rationally connected.[25]

[25] A projective variety X is *rationally connected* if its two general points can be joined by an irreducible rational curve on X.

It is a general fact that a rationally connected variety has no nonzero global holomorphic r-forms, $r > 0$ ([KMM]).

7. Fibre space structure of primitive complex symplectic manifolds

Let Y be a projective, primitive complex symplectic manifold of dimension $2n$. Yau's theorem [Y1] and [Y2] asserts that if we are given a Kähler class η_0, Y carries a unique Ricci flat Kähler metric g whose Kähler form η is cohomologous to the given η_0. The Ricci-flat Kähler metric with holonomy $\mathbf{Sp}(2n)$ furnishes Y with a natural hyperkähler structure, which governs the Hodge-Lefschetz decomposition of the cohomology ring $H^{\bullet,\bullet}(Y,\mathbb{C})$. In view of this special structure of the cohomology, Beauville [Bea1] defined a symmetric bilinear form (*Beauville quadratic form*[26]) $Q(\cdot,\cdot)$ on the Néron-Severi group $\mathrm{NS}(Y)$ such that $D^{2n} = cQ(D,D)^n$, $D \in \mathrm{NS}(Y)$, where c is a constant independent of D.

The existence of the Beauville quadratic form yields non-trivial information on the cone of divisors. If D is a nef divisor $\not\approx 0$ with $D^{2n} = 0$ and H is ample, then we have

$$\sum \binom{2n}{k} t^k D^{2n-k} H^k = (D+tH)^{2n} = cQ(D+tH, D+tH)^n$$
$$= c\big(Q(D,D) + 2tQ(D,H) + t^2 Q(H,H)\big)^n.$$

If we compare the coefficients of t^k in the left-hand side and in the right-hand side, we easily get $Q(D,D) = 0$, $Q(D,H) > 0$, $D^n H^n > 0$ and $D^{n+i}H^{n-i} = 0$, $i > 0$. Namely, a non-zero nef divisor on a primitive complex symplectic manifold Y of dimension $2n$ is either big or looks like a pullback of an ample divisor on an n-dimensional variety.

Starting from this observation, D. Matsushita discovered that any non-trivial fibre space structure of Y must be of very restricted type.

Theorem 7.1 (Matsushita [Mats]). *Let Y be a projective, primitive complex symplectic manifold and let $\pi : Y \to X$ be a morphism onto a normal projective variety X with $0 < \dim X < 2n = \dim Y$ and with $\pi_* \mathcal{O}_Y = \mathcal{O}_X$. Then:*

(1) *X is of dimension n and $\mathbb{Q}$-factorial, i.e., any Weil divisor on X is a Cartier divisor if multiplied by a suitable positive integer. The Picard number of X is one.*

(2) *X has only log-terminal singularities.*

[26] For a systematic account on the Hodge-Lefschetz decomposition on hyperkähler manifolds and the Beauville quadratic form Q, see Fujiki [Fu].

(3) *The anticanonical divisor* $-\mathrm{K}_X$ *is ample (as a* $\mathbb{Q}$*-Cartier divisor). In other words,* X *is a* $\mathbb{Q}$*-Fano variety.*

(4) *Every fibre* Y_x *of* π *is of pure dimension* n *and each of its components is Lagrangian.*

The property (1) specifically means that a nonzero effective Weil divisor D and an effective curve C on X necessarily meet each other.[27]The property (4) implies that a smooth fibre A of π is an abelian variety because Θ_A and the trivial normal bundle $\mathcal{N}_{A/Y} \simeq \mathcal{O}_A^{\oplus n}$ are mutually duals via the symplectic pairing.[28]

Under a certain technical condition, Main Theorem 0.1 shows that the base variety X of a non-trivial fibration of a primitive symplectic manifold Y is a projective space. Namely:

Theorem 7.2. *Let* $\pi : Y \to X$ *be as in Theorem* 7.1 *and assume in addition that* π *admits a section* $\sigma : X \to Y$*. Then* X *is isomorphic to* $\mathbb{P}^n$*.*

The proof of Theorem 7.2 consists of several steps. We start with easy observations.

Lemma 7.3. X *is a smooth uniruled variety. Let* $x \in X$ *be a general point and* $C \subset X$ *a rational curve of minimum degree passing through* x*. If* C *is general, then* C *is smooth and*

$$\Theta_X|_C \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus n-e}.$$

Proof. The projection π and the section σ give a canonical injection $\mathcal{O}_X \hookrightarrow \mathcal{O}_Y$ and a natural surjection $\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_X$, thereby inducing an $\mathcal{O}_X$-linear map $\Omega^1_X \to \Omega^1_Y$ and an $\mathcal{O}_Y$-linear surjection $\Omega^1_Y \to \Omega^1_X$. Therefore Ω^1_X is a direct summand of the $\mathcal{O}_X$-module $\Omega^1_Y|_{\sigma(X)}$. Since $\Omega^1_Y|_{\sigma(X)}$ is locally free by the smoothness of Y, we conclude that Ω^1_X is also locally free; that is, X is smooth. In particular, X is a Fano manifold by Matsushita's theorem. Any Fano manifold (more generally, any $\mathbb{Q}$-Fano variety) is uniruled by [MiMo]. The proof of the second and third statements are given in Theorem 2.8. Q.E.D.

From now on, we fix the notation as follows.

Let x be a general point on X, and $C \subset X$ a general (smooth) rational curve of minimum degree passing through x. By the embedding

[27]We need this fact to prove Theorem 7.2 below.

[28]An alternative proof is by the famous theorem of Liouville on completely integrable Hamiltonian systems (see Arnold [Ar, p.272]).

$\sigma : X \to Y$, we view C and X as closed subschemes of Y. Define $f : \mathbb{P}^1 \to Y$ by fixing an isomorphism $\mathbb{P}^1 \simeq C \subset \sigma(X) \subset Y$.

Lemma 7.4. *We have an isomorphism*

$$f^*\Theta_Y \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e} \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}(-2).$$

Proof. Since X is a Fano manifold, there is no non-zero global 2-form on X.[29] Hence X is a Lagrangian submanifold in Y. Then the symplectic form η defines an isomorphism between

$$f^*\Theta_X \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus n-e}$$

and the dual of $f^*\Theta_Y / f^*\Theta_X$. Q.E.D.

Corollary 7.5. $\mathrm{Hom}(\mathbb{P}^1, Y)$ *is smooth at* $[f]$. *If* $[h] \in \mathrm{Hom}(\mathbb{P}^1, Y)$ *is sufficiently close to* $[f]$, *then* $h^*\Theta_Y$ *has the same decomposition type as* $f^*\Theta_Y$.

Proof. By Lemma 7.4, we have $\dim \mathrm{H}^0(\mathbb{P}^1, f^*\Theta_Y) = 2n+1$, whilst we have the estimate $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) \geq 2n+1$ by Definition 6.5. This shows that $\mathrm{Hom}(\mathbb{P}^1, Y)$ is smooth at $[f]$ and the differential of the universal morphism $\mathrm{Hom}(\mathbb{P}^1, Y) \times \mathbb{P}^1 \to Y$ has rank $2n-e$ at $([f], p)$, where $p \in \mathbb{P}^1$ is general. Therefore, if $[h] \in \mathrm{Hom}(\mathbb{P}^1, Y)$ is sufficiently close to $[f]$, we have

$$\dim \mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y) = \dim_{[h]} \mathrm{Hom}(\mathbb{P}^1, Y) = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Y) = 2n+1$$

and $\mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y)$ generates a subsheaf $\mathcal{E} \subset h^*\Theta_Y$ of rank $2n-e$. The quotient $\mathcal{E}/\Theta_{\mathbb{P}^1}$ is semi-positive, of rank $2n-e-1$, of degree $e-1$, and is isomorphic to $\mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e}$ when $h = f$. This decomposition type is obviously stable under small deformation, and hence

$$\mathcal{E} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \mathcal{O}^{\oplus 2n-2e}.$$

Then by the existence of non-degenerate pairing on $h^*\Theta_Y \supset \mathcal{E}$, we get the assertion. Q.E.D.

Fix a general point $x \in \sigma(X) \simeq X$ and a general rational curve $C \subset \sigma(X)$ of minimum degree through x. Recall that $C \simeq \mathbb{P}^1$ lies on the smooth locus of $\sigma(X)$. Let $f : \mathbb{P}^1 \simeq C \to \sigma(X) \subset Y$ be the embedding given above, M a sufficiently small Zariski open neighbourhood of $[f]$ in $\mathrm{Hom}(\mathbb{P}^1, Y)$, and $\tilde{f} : M \times \mathbb{P}^1 \to Y$ the restriction of the universal

[29] By Kodaira vanishing, we have $\mathrm{H}^2(X, \mathcal{O}_X) = \mathrm{H}^0(X, \Omega_X^2) = 0$.

morphism. Pick up a general point $z = h(p) = \tilde{f}([h], p)$ of $\tilde{f}(M \times \mathbb{P}^1) \subset Y$ and put $M_z = M \cap \mathrm{Hom}(\mathbb{P}^1, Y; p \mapsto z)$. M and M_z are smooth because

$$\dim \mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y) = \dim M = 2n+1,$$
$$\dim \mathrm{H}^0(\mathbb{P}^1, h^*\Theta_Y(-p)) = \dim M_z = n+1.$$

Let $\tilde{f}_z : M_z \times \mathbb{P}^1 \mapsto Y$ denote the restriction of $\tilde{f}$.

By Corollary 7.5, the universal morphism $\tilde{f} : M \times \mathbb{P}^1 \to Y$ is everywhere of constant rank $2n - e$, so that $\tilde{f}(M \times \mathbb{P}^1) \subset Y$ is an immersed locally closed submanifold of dimension $2n - e$ (after shrinking M to a smaller open subset if necessary). In particular, the normalization Z of $\tilde{f}(M \times \mathbb{P}^1)$ is smooth. The morphism $\tilde{f} : M \times \mathbb{P}^1 \to Y$ factors through $\tilde{g} : M \times \mathbb{P}^1 \to Z$ and the natural immersion $Z \to Y$. Notice that $\sigma(X) \subset Y$ is contained in the closure of $\tilde{f}(M \times \mathbb{P}^1)$ because $\Theta_{\sigma(X)}|_C$ is semipositive.

Let Z_z denote the normalization of $\tilde{f}_z(M_z \times \mathbb{P}^1) \subset \tilde{f}(M \times \mathbb{P}^1)$. Let $\tilde{g}_z : M_z \times \mathbb{P}^1 \to Z_z$ and $j : Z_z \to Z$ be the morphisms defined in an obvious manner. Our morphism $h : \mathbb{P}^1 \to \tilde{f}(M_z \times \mathbb{P}^1) \subset \tilde{f}(M \times \mathbb{P}^1) \subset X$ naturally defines morphisms $g_z : \mathbb{P}^1 \to Z_z$, $g = jg_z : \mathbb{P}^1 \to Z$.

Then an immediate consequence of Lemmas 7.3 and 7.4 is the following

Lemma 7.6. *We have*

$$g^*\Theta_Z = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e},$$
$$g_z^*\Theta_{Z_z} \subset \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \subset g_z^*\Theta_{Z_z}(p).$$

The composite natural projection $Z \to Y \xrightarrow{\pi} X$ is everywhere of rank n near the curve $g(\mathbb{P}^1) \subset Z$.

Lemma 7.7. *Let the notation be as above and assume that M is sufficiently small. Then*

(1) *The 2-form η defines a degenerate bilinear form of constant rank $2n - 2e$ on Θ_Z in a natural way.*

(2) *Z_z is smooth and $j : Z_z \to Z$ is an embedding.*

(3) *The subsheaf $\Theta_{Z_z} \subset j^*\Theta_Z$ is determined by the following null-space property:*

$$v \in \Theta_{Z_z} \iff \eta(v, w) = 0 \text{ for arbitrary } w \in j^*\Theta_Z.$$

Thus the subset $Z_z \subset Z$, attached to a general point $z \in Z$, is indeed an integral submanifold of a foliation of rank e on the smooth, locally closed variety Z.

Proof. The rank of Θ_Z is everywhere $2n-e$, while Θ_Y is of rank $2n$. Elementary linear algebra then shows that the bilinear form $\eta|_{\Theta_Z}$ has pointwise rank $\geq 2n-2e$, and the associated null space

$$\mathrm{Null}_\eta(\Theta_{Z,p}) = \{v \in \Theta_{Z,p}; \eta(v, \Theta_{Z,p}) = 0\}$$

has complex dimension $\leq e$. As a function in $p \in Z$, $\dim \mathrm{Null}_\eta(\Theta_{Z,p})$ is uppersemicontinuous (equivalently, rank $\eta|_{\Theta_{Z,p}}$ is lower semicontinuous). Hence the statement (1) follows if we check that $\dim \mathrm{Null}_\eta(\Theta_{Z,p}) = e$ for general $p \in Z$.

On $\mathbb{P}^1$, the maximal positive subbundle $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \subset g^*\Theta_Z$ is of course ample and $g^*\Theta_Z = g_z^* j^* \Theta_Z$ is semipositive, so that any pairing $g_z^*\Theta_{Z_z} \times g_z^* j^* \Theta_Z \to \mathcal{O}$ identically vanishes. This means that the pairing

$$\eta : \Theta_{Z_z} \times j^*\Theta_Z \to \mathcal{O}_{Z_z}$$

vanishes on $g_z(\mathbb{P}^1)$. By deforming h and thereby $g_z(\mathbb{P}^1) \subset Z_z$, we conclude that the pairing $\Theta_{Z_z} \times j^*\Theta_Z \to \mathcal{O}_{Z_z}$ must identically vanish on Z_z. Hence, on an open dense subset of Z_z, the vector bundle Θ_{Z_z} of rank e is pointwise identical with $\mathrm{Null}_\eta(\Theta_{Z,p})$, showing that $\dim \mathrm{Null}_\eta(\Theta_{Z,p})$ is constant on Z_z. Noticing that the family $\{Z_z\}_{z \in Z}$ sweeps out Z, we conclude that the subspaces $\mathrm{Null}_\eta(\Theta_{Z,p})$ have constant dimension e on Z and gives rise to a subbundle $\mathrm{Null}_\eta(\Theta_Z) \subset \Theta_Z$.

The Lie bracket $[\ ,\ ]$ defined on Θ_Z induces an $\mathcal{O}_Z$-homomorphism $\wedge^2 \mathrm{Null}_\eta(\Theta_Z) \to \Theta_Z/\mathrm{Null}_\eta(\Theta_Z)$. If we restrict things to a small deformation of $g_z(\mathbb{P}^1)$, the target is a trivial vectorbundle while the source is ample, and hence any $\mathcal{O}_Z$-homomorphism $\wedge^2 \mathrm{Null}_\eta(\Theta_Z) \to \Theta_Z/\mathrm{Null}_\eta(\Theta_Z)$ is a zero map. This means that the subbundle $\mathrm{Null}_\eta(\Theta_Z)$ is involutive, i.e., $[\mathrm{Null}_\eta(\Theta_Z), \mathrm{Null}_\eta(\Theta_Z)] \subset \mathrm{Null}_\eta(\Theta_Z)$. Thus $\mathrm{Null}_\eta(\Theta_Z)$ defines a foliation on Z.

Furthermore, we have seen that Θ_{Z_z} is identical with the subbundle $\mathrm{Null}_\eta(\Theta_Z)$ (on an open dense subset of Z_z). In other words, Z_z is an integral submanifold of the foliation given by $\mathrm{Null}_\eta(\Theta_Z)$, whence follows (2) and (3). Q.E.D.

Corollary 7.8. *There exist a variety B and a dominant morphism $Z \to B$ of which a general fibre is a manifold of the form $(Z_z)^c \cap Z$, $z \in Z$. (Here $\bullet^c$ denotes the Zariski closure.)*

Proof. Given a point z on Z, the integral variety of a non-singular foliation passing through z is uniquely determined and, in our situation, is a constructible set of the form $Z_{z'}$ for some $z' \in Z$. Hence we have a well-defined morphism $Z \to \mathrm{Chow}(Z^c)$, $z \mapsto [(Z_{z'})^c]$. Q.E.D.

Recall that the closure of $\tilde{f}(M \times \mathbb{P}^1)$ contains $\sigma(X)$. More precisely, $\tilde{f}(M \times \mathbb{P}^1)$ (as well as its normalization Z) contains a smooth small open neighbourhood $U \subset \sigma(X)$ of $C = f(\mathbb{P}^1)$. Hence the natural rational map $\sigma(X) \dashrightarrow Z$ and the fibration $Z \to \mathrm{Chow}(Z^c)$ induce a dominant rational map $\phi : \sigma(X) \dashrightarrow B' \subset \mathrm{Chow}(Z^c)$, well defined as a morphism on U.

Lemma 7.9. $\left(\tilde{f}(M_x \times \mathbb{P}^1)\right)^c = \sigma(X)$, *or equivalently* $e = n$, *the superscript* ^{c} *denoting the closure. In particular,* $\dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, \sigma(X))$ $= 2n + 1$.

Proof. Suppose otherwise. The projective variety B' is then of positive dimension. Take a divisor D' on B' away from the single point

$$\phi(C) = \phi(f(\mathbb{P}^1)) = \phi(\sigma(X) \cap Z_z) \in B',$$

and let D be its inverse image on $\sigma(X)$. (More precisely, blow up $\sigma(X)$ along centres away from U to resolve the indeterminacy of ϕ, and we get the diagram

$$\begin{array}{ccc} \sigma(X)' & \xrightarrow[\phi']{} & B' \subset \mathrm{Chow}(Z) \\ \downarrow{\scriptstyle \mu} & & \\ \sigma(X). & & \end{array}$$

Then we define D to be $\mu(\phi'^* D')$.) D is well-defined on U (i.e., independent of the choice of $\sigma(X)'$) and is away from $\sigma(X) \cap Z_z \supset C$. In other words, there exists an effective Weil divisor D on $\sigma(X)$ which does not meet the effective curve C. This contradicts Matsushita's result that $\sigma(X) \simeq X$ has Picard number one. Q.E.D.

Theorem 7.2 immediately follows from this lemma and our Main Theorem 0.1.

Example 7.10. The following example of non-trivial fibration of primitive complex symplectic manifolds is due to Fujiki ($r = 2$) and Beauville [Bea1] (r general).

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with everywhere non-vanishing 2-form η. $\mathrm{Hilb}^r(S)$, the Hilbert scheme of 0-dimensional closed subschemes of degree r on S, is known to be smooth by a result of Fogarty [Fo]. There is a natural birational morphism from $\mathrm{Hilb}^r(S)$ to the symmetric product $\mathrm{Sym}^r(S) = S \times \cdots \times S/\mathfrak{S}_r$, which is identified with the Chow scheme of the effective 0-cycles of degree r.[30]Thus we have natural

[30]By virture of the normality of the symmetric product and the universal property of the Chow scheme.

morphisms $\mathrm{Hilb}^r(S) \to \mathrm{Sym}^r(S) \to \mathrm{Sym}^r(\mathbb{P}^1) \simeq \mathbb{P}^r$, defining an abelian fibration structure of $\mathrm{Hilb}^r(S)$ over $\mathbb{P}^r$. The $\mathfrak{S}_r$-invariant 2-form $\sum \mathrm{pr}_i^*\eta$ naturally lifts to a symplectic form on $\mathrm{Hilb}^r(S)$, nonzero holomorphic 2-form unique up to non-zero factor.

Examples of this type form a 19-dimensional family because the deformation of the elliptic fibre space structure of S is parameterized by a 19-dimensional space thanks to the global Torelli for K3 surfaces [BaPeVV, VII.11.1].

We can check that the second Betti cohomology group of $\mathrm{Hilb}^r(S)$ is generated by $\mathrm{H}^2(S, \mathbb{C})$ and the exceptional divisor E over the diagonal

$$\Delta = \{(x_1, \dots, x_r); x_i = x_j \text{ for some } (i,j), i \neq j\}/\mathfrak{S}_r \subset \mathrm{Sym}^r(S),$$

so that

$$\mathrm{H}^1(\mathrm{Hilb}^r(S), \Theta_{\mathrm{Hilb}^r(S)}) \simeq \mathrm{H}^1(\mathrm{Hilb}^r(S), \Omega^1) \simeq \mathrm{H}^1(S, \Omega^1) \oplus \mathbb{C}[E] \simeq \mathbb{C}^{21}.$$

The Kuranishi space T of Hilb^r is thus 21-dimensional, and the subspace T_0 which preserves the fibre space structure is of dimension 20. That is to say a general element $t \in T_0$ is not represented as a Hilbert scheme of a K3 surface any more.

Remark 7.11. Let $\pi : Y \to X$ be a fibre space structure of a projective primitive complex symplectic manifold Y over a normal projective variety X. Assume that every scheme theoretic fibre of π has a reduced irreducible component or, equivalently, that π admits analytic local sections at every point $x \in X$. The dual abelian fibration $\pi^\dagger : \mathrm{Pic}^0(Y/X) \to X$ is naturally a (non-proper) group scheme over X, and so is the double dual $\pi^\ddagger : \mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big) \to X$. By choosing a local analytic section $\sigma_U : U \to Y|_U$ defined on a small Stein open subset $U \subset X$, we have a natural identification $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)|_U \simeq Y^\circ|_U$, where $Y^\circ \subset Y$ is the open subset consisting of the non-critical points of π. This isomorphism provides Y° with a $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$-torsor structure over X, or a natural action of $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$ on Y°. The global structure of Y° is recovered from $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$ and the patching data $\eta \in \mathrm{H}^1_{\mathrm{ann}}(X, \mathrm{Pic}^0(\mathrm{Pic}^0(Y/X)/X))$.

Suppose that this $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$-action on Y° extends to an action on the compactification Y. (This is indeed the case when the degenerations are semistable.) Under this additional assumption, we can naturally construct a smooth compactification $\pi^\ddagger : Y^\ddagger \to X$ of the group scheme $\mathrm{Pic}^0\big(\mathrm{Pic}^0(Y/X)/X\big)$ by identifying $Y^\ddagger|_U$ with $Y|_U$ and patching these together via the given data η. On a smooth fibre, the patchig is defined by a suitable translation, which does not affect the relative cotangent sheaf $\Omega^1_{Y/X}$. Therefore we have a canonical isomorphism $\pi_*\Omega^1_{Y/X} \simeq \pi^\ddagger_*\Omega^1_{Y^\ddagger/X}$, implying that $Y^\ddagger$ is again a primitive complex symplectic manifold. Applying Theorem 7.2 to $Y^\ddagger$ instead of Y, we conclude that the base variety X is $\mathbb{P}^n$. In this sense, Theorem 7.2 states something stronger than it sounds.

Theorem 7.2 and Remark 7.11 in mind, we ask the following questions:

Problem 7.12. (1) *Is it possible to completely classify symplectic n-dimensional complex torus fibrations free from multiple fibres over $\mathbb{P}^n$?*

(2) *Is there a fibration $\pi : Y \to X$ from a primitive compact symplectic manifold onto a normal variety which is not $\mathbb{P}^n$?* (If there is any, its projection π will have non-semistable, perhaps multiple, fibres.)

Most primitive symplectic manifolds known so far carry non-trivial fibrations if we suitably deform its complex structure, although we have no idea if that is always the case. For instance, if there is a symplectic manifold with $\mathrm{h}^{1,1} = 1$, it does not allow any non-trivial fibration.

Problem 7.13. (1) Let Y be a compact primitive complex symplectic manifold with $\dim \mathrm{H}^1(Y, \Omega^1_Y) \geq 2$ (or, equivalently, $\dim \mathrm{H}^2(Y, \mathbb{C}) \geq 4$). By suitably deforming the complex structure, we can assume that the Picard number is exactly $\mathrm{h}^{1,1}$. Assume that there is a divisor D with $D^{2n} = 0$. Then, *is it possible to find D such that the linear system $|D|$ defines a non-trivial fibration of Y?*

(2) Since the holonomy group of a primitive symplectic manifold Y is the full symplectic group $\mathbf{Sp}(2n)$, it follows that $\mathrm{H}^0(Y, \Omega^p_Y)$ is $\mathbb{C}$ or 0 according to the parity of $p \leq 2n$. However, the higher cohomology $\mathrm{H}^q(Y, \Omega^p_Y)$ could be highly non-trivial. *Are there a priori dimension estimates (from above and/or from below) for the Betti numbers of primitive symplectic manifolds?*

8. Symplectic resolutions of an isolated singularity

Let $\hat{Z}$ be s normal variety of even dimension $2n$ with a single isolated singularity, and $\pi : Z \to \hat{Z}$ a symplectic resolution. Namely, π is a projective bimeromorphic morphism from a complex manifold Z, which carries an everywhere non-degenerate closed holomorphic 2-form η. Let $E = \bigcup E_i$ denote the exceptional locus of π, each E_i being an irreducible component of E.

Recall that a pure dimensional closed subvariety $W \subset Z$ is called *Lagrangian* if $\dim W = n$ and $\eta|_W$ is identically zero.

Example 8.1. Let $Z = \operatorname{Spec} \operatorname{Sym}\Theta_X$ be the (total space of the) cotangent bundle Θ^*_X of a smooth projective variety X, $\mathrm{pr}_X : Z \to X$

the standard projection, and $0_X \subset Z$ the zero-section. Θ_Z is naturally isomorphic to $\mathrm{pr}_X^*\Theta_X \oplus \mathrm{pr}_X^*\Theta_X^*$, so that it carries a standard symplectic form η defined by

$$\eta\big((\alpha_1, \beta_1), (\alpha_2, \beta_2)\big) = \langle \alpha_1 | \beta_2 \rangle - \langle \alpha_2 | \beta_1 \rangle,$$

where $\alpha_i \in \Theta_X$, $\beta_i \in \Theta_X^*$, and $\langle \cdot | \cdot \rangle$ stands for the canonical pairing between the duals.[31]

The normal bundle of $0_X \simeq X$ is of course isomorphic to Θ_X^*. By Hartshorne-Mori, it is negative if and only if X is $\mathbb{P}^n$. In this case, thanks to a theorem of Grauert [Gra], we can contract 0_X to a point o to get a symplectic resolution $\pi : Z \to \hat{Z}$.

The symplectic resolution $\pi : Z \to \hat{Z}$ described in Example 8.1 is said to be *standard*. The following fact (the existence of "standard flops") is an important feature of standard symplectic resolutions.

Lemma 8.2. *Let $X \subset Z$ be a Lagrangian submanifold isomorphic to $\mathbb{P}^n$ in a complex symplectic manifold. Then a small analytic neighbourhood of X in Z is isomorphic to a standard symplectic resolution described in Example* 8.1 *of an isolated singularity. If $n \geq 2$ (i.e., if $\dim Z \geq 4$), let $\mu : \tilde{Z} = \mathrm{Bl}_X(Z) \to Z$ denote the blowing-up along X. Then the exceptional divisor $\tilde{X} = E_X \subset \tilde{Z}$ is a $\mathbb{P}^{n-1}$-bundle over X and admits another $\mathbb{P}^{n-1}$-fibration over $X' \simeq \mathbb{P}^n$, and accordingly $\tilde{Z}$ has another blowing-down $\mu' : \tilde{Z} \to Z'$ onto a new symplectic manifold.*[32] *Given $p \in X$, the closed subset $\mu'\big(\mu^{-1}(p)\big)$ is a hyperplane in X'.*

Proof. Because of the symplectic paring η, we easily deduce that the normal bundle $\mathcal{N}_{X/Z}$ is naturally isomorphic to the negative vector bundle Θ_X^*. Then we have

$$\mathrm{H}^1(X, \mathrm{Sym}^r \mathcal{N}_{X/Z}^*) = \mathrm{H}^1(X, \Theta_Z \otimes \mathrm{Sym}^r \mathcal{N}_{X/Z}^*) = 0, \quad r > 0,$$

and a theorem of Grauert [Gra, Sect. 4, Satz 7] applies to prove that Z is locally biholomorphic to the standard resolution around X.

[31]While $\Theta_X^* = \mathrm{Spec}\ \mathrm{Sym}\Theta_X$ is a symplectic manifold, its projectivization $W = \mathbb{P}(\Theta_X) = \mathrm{Proj}\ \mathrm{Sym}\Theta_X$ is a *complex contact manifold*, an odd-dimensional analogue of a symplectic manifold (see [Bea2]). Namely there is a subbundle $\mathcal{F} \subset \Theta_W$ of corank one together with a non-degenerate skew symmetric pairing $\mathcal{F} \times \mathcal{F} \to \Theta_W/\mathcal{F} \simeq \mathcal{O}_W(\mathbf{1})$, where $\mathcal{O}_W(\mathbf{1})$ stands for the tautological line bundle on the projective bundle.

[32]The birational map $Z \dashrightarrow Z'$ is a typical flop. We refer the reader to [KM] for flips and flops.

The blown up variety $\tilde{Z}$ is Proj $\bigoplus \mathfrak{I}_X^m$ and the exceptional divisor is given by Proj $\bigoplus (I_X^m/I_X^{m+1}) =$ Proj $\bigoplus \mathrm{Sym}^m \Theta_X$. We have a standard Euler exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \to \Theta_X \to 0,$$

which means that the exceptional divisor $\tilde{X} = E_X$ is a hypersurface in $\mathbb{P}^n \times X \simeq \mathbb{P}^n \times \mathbb{P}^n$ of bidegree $(1,1)$. We can easily check that the first projection $\tilde{X} \to \mathbb{P}^n$ gives another $\mathbb{P}^{n-1}$-fibre space structure, each fibre F of which satisfying $\mathcal{O}_{\tilde{Z}}(\tilde{X})|_F \simeq \mathcal{O}_F(-1)$. Therefore $(\tilde{Z}, \tilde{X})$ can be blown down to (Z', X'), a pair of smooth varieties. The symplectic form η can be viewed as a non-degenerate form on $Z' \setminus X'$. Recalling that the codimension of $X' \subset Z'$ is two or more, the form η naturally extends to a non-degenerate holomorphic 2-form on Z'. Thus we get the assertion. Q.E.D.

The main result of this section is the following

Theorem 8.3. *Let $\hat{Z}$ be a normal projective variety of dimension $2n$ with a single isolated singularity. Assume that there exists a symplectic resolution $\pi : Z \to \hat{Z}$; in other words, π is a birational morphism from the smooth, projective, complex symplectic variety Z onto $\hat{Z}$. Then:*

(1) *The exceptional locus $E \subset \hat{Z}$ of π is a union of Lagrangian submanifolds isomorphic to $\mathbb{P}^n$.*

(2) *When $n = 1$, the exceptional locus E is a tree of smooth $\mathbb{P}^1$'s with configuration of one of the* ADE*-singularities.*

(3) *If $n \geq 2$, then E consists of a single smooth $\mathbb{P}^n$ and $\pi : Z \to \hat{Z}$ is analytically-locally a standard resolution.*

For the proof, we need several easy results.

Lemma 8.4. *Every component E_i of the exceptional locus is uniruled. Furthermore $\mathrm{R}^i\pi_*\mathcal{O}_Z = 0,\ i > 0$.*

Proof. The first statement is a special case of Theorem 1 of [Ka]. (Essentially the adjunction plus Miyaoka-Mori criterion for uniruledness [MiMo].) In order to show the second statement, notice that $\wedge^n\eta$ defines a nowhere vanishing $2n$-form, so that the dualizing sheaf ω_Z is isomorphic to $\mathcal{O}_Z$. Then the Grauert-Riemenschneider vanishing [GraR] yields $\mathrm{R}^i\pi_*\mathcal{O} \simeq \mathrm{R}^i\pi_*\omega = 0,\ i > 0$. Q.E.D.

Corollary 8.5. *Let $\tilde{E}_i$ be a non-singular model of E_i, an irreducible component of the exceptional locus E. Then the pullback of a holomorphic 2-form η on Z to $\tilde{E}_i$ is identically zero. In particular, E_i is isotropic with respect to the symplectic form η, and hence* $\dim E_i \leq n$.

Proof. Take an embedded resolution $\mu : \tilde{Z} \to Z$ of $E \subset Z$. Thus the inverse image $\tilde{E} \subset \tilde{Z}$ of $E \subset Z$ is a divisor of simple normal crossings. Let $D \subset \tilde{E}$ be an arbitrary irreducible component. In order to prove the assertion, it suffices to show that the natural restriction map $\mathrm{H}^0(\tilde{Z}, \Omega^2_{\tilde{Z}}) \to \mathrm{H}^0(D, \Omega^2_D)$ identically vanishes. By Hodge theory, this map is the complex conjugate of the restriction map $\mathrm{H}^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \to \mathrm{H}^2(D, \mathcal{O}_D)$. Take an effective, sufficiently ample divisor $\hat{H}$ on $\hat{Z}$ such that $\mathrm{H}^i\big(\hat{Z}, (\mathrm{R}^j(\pi\mu)_*\mathcal{O}_{\tilde{Z}})(\hat{H})\big) = 0$ for $i > 0, j \geq 0$. Thus, by Leray spectral sequence,

$$\mathrm{H}^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}}(\mu^*\pi^*\hat{H})) \simeq \mathrm{H}^0(\hat{Z}, \mathrm{R}^2(\pi\mu)_*\mathcal{O}_{\tilde{Z}}(\hat{H})).$$

By construction, $\mu^*\pi^*\hat{H}$ is trivial on D. In the meantime, since $\mathrm{R}^j\pi_*\mathcal{O}_{\hat{Z}} = 0$ and Z is non-singular, we have $\mathrm{R}^j(\pi\mu)_*\mathcal{O} = 0,\ j > 0$. Looking at the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2(\tilde{Z}, \mathcal{O}) & \longrightarrow & \mathrm{H}^2(D, \mathcal{O}) \\ \downarrow & & \simeq\downarrow \\ 0 = \mathrm{H}^2(\tilde{Z}, \mathcal{O}(\mu^*\pi^*\hat{H})) & \longrightarrow & \mathrm{H}^2(D, \mathcal{O}(\mu^*\pi^*\hat{H})) \end{array}$$

we conclude that the restriction map in question is zero. Q.E.D.

Lemma 8.6. *Let $p_i \in E_i$ be a point not contained in $\bigcup_{j\neq i} E_j$ and take a rational curve $C_i \subset E_i \subset Z$ such that C_i passes through p_i. Let $g_i : \mathbb{P}^1 \to E_i$ be a standard morphism obtained by normalizing C_i. Then we have:*

$$\dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, E_i) \geq 2n + 1.$$

Furthermore, given any rational curve $C \subset E = \bigcup E_i$, there is some component $E_0 \supset C$ such that $\dim_{[g]} \mathrm{Hom}(\mathbb{P}^1, E_0) \geq 2n+1$, where $g : \mathbb{P}^1 \to E_0 \subset Z$ is induced by the normalization of C.

Proof. Since $\pi(g_i(\mathbb{P}^1)) = \pi(g(\mathbb{P}^1)) = o$ and $\hat{Z}$ is Kähler, every deformation of g_i or g in $\mathrm{Hom}(\mathbb{P}^1, Z)$ maps $\mathbb{P}^1$ to E. Hence

$$\dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, E_i) = \dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, E) = \dim_{[g_i]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n+1,$$
$$\dim_{[g]} \mathrm{Hom}(\mathbb{P}^1, E) = \dim_{[g]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n+1$$

by Definition 6.5. On the other hand, we have a set theoretical equality $\mathrm{Hom}(\mathbb{P}^1, E) = \bigcup \mathrm{Hom}(\mathbb{P}^1, E_i)$. Q.E.D.

If $C_i \subset E_i$ is chosen to have minimum degree among the rational curves passing through p_i, then, locally around $[g_i]$, the scheme

$\operatorname{Hom}(\mathbb{P}^1, E_i)$ is identical with $\bigcup \operatorname{Hom}(\mathbb{P}^1, E_i)$ and has dimension $\leq 2\dim E+1 \leq 2n+1$ by Corollaries 8.5 and 2.5. If C is a rational curve of minimum degree in E, then $\dim_{[g]} \operatorname{Hom}(\mathbb{P}^1, E) = \dim_{[g]} \operatorname{Hom}(\mathbb{P}^1, E_0) \leq 2n+1$ for some $E_0 \supset C$.

Comparing these with Lemma 8.6 and Theorem 0.1, we have proved the following

Corollary 8.7. *$E_i \subset Z$ is Lagrangian, i.e., $\dim E_i = n$. Its normalization $\overline{E}_i$ is isomorphic to a finite quotient of $\mathbb{P}^n$ and there is at least one component, say E_0, such that $\overline{E}_0$ is $\mathbb{P}^n$.*

When $n = 1$, the symplectic manifold Z is a K3 surface (it cannot be an abelian surface because of the existence of the exceptional divisor E) and E is an effective divisor with negative definite intersection matrix. Each E_i is (-2)-curve, while $\mathrm{R}^1\pi_*\mathcal{O}_Z = 0$. Hence the singularity of $\hat{Z}$ is a rational double point and E is a chain of $\mathbb{P}^1$'s of which the dual graph is one of the Dynkin diagrams of type ADE. Thus, in order to complete the proof of Theorem 8.3, we may assume that n is at least two.

For a while, we fix an irreducible component E_0 whose normalization $\overline{E}_0$ is isomorphic to $\mathbb{P}^n$.

Lemma 8.8. *Let $f : C \to \overline{E}_0 \simeq \mathbb{P}^n$ be a morphism from a smooth complete (not necessarily rational) curve. Assume that the normalization morphism $\nu_0 : \overline{E}_0 \to E_0 \subset Z$ is unramified. Then there is a natural exact sequence*

$$0 \to f^*\Theta_{\mathbb{P}^n} \to f^*\nu_0^*\Theta_Z \to f^*\Omega^1_{\mathbb{P}^n} \to 0.$$

If f is non-constant, then we have $\mathrm{H}^0(C, f^*\nu_0^*\Theta_Z) = \mathrm{H}^0(C, f^*\Theta_{\mathbb{P}^n})$. If, in addition, $\mathrm{H}^1(C, f^*\Theta_{\mathbb{P}^n}) = 0$, then $\operatorname{Hom}(C, Z)$ is smooth at $[\nu_0 f]$ and is locally (in Zariski topology) identified with $\operatorname{Hom}(C, \overline{E}_0)$.

Proof. Trivial. Q.E.D.

Let $\nu : \tilde{E} = \coprod \overline{E}_i \to E = \bigcup E_i$ be the normalization morphism, with $\nu_i : \overline{E}_i \to E_i$ being the normalization of each irreducible component.

Corollary 8.9. *Assume that the normalization morphism $\nu_0 : \overline{E}_0 \simeq \mathbb{P}^n \to E_0$ is unramified. Then the singular locus $\operatorname{Sing}(E_0)$ of E_0 and the intersection $E_0 \cap (E \setminus E_0)$ are both zero-dimensional.*

Proof. Consider an irreducible curve $\hat{C}_0$ on $E_0 \subset E$, with the normalization C_0. Let $C_{i\alpha} \subset \overline{E}_i$ be the normalization of a one-dimensional irreducible component[33] of $\nu_i^{-1}(\hat{C}_0)$, $\alpha = 1, \dots, m(i)$. Because of the

[33]Of course $\nu_i^{-1}(\hat{C}_0)$ could contain extra zero-dimensional components.

unramifiedness condition, we see that $\hat{C}_0$ sits in the singular locus of E if and only if $\coprod_{i,\alpha} C_{i\alpha}$ is not isomorphic to C_0.

Take an arbitrary irreducible component C' of the fibre product of the $C_{i\alpha}$ over C_0. Let $\tilde{C}$ be the Galois closure of the finite covering $C' \to C_0$ and let $\tilde{f}_{i\alpha} : \tilde{C} \to C_{i\alpha}$ denote the canonical projection. In short, we take a Galois cover $\tilde{C} \to C_0$ which gives a commutative diagram

$$\begin{array}{ccc} \tilde{C} & \xrightarrow[\tilde{f}_{i\alpha}]{} & C_{i\alpha} \\ \big\downarrow \tilde{f}_{j\beta} & & \big\downarrow \nu_i \\ C_{j,\beta} & \xrightarrow[\nu_j]{} & C_0 \end{array}$$

for any quadruple $(i, j; \alpha, \beta)$. Since $\tilde{C}$ is Galois over C_0 with Galois group Γ, we can write $C_0 = \tilde{C}/\Gamma$, $C_{i\alpha} = \tilde{C}/\Gamma_{i\alpha}$, where $\Gamma_{i\alpha} \subset \Gamma$ is a subgroup.

We consider the modulo p reductions of Z, C_0, $\tilde{C}$ etc., where p is a sufficiently large prime number. Let $\Phi : C \to \tilde{C}$ be the geometric Frobenius and put $f_{i\alpha} = \tilde{f}_{i\alpha}\Phi$. Since $\Theta_{\overline{E}_0}$ is ample, $\mathrm{H}^1(C, f_{0\alpha}\Theta_{\overline{E}_0})$ vanishes provided the prime number p is sufficiently large. Thus Lemma 8.8 applies to show that $\mathrm{Hom}(C, Z)$ is locally irreducible at $\nu_i f_{i\alpha} = \nu_j f_{j\beta}$, and locally identified with the germ $\big(\mathrm{Hom}(C, \overline{E}_0), [\nu_0 f_{0\alpha}]\big)$. The morphism $f = \nu_i f_{i\alpha} : C \to Z$ does not depend on the choice of the indices (i, α) corresponding to $f_{i\alpha}$. This shows that $\mathrm{Hom}(\mathbb{P}^1, \overline{E}_i)$, which is a subset of $\mathrm{Hom}(\mathbb{P}^1, Z)$, is identical with $\mathrm{Hom}(\mathbb{P}^1, \overline{E}_{i,0})$ at $[f_{i\alpha}]$ for $i \neq 0$, where $\overline{E}_{i,0} \subset \overline{E}_i$ is the inverse image of $E_i \cap E_0$.

Recall that $\overline{E}_i$ is a finite quotient $\mathbb{P}^n/G_i$, $G_i \subset \mathrm{Aut}(\mathbb{P}^n)$. If the characteristic p is sufficiently large (for example $p \gg |G_i|, |\Gamma|$), we can construct a finite cover $C^* \to C \simeq C^*/G_i'$, $G_i' \subset G_i$, which completes the commutative diagram

$$\begin{array}{ccccc} C^* & \longrightarrow & \overline{C \times_{\overline{E}_i} \mathbb{P}^n} & \longrightarrow & \mathbb{P}^n \\ & & \big\downarrow & & \big\downarrow \\ & & C & \longrightarrow & \overline{E}_i. \end{array}$$

(Namely, C^* is an irreducible component of the normalization of $C \times_{\overline{E}_i} \mathbb{P}^n$, and $G_i \subset G \subset \mathrm{Aut}(\overline{C \times_{\overline{E}_i} \mathbb{P}^n})$ is the stabilizer of C^*.) Then we have a natural morphism $\mathrm{Hom}(C^*, \mathbb{P}^n)^{G_i} \to \mathrm{Hom}(C, \overline{E}_i)$. Noting the ampleness of $\Theta_{\mathbb{P}^n}$, it turns out there are lot of G_i-invariant (more adequately, G_i-equivariant) morphisms $C^* \to \mathbb{P}^n$ or, equivalently, lot of deformation of $f_{i\alpha} : C \to \overline{E}_i$, when p is sufficiently large. This contradicts $\mathrm{Hom}(C, \overline{E}_i) = \mathrm{Hom}(C, \overline{E}_{i,0})$, the existence of $C_{i\alpha}$, $\tilde{f}_{i\alpha}$, $i \neq 0$ assumed. Thus any curve $\hat{C}_0 \subset E_0$ is not contained in other components E_i, i.e., $E_0 \cap (E \setminus E_0)$ is zero-dimensional.

We check next that E_0 has no self-intersection of positive dimension. Since $\overline{E}_0 \simeq \mathbb{P}^n \to E_0$ is unramified, an irreducible component $C_{0\alpha}$ of $\nu_0^{-1}(\hat{C}_0)$ is an étale cover of C_0. We prove first that $C_{0\alpha}$ is isomorphic to C_0. Assume that $\Gamma = \mathrm{Gal}(\tilde{C}/C_0) \neq \Gamma_{0\alpha} = \mathrm{Gal}(\tilde{C}/C_{0\alpha})$. Then we can find $\tilde{\gamma} \in \Gamma \setminus \Gamma_{0\alpha}$. The automorphism $\tilde{\gamma}$ of $\tilde{C}$ lifts to an automorphism γ of C in a unique manner (recall that $C \to \tilde{C}$ was Frobenius). Thus $f_{0\alpha}\gamma \neq f_{0\alpha}$, while $\nu_0 f_{0\alpha}\gamma = \nu_0 f_{0,\alpha}$ by construction. This implies that a unique morphism $f = \nu_0 f_{0\alpha} \in \mathrm{Hom}(C, Z)$ corresponds to two different morphisms $\in \mathrm{Hom}(\mathbb{P}^1, \overline{E}_0)$, contradicting the local birational isomorphism $\mathrm{Hom}(C, Z) \simeq \mathrm{Hom}(C, \overline{E}_0)$ between smooth schemes. Hence each $C_{0\alpha}$ is isomorphic to C_0.

Finally, if there were two components $C_{0\alpha}$, $C_{0\beta}$, then the same argument shows that $f = \nu_0 f_{0\alpha} = \nu_0 f_{0\beta}$ would correspond to two different element of $\mathrm{Hom}(C, \overline{E}_0)$, another contradiction.

Summing up things together, we conclude that the inverse image of $\hat{C}_0 \subset E$ in the normalization $\coprod \overline{E}_i$ is a single curve $\hat{C}_{01}$ birational to C_0 in E_0, plus (possibly) zero-dimensional components. This shows the assertion. Q.E.D.

Lemma 8.10. *Let the notation be as above. Assume that the normalization map $\nu_i : \overline{E}_i \to E_i$ is unramified whenever $\overline{E}_i$ is isomorphic to $\mathbb{P}^n$. Then every E_i is smooth and isomorphic to $\mathbb{P}^n$. Two components E_i, E_j meet each other transversally at finitely many points and the associated dual graph*[34] *is a tree.*

Proof. First we prove that $\overline{E}_i \simeq \mathbb{P}^n$ for each i.

The index set I of the irreducible components E_i is a disjoint sum $I^+ \cup I^-$, where $\overline{E}_i \simeq \mathbb{P}^n$ if and only if $i \in I^+$. It suffices to derive a contradiction from the hypothesis $I^- \neq \emptyset$. Put $E^+ = \bigcup_{i \in I^+} E_i$, $E^- = \bigcup_{i \in I^-} E_i$. Lemma 8.9 asserts that $E^+ \cap E^-$ is a finite set. Assume that a rational curve $C \subset E^-$ has the minimum degree among all the rational curves in E^-. Let $f : \mathbb{P}^1 \to E^- \subset E \subset Z$ be the morphism obtained by the normalization of C. Since C is not contained in E^+, we have a local (set theoretical) i dentity around $[f]$:

$$\mathrm{Hom}(\mathbb{P}^1, Z) = \mathrm{Hom}(\mathbb{P}^1, E^-) = \mathrm{Hom}(\mathbb{P}^1, E_{i_0}) \text{ for some } i_0 \in I^-.$$

[34] The vertices are the irreducible components E_i and two vertices E_i and E_j are joined by $\sharp(E_i \cap E_j)$ edges.

Thus the deformations of f give an unsplitting, doubly dominant family of rational curves on E_{i_0}, so that $\overline{E}_{i_0} \simeq \mathbb{P}^n$, contradicting the definition of I^-. Thus every $\overline{E}_i$ is isomorphic to $\mathbb{P}^n$ and hence $\mathrm{Sing}(E)$ is zero-dimensional by virtue of Corollary 8.9.

Next we show the second assertion. Let $\mathfrak{I}_E \subset \mathcal{O}_Z$ be the defining ideal of E and t a large integer. The natural short exact sequence

$$0 \to \mathfrak{I}_E/\mathfrak{I}_E^t \to \mathcal{O}_Z/\mathfrak{I}_E^t \to \mathcal{O}_E \to 0$$

induces the cohomology exact sequence

$$\mathrm{H}^1(Z, \mathcal{O}_Z/\mathfrak{I}_E^t) \to \mathrm{H}^1(E, \mathcal{O}_E) \to \mathrm{H}^2(Z, \mathfrak{I}_E/\mathfrak{I}_E^t).$$

Since $(\hat{Z}, o)$ is a rational singularity, the image the first term $\mathrm{H}^1(Z, \mathcal{O}_Z/\mathfrak{I}_E^t)$ in $\mathrm{H}^1(E, \mathcal{O}_E)$ vanishes for sufficiently large t. The sheaf $\mathfrak{I}_E/\mathfrak{I}_E^t$ is a successive extension of the $\mathfrak{I}_E^s/\mathfrak{I}_E^{s+1}$, $s = 1, \ldots, t-1$, which are identical with the $\mathrm{Sym}^s \Theta_{\mathbb{P}^n}$ outside finitely many points. Because of the well known vanishing of $\mathrm{H}^q(\mathbb{P}^n, \mathrm{Sym}^s \Theta_{\mathbb{P}^n})$, $q > 0$, the third term $\mathrm{H}^2(Z, \mathfrak{I}_E/\mathfrak{I}_E^t)$ also vanishes.[35] Thus the middle term $\mathrm{H}^1(E, \mathcal{O}_E)$ is zero.

On the other hand, the normalization $\nu : \coprod \overline{E}_i \to E$, $\overline{E}_i \simeq \mathbb{P}^n$ induces the exact sequence

$$0 \to \mathcal{O}_E \to \bigoplus_i \nu_{i*} \mathcal{O}_{\overline{E}_i} \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf supported by the finitely many singular points of E. The equality $\mathrm{H}^1(E, \mathcal{O}_E) = 0$ holds if and only if $\bigoplus \mathrm{H}^0(\overline{E}_i, \mathcal{O}_{\overline{E}_i})/\mathrm{H}^0(E, \mathcal{O}_E) \simeq \mathbb{C}^{N-1} \to \mathcal{S}$ is a surjection, where N is the number of the irreducible components. Now it is an easy exercise to check that this condition is satisfied only if each component E_i is smooth, meeting other components transversally, and the associated dual graph is a tree. Q.E.D.

Lemma 8.11. *Assume that $n \geq 2$ and that $\nu : \overline{E} = \coprod \overline{E}_i \to E$ is unramified whenever $\overline{E}_i \simeq \mathbb{P}^n$. Then E is a single $\mathbb{P}^n$.*

Proof. Since $\hat{Z}$ is normal, the exceptional locus E is connected by Zariski's main theorem. Therefore, if E is reducible, then there exist irreducible components E_1 and E_2 which mutually meet at an isolated point p. Consider the blowing-up $\mu : \tilde{Z} \to Z$ along E_1. By Lemma 8.2, we can blow down $\tilde{Z}$ in another direction to get a new symplectic manifold Z'. The strict transform $\mu^{-1}_{\mathrm{strict}}(E_2) \subset \tilde{Z}$ contains a

---

[35] N.B.: We cannot conclude the vanishing of $\mathrm{H}^1(Z, \mathfrak{I}_E/\mathfrak{I}_E^t)$ because of the difference between $\mathfrak{I}_E^s/\mathfrak{I}_E^{s+1}$ and $\mathrm{Sym}^s \Theta_{\mathbb{P}^n}$ in dimension zero.

divisor $\mu_{\text{strict}}^{-1}(E_2) \cap \mu^{-1}(E_1)$. Then the blowing-down $\mu' : \tilde{Z} \to Z'$ keeps the $\mu^{-1}(p) \simeq \mathbb{P}^{n-1}$ untouched, so that the strict transform $E_2' \subset Z'$ of $E_2 \subset Z$ is a blown up $\mathbb{P}^n$, a smooth variety not isomorphic to $\mathbb{P}^n$ (note that $n > 1$) nor to a finite quotient of $\mathbb{P}^n$. The naturally induced birational map $Z' \dashrightarrow \hat{Z}$ is well defined as a set theoretic map, and indeed a morphism by the normality of $\hat{Z}$. Then we can apply Corollary 8.7 to Z' to infer that $\overline{E}_2 \simeq E_2'$ must be $\mathbb{P}^n$ or a finite quotient of $\mathbb{P}^n$, a contradiction. Q.E.D.

The proof of Theorem 8.3 is now reduced to the following assertion:

Theorem 8.12. *The normalization map $\nu : \mathbb{P}^n \to X$ is unramified; in other words, the generically injective map $\nu : \mathbb{P}^n \to Z$ is an immersion.*

The proof of this theorem is given in Section 10 below.

Remark 8.13. It would be natural to ask if Theorem 8.3 can be localized. Most steps of our proof in fact apply also to local situations (germs of isolated singularities, say). The compactness of the symplectic manifold Z enters simply to ensure that $\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, Z) \geq 2n+1$ for every non-constant morphism $f : \mathbb{P}^1 \to Z$.

Let $(Z, E) \to (\hat{Z}, o)$ be a projective symplectic resolution of the germ of an isolated singularity of dimension $2n$. The condition of trivial canonical bundle guarantees that $\dim_{[f]} \operatorname{Hom}(\mathbb{P}^1, Z) \geq 2n$ for arbitrary non-constant f, from which we deduce:

a) The exceptional locus E has pure dimension n and each component of it is rationally connected;
b) E is Lagrangian in Z.

These property strongly suggest that the normalization of E is a disjoint union of finite quotients of projective space $\mathbb{P}^n$ or of (possibly singular) hyperquadrics (see Remark 5.3).

9. Symplectic resolution of non-isolated singularities

Let Z be a compact, Kähler, complex symplectic manifold with symplectic form η and let $\pi : Z \to \hat{Z}$ be a projective bimeromorphic morphism onto a normal, Kähler complex space $\hat{Z}$. Let $E_0 \subset Z$ be an irreducible component of the exceptional locus E and B_0 its image in $\hat{Z}$.

Let us fix the notation as follows:

q : a general point $\in B_0$.
X : the general fibre $(\pi|_{E_0})^{-1}(q)$.
p : a general point on X (and hence general in E_0).
$a := \dim X$.
$b := \dim B_0$.

In the notation above, we prove:

Theorem 9.1. *We have $b = 2n - 2a$. An open dense subset U of the smooth locus of B_0 carries a natural symplectic structure induced by η. A general fibre X of the fibration $E_0 \to B_0$ is a union of copies of smooth $\mathbb{P}^a$. If $a = 1$, X is a tree of smooth $\mathbb{P}^1$'s with configuration of type* ADE. *If $a \geq 2$, then $X \simeq \mathbb{P}^a$, so that $E_0|_U$ is an étale $\mathbb{P}^a$-bundle over U.*

Roughly speaking, the exceptional set of a projective bimeromorphic morphism from a complex symplectic manifold is essentially a contraction of projective space and the symplectic structure of the source manifold is inherited by the resulting singular loci.[36]

Most part of the proof of Theorem 9.1 is exactly the same as that of Theorem 8.3 with some flavour of Section 7, and so the proof of the subsequent lemmas will be a little sketchy.

Lemma 9.2. *The exceptional set E_0 is uniruled.* $\mathrm{R}^i\pi_*\mathcal{O}_Z = 0$ *for $i > 0$ and the 2-form η is identically zero as a 2-form on $X \subset E_0$. (More precisely, the pullback of η to a non-singular model of X identically vanishes.)*

The proof is exactly the same as in Lemma 8.4 and Corollary 8.5, and left to the reader.

Let $C \subset X$ be a rational curve passing through a general smooth point $p \in X$, and assume that the degree of C attains minimum among such rational curves. Let $f : \mathbb{P}^1 \to X \subset E_0 \subset Z$ be the morphism induced by the normalization of C.

Lemma 9.3. *Let $\overline{E}_0$ be the normalization of the irreducible variety E_0 and $\overline{f} : \mathbb{P}^1 \to \overline{E}_0$ the map which $f : \mathbb{P}^1 \to E_0$ naturally induces (recall that C, which contains p, does not lie in the singular locus of E_0). If C or, equivalently, $f : \mathbb{P}^1 \to E_0$ is generally chosen, then*

$$\overline{f}^*\Omega^1_{\overline{X}} \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}^{a-e},$$
$$\overline{f}^*\Omega^1_{\overline{E}_0} \simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}^{a+b-e},$$

where $1 \leq e \leq a$.

---

[36](After a comment of D. Barlet) The compact variety B_0 is a *symplectic variety* in the following sense: (a) η can be viewed as a (meromorphic) d-closed 2-form on B_0; (b) If $\pi : E_0 \to B_0$ is flat over $V \subset (B_0)_{\mathrm{smooth}}$, then $\eta|_V$ is a holomorphic symplectic form; and (c) η is holomorphic on B_0, i.e., for any compact 2-chain $\gamma \subset B_0$, we have $|\int_\gamma \eta| < +\infty$. These properties can be easily verified from the subsequent proof of Theorem 9.1.

Proof. Any small deformation f_t of the morphism $f : \mathbb{P}^1 \to Z$ has image inside E_0. In fact, $\pi f(\mathbb{P}^1)$ is a single point, and so is $\pi f_t(\mathbb{P}^1)$ thanks to the Kähler condition on $\hat{Z}$, meaning that a curve $f_t(\mathbb{P}^1) \subset Z$ must stay in E_0 (actually in some closed fibre over a point $\in B_0$).

If we impose the condition that $f_t(\mathbb{P}^1)$ contains $p \in X$, then $f_t(\mathbb{P}^1)$ necessarily sits in X. Since $\overline{f}$ is general, its image $\overline{f}(\mathbb{P}^1)$ does not meet the singular locus of $\overline{E}_0$. Hence Theorem 2.8 applies to get the direct sum decomposition of $\overline{f}^*\Omega^1_{\overline{E}_0}$ as above. In particular, the deformation of f with base condition $\infty \mapsto p$ is unobstructed, and $\mathrm{H}^0(\mathbb{P}^1, \overline{f}^*\Theta_{\overline{E}_0}(-(\infty)))$ generates a subspace of $\overline{f}^*\Theta_X$, meaning that $e \leq a$. Q.E.D.

Lemma 9.4. *The integer e being as in Lemma 9.3, we have the equality $e = 2n-a-b$ or, equivalently,* $\dim E_0 = 2n-e$. *The kernel $\mathcal{K}$ of the natural homomorphism $f^*\Omega^1_Z \to \overline{f}^*\Omega^1_{\overline{E}_0}$ is ample $\simeq \mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus e-1}$. We have an isomorphism*

$$f^*\Omega^1_Z \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1} \oplus \mathcal{O}^{\oplus 2n-2e} \oplus \mathcal{O}(-1)^{\oplus e-1} \oplus \mathcal{O}(-2).$$

Proof. Since a small deformation of C stays in E_0, we have

$$a+b+e+1 = \dim_{[\overline{f}]} \mathrm{Hom}(\mathbb{P}^1, \overline{E}_0) = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, Z) \geq 2n+1,$$

so that

$$\mathrm{rank}\ \mathcal{K} = 2n-a-b \leq e.$$

Then the existence of the symplectic form, or the self-duality of $f^*\Omega^1_Z$, implies the assertion. Q.E.D.

Corollary 9.5. *$f^*\eta$, viewed as a bilinear form on $\overline{f}^*\Theta_{\overline{E}_0}$, is a degenerate form of rank $2n-e$, and determines a non-degenerate bilinear form on $\overline{f}^*\Theta_{\overline{E}}/\mathcal{P}$, where $\mathcal{P} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus e-1}$ is the maximal ample subbundle.*

Proof. An easy exercise. Q.E.D.

Since $\eta_0 = \eta|_{\Theta_{\overline{E}_0}\times\Theta_{\overline{E}_0}}$ is a well-defined skew-symmetric bilinear form, this means

Corollary 9.6. *η_0 is a degenerate skew-symmetric bilinear form of rank $2n-2e$ on an open subset $U \supset \overline{f}(\mathbb{P}^1)$ of $\overline{E}_0$, and the subsheaf*

$$\mathcal{P} = \{v \in \Theta_{\overline{E}_0}; \eta_0(v,*) \equiv 0\} \subset \Theta_{\overline{E}_0}$$

is locally free of rank e on U.

Corollary 9.7. *Let $S\langle p\rangle \subset$ Chow$(\tilde{E}_0)$ be the closed subset parameterizing the rational curves of minimum degree through a general point*

$p \in \tilde{E}_0$, and $F\langle p\rangle \to S\langle p\rangle$ the associated family. (For simplicity we assume $S\langle p\rangle$ is irreducible.) Let $U \subset F\langle p\rangle$ denote a small open neighbourhood of $\{[C]\} \times C$. Then the natural image $\overline{X}_p$ of U on $\overline{X}$ is smooth at p as well as at a general point $p' \in \overline{X}_p$, with tangent spaces exactly $\mathcal{P} \otimes \mathbb{C}(p)$ and $\mathcal{P} \otimes \mathbb{C}(p')$, respectively.

The dimension of $\overline{X}_p$ is clearly e and it is doubly covered by rational curves of minimum degree. Hence by Theorem 0.2, the normalization of $\overline{X}_p$ is a finite quotient of $\mathbb{P}^e$.

Lemma 9.8. *$e = a$, so that the Zariski closure of $\overline{X}_p$ is $\overline{X}$ and* $\mathrm{pr}_Z(F) = E_0$.

Proof. As before, we denote the normalization by putting overlines. Recall that η induces a non-degenerate pairing on $\Theta_{\overline{E}_0}/\mathcal{P}$, a sheaf of rank $2n - 2e$. By Lemma 8.2 above, η identically vanishes on $\Theta_{\overline{X}} \times \Theta_{\overline{X}}$. If $\mathcal{P}$ is strictly smaller than $\Theta_{\overline{X}}$, then η induces a non-zero pairing

$$\Theta_{\overline{X}} \times (\Theta_{\overline{E}_0}|_{\overline{X}}/\Theta_{\overline{X}}) \to \mathcal{O}_{\overline{X}}.$$

Our fibre space structure $\overline{E}_0 \to \overline{B}_0$ gives an isomorphism

$$\mathcal{Q} := \Theta_{\overline{E}_0}|_{\overline{X}}/\Theta_{\overline{X}} \simeq \mathcal{O}^{\oplus b}$$

in an obvious manner. Then there exists a global section θ of $\mathcal{Q}$ such that $\eta(\cdot, \theta)$, viewed as a linear form on $\Theta_{\tilde{X}}$, is not identically zero; in other words, $\eta(\cdot, \theta)$ is a global d-closed 1-form on $\overline{X}$ or, more precisely, on a smooth model $\tilde{X}$ of $\overline{X}$. This is ruled out by the property $\mathrm{R}^1\pi_*\mathcal{O}_Z = 0$ in view of Lemma 9.9 below. This contradiction comes from the hypothesis $\mathcal{P} \neq \Theta_{\overline{X}}$. Q.E.D.

Lemma 9.9. *Let $\pi : W \to \hat{W}$ be a bimeromorphic projective morphism from a manifold W to a normal complex space. Let $T \subset W$ be the inverse image of a point $o \in \hat{W}$ (we equip T with reduced structure). If $\mathrm{R}^1\pi_*\mathcal{O}_W = 0$, then there is no non-zero* d*-closed holomorphic* 1*-form on T.*[37]

Proof. Let $U \subset W$ be a sufficiently small open neighbourhood of T. Consider the truncated De Rham exact sequence

$$0 \to \mathbb{C} \to \mathcal{O} \to \mathrm{d}\mathcal{O} \to 0$$

[37] More precisely, if $\{U_i\}$ is a collection of open subsets of W which covers T and ζ_i is a d-closed holomorphic 1-form on U_i such that $\zeta_i = \zeta_j$ on $T \cap U_i \cap U_j$, then the pullback of ζ_i to a resolution $\tilde{T}$ of any component of T is identically zero.

and the associated commutative diagram of cohomology groups

$$\begin{tikzcd}
\mathrm{H}^0(\mathbb{C}_U) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{H}^0(\mathcal{O}_U) \arrow[r] \arrow[d] & \mathrm{H}^0(\mathrm{d}\mathcal{O}_U) \arrow[r] \arrow[d] & \mathrm{H}^1(\mathbb{C}_U) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{H}^1(\mathcal{O}_U) \arrow[d] \\
\mathrm{H}^0(\mathbb{C}_T) \arrow[r] & \mathrm{H}^0(\mathcal{O}_T) \arrow[r] & \mathrm{H}^0(\mathrm{d}\mathcal{O}_T) \arrow[r] & \mathrm{H}^1(\mathbb{C}_T) \arrow[r] & \mathrm{H}^1(\mathcal{O}_T)
\end{tikzcd}$$

of which the rows are exact. By our hypothesis, $\mathrm{H}^1(U, \mathcal{O}_U) = 0$, so that the edge homomorphism $\mathrm{H}^0(T, d\mathcal{O}_T) \to \mathrm{H}^1(T, \mathbb{C}_T)$ is surjective. It is injective as well because of the isomorphisms $\mathrm{H}^0(\mathbb{C}_T) \simeq \mathrm{H}^0(\mathcal{O}_T) \simeq \mathbb{C}$. Thus we have $\mathrm{H}^0(\mathrm{d}\mathcal{O}_T) \simeq \mathrm{H}^1(\mathbb{C}_T)$.

Suppose that there is a non-zero element $\zeta \in \mathrm{H}^0(T, \mathrm{d}\mathcal{O}_T)$. Then its complex conjugate $\overline{\zeta}$ is of course d-closed, defining a cohomology class $\in \mathrm{H}^1(T, \mathbb{C}_T)$. If $\mathrm{H}^0(T, d\mathcal{O}_T) \neq 0$, then there is a resolution V of some component of T such that the pullbacks of $\mathrm{H}^0(T, d\mathcal{O}_T)$ and $\mathrm{H}^0(T, d\overline{\mathcal{O}}_T)$ are non-zero. Since V is projective, these two spaces are independent in $\mathrm{H}^1(V, \mathbb{C})$, and so are in $\mathrm{H}^1(T, \mathbb{C})$, contracting $\mathrm{H}^1(\mathbb{C}_T) \simeq \mathrm{H}^0(\mathrm{d}\mathcal{O}_T)$. Q.E.D.

Now we are in the position to photocopy the arguments in the previous section to get the following:

Lemma 9.10. *If $a = 1$, X is a tree of $\mathbb{P}^1$'s with configuration of ADE singularities. If $a \geq 2$, then X is a single $\mathbb{P}^a$.*

Proof. Let us check what kind of results were used in the proof of similar statements in the previous section. They were:

(1) The vanishing of $\mathrm{R}^i\pi_*\mathcal{O}_Z$, the counterpart of which was established in Lemma 9.2;
(2) Deformation argument on rational curves, which perfectly works also in this new context;
(3) The existence of a flop Lemma 8.2; and finally
(4) The unramifiedness of the normalization map $\overline{X}_i \simeq \mathbb{P}^a \to X_i$.

The unramifiedness will be proved in the next section. The existence of a flop is still valid by the following lemma. Q.E.D.

Lemma 9.11. *Let (Z, η) be a complex symplectic manifold of dimension $2n$. Suppose that Z contains $E_0 = \mathbb{P}^a \times \Delta^{2n-2a}$ as a closed subset, Δ^{2n-2a} denoting a small polydisc of dimension $2n-2a$, and that $\Theta_{\mathbb{P}^a}$ is exactly the null-space of $(\Theta_{E_0}|_{\mathbb{P}^a}, \eta)$. Assume that $a \geq 2$ and let $\mu : \tilde{Z} \to Z$ be the blow up along E_0. Then $\tilde{Z}$ has a blow down in another direction.*

Analytically locally Z looks like a symplectic product $W \times \Delta^{2n-2a}$ around E_0, and the proof is completely parallel to Lemma 8.2.

Remark 9.12. The vanishing of $\mathrm{R}^i\pi_*\mathcal{O}_Z$ is a fairly restrictive condition if combined with our result $\overline{X} = \mathbb{P}^a/G$. For example, we can easily show that if the normalization map $\nu : \overline{X} \to X$ is unramified outside finitely many points, then ν is everywhere unramified.

Example 9.13. We take up once more Hilbert schemes of a K3 surface.

Let S be a K3 surface and consider the Hilbert scheme $Z = \mathrm{Hilb}^r(S)$ parameterizing the closed subschemes with constant Hilbert polynomial $h(t) = r$. An element of $[z] \in \mathrm{Hilb}^r(S)$ is a zero-dimensional scheme defined by an ideal sheaf I_z such that $\dim_{\mathbb{C}}(\mathcal{O}/I_z) = r$. As was mentioned in Example 7.10, Z is a complex symplectic manifold.

By forgetting the scheme structure of z and viewing it as an effective 0-cycle of degree r, we get a natural morphism $\pi_r : Z \to \hat{Z} = \mathrm{Sym}^r(S) \simeq \mathrm{Chow}_0^r(S)$,

Let us study the structure of the birational morphism π_r for small values of r.

Case $r = 2$. If a 0-cycle $p_1 + p_2$ is supported by two distinct points, the corresponding scheme is uniquely defined by the ideal $\mathfrak{M}_{p_1}\mathfrak{M}_{p_2}$, so that π is an isomorphism over $\mathrm{Sym}^2 S \setminus S$.[38] When $p_1 = p_2 = p$, then a closed subscheme of degree two supported by p is determined by an ideal I such that $\mathfrak{M}_p^2 \subset I \subset \mathfrak{M}_p$ and that $\dim_{\mathbb{C}} \mathfrak{M}_p/I = 1$. It follows that $\pi_2^{-1}([2p])$ is naturally isomorphic to $\mathbb{P}(\mathfrak{M}_p/\mathfrak{M}_p^2) \simeq \mathbb{P}^1$, and π_2 has a $\mathbb{P}^1$-bundle structure over the diagonal $S \subset \mathrm{Sym}^2 S$. Thus the exceptional locus E of the birational contraction π_2 is a smooth $\mathbb{P}^1$-bundle over S.

Suppose that S contains a (-2)-curve. Then we can contract this curve to get a normal surface S'. The symmetric product $\mathrm{Sym}^2 S'$ is a normal variety with a unique point q (of course singular) such that the projection $\mathrm{Sym}^2 S \to \mathrm{Sym}^2 S'$ has $\mathbb{P}^2$ as a fibre over q. Looking at the construction closely, we find that the exceptional locus of the birational contraction $\pi_2' : \mathrm{Hilb}^2(S) \to \mathrm{Sym}^2 S'$ consists of two irreducible components, one of which being a $\mathbb{P}^1$-bundle over S and the other being $\mathbb{P}^2$ over q. The two components meet each other along a smooth quadric in $\mathbb{P}^2$.

Case $r = 3$. Let us look at the fibre of π_3 over a cycle $\gamma = p_1 + p_2 + p_3$ on S.

If the three points are mutually distinct, then the fibre is a single point corresponding the ideal $I_y = \mathfrak{M}_{p_1}\mathfrak{M}_{p_2}\mathfrak{M}_{p_3}$.

[38] Here the image of the diagonal $\Delta_S \subset S \times S$ in $\mathrm{Sym}^2 S$ is identified with S.

When $p = p_1 = p_2 \neq p_3$, the fibre $\pi^{-1}(\gamma)$ is a scheme defined by $I\mathfrak{M}_{p_3}$, where $\mathfrak{M}_p^2 \subset I \subset \mathfrak{M}_p$ and $\dim \mathcal{O}/I = \dim \mathfrak{M}_p/I + 1 = 2$. Hence there is a one-to-one correspondence between $\pi^{-1}(\gamma)$ and the set of one-dimensional subspace in $\mathfrak{M}_p/\mathfrak{M}_p^2 \simeq \mathbb{C}^2$, which is nothing but $\mathbb{P}^1$.

In the most degenerate case $p = p_1 = p_2 = p_3$, the fibre $E_0 = \pi_3^{-1}(\gamma)$ consists of the ideals I such that $\mathfrak{M}_p^3 \subset I \subset \mathfrak{M}_p$ and that $\dim \mathfrak{M}_p/I = 2$. Hence we have an injection from E_0 into the Grassmann variety $\mathrm{Grass}(\mathfrak{M}_p/\mathfrak{M}_p^3, 2)$. There is a naturally marked closed point $[\mathfrak{M}_p^2]$ in E_0. This point represents a (unique) closed subscheme of length three which is supported by p but not locally complete intersection on S.

Assume that the ideal I is not contained in $\mathfrak{M}_p^2$ or, equivalently, that $I/(I \cap \mathfrak{M}_p^2) \neq 0$. By Nakayama's lemma,

$$\dim_{\mathbb{C}}(I + \mathfrak{M}_p^2)/\mathfrak{M}_p^2) = \dim_{\mathbb{C}} \mathfrak{M}_p/\mathfrak{M}_p^2 = 2$$

would mean the absurd equality $I = \mathfrak{M}_p$, and so $I + \mathfrak{M}_p^2/\mathfrak{M}_p^2$ must be one-dimensional. Consequently $(I \cap \mathfrak{M}_p^2)/\mathfrak{M}_p^3$ is necessarily two-dimensional. Let (x, y) be a local coordinate system. Let

$$w = ax + by + q(x, y) \in \mathfrak{M}_p/\mathfrak{M}_p^3$$

be an element of $I/\mathfrak{M}_p^3$ such that

(1) a, b are constants $\in \mathbb{C}$ and $(a, b) \neq (0, 0)$,

(2) $q(x, y)$ is a homogeneous quadratic polynomial in (x, y).

Then this element w uniquely determines

$$I = \mathcal{O}_S w + \mathfrak{M}_p^3 = \big(w, (\bar{b}x - \bar{a}x)^3\big),$$

where $\bar{a}, \bar{b}$ are the complex conjugates of a, b. In fact, the three-dimensional vector space $I/\mathfrak{M}_p^3$ has basis $(ax + by + q(x, y), (ax + by)x, (ax + by)y)$. Given I, the choice of the linear part (a, b) is unique up to non-zero factor. The choice of the quadratic part $q(x, y)$ involves, however, indeterminacy. Indeed, we can change it by adding a quadric of the form $(ax + by)(tx + uy)$, where t, u are arbitrary constants. Thus we have a natural projection $(E_0 \setminus [\mathfrak{M}_p^2]) \to \mathbb{P}^1$, $[I] \mapsto (ax + by) \in \mathbb{P}(\mathfrak{M}_p/\mathfrak{M}_p^2)^*$, of which the fibre is isomorphic to the one-dimensional vector space $\mathfrak{M}_p^2/\big((ax + by)\mathfrak{M}_p + \mathfrak{M}_p^3\big) \simeq \mathbb{C}$. The closed subset $E_0 \subset \mathrm{Hilb}^3(S)$ is therefore a compactification of a $\mathbb{C}$-bundle over $\mathbb{P}^1$ by the single point $[\mathfrak{M}_p^2]$, implying that $E_0 \simeq \mathbb{P}^2$.

In order to have a global picture of the exceptional locus of π_3, we have to check what happens to the point $[I_p\mathfrak{M}_{p_3}]$ $(\mathfrak{M}_p^2 \subsetneqq I_p \subsetneqq \mathfrak{M}_p)$ when p_3 tends to p.

As a reference point $\in \pi_3^{-1}(3p) \subset \mathrm{Hilb}^3(S)$, we take the ideal I generated by (x, y^3) (i.e., $a = 1, b = q(x,y) = 0$), which is a local complete intersection. Hence its universal infinitesimal deformation[39]is given by $\mathrm{Hom}(I/I^2, \mathcal{O}/I) \simeq \mathbb{C}^6$. Recalling the construction of the universal deformation, we have the following explicit description of the family of ideals I_t parameterized by six parameters $t_1, \ldots, t_6$:

$$I_t = (x + t_1 y^2 + t_2 y + t_3, y^3 + t_4 y^2 + t_5 y + t_6).$$

The condition that the scheme is supported by one or two points is thus described by the vanishing of the discriminant $t_4^2 t_5^2 + 18 t_4 t_5 t_6 - 4t_5^3 - 4t_4^3 t_6 - 27 t_6^2$ of the second cubic polynomial in y, thereby defining an locally irreducible hypersurface on the parameter space $\mathbb{C}^6$.

The above observations show that

(1) the exceptional locus of the projection $\pi_3 : \mathrm{Hilb}^3(S) \to S^{(3)}$ consists of a single irreducible component E; that
(2) E is stratified into the disjoint sum of a $\mathbb{P}^1$-bundle over the symplectic 4-fold $S^{(2)} \setminus S$ (the bigger diagonal in $S^{(3)}$) and a $\mathbb{P}^2$-bundle over S (the smaller diagonal); and that
(3) the $\mathbb{P}^a$ bundle structure of the exceptional locus E is not globally defined.

10. Unramifiedness of the normalization map

In this section, we give the deferred proof of the unramifiedness of the normalization map attached to the exceptional locus of a symplectic resolution.

Let $\pi\ Z \to \hat{Z}$ be a projective symplectic resolution of a projective normal variety of dimension $2n$, and let $E \subset Z$ be the exceptional locus with image B in $\hat{Z}$. Take a general point $b \in B$. Then $X = \pi^{-1}(b)$ is a union of a-dimensional Lagrangian subvarieties X_i in a $2a$-dimensional symplectic submanifold $\subset Z$. Their normalizations $\overline{X}_i$ are isomorphic to finite quotients of $\mathbb{P}^a$ and there is at least one component, say X_0, such that $\overline{X}_0 \simeq \mathbb{P}^a$. If $\overline{X}_i$ is not $\mathbb{P}^a$, then X_i necessarily meets another component X_j along a positive dimensional subset.

In the above notation, we show the following

Theorem 10.1. *The normalization map*

$$\nu_0 : \overline{X}_0 \to X_0 \subset X = \pi^{-1}(b) \subset Z$$

is unramified if $b \in B$ is general and if $\overline{X}_0 \simeq \mathbb{P}^a$.

[39]By Theorem 1.6, $\mathrm{Hilb}^r(X)$ is smooth at $[\gamma]$ provided the finite scheme γ is locally complete intersection.

When X has only isolated singularities, this theorem is easy to prove:

Lemma 10.2. *If X has only isolated singularities, then X is a union of smooth $\mathbb{P}^a$.*

Proof. Note that $\overline{X}_i \simeq \mathbb{P}^a$ because the intersection $X_i \cap X_j$ is a finite set for all $j \neq i$.

Consider the short exact sequence $0 \to \mathfrak{I}_X \to \mathcal{O}_Z \to \mathcal{O}_X \to 0$ and the associated long exact sequence

$$0 = \mathrm{R}^1\pi_*\mathcal{O}_Z \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{R}^2\pi_*\mathcal{I}_Z.$$

Outside the finitely many singular points, all the components X_i are isomorphic to $\mathbb{P}^a$ and mutually disjoint. Therefore

$$\mathrm{H}^2(X, \mathfrak{I}_X^i/\mathfrak{I}_X^{i+1}) \simeq \mathrm{H}^2(\mathbb{P}^a, \mathrm{Sym}^i\Theta_{\mathbb{P}^a})^{\oplus s} = 0, \quad i = 1, 2, \ldots,$$

where s is the number of the irreducible components X_i. In particular, $\mathrm{R}^2\pi_*\mathfrak{I}_X/\mathfrak{I}_X^k$ vanishes for k very large, so that $\mathrm{R}^2\pi_*\mathfrak{I}_X = 0$. This shows that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$.

On the other hand, the normalization map gives the exact sequence

$$0 \to \mathcal{O}_X \to \nu_*(\mathcal{O}_{\mathbb{P}^a})^{\oplus s} \to \mathcal{S} \to 0,$$

where $\mathcal{S}$ is a skyscraper sheaf. Taking cohomology, we have the exact sequence

$$0 \to \mathbb{C}^{s-1} \to \mathrm{H}^0(X, \mathcal{S}) \to \mathrm{H}^1(X, \mathcal{O}_X) = 0,$$

which shows that the length of $\mathcal{S} = \nu_*\mathcal{O}_{\overline{X}}/\mathcal{O}_X$ is equal to $s - 1$, s being the number of the irreducible components of X. Since X is connected, this is possible only if each X_i is smooth and any two components are disjoint or meet transversally at a single point. Q.E.D.

The key to the argument above is the simple fact that $\hat{Z}$ has only rational singularities. However, we are not sure if this rationality property directly entails Theorem 10.1 when X has singularities of positive dimension, and we indeed take a completely different approach in this case. While the proof of Lemma 10.2 was done without detailed information on the local structure of X, the core of our proof of Theorem 10.1 below is the local analysis of the singular locus of X. The results Theorem 10.3 through Proposition 10.6 deal with local reductions of singular Lagrangian subvarieties in general. With the aide of this reduction and the spannedness result Lemma 10.7 – Corollary 10.9 for defining ideals of exceptional loci, we show that the non-immersed points of $\nu_0 : \mathbb{P}^n \to Z$ have so special properties described in Lemma 10.10 – Lemma 10.16 that the existence of such points is eventually ruled out in Lemma 10.17.

For simplicity of the notation, we assume that $\hat{Z}$ has a single isolated singularity in what follows. In particular, X is a union of Lagrangian subvarieties in Z. The general case is easily reduced to this special case.

In order to analyze the local property of $X \subset Z$ along the singular locus of X, we use the following *local primitive decomposition theorem,* or the *Lagrangian reduction theorem* for Lagrangian subvarieties, which asserts that a Lagrangian subvariety with smooth normalization is locally a Lagrangian product of a smooth submanifold and a singular Lagrangian subvariety of maximal embedding dimension:

Theorem 10.3. *Let (M, η) be a complex symplectic manifold of dimension $2n$ and L a Lagrangian subvariety in M. Assume that the normalization $\overline{L}$ of L is smooth. Let $o \in L$ be a singular point of L and let e denote the embedding dimension of L at o. If we replace M by a small neibourhood of o, then there exist a symplectic manifold (M', η') of dimension $2n' = 2(e - n)$, a Lagrangian subvariety $L' \subset M'$, an e-dimensional submanifold $N \supset L$ of M and a projection $\sigma : N \to M'$ such that*

(1) *L' has embedding dimension $2n'$ at o'; that*
(2) *σ is a smooth fibration with fibres of dimension $2n - e$; and that*
(3) *$L = \sigma^{-1}(L') \subset N$.*

Proof. Our proof is the induction on $2n - e$. If $2n - e = 0$, we simply let $M' = M$, $L' = L$.

Assume that $e < 2n$. Then we have a smooth hypersurface $D \subset M$ which contains L. The restriction of η to the tangent bundle $\Theta_D \subset \Theta_M|_D$ is everywhere of rank $2n - 1$ and the null space $\mathrm{Null}_\eta \subset \Theta_D$ is everywhere one-dimensional, giving rise to an integrable foliation and a smooth fibration $\sigma_1 : D \to M_1$ with integral submanifolds of Null_η as fibres. Furthermore, the Lagrangian condition on L implies the inclusion relation $\mathrm{Null}_\eta|_L \subset \Theta_L$, and a fibre of σ is either contained in L or away from L. Hence L is the inverse image $\sigma^{-1}(L_1)$ of $L_1 = \sigma_1(L) \subset M_1$. The tangent space of M_1 is isomorphic to $\Theta_N/\mathrm{Null}_\eta$, on which η induces a non-degenerate pairing η_1. It is easy to show that η_1 is d-closed, so that M_1 is a symplectic manifold with Lagrangian subvariety L_1 of embedding dimension $e - 1$. L is locally a product of L_1 and $\mathbb{C}$. Then the induction hypothesis shows the assertions (1) through (3). Q.E.D.

Similarly we can prove the following

Lemma 10.4. *Let (M, o) be a complex symplectic manifold of dimension $2n$ and L a Lagrangian subvariety whose normalization $\overline{L}$ is smooth. Let $(R, o) \subset \overline{L}$ be a smooth subvariety of dimension r and assume that $\nu|_R : R \to M$ is an isomorphism. Take a general smooth* *complete intersection* $T_R \subset M$ *of codimension* r *transversally intersecting* $\nu(R)$ *at* o. *Then, analytically locally around* o, *there exists a smooth fibration* $\phi_R : T_R \to M^\dagger$ *over a symplectic manifold of dimension* $2n^\dagger = 2n - 2r$ *such that*

(1) $L^\dagger = \phi_R(L \cap T_R) \subset M^\dagger$ *is a Lagrangian subvariety whose normalization* $\nu^\dagger : \overline{L}^\dagger \to L^\dagger$ *is a morphism from a smooth manifold and that*

(2) $L \cap T_R$ *is locally isomorphic to* $L^\dagger$ *and hence has embedding dimension* $\leq 2n - 2r$.

The proof is completely parallel to that of Theorem 10.3 and left to the reader.

Corollary 10.5. *In Lemma* 10.4, *let* R *be either an irreducible component of the discriminant locus* $\nu_L^{-1}(\mathrm{Sing}(L)) \subset \overline{L}$ *or an irreducible component of the ramification locus*

$$\mathrm{Ram}(\overline{L}) = \{p \in \overline{L};\ \ \nu_L : \overline{L} \to Z \ \ \textit{is not an immersion at } p\}$$

and o *a general point of* R. *Then the resulting Lagrangian subvariety* $L^\dagger$ *has embedding dimension* $2n^\dagger$.

Proof. The normalization of $L \cap T_R$ is a general complete intersection in $\overline{L}$ and hence smooth. The singular locus or branch locus of $L \cap T_R$ is the finite set $R \cap T_R$. If the embedding dimension e of the reduction $L^\dagger \simeq L \cap T_R$ is less than $2n^\dagger$, apply Theorem 10.3 to $L^\dagger$ to conclude that $L^\dagger$ is locally a product $\mathbb{C} \times$ (Lagrangian subvariety) and hence the discriminant locus or the ramification locus of $\overline{L}^\dagger \to L^\dagger$ cannot be zero-dimensional. Q.E.D.

Proposition 10.6. *In Lemma* 10.4, *assume that* $R \subset \overline{L}$ *is an irreducible component either of the ramification locus* $\mathrm{Ram}(L)$ *or of the discriminant locus* $\nu^{-1}(\mathrm{Sing}(L))$, *and let* o *be a general point in it. Then the resulting Lagrangian subvariety* $L^\dagger \subset M^\dagger$ *is isomorphic with the Lagrangian reduction* $L' \subset M'$ *obtained in Theorem* 10.3. *If* R *and* o *are an irreducible component of* $\mathrm{Ram}(L)$ *and a general point on it, the tangent map* $\mathrm{d}\nu_{L'} : \Theta_{\overline{L}'} \to \nu_{L'}^* \Theta_{M'}$ *is the zero map at the inverse image of the isolated branch points* $\in L'$.

Proof. $L \cap T_R$ is isomorphic to $L^\dagger$ and its embedding dimension is $2n^\dagger$. Since T_R is a smooth complete intersection defined by r equations $f_1 = \cdots = f_r = 0$, we have $\Omega^1_{L\cap T} = \mathcal{O}_{L\cap T} \otimes (\Omega^1_L/(\mathrm{d}f_1, \dots, \mathrm{d}f_r)$ so that the embedding dimension of L is $2n^\dagger + r = \dim T_R < 2n$. Hence L has non-trivial Lagrangian reduction and $L \cap T_R$ is (locally) a section of the reduction, thus giving local isomorphisms $L' \simeq L \cap T_R \simeq L^\dagger$.

Let R be a component of $\mathrm{Ram}(L)$. If $d\nu_{L'}$ does not vanish at o, we can find a smooth curve $\Gamma \subset \overline{L'} \ni o$ such that $\nu|_\Gamma$ is an isomorphism into M'. Then apply Lemma 10.4 to (M', L', Γ) and we conclude that a general hypersurface of L' has embedding dimension $\leq \dim M' - 2$ or, equivalently, that the embedding dimension of L' is $\leq \dim M' - 1$, a contradiction. Q.E.D.

Let us return to our original situation as in Theorem 10.1. Take an irreducible component $X_0 \subset X$ such that $\overline{X_0} \simeq \mathbb{P}^n$. Let $\mathfrak{M} \subset \mathcal{O}_{\hat{Z}}$ be the maximal ideal defining the isolated singular point $o \in \hat{Z}$. Let $l+1$ be the number of the irreducible components X_i of X and let $\mathbf{a} = (a_0, \ldots, a_l)$ be an $(l+1)$-tuple of positive integers. We write $\mathbf{b} \geq \mathbf{a}$ or $\mathbf{b} \gg \mathbf{a}$ if $b_i \geq a_i$ or $b_i \gg a_i$ for all i. For simplicity of the notation, we adopt the convention $s = (s, \ldots, s)$ for a positive integer s. Let $\mathfrak{I}_X^{(\mathbf{a})} \subset \mathcal{O}_Z$ denote the $\mathbf{a}$-th *symbolic power* of the ideal $\mathfrak{I}_X$ (see [Matm, p.56]); *i.e.*, $\mathfrak{I}_X^{(\mathbf{a})} = \bigcap_i \mathfrak{I}_{X_i}^{(a_i)}$ and $\mathfrak{I}_{X_i}^{(a_i)} = \mathfrak{I}_{X_i}^{a} \mathcal{O}_{Z, \mathfrak{I}_{X_i}} \cap \mathcal{O}_Z$.
Then we have the following easy

Lemma 10.7. *Let $\hat{U} \subset \hat{Z}$ be an affine (or Stein) neibourhood of the singular point o and $U \subset Z$ its inverse image. Then*

(1) *Given positive integers $b \geq a$, the $\mathcal{O}_{\hat{U}}$-module $\mathfrak{M}^a/\mathfrak{M}^b$ and the $\mathcal{O}_U$-module $\pi^*(\mathfrak{M}^a/\mathfrak{M}^b)$ are generated by global sections.*

(2) *If $\mathbf{c} \gg b \gg \mathbf{a}$, then $\mathfrak{I}_X^{(\mathbf{c})} \subset \pi^*\mathfrak{M}^b \subset \mathfrak{I}_X^{(\mathbf{a})} \subset \pi^*\mathfrak{M} \subset \mathfrak{I}_X$.*

(3) *There exists $\mathbf{a} \gg 0$ such that $\mathcal{O}_X$-module $\mathfrak{I}_X^{(\mathbf{a})}/\mathfrak{I}_X^{(\mathbf{a}+\mathbf{e}_0))}$ is generated by global sections, where $\mathbf{e}_0 = (1, 0, \ldots, 0)$.*

Proof. (1) and (2) are trivial. The statement (3) easily derives from the following three observations:

(1) $\pi^*\mathfrak{M}/\pi^*\mathfrak{M}^c$ and its quotient $\pi^*\mathfrak{M}/\mathfrak{I}_X^{(\mathbf{b})}$ are generated by global sections;

(2) The $\mathcal{O}_Z$-module $\pi^*\mathfrak{M}/\mathfrak{I}_X^{(\mathbf{b})}$ has filtrations with associated graded module $\bigoplus(\pi^*\mathfrak{M} \cap \mathfrak{I}_X^{(\mathbf{b}_j)})/(\pi^*\mathfrak{M} \cap \mathfrak{I}_X^{(\mathbf{b}_{j+1})})$, where $\mathbf{b}_0 = 1 \leq \mathbf{b}_1 \leq \cdots \leq \mathbf{b}_m = \mathbf{b}$ is an increasing sequence;

(3) Standard multiplications of the ring $\mathcal{O}_Z$ induce a natural $\mathcal{O}_X$-homomorphism $(\mathfrak{I}_X^{(\mathbf{j})}/\mathfrak{I}_X^{(\mathbf{j}+\mathbf{e})}) \otimes (\mathfrak{I}_X^{(\mathbf{k})}/\mathfrak{I}_X^{(\mathbf{k}+\mathbf{e})}) \to \mathfrak{I}_X^{(\mathbf{j}+\mathbf{k})}/\mathfrak{I}_X^{(\mathbf{j}+\mathbf{k}+\mathbf{e})}$, where $0 < \mathbf{e} \leq 1$.

The detail is left to the reader. Q.E.D.

Let $\nu_0 : \mathbb{P}^n \to X_0 \subset X \subset Z$ be the normalization map of an irreducible component $X_0 \subset X$. *From now on until Lemma* 10.16, *we assume that ν_0 is not an immersion so that the ramification locus* $\mathrm{Ram}(X_0) \subset \mathbb{P}^n$ *is non-empty.*

Recall that $\nu_0^*(\mathfrak{I}^{(\mathbf{a})}/\mathfrak{I}^{(\mathbf{a}+\mathbf{e}_0)})/(\text{torsion})$ is a subsheaf of $\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$, where $a = a_0$, $\mathbf{a} = (a_0, a_1, \dots)$. Choose a general point p of an irreducible component R of $\mathrm{Ram}(X_0) \subset \mathbb{P}^n$. Let $(T_0 : T_1 : \cdots : T_n)$ be homogeneous linear coordinates of $\mathbb{P}^n$ such that $p = (1 : 0 : \cdots : 0)$. The linear functions $t_i = T_i/T_0$, $i = 1, \dots, n$, define affine coordinates at p, while t_0 is the constant function 1. The global vector fields on $\mathbb{P}^n$ are generated by

$$\theta_{ij} = T_i\frac{\partial}{\partial T_j} = \begin{cases} t_i\frac{\partial}{\partial t_i} & \text{if } j \neq 0 \\ -t_i(t_1\frac{\partial}{\partial t_1} + \cdots + t_n\frac{\partial}{\partial t_n}) & \text{if } j = 0. \end{cases}$$

This means that if a global vector field θ on $\mathbb{P}^n$ vanishes at p to the second order, then θ is locally contained in the line bundle $\mathcal{L}$ generated by the Euler vector field $-\theta_{00} = t_1\frac{\partial}{\partial t_1} + \cdots + t_n\frac{\partial}{\partial t_n} \in (t_1, \dots, t_n)\Theta_{\mathbb{P}^n}$.

Corollary 10.8. *There exists* $\mathbf{a} = (a, a_1, \dots) \geq 1$ *such that*

$$\nu_0^*(\mathfrak{I}_X^{(\mathbf{a})}/\mathfrak{I}_X^{(\mathbf{a}+\mathbf{e}_0)})/(\text{torsion}) + (t_1, \dots, t_n)^{a-1}\mathcal{L}\cdot\mathrm{Sym}^{a-1}\Theta_{\mathbb{P}^n} \supset (t_1, \dots, t_n)^a\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$$

locally around p.

Proof. The $\mathcal{O}_{\mathbb{P}^n}$-module $\nu^*(\mathfrak{I}_X^{(\mathbf{a})}/\mathfrak{I}_X^{(\mathbf{a}+\mathbf{e}_0)})/(\text{torsion}) \subset \mathrm{Sym}^a\Theta_{\mathbb{P}^n}$ is generated by global sections and, at general points, coincides with the total space $\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$. It is well known that a global section ξ of $\mathrm{Sym}^a\Theta_{\mathbb{P}^n}$ is a sum of the products of θ_{ij}, which have zero of order $\leq a$ at p modulo $\mathcal{L}\cdot\mathrm{Sym}^{a-1}\Theta_{\mathbb{P}^n}$. Q.E.D.

Let R denote the ramification locus of $\nu_0 : \mathbb{P}^n \to Z$ and r its dimension. Choose a general point p of an r-dimensional component of R. By Theorem 10.3 – Proposition 10.6, we find a small neighbourhood $N \subset Z$ of p and a Lagrangian reduction $\phi : (N, X_0, p) \to (M, L, o)$, where $L \subset M$ is a Lagrangian subvariety of dimension $m = n - r$ with a single isolated branch point $o = \phi(p)$. (Here we write X_0 instead of $X_0 \cap N$ by abuse of notation.)

Since X_0 is locally a product, we have local decompositions

$$\Theta_{\mathbb{P}^n} = \mathcal{F} \oplus \phi^*\Theta_{\overline{L}},$$
$$\mathfrak{I}_{X_0}/\mathfrak{I}_{X_0}^2 = \mathcal{F} \oplus \phi^*(\mathfrak{I}_L/\mathfrak{I}_L^2),$$

where $\mathcal{F}$ is a subbundle of $\Theta_{\mathbb{P}^n}$ such that $\mathcal{F}|_R = \Theta_R$. In particular, we have a locally defined sujection $\mathfrak{I}_X^{(a)}/\mathfrak{I}_X^{(a+1)} \to \phi^*(\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)})$. If we let $\nu_L : \overline{L} \to L$ stand for the normalization, there is a natural inclusion

$$\nu_L^*(\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)}) \subset (t_1, \dots, t_{n-r})\mathrm{Sym}^a\Theta_{\overline{L}}$$

and a local surjection $\Theta_{\mathbb{P}^n}|_{\nu_0^{-1}(W)} \to \Theta_{\overline{L}}$, where $t_1, \ldots, t_{n-r}$ are local coordinates of $\overline{L}$ at $\phi(p)$ and $W \subset Z$ is a general smooth complete intersection of codimension r. (In what follows, we write p instead of $\phi(p)$ for simplicity of the notation.) The image of the Euler vector field $-\theta_{00}$ in $\Theta_{\overline{L}}$ is of the form

$$\sum_{i=1}^{n-r} t_i \frac{\partial}{\partial t_i}$$

modulo $(t_1, \ldots, t_{n-r})^2 \Theta_{\overline{L}}$, thus generating a line bundle $\mathcal{L}_L \subset (t_1, \ldots, t_{n-r})\Theta_{\overline{L}}$.

Corollary 10.9. *Let a be a large, sufficiently divisible integer. Then, locally around p, the $\mathcal{O}_{\overline{L}}$-module $(t_1, \ldots, t_{n-r})^a \mathrm{Sym}^a \Theta_{\overline{L}}$ is generated by the submodule $(t_1, \ldots, t_{n-r})^{a-1} \mathcal{L}_L \cdot \mathrm{Sym}^{a-1} \Theta_{\overline{L}}$ and the pullbacks of local sections of the $\mathcal{O}_L$-module $\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)}$.*

Let $z_1, \ldots, z_{2(n-r)}$ be local coordinates of $M \supset L$ at p. There is an integer $d \geq 2$ such that

$$(t_1, \ldots, t_{n-r})^d \supset (z_1, \ldots, z_{2(n-r)})\mathcal{O}_{\overline{L}} \not\subset (t_1, \ldots, t_{n-r})^{d+1}.$$

After performing a linear coordinate change of $(z_1, \ldots, z_{2(n-r)})$ if necessary, we may assume that the equivalence classes $\overline{z}_1, \ldots, \overline{z}_s$ of $z_1, \ldots, z_s$ are linearly independent in $(t_1, \ldots, t_{n-r})^d/(t_1, \ldots, t_{n-r})^{d+1} \simeq \mathbb{C}^{\binom{n-r+d-1}{d}}$ and that $\overline{z}_{s+1} = \cdots = \overline{z}_{n-r} = 0$.

Lemma 10.10. *Let the notation be as above. Then:*

(1) *The congruence relation*

$$\sum_{i=1}^{n-r} t_i \frac{\partial}{\partial t_i} \equiv d \sum_{j=1}^{2(n-r)} z_j \frac{\partial}{\partial z_j} \qquad \mathrm{mod}\ (t_1, \ldots, t_{n-r})^{d+1} \nu_0^* \Theta_M$$

holds.

(2) *The sheaf $(t_1, \ldots, t_{n-r})^a \mathrm{Sym}^a \Theta_{\overline{L}}$, which is viewed as an $\mathcal{O}_M$-module, is generated by*

$$\mathfrak{I}_L^{(a)}/\mathfrak{I}_L^{(a+1)} + (t_1, \ldots, t_{n-r})^{ad} \mathcal{L} \cdot \mathrm{Sym}^{a-1} \Theta_{\overline{L}} + (t_1, \ldots, t_{n-r})^{ad+1} \mathrm{Sym}^a \Theta_{\overline{L}}.$$

(3) *The $\mathbb{C}$-vector space spanned by the homogeneous polynomials of degree a in $\overline{z}_1, \ldots, \overline{z}_s$ is identical with $(t_1, \ldots, t_{n-r})^{ad}/(t_1, \ldots, t_{n-r})^{ad+1}$.*

Proof. Noting that

$$z_{s+1} \equiv \cdots \equiv z_{2(n-r)} \equiv 0 \quad \mathrm{mod}\ (t_1, \ldots, t_{n-r})^{d+1},$$

we derive (1) from the Euler identity

$$\sum_i t_i \frac{\partial z_j}{\partial t_i} \equiv dz_j \qquad \mathrm{mod}\ (t_1, \ldots, t_{n-r})^{d+1},$$

and, in view of this, (2) readily follows from Corollary 10.9.

The identity (2) specifically implies that

$$(*) \qquad \sum_{\sigma \in \mathfrak{S}_a} \frac{\partial z_{j_{\sigma 1}}}{\partial t_{i_1}} \cdots \frac{\partial z_{j_{\sigma a}}}{\partial t_{i_a}} \in (z_1, \ldots, z_{2(n-a)})^a + (t_1, \ldots, t_{n-r})^{ad+1}.$$

The vector space $\big((t_1, \ldots, t_{n-r})/(t_1, \ldots, t_{n-r})^2\big) \otimes \Theta_{\overline{L}}$, naturally identified with the Lie algebra $\mathfrak{gl}(n-r)$, canonically acts on the vector space

$$(t_1, \ldots, t_{n-r})/(t_1, \ldots, t_{n-r})^2 \simeq \mathbb{C}(p) \otimes \Omega_{\overline{L}},$$

and so does

$$\mathrm{Sym}^a \mathfrak{gl}(n-r) = \big((t_1, \ldots, t_{n-r})^a/(t_1, \ldots, t_{n-r})^{a+1}\big) \otimes \mathrm{Sym}^a \Theta_{\overline{L}}$$

on the vector space

$$(t_1, \ldots, t_{n-r})^{ad}/(t_1, \ldots, t_{n-r})^{ad+1} \simeq \mathbb{C}(p) \otimes \mathrm{Sym}^a \Omega^1_{\overline{L}}.$$

It is easy to check that $(t_1, \ldots, t_{n-r})^{ad}/(t_1, \ldots, t_{n-r})^{ad+1}$ contains no nontrivial subspace stable under this action. On the other hand, the relation $(*)$ is nothing but the stabiliy of the subspace $(\overline{z}_1, \ldots, \overline{z}_{2(n-r)})^a$, whence follows (3). Q.E.D.

Corollary 10.11. *Let $\mu_L : \widetilde{L} \to \overline{L}$ and $\mu_M : \widetilde{M} \to M$ be the blowups at $p = (0, \ldots, 0)$ and $\nu_L(p)$. Then $\nu_L : \overline{L} \to M$ naturally lifts to a morphism $\tilde{\nu}_L : \widetilde{L} \to \widetilde{M}$. The exceptional divisor $E_L \simeq \mathbb{P}^{n-r-1} \subset \widetilde{L}$ is isomorphically mapped to a subvariety of $E_M \simeq \mathbb{P}^{2(n-r)-1} \subset \widetilde{M}$ defined by quadratic homogeneous polynomials.*

Proof. By Lemma 10.10(3), we have

$$(z_1, \ldots, z_{2(n-r)})^a \mathcal{O}_{\widetilde{L}} = (t_1, \ldots, t_{n-r})^{ad} \mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{L}}(-adE_L),$$

so that $(z_1, \ldots, z_{2(n-r)})\mathcal{O}_{\widetilde{L}} = \mathcal{O}_{\widetilde{L}}(-dE_L)$, meaning that ν_L lifts to $\tilde{\nu}_L$ by the universal property of monoidal transformations.

We can view $\overline{z}_j$ as a homogeneous polynomial of degree d. Then Lemma 10.10(3) means that the system of polynomials $\overline{z}_1, \ldots, \overline{z}_{2(n-r)}$, as a free linear subsystem of $|\mathcal{O}_{\mathbb{P}^{n-r-1}}(d)|$, defines an embedding $E_L \simeq$

$\mathbb{P}^{n-r-1} \hookrightarrow E_M \simeq \mathbb{P}^{2(n-r)-1}$. In particular, the defining ideal $\overline{\mathfrak{J}}_L$ of E_L in E_M is locally free and satisfies $\overline{\mathfrak{J}}_L^{(a)} = \overline{\mathfrak{J}}_L^a$.

Via the inclusion $\Theta_{\overline{L}} \subset \nu_L^*\Theta_M \simeq \nu_L^*\Omega_M^1$ and Lemma 10.10(3), we have

$$\prod_{\alpha=1}^{a} t_{i_\alpha}\frac{\partial}{\partial t_{j_\alpha}} = \prod_{\alpha=1}^{a}\Big(\sum_{k=1}^{2(n-r)} t_{i_\alpha}\frac{\partial z_k}{\partial t_{j_\alpha}}\frac{\partial}{\partial z_k}\Big) \subset \Big(t_i\frac{\partial z_k}{\partial t_j}\Big)^a \nu_L^*\Omega_M^1$$
$$\equiv (t_1,\ldots,t_{n-r})^{ad}\nu_L^*\Omega_M^1 = (z_1,\ldots,z_s)^a\nu_L^*\Omega_M^1 \bmod (t_1,\ldots,t_{n-r})^{ad+1}.$$

This means that a section of $\mathfrak{J}_L^{(a)}/\mathfrak{J}_L^{(a+1)}$ has image of the form $\sum_{IJ} a_{IJ}z^I\mathrm{d}z^J$ in $\mathrm{Sym}^a\Omega_M^1/(z_1,\ldots,z_{2(n-r)})^{a+1}\mathrm{Sym}^a\Omega_M^1$, where I, J are multi-indices with $|I| = |J| = a$. We note that, if the section does not vanish at $\nu_L(p)$, at least one of the holomorphic functions $a_{IJ}(z)$ does not vanish at $(0,\ldots,0)$. Indeed, a non-zero local section determines a non-zero element of $(t_1,\ldots,t_{n-r})^a\mathrm{Sym}^a\Theta_{\overline{L}}$, which in turn defines a non-zero linear transformation of

$$(z_1,\ldots,z_{2(n-r)})^a + (t_1,\ldots,t_{n-r})^{ad+1}/(t_1,\ldots,t_{n-r})^{ad+1}$$
$$= (t_1,\ldots,t_{n-r})^{ad}/(t_1,\ldots,t_{n-r})^{ad+1}.$$

Thus the sheaf $\mathfrak{J}_L^{(a)}/(z_1,\ldots,z_n)\mathfrak{J}_L^{(a)}$ is generated by homogeneous polynomials precisely of degree $2a$ (degree a from the part $(z_1,\ldots,z_{2(n-r)})^a$ and degree a from $\mathrm{Sym}^a\Omega_M^1$). Of course this sheaf is the a-th power $\overline{\mathfrak{J}}_L^a$ of the ideal sheaf of E_L, and E_L is necessarily defined by quadratic equations in E_M. Q.E.D.

The embedding $\mathbb{P}^{n-r-1} = E_L \hookrightarrow E_M = \mathbb{P}^{2(n-r)-1}$ is a linear projection from the d-th Veronese embedding

$$\mathbb{P}^{n-r-1} \hookrightarrow \mathbb{P}^{\binom{d+n-r-1}{n-r-1}-1}$$

to a linear subsystem. Then a classical lemma of Terracini (see [Zak], p.2) yields the following

Lemma 10.12. *Assume that $n-r \geq 3$ or $d \geq 3$. Then $\overline{z}_1, \ldots, \overline{z}_{2(n-r)}$ are linearly independent in $(t_1,\ldots,t_{n-r})^d/(t_1,\ldots,t_{n-r})^{d+1}$.*

Proof. The embedding $E_L \simeq \mathbb{P}^{n-r-1} \hookrightarrow \mathbb{P}^{2(n-r)-1}$ is a linear projection of the d-th Veronese embedding. The Terracini lemma states that the dimension of the secant variety SX of the subvariety $E_L \subset \mathbb{P}^{\binom{d+n-r-1}{n-r-1}}$ is given by the dimension of the linear span

$$\langle T_{E_L,x}, T_{E_L,y}\rangle \subset \mathbb{P}^{\binom{d+n-r-1}{n-r-1}}$$

of the embedded tangent spaces $T_{E_L,x}, T_{E_L,y} \subset \mathbb{P}^{\binom{d+n-r-1}{n-r-1}}$ of E_L at general points $x, y \in E_L$. In case $d > 2$, this shows that $\dim SX = 2(n-r) - 1$ and we cannot isomorphically project E_L to $\mathbb{P}^{2(n-r-1)}$, meaning the linear independence of $z_1, \ldots, z_{2(n-r)}$.

Suppose that $d = 2$ and that $z_1, \ldots, z_{2(n-r)}$ are linearly dependent. Then we have an embedding $\mathbb{P}^{n-r-1} \subset \mathbb{P}^{2(n-r-1)}$, defining equations of which are hyperquadrics by Corollary 10.11. It follows that $E_L = \mathbb{P}^{n-r-1}$ is a complete intersection of quadrics. Indeed, the degree of E_L is 2^{n-r-1} and there are at least $n-r-1$ independent quadrics that contains E_L. In view of the adunction formula, the subvariety $\mathbb{P}^{n-r-1} \subset \mathbb{P}^{2(n-r-1)}$ is a complete intersection of quadrics if and only if $n - r - 1 = 1$. Q.E.D.

Lemma 10.13. *If* $n - r \geq 2, d \geq 3$, *then* $d = 3, n - r = 2$.

Proof. By construction, $E_L = \mathbb{P}^{n-r-1} \subset \mathbb{P}^{2(n-r)-1}$ has degree d^{n-r-1} and is defined as an intersection of quadrics. Hence we have a trivial inequality $2^{n-r} \geq d^{n-r-1}$. If the equality holds, then $d = 4$, $n - r = 2$ and $\mathbb{P}^1 \subset \mathbb{P}^3$ would be a complete intersection of two quadrics, which would be a curve of arithmetic genus one. Thus we have only two possibilities: $d = 2$ and $d = 3, n - r = 2$. Q.E.D.

Lemma 10.14. *The case* $n - r = 2, d = 3$ *does not occur.*

Proof. Let (x, y) be a local coordinate system on $\overline{L}$. Under the assumption of the lemma, we can assume that $z_1 \equiv x^3, z_2 \equiv x^2y, z_3 \equiv xy^2, z_4 \equiv y^3$ modulo $(x, y)^4$. Blow up $\overline{L}$ and M at p and $\nu_L(p)$. Then, by taking a suitable holomorphic function u on $\tilde{L}$, we get a local coordinate system (x, u) on $\tilde{L}$ such that $z_1 = x^3, z_2 = x^3u, z_3 = x^3u^2 + x^4g, z_3 = x^3u^3 + x^4h$, where g, h are holomorphic. Let us prove that z_3 and z_4 are functions in (x^3, u) and hence $\nu_L : \overline{L} \to M$ cannot be bimeromorphic.

Recall that E_L is defined by three quadrics and $\mathfrak{I}_L^{(a)}$ is generated by elements congruent to the products of these quadrics modulo terms of degree $\geq 2a + 1$. We have therefore relations of functions in (x, u):

$$(z_iz_j - z_kz_l)^a \equiv 0 \mod (z_1, \ldots, z_4)^{2a+1} \text{ if } i + j = k + l.$$

Assume that z_3 or z_4 contains a term $x^{3(m+1)+c}$, where m is a non-negative integer and $0 < c < 3$. Let $3(m+1) + c$ be the minimum of such exponents and assume that the minimum is attained by a term in z_3. Then $(z_1z_3 - z_2^2)^a$ contains the non-zero x^{3a+m+c}-term, while the terms of degree $\geq 2a + 1$ terms do not. This means that $(z_1z_3 - z_2^2)^a +$ (terms of higher order) cannot vanish on L, contradicting our assumption. In case z_4 contains the term x^{3m+c} and z_3 does not, then look at a second equation $(z_2z_4 - z_3^2)^a + (\text{terms of order } 2a+1) = 0$ and we similarly get the absurd conclusion that z_4 is a function in (x^3, u). Q.E.D.

Lemma 10.15. *If $d = 2$, then $n - r \leq 2$.*

Proof. Let $\mathcal{N}^*$ be the conormal bundle of $E_L = \mathbb{P}^{n-r-1} \subset E_M = \mathbb{P}^{2(n-r)-1}$. A quadric relation defines an injection

$$\tilde{\nu}_L^* \mathcal{O}_{E_M}(-2) = \mathcal{O}_{E_L}(-2d) = \mathcal{O}_{E_L}(-4) \subset \mathcal{N}^*$$

and hence an element of $\mathrm{H}^0(\mathcal{N}^*(4))$. The assumption that $E_L = \mathbb{P}^{n-r-1}$ is defined by quadratic equations in $\mathbb{P}^{2(n-r)-1}$ is rephrased as the global generation of $\mathcal{N}^*(4)$, meaning that $\mathcal{N}^*(4)$ is a direct sum of trivial line bundles $\mathcal{O}$ and a nef big vector bundle $\mathcal{P}$. By Serre duality, $\mathrm{H}^1(\mathbb{P}^{n-r-1}, \mathcal{P}) \simeq \mathrm{H}^1(\mathbb{P}(\mathcal{P}), \mathcal{O}(\mathbf{1}))$ is the dual of

$$\mathrm{H}^{\operatorname{rank}\mathcal{P}+2(n-r)-3}(\mathbb{P}(\mathcal{P}), \mathcal{O}(-(n-r+1)\mathbf{1}))$$

and vanishes by the Kawamata-Viehwg vanishing theorem. Consequently we have $\mathrm{H}^1(\mathbb{P}^{n-r-1}, \mathcal{N}^*(4)) = 0$.

In view of the standard dual Euler exact sequences

$$0 \to \tilde{\nu}_L^* \Omega^1_{E_M} \to \mathcal{O}(-2)^{\oplus 2(n-r)} \to \mathcal{O} \to 0,$$
$$0 \to \Omega^1_{E_L} \to \mathcal{O}(-1)^{\oplus n-r} \to \mathcal{O} \to 0,$$

the conormal bundle $\mathcal{N}^*$ is explicitly described by the exact sequence

$$0 \to \mathcal{N}^* \to \mathcal{O}(-2)^{\oplus 2(n-r)} \to \mathcal{O}(-1)^{\oplus n-r} \to 0.$$

In view of the vanishing of $\mathrm{H}^1(\mathbb{P}^{n-r-1}, \mathcal{N}^*(4))$, we infer that

$$2(n-r)\dim \mathrm{H}^0(\mathcal{O}(2)) - (n-r)\dim \mathrm{H}^0(\mathcal{O}(3)) = \dim \mathrm{H}^0(\mathcal{N}^*(4)).$$

On the other hand, $\dim \mathrm{H}^0(\mathcal{N}^*(4))$ is nothing but the number of quadratic equations of $E_L \subset E_M$, which was at least

$$\operatorname{length}(t_1, \ldots, t_{n-r})\Theta_{\overline{L}}/(\mathcal{L}_L + (t_1, \ldots, t_{n-r})^2\Theta_{\overline{L}}) = (n-r)^2 - 1.$$

Hence

$$(n-r)^2\{(n-r+1) - \frac{1}{6}(n-r+2)(n-r+1) - 1\} \geq -1,$$

yielding the inequality $n - r \leq 2$. Q.E.D.

Lemma 10.16. $n - r = 1$.

Proof. It suffices to exclude the case $d = n - r = 2$. In this case, we may well assume that $z_1 \equiv x^2, z_2 \equiv xy, z_3 \equiv y^2, z_4 \equiv 0$ modulo $(x, y)^3$. Let η_{ij} be the $\mathrm{d}z_i \wedge \mathrm{d}z_j$-coefficient of the symplectic form η on M. Then, since $L \subset M$ is Lagrangian,

$$0 = \nu_L^* \eta \equiv (2x^2\eta_{12} + 4xy\eta_{13} + 2y^2\eta_{23})\mathrm{d}x \wedge \mathrm{d}y \qquad \mathrm{mod}\ (x, y)^3,$$

and hence

$$\eta_{12} \equiv \eta_{13} \equiv \eta_{23} \equiv 0 \quad \mathrm{mod}\ (z_1, z_2, z_3, z_4),$$

meaning that η at $(0, 0, 0, 0)$ has the three-dimensional isotropic subspace spanned by $\frac{\partial}{\partial z_i}$, $i = 1, 2, 3$, thus contradicting the nondegeneracy of η. Q.E.D.

Lemma 10.17. *The case $n - r = 1$ is impossible.*

Proof. In this case, $\mathfrak{I}_L \subset \mathcal{O}_M$ is an invertible sheaf with $\mathfrak{I}_L^{(a)} = \mathfrak{I}_L^a$. Thus it suffices to show that $\nu_L^* \mathfrak{I}_L / \mathfrak{I}_L^2 \not\ni t\Theta_{\overline{L}}$, where t is a local parameter of the curve $\overline{L}$. If d is the multiplicity of L at o, then there exists an integer $e > d$ such that $z_1 = t^d$, $z_2 = (\mathrm{unit})t^e$. The image of $t(\partial/\partial t)$ is

$$\begin{aligned} t\frac{\mathrm{d}z_1}{\mathrm{d}t}\frac{\partial}{\partial z_1} + t\frac{\mathrm{d}z_2}{\mathrm{d}t}\frac{\partial}{\partial z_2} &= \eta_{12} t(\frac{\mathrm{d}z_1}{\mathrm{d}t}\mathrm{d}z_2 - \frac{\mathrm{d}z_2}{\mathrm{d}t}\mathrm{d}z_1) \\ &\equiv \eta_{12}(0)(dz_1\mathrm{d}z_2 - ez_2\mathrm{d}z_1) \quad \mathrm{mod}\ t^{d+1}\mathcal{O}_{\overline{L}}\mathrm{d}z_2 + t^{e+1}\mathcal{O}_{\overline{L}}\mathrm{d}z_1. \end{aligned}$$

If this comes from $(\mathfrak{I}_L/\mathfrak{I}_L^2)$, then the defining equation of L contains an element $\in \eta_{12}(d - e)z_1z_2 + (z_2^2, z_1^3, z_1^2z_2)$, which we can easily rule out by comparing the degrees in t. Q.E.D.

Thus we have eventually excluded the possibility that $\nu_0 : \mathbb{P}^n \to X$ is not an immersion, completing the proof of Theorem 10.1.

References

[Ar] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Math., **60**, Springer-Verlag, New York-Heidelberg-Berlin, 1978.

[Au] T. E. Aubin, *Equations de type Monge-Ampère sur les variétés kähleriennes compactes*, Bull. Math. France, **102** (1978), 63–95.

[BaPeVV] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Ergeb. Math. Grenz., III-4, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[Bea1] A. Beauville, *Variétés kähleriennes dont la première classe de Chern est nulle*, J. Differential Geom., **18** (1983), 755–782.

[Bea2] A. Beauville, *Fano contact manifolds and nilpotent orbits*, alg-geom 9707015.

[Ber] M. Berger, *Sur les groupe d'holonomie homogène des variétés à connecxion affine et des variétés riemanniennes*, Bull. Soc. Math. France, **83** (1955), 279–330.

[Bo] F. Bogomolov, *On the decomposition of Kähler manifolds with trivial canonical class*, Math. USSR Sb., **22** (1974), 580–583.

[Br] E. Brieskorn, *Ein Satz über die komplexen Quadriken*, Math. Ann., **155** (1964), 184–193.

[CO] B.-Y Chen and K. Ogiue, *Some characterizations of complex space forms in terms of Chern classes*, Quat. J. Math. Oxford, **26** (1975), 459–464.

[CS1] K. Cho and E. Sato, *Smooth projective varieties dominated by smooth quadric hypersurfaces in any characteristic*, Math. Zeitschrift, **217** (1994), 553–565.

[CS2] K. Cho and E. Sato, *Smooth projective varieties with the ample vector bundle $\wedge^2 T_X$ in any characteristic*, J. Math. Kyoto Univ., **35** (1995), 1–33.

[FGA] A. Grothendieck, *Fondement de Géométrie Algébrique*, Sécretariat Mathématique, Paris, 1962.

[Fo] J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math., **90** (1968), 511–521.

[Fr] T. Frankel, *Manifolds with positive curvature*, Pacific J. Math., **11** (1961), 165–174.

[Fu] A. Fujiki, *On the de Rham cohomology group of a compact Kähler symplectic manifold*, Algebraic Geometry, Sendai 1985 (Adv. Study in Pure Math., vol. 10), Kinokuniya, Tokyo and North-Holland, Amsterdam, 1987, pp. 105–165.

[Gra] H. Grauert, *Über Modifikationen und exzeptionelle anatyische Mengen*, Math. Ann., **146** (1962), 331–368.

[GraR] H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Inv. Math., **11** (1970), 263–292.

[Gro] A. Grothendieck *et al.*, *SGA I: Revêtements étales et groupe fondemental*, Lect. Notes in Math., No. 224, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[Ha1] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Lect. Notes in Math., **156**, Springer-Verlag, Heidelberg, 1970.

[Har2] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, **52**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[Hi] F. Hirzebruch, *Neue topologische Methoden in der algebraischen Geometrie*, Ergebnisse der Mathematik u. ihrer Grenzgeb., N. F. Heft, **9**, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1953.

[HK] F. Hirzebruch and K. Kodaira, *On the complex projective spaces*, J. Math. Pures Appl., **36** (1957), 201–216.

[Io] P. Ionescu, *Generalized adjunction and applications*, Math. Proc. Cambridge Phil. Soc., **99** (1986), 457–472.

[Is1] V. A. Iskovskih, *Fano* 3*-folds, I*, Math. USSR.-Izv., **11** (1978), 485–527.

[Is2] V. A. Iskovskih, *Fano* 3*-folds, II*, Math. USSR.-Izv., **12** (1979), 496–506.

[Ka] Y. Kawamata, *The cone of curves of algebraic varieties*, Ann. Math., **119** (1984), 95–110.

[Ke1] S. Kebekus, *Families of singular rational curves*, AG/0004023.

[Ke2] S. Kebekus, *Projective bundles of singular plane cubics*, AG/0009083.

[KO] S. Kobayashi and T. Ochiai, *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ., **13** (1973), 31–47.

[Kob] S. Kobayashi, *On compact Kaehler manifolds with positive definite Ricci tensor*, Ann. of Math., **74** (1961), 570–574.

[Kod] K. Kodaira, *On a differential-geometric method in the theory of analytic stacks*, Proc. Nat. Acad. Sci. U. S. A., **39** (1953), 1268–1273.

[Kol] J. Kollár, *Rational Curves on Algebraic Varieties*, (Ergebnisse der Math. u. ihrer Grenzgeb. III-32), Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1996.

[KM] J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge Univ. Press, Cambridge, 1998.

[KMM] J. Kollár, Y. Miyaoka and S. Mori, *Rationally connected varieties*, J. Algebraic Geometry, **1** (1992), 429–448.

[La] R. Lazarsfeld, *Some applications of positive vector bundles*, Complete Intersections, Acireale 1983 (ed: S. Greco and R. Strano), Lecture Notes in Math., **1092**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984, pp. 29–61.

[Matm] H. Matsumura, *Commutative Algebra*, Benjamin, New York, 1970.

[Mats] D. Matushita, *On fibre space structure of a projective irreducible symplectic manifold*, Topology, **38** (1999), 72–83.

[MiMo] Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Ann. Math., **124** (1986), 65–69.

[Mo1] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. Math., **110** (1979), 593–606.

[Mo2] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. Math., **116** (1982), 133–176.

[Mo3] S. Mori, *Cone of curves, and Fano* 3*-folds*, Proceedings of the International Congress of Mathematicians, August 16-24, 1983, Warszawa, North-Holland, Amsterdam-New York-Oxford, 1984.

[MoMu] S. Mori and S. Mukai, *Classification of Fano threefolds with* $B_2 \geq 2$, Manuscr. Math., **36** (1981), 147–162.

[RV] R. Remmert and A. Van de Ven, *Über holomorphe Abbildingen projekiv-algebraischen Mannigfaltigkeiten auf komplexe Räume*, Math. Ann., **142** (1961), 453–486.

[SaU] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of two-spheres*, Ann. of Math., **113** (1981), 1–24.

[Si] Y.-T. Siu, *Curvature characterization of hyperquadrics*, Duke Math. J., **47** (1980), 641–654.

[SY] Y.-T. Siu and S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Inv. Math., **59** (1980), 189–204.

[Ti] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Weil-Petersson metric*, Mathematical Aspects of String Theory (S.-T. Yau, ed.), World Scientific, Singapore-New Jersey-Hong Kong, 1987, pp. 629–646.

[To] A. Todorov, *Spinors and moduli of Einstein metrics on Kähler simply connected manifolds with a canonical class* $K \equiv 0$, Lectures on supermanifolds, geometrical methods and conformal groups, Varna, World Scientific, Teaneck, NJ, 1989, pp. 39–43.

[W] J. A. Wiśniewski, *On contraction of extremal rays of Fano manifolds*, J. Reine Angew. Math., **417** (1991), 141–157.

[Y1] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U. S. A., **74** (1977), 1798–1799.

[Y2] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I*, Comm. Pure Appl. Math., **31** (1978), 339–411.

[Zak] F. L. Zak, *Tangents and Secants of Algebraic Varieties*, (Translations of Mathematical Monographs, 127), American Mathematical Society, Providence, Rhode Island, 1993.

Koji Cho
School of Mathematics
Kyushu University
Hakozaki 6-10-1, Fukuoka, Japan

Yoichi Miyaoka
School of Mathematics
University of Tokyo
Komaba 3-8-1, Meguro, Tokyo, Japan

N. I. Shepherd-Barron
DPMMS, University of Cambridge
Cambridge CB3 0WB, UK