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A characterization of $^2\mathbf{E}_6(2)$

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§1. Introduction

This paper is part of a program to provide a uniform, self-contained treatment of part of the foundations of the theory of the sporadic finite simple groups. More precisely our eventual aim is to provide complete proofs of the existence and uniqueness of the twenty-six sporadic groups and to derive the basic structure of each sporadic. The two books [SG] and [3T] make a beginning on that program.

In this paper we provide a uniqueness proof for the group $^2E_6(2)$. Of course $^2E_6(2)$ is a group of Lie type, not a sporadic group, but in order to treat the Monster and the Baby Monster, one first needs to treat $^2E_6(2)$. Thus this paper begins that part of the program dealing with the large sporadics.

Suzuki was one of the pioneers in identifying finite groups from information on subgroup structure. His characterization of $L_3(2^n)$ in [S] identifies those groups by producing a BN-pair. That approach is not so different from the one adopted in our program. Indeed in the work of S. Smith and the author on quasithin groups, the groups $L_3(2^n)$, n even, can not quite be handled using our standard methods, so we appropriate a clever counting argument of Suzuki's from [S] to fill the gap. Hopefully Suzuki would regard this paper as continuing a tradition which he pioneered.

Define a finite group G to be of *type* $^2E_6(2)$ if G possesses an involution z such that $F^*(C_G(z)) = O_2(C_G(z))$ is extraspecial of width 10, $C_G(z)/O_2(C_G(z)) \cong U_6(2)$, and z not weakly closed in $O_2(C_G(z))$ with respect to G.

Define G to be of *type* $\mathbf{Z}_2/{}^2E_6(2)$ if G possesses an involution z such that $F^*(C_G(z)) = O_2(C_G(z))$ is extraspecial of width 10 and $C_G(z)$ has a subgroup H of index 2 such that $H/O_2(C_G(z)) \cong U_6(2)$, and z is not weakly closed in $O_2(C_G(z))$ with respect to G.

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Our main theorems are:

Theorem 1. *Each group of type* ${}^2E_6(2)$ *is isomorphic to* ${}^2E_6(2)$.

Theorem 2. *If* G *is of type* $\mathbf{Z}_2/{}^2E_6(2)$ *then* $F^*(G)$ *is of index* 2 *in* G *and* $F^*(G) \cong {}^2E_6(2)$.

Theorems 1 and 2 are proved in sections 8 and 9, respectively, where they appear as Theorems 8.7 and 9.1. Many lemmas are included in the paper which are not used in the proof of the main theorems. They will be used later in the program and appear here because it is convenient to provide an exposition of related results in one place. Similarly the proof of the following two lemmas will appear in later papers in this series for the same reason, as will the proof of the third part of lemma 5.8.

(1.1) *Let* Γ *be a building of type* F_4 *and* Δ *the collinearity graph of* Γ. *Then* Δ *is simply connected.*

(1.2) *Let* G *be a group and* V *a faithful finite dimensional* $\mathbf{F}_2G$-*module. Assume* $u \in V^{\#}$ *such that the full group* T *of transvections on* V *with center* u *is contained in* G. *Let* $U = \langle u^G \rangle$ *and* $L = \langle T^G \rangle$. *Then* $Aut_L(U) = GL(U)$.

§2. Presentations for modules

In this section Ω is a graph with vertex set Ω and $\Omega(x)$ denotes the set of vertices adjacent to a vertex x of Ω. Assume G is a group of automorphism of Ω transitive on the vertices of the graph and let V be the permutation module for G on Ω over $\mathbf{F}_2$. Thus Ω is a basis for the $\mathbf{F}_2$-space V and $G \leq GL(V)$ is transitive on the basis Ω.

Define a bilinear form β on V by

$$\beta(x, y) = 0 \text{ if and only if } y \in \Omega(x) \cup \{x\} \text{ for } x, y \in \Omega.$$

As the relation defining the graph Ω is symmetric, the bilinear form β is symmetric.

Let $R = \mathrm{Rad}(\beta)$ be the radical of the bilinear form β; that is

$$R = \{v \in V : \beta(u, v) = 0 \text{ for all } u \in V\}.$$

Finally let $\bar{V} = V/R$ and write $\bar{\beta}$ for the bilinear form induced by β on $\bar{V}$. That is

$$\bar{\beta}(\bar{v}, \bar{u}) = \beta(u, v)$$

which is well defined as R is the radical of β. Further as R is the radical of β, the induced form $\bar{\beta}$ is nondegenerate, so $\bar{\beta}$ is a symplectic form on $\bar{V}$. As G is a group of automorphisms of the graph Ω, G preserves the form β, and hence also the induced form $\bar{\beta}$. We summarize all this as:

(2.1) $(\bar{V}, \bar{\beta})$ *is a symplectic space over* $\mathbf{F}_2$ *and* $G \leq Sp(\bar{V})$ *is a group of isometries of this symplectic space transitive on the generating set* $\bar{\Omega}$ *of* $\bar{V}$.

(2.2) *Assume* U *is an* $\mathbf{F}_2G$*-module and* $\rho : \Omega \to U$ *is a map such that* $U = \langle \rho(\Omega) \rangle$ *and* $\rho : \Omega \to \rho(\Omega)$ *is an equivalence of* G*-sets. Assume further that* γ *is a symplectic form on* U *with*

$$\beta(x, y) = \gamma(\rho(x), \rho(y)) \text{ for all } x, y \in \Omega.$$

Then ρ *extends to an* $\mathbf{F}_2G$*-isometry* $\bar{\rho} : (\bar{V}, \bar{\beta}) \to (U, \gamma)$.

Proof. As $U = \langle \rho(\Omega) \rangle$, the map ρ extends to a surjective $\mathbf{F}_2G$-homomorphism $\rho : V \to U$. Let $v \in V$; then $v = \sum_{y \in S(v)} y$, where $S(v)$ is the support of v with respect to the basis Ω. Further for $x \in \Omega$, $\beta(v, x) = |\Gamma(x) \cap S(v)| \mod 2$, where $\Gamma(x) = \Omega - x^{\perp}$. Now $\rho(v) = \sum_{y \in S(v)} \rho(y)$ and

$$\gamma(\rho(v), \rho(x)) = \sum_{y \in S(v)} \gamma(\rho(y), \rho(x)) = |\Gamma(x) \cap S(v)| \mod 2 = \beta(v, x)$$

as $\beta(x, y) = \gamma(\rho(x), \rho(y))$ for all $x, y \in \Omega$. Therefore $v \in R$ if and only if $\beta(v, x) = 0$ for all $x \in \Omega$ if and only if $\gamma(\rho(v), \rho(x)) = 0$ for all $x \in \Omega$ if and only if $\rho(v) \in U^{\perp} = 0$, since $U = \langle \rho(\Omega) \rangle$. Therefore $R = \ker(\rho)$, so ρ induces the isometry $\bar{\rho} : (\bar{V}, \bar{\beta}) \to (U, \gamma)$. Q.E.D.

(2.3) *Assume* (U, q) *and* (W, Q) *are orthogonal spaces over* $\mathbf{F}_2$ *with* G *irreducible on* U, $G \leq O(U, q)$, *and* $G \leq O(W, Q)$. *Let* γ *and* α *be the bilinear forms of* q *and* Q, *respectively, and assume* $\rho : (U, \gamma) \to (W, \alpha)$ *is an* $\mathbf{F}_2G$*-isometry. Then* $\rho : (U, q) \to (W, Q)$ *is also a* $\mathbf{F}_2G$*-isometry.*

Proof. As G is irreducible on U, there is at most one quadratic form on U preserved by G with bilinear form γ. (cf. 4.9 in [A]; the argument is easy.) Therefore q is that unique form. Similarly as $\rho : U \to W$ is an equivalence of $\mathbf{F}_2G$-representations, G is irreducible on W, so Q is the unique quadratic form on W preserved by G with bilinear form α, so that ρ is also an isometry of the corresponding orthogonal spaces.

Q.E.D.

(2.4) *Assume (U,q) and (W,Q) are orthogonal spaces over $\mathbf{F}_2$ with G irreducible on U, $G \leq O(U,q)$, and $G \leq O(W,Q)$. Assume further that $u \in U$, $w \in W$, with $G_u = G_w$, $U = \langle uG \rangle$, $W = \langle wG \rangle$, and $\gamma(u,ug) = \alpha(w,wg)$ for all $g \in G$, where γ and α are the bilinear forms of q and Q, respectively. Then there exists an $\mathbf{F}_2G$-isometry $\rho : (U,q) \to (W,Q)$ with $\rho(u) = w$.*

Proof. As $G_u = G_w$, the map $\rho : uG \to wG$ defined by $\rho(ug) = wg$ is a well defined equivalence of permutation representations. Now take Ω_U to be the graph on uG with $\Omega_U(u) = \Omega_U \cap u^{\perp}$. As $\gamma(u,ug) = \alpha(w,wg)$, ρ defines a G-equivariant isomorphism of Ω_U with the corresponding graph Ω_W on wG. Now apply 2.2 to get $\mathbf{F}_2G$-sometries $\rho_U : (U,q) \to (\bar{V}_U, \bar{q})$ and $\rho_W : (W,Q) \to (\bar{V}_W, \bar{Q})$, where $\bar{V}_U$ and $\bar{V}_W$ are modules of the graphs Ω_U and Ω_W, respectively, and $\bar{q}$ and $\bar{Q}$ are the transfer of the forms q and Q via ρ_U and ρ_W. As $\rho : \Omega_U \to \Omega_W$ is a G-isomorphism, ρ induces an $\mathbf{F}_2G$-isometry $\bar{\rho} : (\bar{V}_U, \bar{\beta}_U) \to (\bar{V}_W, \bar{\beta}_W)$, and hence also an $\mathbf{F}_2G$-isometry $\bar{\rho} : (\bar{V}_U, \bar{q}) \to (\bar{V}_W, \bar{Q})$ by 2.3. Then the composition $\rho_W^{-1} \circ \bar{\rho} \circ \rho_U$ agrees with ρ on uG and is the required extension. Q.E.D.

§3. Some central extensions

We adopt the notation of section 33 of [FGT] and section 23 of [3T] in discussing central extensions. In particular if G is a perfect finite group then $\mathrm{Cov}(G)$ is the universal covering group of G and $\mathrm{Schur}(G)$ is the Schur multiplier of G. In particular $\mathrm{Schur}(G) \leq Z(\mathrm{Cov}(G))$ with $\mathrm{Cov}(G)/\mathrm{Schur}(G) \cong G$. In addition if p is a prime define

$$\mathrm{Cov}_p(G) = \mathrm{Cov}(G)/O^p(\mathrm{Schur}(G))\Phi(O_p(\mathrm{Schur}(G)))$$

and

$$\mathrm{Schur}_p(G) = \mathrm{Schur}(G)/O^p(\mathrm{Schur}(G))\Phi(O_p(\mathrm{Schur}(G)))$$

That is $\mathrm{Cov}_p(G)$ is the largest perfect central extension of an elementary abelian p-subgroup by G.

Let $\mathcal{H}$ be the class of finite groups H such that $F^*(H)$ is an extraspecial 2-group and $H/O_2(H))$ is irreducible on $F^*(H)/Z(F^*(H))$. Our notational convention will be to write $Q = F^*(H)$, $\tilde{H} = H/Z(Q)$, and $H^* = H/Q$. We recall from section 8 of [SG] that the commutator map and power map define a nondegenerate bilinear form and quadratic form on $\tilde{Q}$ preserved by H^*. By Exercise 8.5 in [FGT], $\mathrm{Out}(Q) = O(\tilde{Q})$ is the isometry group of this quadratic form.

(3.1) *Let $H_i \in \mathcal{H}$, $i = 1, 2$, with $Q_1 \cong Q_2$ and assume $\tilde{Q}_i$ is absolutely irreducible as an $\mathbf{F}_2 H_i^*$-module. Then $\tilde{H}_1 \cong \tilde{H}_2$ if and only if the induced representations of H_i^* on $\tilde{Q}_i$ are quasiequivalent for $i = 1, 2$.*

Proof. Identifying Q_1 and Q_2 via our isomorphism, we may take $Q_1 = Q_2 = Q$. Then identifying $\tilde{H}_i$ with $\mathrm{Aut}_{H_i}(Q)$, we have $\tilde{H}_i \leq \mathrm{Aut}(Q) = A$ and $H_i^* \leq A/\tilde{Q} = Out(Q) \cong O(\tilde{Q})$.

The representations of H_1^* and H_2^* on $\tilde{Q}$ are quasiequivalent if and only if H_1^* and H_2^* are conjugate in $GL(\tilde{Q})$. Further as $\tilde{Q}$ is an absolutely irreducible $\mathbf{F}_2 H_i^*$-module, the quadratic form on $\tilde{Q}$ is the unique one preserved by $\tilde{H}_i$, (cf. 4.9 in [A]), so H_1^* is conjugate to H_2^* in $GL(\tilde{Q})$ if and only if the groups are conjugate in $O(\tilde{Q})$. Thus the representations are quasiequivalent if and only if $\tilde{H}_1$ is conjugate to $\tilde{H}_2$ in A, establishing the lemma. Q.E.D.

(3.2) *Let $H \in \mathcal{H}$ be perfect and let $\hat{H} = \mathrm{Cov}_2(H)$, $\hat{Q} = O_2(\hat{H})$, and $P = [\hat{Q}, \hat{H}]$. Then*

(1) *$\hat{H}/P \cong \mathrm{Cov}_2(H^*)$ and $\hat{Q}/P \cong \mathrm{Schur}_2(H^*)$.*

(2) *$P \cong Q \times H^1(H^*, \tilde{Q})$.*

(3) *If H_1 is a perfect central extension of $\tilde{H}$ then the representation of $\mathrm{Aut}(H_1)$ on H_1 by conjugation factors through $\mathrm{Aut}(\hat{H})$.*

(4) *$D = C_{\mathrm{Aut}(\hat{H})}(P/Z(P))$ is elementary abelian and centralizes $P/\Phi(P)$, and $D/\mathrm{Aut}_P(\hat{H})$ acts faithfully as the full group of transvections on $Z(P)$ with center $\Phi(P)$.*

(5) *$D/\mathrm{Aut}_P(\hat{H})$ is regular on the complements to $\Phi(P)$ in $Z(P)$, so if U is such a complement then $\mathrm{Aut}(\hat{H}) = DN_{\mathrm{Aut}(\hat{H})}(U)$ with $\mathrm{Aut}_P(\hat{H}) = N_D(U)$.*

(6) *If $H_0 \in \mathcal{H}$ with $F^*(H_0) \cong F^*(H)$ then $H_0/Z(H_0) \cong H/Z(H)$ if and only if $H_0 \cong \hat{H}/V$ for some complement V to $\Phi(P)$ in $Z(\hat{H})$ containing U.*

Proof. This is an extension of 8.17 in [SG], where the result is essentially proved under the extra hypotheses that $H^1(H^*, \tilde{Q}) = 0$ and H^* is absolutely irreducible on $\tilde{Q}$. Much of the same proof works. In particular if $\rho : \hat{H} \to H$ is the universal covering of H and $\hat{Z} = \ker(\rho)$ then $\hat{Q} = \rho^{-1}(Q)$ is of class 2 with center $Z = \rho^{-1}(Z(Q))$, $Z = Z(\hat{H})$, and $|Z : \hat{Z}| = 2$. As $\hat{Z} = \mathrm{Shur}_2(H)$, $\hat{Z}$ is elementary abelian. Arguing as in the proof of 8.17 of [SG], $\Phi(P)$ is elementary abelian, so as $Z = \Phi(P)\hat{Z}$, Z is elementary abelian. Similarly the proof of 8.17 in [SG] shows that (1) holds. Part (3) follows from the universal property of ρ; cf. 33.7 and 33.8 in [FGT].

Let $x \in \hat{Q}$ with $x\rho$ of order 4 in Q. Then $x^2 \in Z = Z(\hat{H})$, so $(x^g)^2 = x^2$ for all $g \in \hat{H}$. But as H^* is irreducible on $\tilde{Q}$, $\tilde{Q} = \langle \tilde{x}^{H^*} \rangle$, so $\hat{Q} = \langle x^{\hat{H}}, \hat{Z} \rangle$ and then as $\Phi(Q) = \langle x^2\rho \rangle$, $\Phi(\hat{Q}) = \langle x^2 \rangle$ is of order 2. Therefore $\hat{Q} \cong Q \times E_{2^m}$ as Z is elementary abelian. Then as $\hat{Q} = P\hat{Z}$, $\Phi(\hat{Q}) = \Phi(P)$ and $P \cong Q \times E_{2^n}$.

As $H^* \leq O(\tilde{Q})$, $\tilde{Q}$ is self dual as an H^*-module. Therefore as $P = [P, \hat{H}]$ and $H^* \cong \hat{H}/\hat{Q} = \hat{H}/C_{\hat{H}}(P/\Phi(P))$ with $P/Z(P) \cong \tilde{Q}$ self dual as an H^*-module, $n \leq \dim_{\mathbf{F}_2}(H^1(H^*, \tilde{Q})) = k$. (cf. 17.12 in [FGT].) So (2) will be established once we show $n \geq k$.

Let $A = \mathrm{Aut}(\hat{H})$ and $D = C_A(P/Z(P))$. Then $[\hat{H}, D] \leq C_{\hat{H}}(P/Z(P)) = \hat{Q}$, so as $\hat{Q}/Z$ is of exponent 2, so is D. Suppose $d \in D - \hat{Q}/Z$ and let $\tilde{P} = \hat{P}/\Phi(P)$, and form the product $E = \tilde{P}\langle d \rangle$. As d centralizes $\hat{H}/\hat{Q}$ and $\hat{H}/P$ is perfect, d centralizes $\hat{H}/P$, so $\hat{H}$ acts on E. Claim E is abelian. If not, as $\tilde{P}$ is abelian, $C_{\tilde{P}}(d) = Z(E)$ is $\hat{H}$ invariant, so as H^* is irreducible on $\tilde{Q} = \tilde{P}/\tilde{Z}(P)$ and $\tilde{P} = [\tilde{P}, \hat{H}]$, either $Z(E) \leq \tilde{Z}(P)$ or $\tilde{P} = Z(E)$, with the latter impossible as E is nonabelian. So $C_{\tilde{P}}(d) \leq \tilde{Z}(P)$. Let $x \in P - Z(P)$, $U = \langle [x, d] \rangle$, and $\bar{E} = E/\tilde{U}$. Then $\bar{x} \in C_{\bar{P}}(d) - \bar{Z}(P)$, so the argument above shows $\bar{E}$ is abelian, and hence $\tilde{U} = [\tilde{P}, d]$. Therefore $|\tilde{P} : C_{\tilde{P}}(d)| = |\tilde{U}| = 2$, so as $C_{\tilde{P}}(d) \leq \tilde{Z}(P)$, $\tilde{Q}$ is of order 2, a contradiction.

We have shown that E is abelian and hence that D centralizes $P/\Phi(P)$. On the other hand $[C_A(P), \hat{H}] \leq C_{\hat{H}}(P) = Z$, so as $\hat{H}$ is perfect, $C_A(P) = 1$. Thus D is faithful on P. But $P = P_0Z(P)$ with $P_0 \cong Q$ and as D centralizes $P/\Phi(P)$, D centralizes $P_0/\Phi(P_0)$. Hence as $\mathrm{Inn}(P_0) = C_{\mathrm{Aut}(P_0)}(P_0/\Phi(P_0))$, $D/\mathrm{Inn}(P)$ is faithful on $Z(P)$. That is $D/\mathrm{Inn}(P)$ acts faithfully as a group of transvections on $Z(P)$ with center $\Phi(P)$. So to complete the proof of (2) and (4), it remains to show $m(D/\mathrm{Inn}(P)) \geq k$.

Let W be the largest $\mathbf{F}_2H^*$-module with $C_W(H^*) = 0$ and $V = [W, H^*] \cong \tilde{Q}$.(cf. section 17 in [FGT].) Let $x \mapsto \dot{x}$ be an H^*-isomorphism of $\tilde{Q}$ with V. The representation of H^* on W induces a representation $\pi : \tilde{H} \to GL(W)$ of $\tilde{H}$ on W. Form the semidirect product $G = \tilde{H}W$ of W by $\tilde{H}$ with respect to the representation π and let $V_0 = \{x\dot{x} : x \in \tilde{Q}\} \leq G$. As $\tilde{Q}$ centralizes W, V_0 is a normal subgroup of G and in G/V_0, $x \in \tilde{Q}$ is identified with $\dot{x}$, so G/V_0 has normal subgroups $\tilde{H}V_0/V_0 \cong \tilde{H}$ and $WV_0/V_0 \cong W$ with $(\tilde{H}V_0/V_0) \cap (WV_0/V_0) = \tilde{Q}V_0/V_0 \cong \tilde{Q}$. Hence

W induces a faithful group of automorphism on $\tilde{H}$ centralizing $\tilde{Q}$ and by part (3), W factors through D, so $m(D/\mathrm{Inn}(P)) \geq m(W/V) = k$, completing the proof of (2) and (4).

Notice that (4) implies (5). Finally (5) and the argument in the penultimate paragraph of the proof of 8.17 in [SG] establishes (6).

Q.E.D.

(3.3) *Let $H \in \mathcal{H}$ be perfect with* $\mathrm{Schur}_2(H^*) = 1$. *Then each $H_0 \in \mathcal{H}$ with $F^*(H_0) \cong F^*(H)$ and $H_0/Z(F^*(H_0)) \cong H/Z(F^*(H))$ is isomorphic to H.*

Proof. Adopt the notation of 3.2. As $\mathrm{Schur}_2(H^*) = 1$, $P = \hat{Q}$ by 3.2.1. Then by 3.2.6, $H \cong \hat{H}/U \cong H_0$ for some fixed complement U to $\Phi(P)$ in $Z(P)$. Q.E.D.

§4. Large extraspecial 2-subgroups

In this section we assume the following hypotheses:

Hypothesis 4.1. G is a finite group, z is an involution in G, $H = C_G(z)$, and $Q = F^*(H)$ is an extraspecial 2-group.

In addition we adopt the following notational conventions: Let $\tilde{H} = H/\langle z\rangle$ and $H^* = H/Q$. From section 8 in [SG], $\tilde{Q}$ has the structure of an orthogonal space over $\mathbf{F}_2$ when we identify $\mathbf{F}_2$ with $\{1, z\}$ and take $q(\tilde{u}) = u^2$ and $(\tilde{u}, \tilde{v}) = [u, v]$ for $u, v \in Q$. Of course H^* is embedded into $O(\tilde{Q})$ via its action by conjugation.

The *width* of an extraspecial 2-group Q is the integer w such that $|Q| = 2^{2w+1}$.

Example 4.2. Let w be a positive integer and L a finite group. A pair (G, z) satisfies Hypothesis $\mathcal{H}(w, L)$ if (G, z) satisfies Hypothesis 4.1 with Q of width w, $H^* \cong L$, and z not weakly closed in Q with respect to G. In [SG] the Monster and Baby Monster are constructed as groups satisfying Hypotheses $\mathcal{H}(12, Co_1)$ and $\mathcal{H}(11, Co_2)$, respectively.

(4.3) *Assume no element of H induces a transvection on $\tilde{Q}$, and let x be an involution in Q with $x \notin z^G$ and $T \in Syl_2(C_H(x))$. Then*

(1) *$\langle x, z\rangle = Z(T) = C_G(C_Q(x))$, z is weakly closed in $Z(T)$ with respect to G, and $T \in Syl_2(C_G(x))$.*

(2) $x^G \cap Q = x^H$.

Proof. Let $X = \langle z, x\rangle$. Then $Z(T)^* \leq C_H(C_Q(x))^* = Y^*$, and Y^* centralizes the hyperplane $\widetilde{C_Q(x)})$ of $\tilde{Q}$, so as no element of H induces a transvection on $\tilde{Q}$, $Y \leq Q$. Then as $X = Z(C_Q(x))$, $X = Y = Z(T)$. As $xz \in x^Q$, z is weakly closed in X with respect to G. Hence $T \in Syl_2(C_G(x))$, establishing (1).

Let $x^g \in Q$ and $S \in Syl_2(C_H(x^g))$. Then by (1), $T, S^{g^{-1}}$ are Sylow in $C_G(x)$, so there is $c \in C_G(x)$ with $T^c = S^{g^{-1}}$. Then $z^{cg} = z$ as z is weakly closed in $Z(S)$, so $h = cg \in H$ with $Z(T)^h = Z(S)$, and hence replacing h by kh with $k \in Q - C_Q(x)$ if necessary, $x^h = x^g$, establishing (2). Q.E.D.

In the remainder of this section we assume the following hypothesis:

Hypothesis 4.4. Hypothesis 4.1 holds with z not weakly closed in Q with respect to G. In addition $T \in Syl_2(H)$ and $J(T^*) \cong E_{2^{w-1}}$, where $w > 2$ is the width of Q.

We adopt the following notational conventions: Let $g \in G - H$ with $s = z^g \in Q$, $E = Q \cap Q^g$, and $R = (Q^g \cap H)(Q \cap H^g) \leq T$.

Remark. Note that by Hypothesis 4.1, hypotheses (L1)-(L3) of section 8 of [SG] are satisfied by Q. Further as $w \geq 2$ and z is not weakly closed in Q with respect to G, the hypotheses of 8.7.3 in [SG] are satisfied, so by that result, Q is a large extraspecial subgroup of G, as defined in section 8 of [SG]. In particular we can appeal to the lemmas in that section.

(4.5) (1) *$E \cong E_{2^{w+1}}$.*
(2) *$C_{H^*}(\tilde{s}) = N_{H^*}(R^*)$.*
(3) *$R^* = J(T^*)$.*
(4) *Let $X_2 = \langle Q, Q^g\rangle$ and $V = \langle z, s\rangle$. Then $P_2 = N_G(V) = X_2 C_H(V)$ with $R = C_{X_2}(V)$, $P_2/R = X_2/R \times C_G(V)/R$, $X_2/R \cong S_3$, and $C_G(V)/R \cong N_{H^*}(R^*)/R^*$.*
(5) *$E/V \leq Z_2(R)$ is centralized by X_2 and is isomorphic to the dual of R^* as a module for $C_G(V)/R$.*
(6) *$R/E \cong E_{2^{2w-2}}$ is the tensor product of the natural module for X_2/R and the module R^* for $C_G(V)/R$. In particular $C_Q(s)/E$ is isomorphic to R^* as a $C_H(V)$-module.*
(7) *R^* induces the full group of transvections with center $\tilde{s}$ on $\tilde{E}$ and the full group of transvections with axis $C_Q(s)/E$ on Q/E.*

(8) *If $N_{H^*}(R^*)$ is irreducible on R^* then $N_{H^*}(E) = C_{H^*}(\tilde{s})$ and H^* is absolutely irreducible on $\tilde{Q}$.*

Proof. By 8.15 in [SG], $m_2(E) = m + 1$ with $m \leq w$ and R^* is elementary abelian of rank $2w - m - 1$. Let $R \leq T \in Syl_2(H)$. By Hypothesis 4.4, $J(T^*) \cong E_{2^{w-1}}$, so $2w - m - 1 = m(R^*) \leq m(T^*) = w - 1$, and hence $w \leq m$. We conclude $m = w$ and $R^* = J(T^*)$. In particular (1) and (3) hold.

Next by (1) and 8.15 in [SG], (4) and (5) hold, and R/E is the tensor product of the natural module for $X_2/R \cong L_2(2)$ with the $C_G(V)/R$-module isomorphic to R^*, E/V is dual to R^* as a $C_G(V)/R$-module, and R^* induces the full group of transvections on $\tilde{E}$ with center $\tilde{s}$. Then as Q/E is dual to $\tilde{E}$ as a $N_{H^*}(E)$-module, R^* induces the full group of transvections with axis $C_Q(s)/E$ on Q/E, establishing (7).

For $e \in E$, $[RQ, \tilde{e}] \leq \langle \tilde{s} \rangle$ and for $q \in C_Q(s) - E$, $[RQ, \tilde{q}] \leq \tilde{E}$. Finally for $u \in Q - C_Q(s)$, $C_Q(s) \leq [RQ, u]E$, so $qe \in [RQ, u]$ for some $e \in E$. Then $[RQ, qe] \leq [RQ, u]$ and as RQ centralizes E/V, $m([RQ, \tilde{q}\tilde{e}]) \geq m([RQ, \tilde{q}]) - 1$, so

$$m([RQ, \tilde{u}]) \geq w - 1 + m([RQ, \tilde{q}]) - 1 > m([RQ, \tilde{q}]).$$

Therefore $m([RQ, \tilde{u}]) \geq m([RQ, \tilde{y}])$ for all $y \in R \cap Q$, so $R \cap Q \trianglelefteq N_H(RQ)$. Hence $V = Z(R \cap Q) \trianglelefteq N_H(RQ)$, so $N_H(RQ) = QC_H(s)$. This completes the proof of (2).

Finally assume $N_{H^*}(R^*)$ is irreducible on R^*. Then by (4)-(7), $C_H(V)/R \cong N_{H^*}(R^*)/R^*$ has chief series

$$0 < \tilde{V} < \tilde{E} < C_Q(t)/\langle z \rangle < Q$$

and the stabilizers in H^* of each of the nontrivial members of this series, other than $\tilde{E}$, also stabilizes V. Further as $F^*(H) = Q$ and $1 \neq R^* \trianglelefteq N_{H^*}(R^*) = C_{H^*}(\tilde{V})$, $C_{H^*}(\tilde{V})$ is proper in H^*, so either H^* is irreducible on $\tilde{Q}$ or $C_{H^*}(\tilde{s}) < N_{H^*}(E)$. Indeed in the former case as $\tilde{V}$ is of order 2 and $C_{GL(\tilde{Q})}(H^*)$-invariant, the representation is even absolutely irreducible.

So we may assume $C_{H^*}(\tilde{s}) < N_{H^*}(E)$, and it remains to derive a contradiction. Then $N_{H^*}(E)$ is irreducible on $\tilde{E}$, so by 1.2, $N_{H^*}(E)$ induces $GL(\tilde{E})$ on $\tilde{E}$. Further as R^* is faithful on $\tilde{E}$ and normal in $N_{H^*}(V) = C_{H^*}(\tilde{s})$ and $R^* = J(T^*)$, $N_{H^*}(E)$ is faithful on $\tilde{E}$. Then as $E_{2^{w-1}} \cong R^* = J(T^*)$ while $N_{H^*}(E) \cong GL(\tilde{E}) \cong GL_w(2)$, it follows that $w \leq 2$, contrary to Hypothesis 4.4. Namely $m_2(GL_w(2)) > w - 1$ for $w > 3$ and $J(T^*) = T^* \cong D_8$ when $N_{H^*}(E) \cong GL_3(2)$. Q.E.D.

(4.6) *If $C_{R^*}(N_{H^*}(R^*) = 1$ then*
(1) $\langle\tilde{s}\rangle = C_{\tilde{Q}}(N_{H^*}(R^*))$, *and*
(2) $z^G \cap Q = \{z\} \cup s^H$.

Proof. By 4.5.2 and 4.5.6, $C_Q(s)/E$ is isomorphic to R^* as a $N_{H^*}(R^*)$-module, while by hypothesis, $C_{R^*}(N_{H^*}(R^*)) = 1$, so $N_{H^*}(R^*)$ has no fixed points on $C_Q(s)/E$. Hence (1) follows from 4.5.2 and 4.5.7.

Let $y \in G - H$ and $t = z^y \in Q$. By (1), 4.5.3, and symmetry between s and t, $\langle\tilde{t}\rangle = C_{\tilde{Q}}(N_{H^*}(J(S^*)))$ for some $S^* \in Syl_2(H^*)$. Then by Sylow's Theorem, $J(S^*)$ is H^*-conjugate to $J(T^*)$, so t is H-conjugate to s. Q.E.D.

(4.7) *Assume $R^* = C_{H^*}(R^*)$. Then*

(1) *No element of H^* induces a transvection on $\tilde{Q}$.*

(2) *If in addition $C_{R^*}(N_{H^*}(R^*)) = 1$, then $x^G \cap Q = x^H$ for each involution $x \in Q$ with $x \notin \{z\} \cup s^H$.*

Proof. Part (2) follows from (1), 4.3, and 4.6. If $h^* \in H^*$ induces a transvection on $\tilde{Q}$ then h^* is an involution, to we may take $h \in T$. By 4.5.5, E/V is dual to $R^* \cong C_Q(s)/E$ as a T^*-module and $C_Q(s)/E$ is isomorphic to R^* by 4.5.6, so if $[R^*, h^*] \neq 1$ then $m([\tilde{Q}, h^*]) \geq 2m([R^*, h^*]) > 1$, a contradiction. Hence $h^* \in C_{H^*}(R^*) = R^*$. Then by 4.5.7, $m([\tilde{Q}, h^*]) > 1$. Q.E.D.

(4.8) *Assume H^* is irreducible on $\tilde{Q}$. Then*

(1) *The regular orbits of R^* on $\tilde{Q}/\langle\tilde{s}\rangle$ are those in $\tilde{Q}/\langle\tilde{s}\rangle - \widetilde{\overline{C_Q(s)}}/\langle\tilde{s}\rangle$.*

(2) *If (G_1, z_1) satisfies Hypothesis $\mathcal{H}(w, H^*)$ and $C_{R^*}(N_{H^*}(R^*)) = 1$ then $\tilde{H}_1 \cong \tilde{H}$.*

Proof. Let $V = \langle s, z\rangle$ and $\bar{Q} = Q/V$. By 4.5.7, R^* induces the group of transvections with axis $C_Q(s)/E$ on Q/E, so all orbits of R^* on $\bar{Q} - \overline{C_Q(s)}$ are regular. Hence to prove (1) it suffices to show $C_{R^*}(\bar{u}) \neq 1$ for each $u \in C_Q(s)$. If $u \in E$ this follows from 4.5.7, so assume $u \in C_Q(s) - E$ with $C_{R^*}(\bar{u}) = 1$. Then $m([R^*, \bar{u}]) = m(R^*) = w - 1 = m(\bar{E})$, while by 4.5.7, $[R, u] \leq E$, so $[R^*, \bar{u}] = \bar{E}$. By symmetry between z and s, we may assume there is $v \in Q^g \cap H - E$ with $[v, Q \cap H^g]V = E$. But as v^* induces an involutory automorphism on $\tilde{Q}$, $[\tilde{Q}, v^*] \leq C_{\tilde{Q}}(v^*)$, so v^* centralizes $\tilde{E}$, contrary to 4.5.7. This completes the proof of (1).

Let $K^* = N_{H^*}(R^*)$ and Ω the graph on H^*/K^* with K^* adjacent to K^*h^* if $K^*h^*R^*$ is not a regular orbit for R^*. Let β be the bilinear form on $\tilde{Q}$. By (1), $\beta(\tilde{s}, \tilde{s}^h) = 0$ if and only if $K^*h^* \in \Omega(K^*)$.

Assume the hypotheses of (2) and let γ be the bilinear form on $\tilde{Q}_1$. Then there is an isomorphism $H^* \cong H_1^*$ which induces a representation of H^* on $\tilde{Q}_1$. By 4.5.2, $K^* = C_{H^*}(\tilde{s}_1)$ for some $s_1 = z_1^{g_1} \in Q_1$ and by (1) applied to G_1, $\gamma(\tilde{s}_1, \tilde{s}_1^h) = 0$ if and only if $K^*h^* \in \Omega(K^*)$. Therefore by 2.4, the representations of H^* on $\tilde{Q}$ and $\tilde{Q}_1$ are equivalent and $\tilde{Q}$ is isometric to $\tilde{Q}_1$. As $\tilde{Q}$ and $\tilde{Q}_1$ are isometric, $Q \cong Q_1$. As H^* is irreducible on $\tilde{Q}$ and $C_{\tilde{Q}}(K^*) = \langle \tilde{s} \rangle$ is 1-dimensional by 4.6.1, $\tilde{Q}$ is an absolutely irreducible $\mathbf{F}_2H^*$-module. Hence by 3.1, $\tilde{H} \cong \tilde{H}_1$. Q.E.D.

(4.9) *Assume $N_{H^*}(R^*)$ is irreducible on R^* and (G_1, z_1) satisfies Hypothesis $\mathcal{H}(w, H^*)$. Then $\tilde{H}_1 \cong \tilde{H}$.*

Proof. As $N_{H^*}(R^*)$ is irreducible on R^*, H^* is irreducible on $\tilde{Q}$ by 4.5.8, and $C_{R^*}(N_{H^*}(R^*)) = 1$. Hence the lemma follows from 4.8.2. Q.E.D.

§5. $Sp_6(2)$ and $U_6(2)$

(5.1) *Let V be a $2m$-dimensional symplectic space over a perfect field F of characteristic 2 and $G = Sp(V)$. The the conjugacy classes of involutions of G are a_k, b_k, and c_k, $1 \le k \le m$, where for $d = a, b, c$ and $t \in d_k$, $m([V,t]) = k$, k is odd if and only if $d = b$, and $V(t) = \{v \in V : (v, v^t) = 0\} = V$ if $d = a$, while $V(t)$ is a hyperplane of V if $d = b$ or c.*

Proof. This is contained in section 7 of [ASe], but we repeat the proof here for completeness. Let t be an involution in G. For $u, v \in V$, $(v, u^t) = (u, v^t)$, so the map $v \mapsto (v, v^t)$ is a linear map from V into F with kernel $V(t)$. In particular $\dim(V/V(t)) \le 1$.

Suppose $V = V(t)$. Pick $y_1 \in V - C_V(t)$, $x_1 \in (y_1^t)^\perp - y_1^\perp$, and let $V_1 = \langle y_1, y_1^t, x_1, x_1^t \rangle$. Multiplying x_1 by a suitable scalar, we may take $(y_1, x_1) = 1$. Then $\{y_1, x_1, y_1^t, x_1^t\}$ is a hyperbolic basis for V_1. (cf. section 19 in [FGT]) In particular V_1 is nondegenerate so $V = V_1 \oplus V_1^\perp$, and proceeding by induction on m,

$$V = V_1 \perp \cdots V_r \perp W$$

where $W \le C_V(t)$ and V_i has a hyperbolic basis $\{y_i, x_i, y_i^t, x_i^t\}$. Notice $[V,t]$ has basis $\{y_i + y_i^t, x_i + x_i^t : 1 \le i \le r\}$, so $\dim([V,t]) = 2r$ and G is transitive on the set a_{2r} of involutions t with $V = V(t)$ and $\dim([V,t]) = 2r$ by Witt's Lemma.

So assume $V \ne V(t)$. Then $V(t)$ is a hyperplane of V, so $V(t) = V_0^\perp$ for the point $V_0 = V(t)^\perp$. Pick $u \in V - V(t)$, $a \in F$ with $a^2 = (u, u^t)^{-1}$, and let $x_1 = au$. Then $\{x_1, x_1^t\}$ is a hyperbolic basis for $V_1 = \langle x_1, x_1^t\rangle$ and $V = V_1 \oplus V_1^{\perp}$. Continuing in this fashion we write

$$V = V_1 \perp \cdots \perp V_s \perp W$$

where V_i has hyperbolic basis $\{x_i, x_i^t\}$ and $W \leq V(t)$. Then $V_0 = \langle v_0\rangle$, where $v_0 = \sum_{i=1}^{s} x_i + x_i^t$. If s is odd let $x = \sum_{i=1}^{s} x_i$ and observe $\{x, x^t\}$ is a hyperbolic basis for $U = \langle x, x^t\rangle$ with $U^{\perp} = V(t) \cap x^{\perp} \leq V(t)$, so by the a_{2r} case, the restriction of t to $U^{\perp}$ is of type a_{2r} and G is transitive on the set b_{2r+1} of involutions t with $m([V, t]) = 2r + 1$.

Finally if s is even let $x = x_s$ and $y = \sum_{i<s} x_i$. Then $\{x, x^t, y, y^t\}$ is a hyperbolic basis for $U = \langle x, x^t, y, y^t\rangle$ with $V_0 \leq U$, so again $U^{\perp} \leq V(t)$ and by the a_{2r} case, G is transitive on the set c_{2r} of involutions with $V \neq V(t)$ and $m([V, t]) = 2r$. Q.E.D.

As an immediate corollary to 5.1 we have:

(5.2) *$Sp_6(2)$ has four classes b_1, a_2, c_2, and b_3 of involutions.*

(5.3) *Let $G = Sp_6(2)$. Then* $\mathrm{Schur}_2(G) \cong \mathbf{Z}_2$ *and involutions of type b_1 and c_2 in G lift to elements of order* 4 *in* $\mathrm{Cov}_2(G)$.

Proof. The centralizer of an involution in Co_3 is a covering of $Sp_6(2)$ over $\mathbf{Z}_2$, so it remains to show $|\mathrm{Schur}_2(G)| \leq 2$ and to establish the statement about lifts of involutions. Let b be a transvection in G, $H = C_G(b)$, and $A = O_2(H)$. Then b is of type b_1 and A is the core of the permutation module for the Levi factor $L \cong S_6$ for H, with each coset of $\langle b\rangle$ in A containing one involution of type a_2 and one of type c_2.

Let $\hat{G}$ be a covering of G over a center $Z = \langle z\rangle$ of order 2 and for $B \leq G$ write $\hat{B}$ for the preimage of B in $\hat{G}$. From the representation of L on A, either $\Phi(\hat{A}) = 1$ or $\hat{A} \cong \mathbf{Z}_4 * 2^{1+4}$. Assume the former. Then as $H^1(L, A/\langle b\rangle) \cong \mathbf{Z}_2$, $\hat{A}$ splits over Z. Further all involutions in L are of type b_1, a_2, or c_2, and hence lift to involutions as $\Phi(\hat{A}) = 1$. Therefore $\hat{L} = Z \times \hat{L}_0$ and then $\hat{H} = \hat{L}_0[\hat{A}, \hat{L}_0] \times Z$ splits over Z. But then as H contains a Sylow 2-subgroup of G, $\hat{G}$ splits over Z, a contradiction.

So $\hat{A} = \mathbf{Z}_4 * 2^{1+4}$ and in particular $\langle \hat{b}\rangle = \langle \beta\rangle$ so that involutions of type b_1 lift to element of order 4. Next G has a parabolic P with $P/O_2(P) \cong L_3(2)$ and possessing a P-submodule R of $O_2(P)$ which is the natural module for $P/O_2(P)$ with each involution in R of type a_2. As P is transitive on $R^{\#}$, $\Phi(\hat{R}) = 1$, so elements of type a_2 lift to involutions. Thus if $\sigma \in \hat{A}$ is the lift of an involution of type a_2 then σ is an involution, so the lift $z\sigma$ of an involution of type c_2 is of order 4.

Now let $\tilde{G} = \mathrm{Cov}_2(G)$. Then $\hat{G} = \tilde{G}/U$ for some hyperplane U of $V = Z(\tilde{G})$. Further if $\alpha \in \tilde{G}$ with α of type b_1 then $\alpha^2 \in V - U$. But if $U \neq 1$ there is a hyperplane W of V with $\alpha^2 \in W$, so that $\tilde{G}/W$ is a covering of G over $\mathbf{Z}_2$ in which transvections lift to involutions, a contradiction. Q.E.D.

(5.4) *Up to isomorphism the spin module for $Sp_6(2)$ is the unique 8-dimensional irreducible $\mathbf{F}_2Sp_6(2)$-module.*

Proof. Let $G = Sp_6(2)$ and $0 \neq M$ an irreducible $\mathbf{F}_2G$-module. As $\mathbf{F}_2$ is a splitting field for G, $M = M(\lambda)$ for some restricted dominant weight $\lambda \neq 0$. Next the Weyl group W for G is of type C_3, so the orbit λW of λ under W is of length $|W : W_\lambda|$ where W_λ is the parabolic stabilizing λ, so either $|\lambda W| > 8$ or $\lambda = \lambda_1$ or λ_3 and $|\lambda W| = 6$ or 8, respectively, where λ_i is the ith fundamental dominant weight. As $M(\lambda_1)$ is the natural module of dimension 6 and $M(\lambda_3)$ the spin module of dimension 8, the lemma follows. Q.E.D.

(5.5) *Let $G \cong U_6(2)$ and V an absolutely irreducible 20-dimensional $\mathbf{F}_2G$-module such that $G_v \cong L_3(4)/E_{2^9}$ for some $v \in V$. Let $M = V \otimes_{\mathbf{F}_2} \mathbf{F}_4$ regarded as a $\mathbf{F}_4G$-module. Then $M = \bigwedge^3(N)$, where N is the natural module of dimension 6 for the covering $\hat{G} \cong SU_6(2)$ of G. In particular the $\mathbf{F}_2G$-module V is determined up to equivalence.*

Proof. As V is an absolutely irreducible $\mathbf{F}_2G$-module of dimension 20, M is an irreducible $\mathbf{F}_4G$-module of dimension 20. Next $\hat{G} \leq S \leq GL(M)$ with $S \cong SL_6(4)$ and if σ is the graph-field automorphism of S with $C_S(\sigma) = \hat{G}$ then σ acts on M too. As v is fixed by the maximal parabolic G_v of G, v is a high weight vector for M as an $\mathbf{F}_4S$-module, so $\mathbf{F}_4v$ is stabilized by a parabolic P of S containing $\hat{G}_v$ and invariant under σ. It follows that P is the parabolic of S corresponding to the middle node of the Dynkin diagram of S. Thus if λ is the high weight vector of M and W is the Weyl group of S then W_λ is the parabolic of W corresponding to the middle node, so $W_\lambda \cong S_3 \times S_3$ and λW is of length $|W : W_\lambda| = 20$. Hence as $20 = \dim_{\mathbf{F}_4}(M)$, λ is the unique dominant weight of M, so $\lambda = \lambda_3$ is the third fundamental dominant weight for S and $M = M(\lambda_3)$ is the corresponding high weight module. Hence $M = \bigwedge^3(N)$. Q.E.D.

In the next three lemmas in this section let $G \cong U_6(2)$, V, M, S, and N be as in Lemma 5.5. We discover in section 7 that a module satisfying the hypothesis of V admits the structure of an orthogonal space over $\mathbf{F}_2$ preserved by G, so as V is determined up to equivalence, V has that structure and $G \leq O(V)$.

(5.6) *Let $G_0 = G_1 \times G_2$ be the stabilizer in G of a nondegenerate 2-dimensional subspace of the natural module N for $\hat{G}$, with $G_1 \cong U_2(2)$ and $G_2 \cong U_4(2)$. Then as an orthogonal space over $\mathbf{F}_2$, $V = (V_1 \oplus V_2) \perp V_3$, where V_1 and V_2 are copies of the $O_6^-(2)$-module for G_2, $V_1 = [V, j]$ for some involution $j \in G_1$, and V_3 is isomorphic to the $U_4(2)$-module for G_2.*

Proof. Let G_0 be the stabilizer of a nondegenerate 2-subspace N_0 of N. Pick an orthonormal basis $\{x_1, \ldots, x_6\}$ for N with $x_1, x_2 \in N_0$. By 5.5 we may regard M as $\bigwedge^3(N)$. Let M_3 be the subspace of M spanned by $m_i = x_1 \wedge x_2 \wedge x_i$, $3 \leq i \leq 6$. Then G_1 centralizes M_3 and the map $m_i \mapsto x_i$ induces an isomorphism of M_3 with $N_0^\perp$ as an $\mathbf{F}_4 G_2$-module, so M_3 is the natural module for $G_2 \cong U_4(2)$.

Next we can choose j to interchange x_1 and x_2, so $[M, j] = M_1$ is spanned by $m_{r,s} = (x_1 + x_2) \wedge x_r \wedge x_s$, $3 \leq r < s \leq 6$, and the map $m_{r,s} \mapsto x_r \wedge x_s$ is an isomorphism of M_1 with $\bigwedge^2(N_0^\perp)$ as an $\mathbf{F}_4 G_2$-module. Therefore as $\bigwedge^2(N_0^\perp)$ is the $O_6^-(2)$-module for G_2 tensored up to $\mathbf{F}_4$, M_1 is that module. Similarly $G_1 = \langle j, i \rangle$ for i a conjugate of j and $M_2 = [M, i]$ is isomorphic to M_1 as an $\mathbf{F}_4 G_2$-module and $M = M_1 \oplus M_2 \oplus M_3$. Recall $G = C_S(\sigma)$ with σ acting on M_i, so $M_i = V_i \otimes_{\mathbf{F}_2} \mathbf{F}_4$ for some $\mathbf{F}_2 G_0$-submodule V_i of V satisfying the conclusions of this lemma. Q.E.D.

(5.7) *Let z be a long root element of G, $L \cong U_4(2)$ a Levi factor of $C_G(z)$, and W a $\mathbf{F}_2 G$-module with $C_W(G) = 0$ and $[W, G] = V$. Then $W = W_1 \oplus W_2 \oplus W_3$ as a $\mathbf{F}_2 L$-module, with $W_i \leq V$ of dimension 6 for $i = 1, 2$, $V_3 = V \cap W$ of dimension 8, and $C_W(L) = 0$.*

Proof. First $K = C_G(L) \cong S_3$ with KL the stabilizer in G of a nondegenerate 2-subspace N_0 of N. Thus by the previous lemma, $V = V_1 \oplus V_2 \oplus V_3$ with $V_1 + V_2 = [V, K]$, $\dim(V_1) = \dim(V_2) = 6$, and $V_3 = C_V(K)$ of dimension 8. Let Y be of order 3 in K. Then $V_1 + V_2 = [W, K]$. Let $W_3 = C_V(Y)$. Then $V_3 = V \cap W_3$ and it remains to show $C_W(L) = 0$. Assume not and let U be a point in $C_W(L)$. Replacing W by $V + U$ we may assume V is a hyperplane of W. Now $C_W(L) = C_{W_3}(L) = C_W(LK) = U$.

Let $E_{27} \cong E \leq L$ and $A = EY$. Then $A = J(T)$ for $T \in Syl_3(G)$ and $N_G(A)/A \cong S_6$. As V_3 is the $U_4(2)$-module for L, $V_3 = [V_3, E]$, so as $V_1 + V_2 = [W,Y]$, $V = [W,A]$ and $U = C_W(A)$. Therefore $X = \langle N_G(A), LK\rangle$ centralizes U, so to derive a contradiction, it remains to prove $X = G$.

Now X is a group generated by the class $D = z^X$ of 3-transpositions. Further as $C_G(z)$ is a maximal parabolic of G with L irreducible on $O_2(C_G(z))/\langle z\rangle$, $C_X(z) = \langle z\rangle \times L$. By Exercise 3.3 in [3T], $O_3(X) \leq Z(X) \geq O_2(X)$. Let $B = N_G(A)$; then $B = \langle C_B(z), C_B(d)\rangle$ for $d \in z^B - K$, so $X = \langle L, B\rangle = \langle C_X(z), C_B(d)\rangle$, and hence the commuting graph on D is connected. Therefore by 9.4.4 in [3T], X is primitive on D. Then by Theorem 9.5.4, X is rank 3 on D, and hence $C_X(z)$ is maximal in X, contradicting $C_X(z) < KL$. This completes the proof of the lemma. Q.E.D.

(5.8) (1) $\dim_{\mathbf{F}_2} H^1(G,V) = 2$.

(2) *Let $L \cong U_4(2)$ and U the natural module for L regarded as an 8-dimensional $\mathbf{F}_2$-module. Then* $\dim_{\mathbf{F}_2} H^1(L,U) = 2$.

(3) *Let D be the largest $\mathbf{F}_2G$-module such that $D = [D,G]$ and $D/C_D(G) = V$, G_v a $L_3(4)/E_{2^9}$ parabolic of G, and $E/C_D(G)$ the 10-dimensional G_v-submodule of V. Then $C_D(G) \leq [E,G_v]$.*

Proof. By 5.7, $\dim_{\mathbf{F}_2} H^1(G,V) \leq \dim_{\mathbf{F}_2} H^1(L,U)$. Further we find in a later paper in this series that $\dim_{\mathbf{F}_2} H^1(G,V) \geq 2$ and that (3) holds, so it remains to show $\dim_{\mathbf{F}_2} H^1(L,U) \leq 2$. Let W be the largest $\mathbf{F}_2L$-module with $[W,L] = U$ and $C_W(L) = 0$. (cf. 17.11 of [FGT]) As U is a $\mathbf{F}_4L$-module, so is W by the universal property of W, and it remains to show $\dim_{\mathbf{F}_4}(W/U) \leq 1$. Let $S \in Syl_3(L)$. Then $A = J(S) \cong E_{27}$ and $Z = Z(S)$ is of order 3 with $O_3(C_L(Z)) = P \cong 3^{1+2}$ and $C_G(Z)/P \cong SL_2(3)$. Now $U = [U,A]$ so $W = U \oplus C_W(A)$ and $N_L(A)$ centralizes $C_W(A)$. On the other hand $C_U(Z)$ is a point centralized by $O^3(C_L(Z))$, so the involution t inverting P/Z acts on S and hence centralizes $C_W(A)$ and then also $C_W(Z) = C_U(Z) + C_W(A)$. Then if x is of order 4 in $C_L(Z)$ with $x^2 = t$, x induces a $\mathbf{F}_4$-transvection on $C_W(Z)$ with center $C_U(Z)$, so if $\dim_{\mathbf{F}_4}(W/U) > 1$, then the hyperplanes $C_W(Z\langle x\rangle)$ and $C_W(A)$ of $C_W(Z)$ intersect nontrivial, so $C_W(X) \neq 0$, where $X = \langle N_L(A), x\rangle$. Finally as $N_L(A)$ is a maximal parabolic of $L \cong PSp_4(3)$ and $x \notin N_L(A)$, $X = L$, contradicting $C_W(L) = 0$.

Q.E.D.

(5.9) *Let V be a 6-dimensional unitary space over $\mathbf{F}_4$ and Δ the graph on the totally singular 3-subspaces of V with distinct $x, y \in \Delta$ adjacent if $x \cap y \neq 0$. Then $Aut(\Delta) = P\Gamma(V) \cong Aut(U_6(2))$ is the group of projective semilinear unitary maps on V.*

Proof. Let $G = P\Gamma(V)$ and $A = Aut(\Delta)$, so that $G \leq A$. For $x \in \Delta$, $G_x = LR$, where $R \cong E_{2^9}$ is the radical of G_x and L is a Levi factor isomorphic to $PGL_3(4)$ extended by a field automorphism. Further $\Delta(x) = \Delta_1(x) \cup \Delta_2(x)$ where

$$\Delta_i(x) = \{y \in \Delta : \dim(x \cap y) = i\}$$

with $|\Delta_1(x)| = 336$ and $|\Delta_2(x)| = 42$. Also $\Delta - x^{\perp} = \Gamma(x)$ is of order 512 with R regular on $\Gamma(x)$ and $L = G_{x,z}$ for suitable $z \in \Gamma(x)$.

For $y \in \Delta(x)$, let

$$\theta(y) = \{u \in \Delta(x) : x \cap y = x \cap u\}$$

and let $\theta = \{\theta(y) : y \in \Delta(x)\}$ and $\theta_i = \{\theta(y) : y \in \Delta_i(x)\}$. Notice $u \in \Delta(x,z)$ if and only if $u = (u \cap x) + (u \cap z)$ with $u \cap z = (u \cap x)^{\perp} \cap z$, so $|\Delta(x,z) \cap T| = 1$ for each $T \in \theta$. Thus if $m_i = |\Delta(y) \cap \Gamma(x)|$ for $y \in \Delta_i(x)$, then

$$m_i \cdot |\Delta_i(x)| = 512 \cdot 21$$

so $m_1 = 2^5$ and $m_2 = 2^8$. Therefore A_x acts on $\Delta_i(x)$ for $i = 1, 2$. Also for $y \in \Delta_2(x)$, $21 \cdot |\theta(y)| = |\Delta_2(x)| = 42$, so $\theta(y)$ is of order 2.

As R is regular on $\Gamma(x)$, $A_x = RA_{x,z}$. Now for $u \in \Delta_1(x,z)$ and $v \in \Delta_2(x,z)$, $u \in \Delta(v)$ if and only if $u \cap x \leq v \cap x$, so $\Delta(x,z)$ has the structure of the projective plane π on x, and that structure is preserved by $A_{x,z}$. Let B be the kernel of the action of $A_{x,z}$ on $\Delta(x,z)$. As $Aut(\pi) \cong L$ and L is faithful on $\Delta(x,z)$, $A_{x,z} = LB$. Further for $T \in \theta_2$, $|\Delta(x,z) \cap T| = 1$ and $|T| = 2$, so B fixes both points of T. Therefore B is trivial on $\Delta_2(x)$. However as L is irreducible on R, L is maximal in $G_x = LR$, so as R is regular on $\Gamma(x)$, G_x is primitive on $\Gamma(x)$, and hence for $z \neq w \in \Gamma(x)$, $\Delta_2(x,z) \neq \Delta_2(x,w)$. Therefore as B is trivial on $\Delta_2(x)$, B is also trivial on $\Gamma(x)$. Hence B fixes $\Delta(x,w) \cap T$ for each $T \in \theta_1$, so B is trivial on $\Delta_1(x)$, and therefore $B = 1$.

We have shown $A_{x,z} = LB = L$, so $A_x = RA_{x,z} = RL = G_x$. Then as G is transitive on Δ, $A = GA_x = G$, completing the proof. Q.E.D.

§6. Groups of type ${}^2E_6(2)$

Define a group G to be of *type* ${}^2E_6(2)$ if G possesses an involution z such that (G, z) satisfies Hypothesis $\mathcal{H}(10, U_6(2))$, in the language of Example 4.2. Throughout this short section, assume G is of type ${}^2E_6(2)$ and let z be an involution in G such that $H = C_G(z)$ and $Q = F^*(H)$ satisfy our hypotheses. Therefore Hypothesis 4.1 is satisfied, and indeed in a moment we see that Hypothesis 4.4 is also satisfied. Thus we adopt the notation of section 4, except that we write $t = z^g$ for our distinguished element of $z^G \cap Q - \{z\}$. In particular $H = C_G(z)$ satisfies $Q = F^*(H) \cong 2^{1+20}$, $H^* = H/Q \cong U_6(2)$, and z is not weakly closed in Q with respect to G. Recall also that $E = Q \cap Q^g$ and $R = (Q^g \cap H)(Q \cap H^g)$.

(6.1) (1) $E \cong E_{2^{11}}$.

(2) $N_{H^*}(E) = C_{H^*}(\tilde{t}) = N_{H^*}(R^*)$ *is the parabolic of* H^* *which is the split extension of* $R^* \cong E_{2^9}$ *by* $L_3(4)$ *with* R^* *the Todd module for* $L_3(4)$.

(3) $R^* = J(T^*)$ *for* $T \in Syl_2(H)$.

(4) *Let* $X_2 = \langle Q, Q^g \rangle$ *and* $V = \langle z, t \rangle$. *Then* $P_2 = N_G(V) = X_2 C_H(V)$ *with*

$$R = O_2(P_2) = C_{X_2}(V),$$

$P_2/R = X_2/R \times C_G(V)/R$, $X_2/R \cong S_3$, *and* $C_G(V)/R \cong L_3(4)$.

(5) $E/V = Z_2(R)$ *is centralized by* X_2 *and is the dual of the Todd module for* $C_G(V)/R$.

(6) $R/E \cong E_{2^{18}}$ *is the tensor product of the natural module for* X_2/R *and the Todd module for* $C_G(V)/R$.

(7) H^* *is absolutely irreducible on* $\tilde{Q}$.

Proof. Let $R \le T \in Syl_2(H)$. By 23.4 in [3T], $J(T^*) \cong E_{2^9}$, so Hypothesis 4.4 is satisfied. Indeed $N_{H^*}(J^*)$ is the parabolic of $H^* \cong U_6(2)$ which is the split extension of $J(T^*)$ by $L_3(4)$ with $J(T^*)$ the Todd module. Therefore the lemma follows from 4.5. Q.E.D.

(6.2) $\tilde{Q} \otimes_{\mathbf{F}_2} \mathbf{F}_4$ *is isomorphic as a* $\mathbf{F}_4 H^*$*-module to* $\bigwedge^3(N)$, *where* N *is the natural module of dimension* 6 *for the covering* $\hat{H}^* \cong SU_6(2)$ *of* H^*. *In particular the representation of* H^* *on* $\tilde{Q}$ *is determined up to equivalence.*

Proof. By 6.1.7, $\tilde{Q}$ is an absolutely irreducible $\mathbf{F}_2 H^*$-module of dimension 20, while by 6.1.2, $H^*_{\tilde{t}} \cong L_3(4)/E_{2^9}$. So as $H^* \cong U_6(2)$, the lemma follows from 5.5. Q.E.D.

§7. $^2E_6(2)$

In this section $G = {}^2E_6(2)$ and z is a long root involution in G. It is well known that:

(7.1) *The group* G *is of type* $^2E_6(2)$ *with* z 2*-central in* G.

Thus we adopt the notation of section 6. In particular $H = C_G(z)$, $Q = O_2(H)$, and $T \in Syl_2(H)$ with $R \leq T$. Let $\Delta = z^G$, and let $P_1 = H$, P_2, P_3, P_4 be the four maximal parabolics of G containing T ordered so that we have the diagram

For $J \subseteq \{1,2,3,4\}$ let L_J be the standard Levi factor in the parabolic $P_J = \bigcap_{j \in J} P_j$ and $R_J = O_2(P_J)$ the unipotent radical of P_J. In particular $R = R_2$. Let W be the Weyl group of G.

(7.2) *H has the following 5 orbits on Δ:*
(1) $\Delta^0(z) = \{z\}$.
(2) $\Delta^1(z) = Q \cap \Delta - \{z\}$.
(3) $\Delta^2_1(z) = \Delta \cap H - Q$.
(4) $\Delta^2_2(z) = \{d \in \Delta : [z,d] \in \Delta\}$.
(5) $\Delta^3(z) = \{d \in \Delta : |zd| = 3\}$.

Proof. We sketch the proof in section 12 of [ASe] for completeness. The subgroup $W_1 = W \cap P_1$ has 5 orbits on W/W_1 so $H = P_1$ has 5 orbits on $G/H \cong \Delta$; cf. Exercise 14.6.1 in [FGT]. Now $z = U_\alpha(1)$, where α is the highest root in the root system Φ determining T. There is a long root $\beta \neq \alpha$ with $t = U_\beta(1) \in Q$; then $t \in \Delta^1(z)$. Similarly there is a long root γ such that $U_\gamma(1) \in L_1$, long roots ϵ_i, $i = 1,2$ with $U_{\epsilon_i}(1) \in L_1$, and $h \in H$ with $t^h \in C_Q(t)$, so that

$$[t, t^h] = z \qquad \text{and } |U_{\epsilon_1}(1)U_{\epsilon_2}(1)| = 3$$

so $U_\gamma(1) \in \Delta^2_1(z)$ and $\Delta^2_2(z) \neq \varnothing \neq \Delta^3(z)$. Q.E.D.

(7.3) (1) *$L_1 \cong U_6(2)$ is a complement to Q in H.*

(2) *L_1 has 3 classes of involutions with representatives j_1, j_2, j_3, where j_i is the product of i transvections in $U_6(2)$. In particular j_1 is a long root involution of L_1 and j_2 is a short root involution.*

(3) *$A = J(T \cap L_1) \cong E_{2^9}$ is the unipotent radical of the parabolic $P_2 \cap L_1$ of L_1, $P_2 \cap L_1 = L_{1,2}A$ with $L_{12} \cong L_3(4)$, and A is the 9-dimensional Todd module for $L_{1,2}$.*

(4) *All involutions in L_1 are fused into A and if $a \in A \cap j_3^{L_1}$ then $C_{L_{1,2}}(a) \cong U_3(2)$.*

Proof. As L_1 is the standard Levi factor for P_1, L_1 is a complement to $R_1 = Q$ in $P_1 = H$. By 7.1, $L_1 \cong U_6(2)$. Then 23.2 in [3T] implies (2), 6.1 implies (3), and 23.3, and 22.2 in [3T] imply (4). Q.E.D.

(7.4) (1) $\dim([\tilde{Q}, j_i]) = 6, 8, 10$ *for* $i = 1, 2, 3$, *respectively.*
(2) Q *is transitive on the involutions in* j_3Q.

Proof. Let $M = N_{L_1}(L_{1,4})$. Then M is the stabilizer in L_1 of a nondegenerate 2-dimensional subspace of the natural module for $L_1 \cong U_6(2)$, so by 5.6, $M = M_1 \times M_2$ with $M_2 = L_{1,4} \cong U_4(2)$ and $M_1 = C_{L_1}(M_2) \cong L_2(2)$ with $j_1 \in M_1$. Further (again by 5.6) as an orthogonal space over $\mathbf{F}_2$, $\tilde{Q} = (\tilde{Q}_1 \oplus \tilde{Q}_2) \perp \tilde{Q}_3$, where $\tilde{Q}_1$ and $\tilde{Q}_2$ are copies of the $O_6^-(2)$-module for M_2, $\tilde{Q}_1 = [\tilde{Q}, j_1]$, and $\tilde{Q}_3$ is isomorphic to the $U_4(2)$-module for M_2. Thus $6 = \dim(\tilde{Q}_1) = \dim([\tilde{Q}, j_1])$. Next we can take $j_2 = ab$, where a, b are L_1 conjugates of j_1 in M_2, so $\dim([\tilde{Q}_3, j_2]) = 4$ and $\dim([\tilde{Q}_i, j_2]) = 2$ for $i = 1, 2$, and hence $\dim([\tilde{Q}, j_2]) = 8$. Finally we can take $j_3 = j_1j_2$. Then j_3 interchanges two of the three M_2-irreducibles on $\tilde{Q}_1 \oplus \tilde{Q}_2$, so $\dim([\tilde{Q}_1 \oplus \tilde{Q}_2, j_3]) = 6$ and $\dim([\tilde{Q}_3, j_3]) = \dim([\tilde{Q}_3, j_2]) = 4$. That is (1) holds.

As $\dim([\tilde{Q}_3, j_3]) = 10 = \dim(\tilde{Q})/2$, $C_{\tilde{Q}}(j_3) = [\tilde{Q}, j_3]$, so $\tilde{Q}$ is transitive on the involutions in $\tilde{j}_3\tilde{Q}$; cf. Exercise 2.8.1 in [SG]. Hence all involutions in j_3Q are conjugate to j_3 or j_3z. Next we have a symplectic form α on $\tilde{Q}_2$ defined by $\alpha(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}j_1)$ and there exists $\tilde{u} \in \tilde{Q}_2$ with $\alpha(\tilde{u}, \tilde{u}j_2) \neq 0$ as j_2 is of type c_2 in M_2 and $\tilde{Q}_2$ is the $O_6^-(2)$-module for M_2. Therefore $(\tilde{u}, \tilde{u}j_3) = (\tilde{u}, \tilde{u}j_2j_1) = \alpha(\tilde{u}, \tilde{u}j_2) \neq 0$, and hence $\tilde{u} + \tilde{u}j_3 \in C_{\tilde{Q}}(j_3)$ is nonsingular, so $j_3^u = j_3z$, establishing (2). Q.E.D.

(7.5) (1) $j_1 \in \Delta$ *is a long root involution so* $j_1 \in z^G$ *and* $H = C_G(z) \cong C_G(j_1)$.

(2) j_2 *is a short root involution, there is* $x \in j_2^G \cap Q \cap Z(R_4)$, *and* $C_G(x) \leq P_4$, $C_G(x) = R_4C_{L_4}(x)$, *where* $C_{P_4}(x) \cong Sp_6(2)$ *is the stabilizer in* $L_4 \cong \Omega_8^-(2)$ *of* x *regarded as a nonsingular point of the* 8-*dimensional orthogonal space* $Z(R_4)$ *for* L_4, *with* $Q \cap Z(R_4)$ *the subspace orthogonal to* z.

(3) *There is* $y \in j_3^G \cap Q \cap Q^g$ *for* $g \in P_2 - H$, $C_G(y) \leq P_2$ *with* $|R_2 : C_{R_2}(y)| = 4$ *and* $C_{L_2}(y) \cong L_2(2) \times U_3(2)$.

(4) $z, t = z^g, x, y$ *are representatives for the orbits of* H *on involutions of* Q, *with* $C_{L_1}(\tilde{t}) \cong L_3(4)/E_{2^9}$, $C_{L_1}(\tilde{x}) \cong Sp_4(2)/2^9$, *and* $C_{L_1}(\tilde{y}) \cong U_3(2)/2^8$.

Proof. First j_1 is a long root involution of L_1 by 7.3.2, so $j_1 \in \Delta$ and (1) holds.

Similarly by 7.3.2, j_2 is a short root involution of L_1 and hence of G. Let $Z_4 = Z(R_4)$; it is well known (cf. [CKS]) that Z_4 is the natural module for $L_4 \cong \Omega_8^-(2)$ with long root involutions in Z_4 the singular points and short root involutions in Z_4 the nonsingular points. Further $Q \cap Z_4$ is the subspace of Z_4 orthogonal to z. So if $x \in Q \cap Z_4$ is a short root involution then $C_{L_4}(x) \cong Sp_6(2)$. Now $C_G(x) \leq P$ for some parabolic P by Borel-Tits; cf. 47.8.2 in [FGT]. But the only parabolics of G containing subgroups of the form $C_{L_4}(x)R_4 = Sp_6(2)/2^{24}$ are conjugates of P_4, so $P = P_4^h$ for some $h \in G$. Then $O_2(P) = O_2(C_{P_4}(x)) = R_4$, so $P = P_4$ and (2) is established.

Let $g \in P_2 - H$, $t = z^g$, and $E = Q \cap Q^g$. By 7.3, $A = J(T \cap L_1) = R_{1,2} \cap L_1 \cong E_{2^9}$ contains a conjugate of j_3. Further from 6.1.6, $L_{1,2}$ has three irreducibles on R_2/E, all fused under P_2, so AE/E is one of those irreducibles and $(Q \cap R_2)/E$ is another, and A is fused to $A^w \leq Q \cap R_2$ under P_2. Next A^w and $[E, L_{1,2}]$ are dual irreducibles for $L_{1,2}$ and there is $l \in N_{L_1}(L_{1,2})$ inducing a graph automorphism on $L_{1,2}$, so $A^{wl} = [E, L_{1,2}]$, and hence there is $y \in j_3^G \cap E$. Next $C_{P_2}(y) = C_{L_2}(y)C_{R_2}(y)$ with $C_{L_2}(y) = L_{2,3,4} \times C_{L_{1,2}}(y)$ and by 7.3.4 and 6.1.5, $C_{L_{1,2}}(y) \cong U_3(2)$ with $C_A(y)$ a hyperplane of A and $|R_2 : C_{R_2}(y)| = 4$. Thus to complete the proof of (3) it remains to show $C_G(y) \leq P_2$. Again by Borel-Tits, $C_G(y) \leq P$ for some parabolic P of G and by 4.3, z is weakly closed in the center of a Sylow 2-subgroup of $C_G(y)$, so $P \cap H$ is a parabolic of G. Then $C_H(y) \leq P \cap H$.

Let $B = C_A(y)$. Observe first that $C_{\tilde{Q}}(B) = \langle \tilde{t}, \tilde{y} \rangle$. For $C_{L_{1,2}}(y)$ is irreducible on the hyperplane $[Q/E, B]$ of Q/E and as $L_{1,2}$ is irreducible on A, B contains a conjugate b of j_3. By 7.4.1, $C_{\tilde{Q}}(b)E/E = [\tilde{Q}, b]E/E \leq [Q/E, B]$, so as $C_{L_{1,2}}(y)$ is irreducible on $[Q/E, B]$, so $C_{\tilde{Q}}(B) \leq E$, and then by 4.5.7 completes the proof of the observation.

Next as $|R_2 : C_{R_2}(y)| = 4$, R_2 is transitive on $\langle z, t, y \rangle - \langle z, t \rangle$, so $\tilde{t}$ is weakly closed in $C_{\tilde{Q}}(B) = \langle \tilde{t}, \tilde{y} \rangle$, and therefore $N_{L_1}(S) \leq L_1 \cap P_2 = N_{L_1}(A)$, for each 2-subgroup S of L_1 containing B. Hence $P_2 \cap P \cap L_1$ contains a Sylow 2-subgroup of $P \cap L_1$, so as $A = J(T \cap L_1)$, $A \leq P \cap H$. Then as $A = O_2(A(C_{L_1}(y) \cap N_{L_1}(A)))$ and $C_{L_1}(y)$ is irreducible on B, $B \leq O_2(P \cap L_1) \leq A$, so that $P \cap L_1 = P_2 \cap L_1$ and then $P \cap H = P_{1,2}$. Therefore $P_2 = \langle P_{1,2}, P_{2,3,4} \rangle \leq P$, so $P = P_2$ and (3) holds.

Now $|L_1| = 2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$ and $|C_{H^*}(\tilde{y})| = 2^{11} \cdot 3^2$, so $|\tilde{y}^H| = 2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$. Similarly $|C_H(\tilde{t})| = 2^{15} \cdot 3^2 \cdot 5 \cdot 7$, so $|\tilde{t}^H| = 3^4 \cdot 11$. Finally

$$C_H(x) = C_{P_4}(z) \cap C_{P_4}(x) = R_4 C_{L_4}(\langle z, x \rangle)$$

with $C_{L_4}(\langle z, x \rangle) \cong Sp_4(2)/2^5$, so $|C_H(x)| = 2^{33} \cdot 3^2 \cdot 5$. Then as $|C_Q(x)| = 2^{20}$, $|C_{L_1}(\tilde{x})| = 2^{13} \cdot 3^2 \cdot 5$, so $|\tilde{x}^H| = 2^2 \cdot 3^4 \cdot 7 \cdot 11$. Now the sum of the lengths of these three orbits is

$$3^4 \cdot 11 \cdot (1 + 2^2 \cdot 7 + 2^4 \cdot 5 \cdot 7) = 3^4 \cdot 11 \cdot 588 = 3^4 \cdot 11 \cdot 19 \cdot 31 = (2^9 + 1)(2^{10} - 1).$$

But $(2^9+1)(2^{10}-1)$ is the number of singular points in a 20-dimensional orthogonal space of maximal Witt index over $\mathbf{F}_2$, so (4) is established. Q.E.D.

(7.6) *Q is regular on $\Delta^3(z)$ and for $d \in \Delta^3(z)$, $C_G(\langle z, d\rangle)$ is conjugate under Q to L_1.*

Proof. By 7.2, we may take $z = U_\alpha(1)$ and $d = U_{-\alpha}(1)$. Then $C_G(\langle z, d\rangle) = H \cap H^{w_0} = P_1 \cap P_1^{w_0} = L_1$, where w_0 is the long word in W, as $\alpha W_0 = -\alpha$, so $z^{w_0} = d$. Thus as L_1 is a complement to Q, Q is regular on $\Delta^3(z)$. Q.E.D.

(7.7) *j_1, j_2, and j_3 are representatives for the three conjugacy classes of involutions in G.*

Proof. We first observe that if j is an involution in G then $z^i \in \Delta^3(z)$ for some $i \in j^G$. This is Lemma 12.2 in [ASe], but we sketch a proof for completeness. Without loss, $j \in H$. By 7.5, each involution in Q is fused into L_2, so we may assume $j \notin Q$. Let $H^* = H/Q$. It is easy to check that $|k^* k^{*j}| = 3$ for some root involution $k \in L_1$, so by 7.2, $k^j \in \Delta^3(k)$, completing the proof of the observation.

So each involution in G is fused to $s \in L_1 \cup L_1 z$, so s is fused to j_i or $j_i z$. Finally $z j_i$ centralizes a conjugate of $\langle z, d\rangle$ in L_1 unless $i = 3$, so it remains to observe that $z j_3$ is conjugate to j_3 by 7.4.2.

We have shown each involution in G is conjugate to j_i for $i = 1$, 2, or 3. But by 7.5.4 and 4.7.2, these involutions are not fused in G. Q.E.D.

(7.8) *Let $g \in P_2 - H$, $t = z^g$, and $E = Q \cap Q^g$. Then*

(1) *For $h \in P_{1,3,4} - P_2$, $t^h \in E$.*

(2) *$U_3 = Q \cap Q^g \cap Q^{gh} \cong E_{2^7}$.*

(3) *Let $V_3 = \langle z, t, t^h\rangle$. Then $C_H(V_3)/O_2(C_H(V_3)) \cong L_2(4)$ has chief series*

$$0 < \tilde{V} < \tilde{V}_3 < \tilde{U}_3 < \tilde{E}$$

on $\tilde{E}$ with E/U_3 the $\Omega_4^-(2)$-module and U_3/V_3 the $L_2(4)$-module. Further $C_H(V_3)$ has four $L_2(4)$-sections and three $\Omega_4^-(2)$-sections on R_3.

Proof. First by 7.5.2, $Z_4 = Z(R_4)$ is the orthogonal space for $L_4 \cong \Omega_8^-(2)$ with $Q \cap Z_4$ the hyperplane orthogonal to z. Further the parabolic $P_{3,4}$ is the stabilizer in P_4 of the totally singular 3-subspace $V_3 = \langle z, t, t^h\rangle$. Thus $t^h \in E$ and indeed $V_3 = Z(P_3)$ with $C_H(V_3) = L_{1,2,3}R_3$ and $L_{1,2,3} \cong L_2(4)$ has chief series on $\tilde{E}$ has described in (3), except we have not shown that $U_3 = E_3$, where E_3 is the penultimate term in the series. But as U_3 is $C_H(V_3)$-invariant, $U_3 = E_3$ or V_3, and the latter is impossible as $U_3 \cap Z_4$ is of dimension 5.

Finally the chief sections can be retrieved as follows. Let $A = R_2 \cap L_1$ be as in 7.3. The nontrivial chief sections of $L_{1,2,3}$ on R_4 are those in $(R_{1,2,3} \cap L_1)/A$, A, E/V, and $C_Q(t)/E$, and by 6.1, A is isomorphic to $C_Q(t)/E$ and to the dual of E/V as an $L_{1,2,3}$-module. Finally $(R_{1,2,3} \cap L_1)/A$ is the $L_2(4)$-module, while A has one $L_2(4)$ chief section and one $\Omega_4^-(2)$-chief section. Q.E.D.

(7.9) *Let Δ be the graph with vertex z^G and z adjacent to t if $z \neq t \in Q$. Then Δ is simply connected.*

Proof. This follows from 1.1, since the building for G is of type F_4 and Δ is the collinearity graph of the building. Q.E.D.

(7.10) (1) *G has an involutory outer automorphism σ with $C_G(\sigma) \cong F_4(2)$, and we may choose σ so that:*

(2) *$C_{L_1}(\sigma) \cong Sp_6(2)$ and $C_Q(\sigma) = D_1 D_2$ where $D_1 \cap D_2 = \langle z \rangle$, $[D_1, D_2] = 1$, $\tilde{D}_1 = [\tilde{Q}, \sigma]$, D_1 is isomorphic to the stabilizer of a non-singular point in an 8-dimensional orthogonal space over $\mathbf{F}_2$ as a $C_{L_1}(\sigma)$-module, with singular points in j_2^G, and $D_2 \cong 2^{1+8}$ with $C_Q(\sigma)/D_1$ the spin module for $C_{L_1}(\sigma)$.*

(3) *$C_{L_2}(\sigma) \cong S_3 \times L_3(2)$ and σ centralizes $Z(R_2)$.*

(4) *For $S \in Syl_2(C_G(\sigma))$, $Z(S) = Z(S) \cap Q \cong E_4$.*

(5) *σ and σz are representatives for the orbits of G on involutions in σG and $C_G(\sigma z) = C_H(\sigma)$.*

(6) *Let Y be a diagonal group of outer automorphisms of G of order 3. Then $C_G(Y)$ is of even order and if all involutions in $C_G(Y)$ are in j_3^G then $N_{Aut(G)}(Y)/Y \cong Aut(U_3(8))$.*

Proof. This is well known; indeed σ is a graph-field automorphism of G. See for example section 4 of [CKS] for parts (1)-(5). Part (6) can be retrieved from the Springer-Steinberg theory of semisimple elements of finite groups of Lie type. Q.E.D.

(7.11) (1) $|\mathrm{Schur}_2(G)| = 4$.

(2) *The outer automorphism group of G is faithful on* $\mathrm{Schur}_2(G)$.

Proof. Let $\hat{G} = Cov_2(G)$ and $Z = Z(\hat{G})$. For $Y \leq G$, write $\hat{Y}$ for the preimage of Y in $\hat{G}$.

As $T \leq H$, $\hat{H}$ is a covering of H, and hence an image of $\mathrm{Cov}_2(H)$, described in 3.2. In particular $\hat{Q} \cong Q \times Z$ by 3.2, so $[\hat{Q}, \hat{E}] = \Phi(\hat{Q}) \cong \mathbf{Z}_2$. Then as $\hat{X}_2 = \langle \hat{Q}, \hat{Q}^g \rangle$, $[\hat{X}_2, \hat{E}] = \Phi(\hat{Q})\Phi(\hat{Q})^g \cong E_4$.

Next $L_2 = L_{234} \times L_{12}$ with $L_{234} = X \cap L_2 \cong S_3$ and $L_{12} \cong L_3(4)$. Let $\hat{Y}$ be of order 3 in $\hat{L}_{234}$. Then $\hat{V}_Y = [\hat{V}_2, \hat{Y}] = [\hat{X}_2, \hat{E}]$ is a complement to Z in $\hat{V}_2$ and $[\hat{R}_2, \hat{E}] = \hat{V}_Y$ as $R_2 = O_2(X_2)$. Therefore $\hat{R}_2$ centralizes $\hat{E}/\hat{V}_Y$, so setting $\hat{E}_Y = [\hat{E}, \hat{P}_2]$, it follows that $\hat{E}_Y = [\hat{E}, \hat{L}_{1,2}]\hat{V}_Y$.

Next $\hat{E}/\hat{V}_2 \cong E/V$ is quasiequivalent to the Todd module for L_{12} by 6.1.5. Therefore

$$|(\hat{V}_2 \cap \hat{E}_Y)/\hat{V}_Y| \leq |H^1(L_{12}, E/V)| = 4$$

with the last equality following from 23.6 in [3T]. Hence $U = Z \cap \hat{E}_Y$ is of order at most 4 and as $Out(G)$ induces a group of outer automorphisms on L_{12}, $Out(G)$ is faithful on U if $U \neq 1$ by 23.6 in [3T]. So it remains to show $U = Z$, since we will find in a later paper in this series that $\mathrm{Schur}_2(G) \neq 1$.

Let $G^* = \hat{G}/U$; it remains to show $Z^* = 1$. Now $R_2 = [R_2, Y]$ so $\hat{R}_2^*/\hat{E}_Y^* = [\hat{R}_2^*/\hat{E}_Y^*, \hat{Y}^*] \times Z^*$. Therefore $\hat{P}_2^*/[\hat{R}_2^*, \hat{Y}^*] \cong \hat{L}_{234}^* \times \hat{L}_{12}^*$ with $\hat{L}_{12}^*$ quasisimple with center Z^*. Next $Q \leq L_{234}R$ by 6.1, so $Q \cong \hat{Q}^*$ and $\hat{H}^*/\hat{Q}^*$ is quasi simple with center Z^*. Indeed

$$\hat{R}_2^*\hat{Q}^*/\hat{Q}^* = [\hat{R}_2^*, \hat{Y}^*]\hat{Q}^*/\hat{Q}^* \times Z^*$$

so by 23.5.5 in [3T], $Z^* = 1$, completing the proof. Q.E.D.

(7.12) *Assume $M(22) \cong M \leq G$ such that the set D of 3-transpositions of M is contained in Δ. Then $C_D(a) \neq \varnothing$ for each $a \in \Delta$, and indeed M has the following four orbits, Δ_i, $1 \leq i \leq 4$, on Δ:*

(1) $\Delta_1 = D$ *of order* 3, 510.

(2) $\Delta_2 = \{a \in \Delta : C_D(a) \subseteq O_2(C_G(a))\}$ *of order* 142, 155, *with* $C_M(a) \cong M_{22}/E_{2^{10}}$ *and* $C_D(a)$ *of order* 22 *generating* $O_2(C_M(a))$.

(3) $\Delta_3 = \{a \in \Delta - D : |D \cap O_2(C_G(a))| = 1\}$ *of order* 3, 127, 410, *with* $C_M(a) \cong L_3(4)/E_{2^{10}}$ *and* $C_D(a)$ *of order* 22 *generating* $O_2(C_M(a))$.

(4) $\Delta_4 = \{a \in \Delta : D \cap O_2(C_G(a)) = \varnothing\}$ *of order* 694, 980, *with* $C_M(a) = \langle C_D(a)\rangle \cong Sp_6(2)/E_{64}$.

Proof. First $\Delta_1 = D$ is an orbit of M on Δ of length 3, 510 by 16.7 in [3T].

As $D \subseteq \Delta$, we may take $z \in D$. Then $K = C_M(d)$ is quasisimple with $K/\langle d\rangle \cong U_6(2)$, so $H = KQ$ with $K \cap Q = \langle z\rangle$. Claim

(5) K has the following six orbits on $\Delta \cap H$:

(i) $\{z\}$.

(ii) $D_z = H \cap D - \{z\}$.

(iii) $\Delta_i(z)$, $i = 1, 2$ with $\Delta_1(z) \cup \Delta_2(z) = \Delta(z) = \Delta \cap Q - \{z\}$,

$$\Delta_2(z) = \{za : a \in \Delta_1(z)\},$$

and $C_K(a) \cong L_3(4)/E_{2^{10}}$ for $a \in \Delta(z)$.

(iv) $\Delta_3(z)$ with $C_K(a) \cong A_5/E_{16}/E_{2^{10}}$ for $a \in \Delta_3(z)$.

(v) $\Delta_4(z)$ with $C_K(a) \cong Sp_4(2)/2^{1+8}/\mathbf{Z}_2$ for $a \in \Delta_4(z)$.

Namely by 7.2, H has three orbits on $\Delta \cap H$: $\{z\}$, $\Delta(z) = H \cap Q - \{z\}$, and $\Delta_1^2(z) = H \cap \Delta - Q$. As $H = KQ$ with $K \cap Q = \langle z \rangle$, K has two orbits $\Delta_i(z)$, $i = 1, 2$ on $\Delta(z)$, with $\Delta_2(z) = \{za : a \in \Delta_1(z)\}$, and by 6.1.2 and 23.5 in [3T], $C_K(a) \cong L_3(4)/E_{2^{10}}$ for $a \in \Delta(z)$.

Next let $b \in D_z$. Then $b \in \Delta_1^2(z)$ and each member of $\Delta_1^2(z)$ is K-conjugate to bu for some $u \in [Q, b]$. Now $[\tilde{Q}, b]$ is the natural module for $C_K(b)/O_2(C_K(b)) \cong \Omega_6^-(2)$ with $O_2(C_K(b)) \cong 2^{1+8}/\mathbf{Z}_2$, (cf. 7.3 and the proof of 7.4) so K has two orbits $\Delta_3(z)$ and $\Delta_4(z)$ on $\Delta_1^2(z) - D_z$, with representatives bu and bv, where $u, v \in [\tilde{Q}, b]$ with $\tilde{u}$ a singular point of the orthogonal space $[\tilde{Q}, b]$ and $\tilde{v}$ a nonsingular point. Then $C_K(bu) = C_K(b) \cap C_K(u) \cong A_5/E_{16}/2^{1+8}/\mathbf{Z}_2$ and $C_K(bv) = C_K(b) \cap C_K(v) \cong Sp_4(2)/2^{1+8}/\mathbf{Z}_2$. Indeed $C_K(u)$ is the parabolic $N_K(T \cap D) \cong L_3(4)/E_{2^{10}}$ with $O_2(C_K(u)) = \langle T \cap D \rangle$, so $C_K(bu) \cong A_5/E_{16}/E_{2^{10}}$, completing the proof of the claim.

Let $z^\perp = \{z\} \cup D_z$. If $a \in \Delta(z)$ or $\Delta_3(z)$ then $z^\perp \cap C_G(a) = T \cap D$ is of order 22 and hence is of the form $S \cap D$ for some $S \in Syl_2(M)$, with $A = \langle S \cap D \rangle \cong E_{2^{10}}$. Further if $a \in \Delta(z)$ then by 6.1, $C_K(a)$ has 3 irreducibles on $(Q \cap H_a)(Q_a \cap H)/(Q \cap Q_a)$, and one of them is $A(Q \cap Q_a)/(Q \cap Q_a)$, so $A(Q \cap Q_a) = Q_a \cap H$ or $Q_{az} \cap H$. In the first case, $A \leq Q_a$. Therefore for each $b \in A \cap D$, $a \in \Delta(b)$, so $A \cap D = b^\perp \cap C_G(a)$ and $C_M(\langle a, b \rangle)$ acts 2-transitively as $L_3(4)$ on $A \cap D - \{b\}$. Therefore $N_M(A) \leq C_M(a)$ with $N_M(A)/A \cong M_{22}$ by 25.7 in [3T]. As $A \cap D$ is a connected component of $C_D(a)$, it follows (cf. 24.3 in [3T] and its proof) that $A \cap D = C_D(a)$, so that $N_M(a) = C_M(a)$. That is $a \in \Delta_2$.

In the second case, $za \in \Delta_2$ and $A \cap Q_a = \langle z \rangle$, so for $b \in A \cap D - \{z\}$, $b \notin Q_a$, and hence $a \notin Q_b$, so $a \in \Delta_i(b)$ for $i = 3$ or 4. As $C_K(\langle a, b \rangle) \cong A_5/E_{16}/E_{2^{10}}$, and $C_M(\langle a, b \rangle)$ contains no such subgroup if $a \in \Delta_4(b)$, we conclude $a \in \Delta_3(b)$. Therefore $S \cap D = b^\perp \cap C_G(a)$ for each $b \in S \cap D$, so as above, $S \cap D = C_D(a)$ and $\{z\} = D \cap Q_a$. Hence $C_M(a) = C_K(a)$ and $a \in \Delta_3$ in this case.

So Δ_2 and Δ_3 are the orbits of M on $\Delta - D$ consisting of elements

a with $D \cap Q_a \neq \varnothing$. This leaves

$$\Delta_4' = \{a \in \Delta : C_D(a) \neq \varnothing = D \cap Q_a\} = \bigcup_{d \in D} \Delta_4(d)$$

as an orbit under M.

Pick $a \in \Delta_4(z)$ and let $D(a) = C_D(a)$. Then $D(a)$ is a set of 3-transpositions of $M_a = \langle D(a) \rangle$. Now $C_K(a) \cong Sp_4(2)/2^{2+8}$ and $C_K(a) = \langle z^\perp \cap D(a) \rangle$. Indeed for each $b \in D(a)$, $a \in \Delta_4(b)$ as $a \notin \Delta_3 \cup \Delta_4$, so by 8.2.2 in [3T], M_a is transitive on $D(a)$. Then by a Frattini argument, $C_M(a) = M_a C_K(a) = M_a$. Also in the language of [3T], $V_z = \{z, d\}$, where d is the unique member of $D \cap aQ$, so by 9.2 in [3T], $\{z, d\} = z^{O_2(M_a)}$, so $[z, O_2(M_a)] = \langle zd \rangle \leq Z(O_2(M_a))$. Therefore $U = \langle (zd)^{M_a} \rangle$ is elementary abelian and z induces a transvection on U. Let $\bar{M}_a = M_a/U$. As $O_2(C_K(a))/\langle zd \rangle \cong 2^{1+8}$ and $O_2(C_K(a))/\langle z, d \rangle$ is the sum of two 4-dimensional irreducibles for $C_K(a)$, $m(C_U(z)) = 5$, $m(U) = 6$, and $C_{\bar{M}_a}(\bar{z}) \cong C_K(a)/C_U(z) \cong Sp_4(2)/E_{32}$. As $O_2(C_{\bar{M}_a}(\bar{z})) \not\leq Z(C_{\bar{M}_a}(\bar{z}))$, $O_3(\bar{M}_a) \leq Z(\bar{M}_a)$ by Exercise 3.2 in [3T], while as $[z, O_2(M_a)] \leq U$, $O_2(\bar{M}_a) \leq Z(\bar{M}_a)$. Then by Theorem Q in section 14 of [3T], $\bar{M}_a \cong Sp_6(2)$.

To complete the proof we calculate the order of $\mathcal{O} = \Delta_2$, Δ_3 and Δ_4' via $|\mathcal{O}| = |M : C_M(a)|$, for $a \in \mathcal{O}$, and determine they are as indicated in the statement of the lemma. Then we calculate that

$$|\Delta_1| + |\Delta_2| + |\Delta_3| + |\Delta_4'| = 3,968,055 = |\Delta|$$

so $\Delta_4' = \Delta_4$ and the proof of the lemma is complete. Q.E.D.

(7.13) *Let* $\hat{H} = \mathrm{Cov}_2(H)$, $\rho : \hat{H} \to H$ *the universal covering,* $V = \ker(\rho)$, $\hat{Q} = O_2(\hat{H})$ *and* $P = [\hat{Q}, \hat{H}]$. *Let* H_+ *be a group with* $Q_+ \cong F^*(H_+) \cong Q$ *and* $H_+/Z(H_+) \cong \tilde{H}$. *Then*

(1) $\hat{H} = \hat{L}P$ *with* $P \cap \hat{L} = 1$, $\hat{L} \cong Cov_2(L_1)$ *and* $\rho(\hat{L}) = L_1$.

(2) $P \cong E_4 \times Q$, $Z(\hat{L}) \cong E_4$, $\hat{Q} = Z(\hat{L}) \times P$ *and* $Z(\hat{H}) = Z(\hat{L}) \times Z(P)$.

(3) $Z(\hat{L}) \leq V$ *and* $V = [\tau, Z(\hat{H})]$ *is a complement to* $\Phi(P)$ *for some automorphism* τ *of order* 3 *inducing an outer automorphism on* $\hat{L}$.

(4) $H_+ \cong H$ *if and only if* H_+ *possesses a complement* L_+ *to* Q_+ *such that* $E_+/\langle t_+ \rangle$ *splits over* $\langle z_+, t_+ \rangle/\langle t_+ \rangle$ *as a* J_+*-module, where* x_+ *is the image of* $x = z, t, E$ *under the isomorphism* $Q \cong Q_+$, *and* $J_+ = C_{L_+}(t_+)$.

(5) *If* $H_+ = C_{G_+}(z_+)$ *for some group* G_+ *of type* $^2E_6(2)$ *and* H_+ *splits over* Q_+ *then* $H_+ \cong H$.

Proof. By 6.2, $\tilde{Q} \otimes_{\mathbf{F}_2} \mathbf{F}_4 \cong \bigwedge^3(N)$ as a $\mathbf{F}_4 L_1$-module. Then by 5.8, $H^1(L_1, \tilde{Q}) \cong E_4$. By 23.7 in [3T], $\mathrm{Schur}_2(L_1) \cong E_4$. Therefore (1) and (2) follow from 3.2.

Let $D = C_{Aut(\hat{H})}(P/\Phi(P))$ and $\hat{H}D$ the semidirect product of $\hat{H}$ by D. By 3.2, V is a complement to $\Phi(P)$ in $Z(\hat{H})$ and $D/Inn(P) \cong H^1(L_1, \tilde{Q}) \cong E_4$ is regular on complements to $\Phi(P)$ in $Z(P)$. Indeed by 3.2.6, $H_+ \cong \hat{H}/V_+$ for some complement V_+ to $\Phi(P)$ in $Z(\hat{H})$.

As $\tilde{Q} \otimes_{\mathbf{F}_2} \mathbf{F}_4 \cong \bigwedge^3(N)$ and the representation of L_1 on $\bigwedge^3(N)$ extends to $PGU_6(2) = L_1\langle\tau\rangle$ for some τ of order 3, the representation of L_1 on $\tilde{Q}$ extends to $L_1\langle\tau\rangle$. Thus τ is an automorphism of $\hat{H}$ by 3.2.3, so as τ is faithful on $\mathrm{Schur}_2(L_1)$ and $H^1(L_1, \tilde{Q})$, τ is faithful on $Z(\hat{H})/Z(P)$ and $Z(P)/\Phi(P)$. As some outer automorphism of G of order 3 acts on Q and L_1 and induces an outer automorphism of L_1, we may take τ to act on $\hat{L}$, τ induces an outer automorphism on $\hat{L}$, and $V = [Z(\hat{H}), \tau]$ is the unique τ-invariant complement to $\Phi(P)$, so that (3) holds.

Notice that D is transitive on the complements to $\hat{Q}/Z(\hat{H})$ in $\hat{H}/Z(\hat{H})$.

Let $L_+ = \hat{L}V_+/V_+$ be the image of $\hat{L}$ in H_+. We next prove

(6) Under the hypothesis of (5), we can pick L_+ with $O_2(J_+) \leq Q_{t_+} = O_2(C_{G_+}(t_+))$.

To simplify notation we argue in G. Now J has three 9-dimensional irreducibles on $O_2(J)Q/E$: $C_Q(t)/E$, $(Q_t \cap H)/E$, and $(Q_{tz} \cap H)/E$, so as $O_2(J)E/E$ is one of these irreducibles, conjugating L_+ by an element of $Q - C_Q(t)$ if necessary, we may take $O_2(J)E = Q_t \cap H$, establishing (6). We also prove

(7) Under the hypothesis of (5), there is a complement I_+ to $O_2(J_+)$ in J_+ such that E_+ splits over $\langle z_+, t_+\rangle$ as an I_+-module.

First if $G_+ = G$ then $I = L_{12}$ works as L_{12} acts on the complement $C_E(L_{234})$ to $\langle z, t\rangle$ in E. Moreover $\tilde{Q}$ is a semisimple L_{12}-module and $L_{12} = N_{L_1}([\tilde{E}, L_{12}])$.

In the general case $\tilde{H}_+ \cong \tilde{H}$ (cf. 8.1) so the preimage I_+ in L_+ of the image of $\tilde{L}_{12}$ in $\tilde{H}_+$ under this isomorphism acts semisimply on $\tilde{Q}_+$ as L_{12} is semisimple on $\tilde{Q}$. In particular $C_{Q_+}(I_+) \cong D_8$. Similarly as the image F of $[E_+, I_+]$ in $\tilde{Q}_{t_+}$ is a simple I_+-module, and as $\tilde{H}_{t_+}$ is isomorphic to $\tilde{H}_t$, $I_+Q_{t_+} = N_{H_{t_+}}(F)$ and then $\tilde{Q}_{t_+}$ is a semisimple I_+-module and $C_{Q_{t_+}}(I_+) \cong D_8$. Therefore $\langle C_{Q_+}(I_+), C_{Q_{t_+}}(I_+)\rangle$ contains an element X_+ of order 3 such that $C_{E_+}(X_+)$ is an I_+-invariant complement to $\langle z_+, t_+\rangle$ in E_+, completing the proof of (7).

Observe next that

(8) L_+ is a complement to Q_+ in H_+ if and only if $Z(\hat{L}) \leq V_+$.

We also claim

(9) If L_+ is a complement to Q_+ then $V_+ = V$ if and only if the following splitting property holds: $E_+/\langle t_+\rangle$ splits over $E_+/\langle z_+, t_+\rangle$ as a J_+-module.

If $V_+ = V$ this follows from (6) and (7). Namely by (6), we may choose L_1 so that $O_2(J) \leq Q_t$, where $J = C_{L_1}(t)$. Therefore $O_2(J)$ centralizes $E/\langle t\rangle$ as $E \leq Q_t$. Further by (7), E splits over $\langle z, t\rangle$ as an I-module, so as $J = O_2(J)I$, we have the splitting property.

Notice this argument only depended upon the hypothesis of (5). Thus (9) will imply (5), since under the hypothesis of (5), as D is transitive on complements to $\hat{Q}/Z(\hat{Q})$, we may assume the complement to Q_+ is the image of $\hat{L}$. Thus, as we just observed, H_+ has the splitting property, so $H \cong H_+$ by (9). Similarly (8) and (9) imply (4), so it remains to assume the splitting property and show $V_+ = V$. Let $\hat{E} = \rho^{-1}(E) \cap P$ and $\hat{J} = \rho^{-1}(C_{L_1}(t))$. We show $Z(P) \leq [\hat{E}, \hat{J}]\Phi(P)$, so that as H_+ has the splitting property,

$$V_+ \cap Z(P) = [\hat{E}, \hat{J}] = V \cap Z(P)$$

and then

$$V_+ = Z(\hat{L}) + (V_+ \cap Z(P) = Z(\hat{L}) + (V \cap Z(P) = V$$

as desired.

Now $P/\Phi(P)$ is the largest module $M = [M, L_1]$ for L_1 such that $M/C_M(L_1) \cong \tilde{Q}$. Further $J = C_{L_1}(\tilde{t})$ and $\tilde{E}$ is the unique 10-dimensional L_1-submodule of $P/Z(P)$, so $Z(P) \leq [\hat{E}/\Phi(P), J]$ by 5.8.3, completing the proof of the lemma. Q.E.D.

(7.14) *Let $\check{G}$ be the extension of G by the graph-field automorphism σ of 7.10 and $\check{H} = C_{\check{G}}(z)$. Assume $\check{H}_1$ is a group with $F^*(\check{H}_1) = Q_1 \cong Q$ and with a subgroup H_1 of index 2 containing Q_1 such that $H_1/Z(H_1) \cong \tilde{H}$. Then $\check{H}/\langle z\rangle \cong \check{H}_1/Z(H_1)$.*

Proof. As $F^*(\check{H}_1) = Q_1$ and H_1 is of index 2 in $\check{H}_1$ with $H_1/Z(H_1) \cong \tilde{H}$, $F^*(\check{H}_1/Q_1) = H_1/Q_1 \cong H^* \cong U_6(2)$. Therefore as $Out(U_6(2)) \cong S_3$, $\check{H}_1/Q_1 \cong \check{H}/Q$. As $H_1/Z(H_1) \cong \tilde{H}$, the representation of H_1/Q_1 on $\tilde{Q}_1 = Q_1/Z(Q_1)$ is quasiequivalent to that of H^* on $\tilde{Q}$ by 3.1. By 6.1.7, H^* is absolutely irreducible on $\tilde{Q}$, so $N_{GL(\tilde{Q})}(H^*) \cong Aut(U_6(2))$, and hence as $\check{H}_1/Q_1 \cong \check{H}/Q$, the representation of $\check{H}_1/Q_1$ on $\tilde{Q}_1$ is quasiequivalent to that of $\check{H}/Q$ on $\tilde{Q}$, so 3.1 completes the proof of the lemma. Q.E.D.

(7.15) (1) *For $p \neq 2$ or 11, p prime, and $P \in Syl_p(G)$, $C_G(P) \leq P$ and if $p = 3$ then $N_G(P)$ is a $\{2,3\}$-group.*

(2) *If $Y \leq G$ is of order 11 then $C_G(Y) \cong \mathbf{Z}_{11} \times S_3$.*

(3) *If $Y \leq G$ is of order 7 then $C_G(Y) = Y \times E(C_G(Y))$ with $E(C_G(Y)) \cong L_3(2)$ or $L_3(4)$.*

(4) *If $Y \leq G$ is of order 5 then $C_G(Y) \cong \mathbf{Z}_5 \times A_5$.*

(5) *If Y is a 3-central subgroup of G of order 3 then $C_G(Y)$ is a $\{2,3\}$-group.*

(6) *If $S \in Syl_3(G)$ then $J(S) \cong E_{3^5}$ and $N_G(J(S))/J(S) \cong O_6^-(2)$.*

Proof. This is well known and follows from the Springer-Steinberg theory of semisimple elements of finite groups of Lie type. Q.E.D.

(7.16) *If $M \leq G$ is of odd order then $|M| < 10^5$.*

Proof. Let $F = F(M)$. As M is of odd order, M is solvable, so $C_M(F) \leq F$. (cf 31.10 in [FGT]) Let p be a prime divisor of $|F|$ and $P = O_p(M)$. If $p \neq 3$ or 11 and $P \in Syl_p(G)$, then by 7.15.1, $O^p(F) \leq C_G(P) \leq P$, so $P = F$. Thus $|M| \leq n_p|P|$, where n_p is the maximal order of a subgroup X of $GL(P/\Phi(P))$ of odd order with $O_p(X) = 1$. In each case $n_p|P| < 10^5$.

Further if $p = 11$ then $F \leq O^2(C_G(P)) \cong \mathbf{Z}_{33}$ by 7.15.2, so

$$|M| \leq |F| \cdot |O(Aut(\mathbf{Z}_{11}))| \leq 33 \cdot 5 < 10^5$$

Similar arguments work if P is of order 5 or 7, using 7.15.3 and 7.15.4.

Therefore we may assume $F = O_3(M)$. Now if $P \in Syl_3(FC_G(F))$ then by a Frattini argument, $M \leq N_G(F) = C_G(F)(N_G(P) \cap N_G(F)$, so as $C_M(F) \leq F$, $N_G(P) \cap N_G(F)$ contains a subgroup M_0 of odd order with $|M_0| \geq |M|$. Hence replacing M by M_0 if necessary, we may assume $P = F$. In particular taking $F \leq S \in Syl_3(G)$, $Z = Z(S) \leq F$. Let $U = \langle Z^M \rangle$, so that $Z \cong E_{3^n}$ for some n. Then $C_M(U) \leq C_M(Z)$, and $C_M(Z)$ is a 3-group by 7.15.5. Therefore $C_M(U) \leq O_3(M) = F$. Hence $|M| \leq |F|N_n$, where N_n is the maximal order of a subgroup X of odd order in $GL_n(3)$ with $O_3(X) = 1$.

By 7.15.6, $n \leq 5$, so $|M|_{3'}$ divides $5 \cdot 11 \cdot 13$. Indeed if 11 divides $|M|$ then $n = 5$, so $U = J(S)$ for $S \in Syl_3(G)$ by 7.15.6, whereas by the last remark in 7.15.6, 11 does not divide the order of $N_G(J(S))$. So 11 does not divide the order of M. Further by 7.15.4, G has no subgroup of order $13 \cdot 5$, so by Hall's Theorem, (cf. 18.5 in [FGT]) $|M|_{3'} = 1$, 5, or 13. But $|G|_3 = 3^9$ and $3^9 \cdot 5 < 10^5 > 3^8 \cdot 13$, so we are left with the case $|M| = 3^9 \cdot 13$.

By 7.15.1, if Y is of order 13 in M then $C_F(Y) = 1$ and $|N_M(Y)| = 1$ or 3. Therefore $|F| = 3^{3k}$ for some k and hence $F \in Syl_3(G)$, contradicting 7.15.1. Q.E.D.

§8. Groups of type $^2E_6(2)$ are isomorphic to $^2E_6(2)$

In this section we assume the hypotheses and notation of section 6. In particular G is of type $^2E_6(2)$, z is a 2-central involution in G, $H = C_G(z)$, etc. Further let $G_0 = {}^2E_6(2)$ and z_0 a long root involution of G_0. By 7.1, G_0 is of type $^2E_6(2)$ with z_0 2-central in G_0. Let $H_0 = C_{G_0}(z_0)$, $Q_0 = O_2(H_0)$, etc.

(8.1) $\tilde{H} \cong H_0/\langle z_0 \rangle$.

Proof. First $Q_0 \cong Q$, so we may identify the two groups. Further by 6.2, the representation of H_0^* on $\tilde{Q}_0$ is quasiequivalent to that of H^* on $\tilde{Q}$, so $\tilde{H} \cong \tilde{H}_0$ by 3.1. Q.E.D.

By 8.1, $\tilde{H}_0 \cong \tilde{H}$, so by 7.8 there is $h \in H - C_H(\tilde{t})$ with $t^h \in E$. Let $k = gh$, $V_3 = \langle z, t, z^k \rangle$, $U_3 = Q \cap Q^g \cap Q^k$, $X_3 = \langle Q, Q^g, Q^k \rangle$, $R_3 = C_{X_3}(V_3)$,

$$S_3 = (Q \cap Q^g)(Q \cap Q^k)(Q^g \cap Q^k),$$

and $P_3 = N_G(V_3)$. By 8.16 in [SG],

$$R_3 = C_Q(V_3)C_{Q^g}(V_3)C_{Q^k}(V_3) = O_2(X_3),$$

$X_3/R_3 = GL(V_3) \cong L_3(2)$, $[X_3, U_3] \leq V_3$, $\Phi(U_3) = 1$, $P_3 = X_3C_H(V_3)$, and $P_3/R_3 = X_3/R_3 \times C_H(V_3)/R_3$.

By 7.8, $C_H(V_3)/R_3 \cong A_5$, so $P_3/R_3 \cong L_3(2) \times A_5$. Again by 7.8, $m(U_3) = 6$, so by 8.16 in [SG], S_3/U_3 is the sum of 4 copies of the dual V_3^* of V_3 as an X_3/R_3-module, and R_3/S_3 is the sum of 4 copies of V_3 as an X_3/R_3-module. By 7.8, $C_H(V_3)$ has chief series $0 < \tilde{V} < \tilde{V}_3 < \tilde{U}_3 < \tilde{E}$ on $\tilde{E}$ with E/U_3 the $\Omega_4^-(2)$-module and U_3/V_3 the $L_2(4)$-module for $C_H(V_3)$. Finally by 7.8, $C_H(V_3)$ has four $L_2(4)$-sections and three $\Omega_4^-(2)$-sections on R_3. We summarize all this as:

(8.2) (1) *$P_3/R_3 = X_3/R_3 \times C_H(V_3)/R_3$ with $X_3/R_3 \cong L_3(2)$ and $C_H(V_3)/R_3 \cong A_5$.*

(2) *R_3 has chief series*

$$0 < V_3 < U_3 < S_3 < R_3$$

with V_3 the natural module for X_3/R_3, $[X_3, U_3] \leq V_3$ and U_3/V_3 is the $L_2(4)$-module for $C_H(V_3)/R_3$, S_3/U_3 is the tensor product of the dual of V_3 as an X_3/R_3-module with the $\Omega_4^-(2)$-module for $C_H(V_3)/R_3$, and R_3/S_3 is the tensor product of V_3 as an X_3/R_3-module with the $L_2(4)$-module for $C_H(V_3)/R_3$.

(8.3) *There exists $s \in z^G$ with sz of order 3, $C_G(\langle s, z\rangle) \cong U_6(2)$, and $N_G(\langle sz\rangle) = \langle s, z\rangle \times C_G(\langle s, z\rangle)$.*

Proof. Let $X_2 = \langle Q, Q^g\rangle$. Then $X_2 \leq X_3$ so there is x of order 3 in X_2 fused to $y \in X_3 \cap H$. Notice y^* is inverted by a transvection in H^* as $\tilde{H}_0 \cong \tilde{H}$ and the remark holds in H_0^* since y is inverted by some conjugate $c \in Q^g$ of z in H_0 and c^* is a transvection in H_0^* by 7.2 and 7.3.2. Therefore $C_Q(y) \cong D_8^4$ and $C_H(y)/C_Q(y)\langle y\rangle \cong U_4(2)$. Let $T_y \in Syl_2(C_H(y))$; then $\langle z\rangle = Z(T_y)$ and T_y is of order 2^{15}. As $\langle z\rangle = Z(T_y)$, $T_y \in Syl_2(C_G(y))$.

Next let $T_x \in Syl_2(C_{P_2}(x))$. From the structure of P_2 described in 6.1,

$$C_{P_2}(x)/\langle x\rangle \cong L_3(4)/E_{2^9}$$

with $O_2(C_{P_2}(x))$ quasiequivalent to the Todd module for $C_{P_2}(x)/O_2(C_{P_2}(x))\langle x\rangle$. In particular T_x is of order 2^{15} and hence as x and y are conjugate, the previous paragraph says that $T_x \in Syl_2(C_G(x))$ and $Z(T_x)$ is generated by a conjugate of z. Now the hypotheses of Theorem 30.1 in [3T] are satisfied, so by that Theorem, $C_G(x)/\langle x\rangle \cong C_G(y)/\langle y\rangle \cong U_6(2)$.

Next x is inverted by an involution $u \in Q$ with $[C_{P_2}(x), u] = \langle x\rangle$, so u induces an automorphism of $C_G(x)/\langle x\rangle \cong U_6(2)$ centralizing the parabolic $C_{P_2}(x)/\langle x\rangle$, and hence centralizing $C_G(x)/\langle x\rangle$. Therefore $N_G(\langle x\rangle) = \langle x, u\rangle \times E(C_G(x))$ with $E(C_G(x)) \cong U_6(2)$.

Finally $u \in Q$ centralizes a $L_3(4)$-section of H, so as $\tilde{H} \cong \tilde{H}_0$, 7.5 says that $u \in t^H \subseteq z^G$. Hence there exists $s \in z^G$ with $\langle s, z\rangle$ conjugate to $\langle u, x\rangle$, completing the proof. Q.E.D.

(8.4) $H \cong H_0$.

Proof. By 8.3 there is $s \in z^G$ with $C_G(\langle s, z\rangle)$ a complement to Q in H. Hence 7.13.5 completes the proof. Q.E.D.

By 8.4 there is an isomorphism $\alpha : H \to H_0$. Let $t_0 = t\alpha$, $t_0 = t^{g_0}$, $h_0 = h\alpha$ where $k = gh$, $V_3^0 = V_3\alpha$, and $P_3^0 = N_{G_0}(V_3^0)$.

(8.5) *There exist an isomorphism $\zeta : P_3 \to P_3^0$ such that $\alpha = \zeta$ on $H \cap P_3$.*

Proof. We appeal to 21.12 in [3T]. The P_3-chief series required in that lemma is:

$$1 < V_3 < U_3 < S_3 < R_3$$

and by 8.2, the image of this series under α is the corresponding series in R_3^0. Namely by definition, $V_3^0 = V_3\alpha$. Also as $t_0 = t\alpha$, $V_0 = V\alpha$ and then as $E/V = C_{Q/V}(O_2(C_H(\tilde{V})))$,

$$(Q \cap Q^g)\alpha = E\alpha = E_0 = Q_0 \cap Q_0^{g_0}.$$

Therefore $U_3\alpha = (E \cap E^h)\alpha = E_0 \cap E_0^{h_0} = U_3^0$.

Next $(Q \cap H^g)/E$, $(Q^g \cap H)/E$, and $(Q^{gu} \cap H)/E$, $u \in Q - C_Q(t)$, are the three $C_H(\tilde{t})$-invariant subspaces of $O_2(C_H(\tilde{t}))/E$, with $Q^g \cap H$ distinguished by $\Phi(Q^g \cap H) = \langle t\rangle$, so $(Q^g \cap H)\alpha = Q_0^{g_0} \cap H_0$. Then

$$(Q^g \cap Q^{gh})\alpha = Q_0^{g_0} \cap H_0 \cap Q_0^{g_0 h_0} \cap H_0 = Q_0^{g_0} \cap Q_0^{g_0 h_0},$$

so

$$S_3\alpha = (Q \cap Q^g)((Q \cap Q^{gh})(Q^g \cap Q^{gh})\alpha = S_3^0.$$

Finally $R_3 = O_2(C_H(V_3))$, so $R_3\alpha = R_3^0$.

Next 8.2 says that hypotheses (2), (3), (5) and (6) of 21.12 in [3T] are satisfied. To check hypothesis (4) of that lemma, use Remark 21.9 and Lemma 21.13 of [3T]. Now 21.12 in [3T] supplies the extension $\zeta : P_3 \to P_3^0$ of $\alpha : P_3 \cap H \to P_3^0 \cap H_0$. Q.E.D.

(8.6) $G = \langle H, P_3 \rangle$.

Proof. Let $K = \langle H, P_3 \rangle$ and assume that $K \neq G$. Then by induction on the order of G, $K \cong {}^2E_6(2)$. By 7.7, K has 3 classes of involutions with representatives j_i, $1 \leq i \leq 3$, while by 7.5, each class is fused into Q under K. By 4.7.2,

$$z^G \cap Q = \{z\} \cup t^H = z^K \cap Q,$$

so $z^G \cap K = z^K$. Hence as also $C_G(z) = H \leq K$, 7.3 in [SG] says K is the unique point of G/K fixed by z. We show K is strongly embedded in G; then 7.6 in [SG] contradicts the fact that K has more than one class of involutions.

To show K is strongly embedded in G it remains to show $C_G(j) \leq K$ for each involution $j \in K$. So assume $Y = C_G(j) \not\leq K$ for some involution $j \in K$ and let $Y^* = Y/\langle j \rangle$. We have seen $j \neq j_1 = z$. If $j = j_2$, then from 7.5, we may take $j \in Z_4 = Z(P_4)$ with $R_4 \leq C_K(j) \leq P_4$ and $C_K(j)/R_4 \cong Sp_6(2)$. By 7.4 in [SG], $C_K(j)$ controls 2-fusion in $C_K(j)$, so Z_4^* is a strongly closed abelian subgroup of $C_K(j)^*$ in Y^*. From 7.5, Z_4 has the structure of an 8-dimensional orthogonal space over $\mathbf{F}_2$ with $z^G \cap Z_4$ the singular points and $j^G \cap Z_4$ the nonsingular points. The subspace U_4 of this orthogonal space orthogonal to j is $C_K(j)$ invariant.

Pick $u \in Y - K$ to be fused to an element of $z^G \cap Z_4 - U_4$ under Y. As $C_K(j)$ controls 2-fusion in $C_K(j)$, z^* and u^* are not conjugate in Y^*, so z^*u^* has even order. Let i^* be the involution in $\langle z^*u^* \rangle$. Then $i^* \in C_{Y^*}(z^*) \leq C_K(j)^*$ and z^*i^* is fused to z^* or u^*, and hence is in Z_4^*, so $i^* \in Z_4^*$. Then as $C_{Y^*}(i^*) \not\leq C_K(j)^*$, it follows that $\langle i, j \rangle = J$ contains no conjugate of z, so J is a definite line in Z_4. Then $R_4 \leq C_K(J) \leq P_4$ with $C_K(J)/R_4 \cong \Omega_6^+(2)$ and $X = C_G(J) \not\leq K$.

Let $X' = X/J$. Again $C_K(J)'$ controls 2-fusion in $C_K(J)'$, so Z_4' is a strongly closed abelian subgroup of $C_K(J)'$ in X'. This time there are two X'-classes of involutions z' and v' in Z_4' corresponding to the singular and nonsingular points of the orthogonal space Z_4'. As both zJ and vJ contain a member of z^G, both z' and v' fix a unique point of $X'/C_K(J)'$. But now the argument of the previous paragraph applied to $u \in X - K$ fused under Y to v supplies a contradiction.

So $C_G(j_2) \leq K$ and $j = j_3$. By 7.5 we may take $j \in E$ and $C_K(j) \leq P_2$. Then V^* and E^* are strongly closed abelian subgroups of $C_K(j)^*$ and we argue as above on $u \in Y - K$ fused under Y to a conjugate of z in $E - V$ to obtain a contradiction and complete the proof. Q.E.D.

Theorem 8.7. *Each group of type* ${}^2E_6(2)$ *is isomorphic to* ${}^2E_6(2)$.

Proof. We must show G is isomorphic to G_0. We use the machinery of Section 37 of [SG] to do so. In particular we construct uniqueness systems $\mathcal{U}$ and $\mathcal{U}_0$ for G and G_0.

Let Δ be the graph with vertex set z^G and $\Delta(z) = t^H$. Then G is an edge and vertex transitive group of automorphisms. Define Δ_0 for G_0 similarly. By 7.9, Δ_0 is simply connected.

Let θ be the complete graph with vertex set z^{P_3}. Then θ is a subgraph of Δ and P_3 is vertex and edge transitive on θ. Define θ_0 for G_0 similarly. As $C_H(t)$ is transitive on $t^H \cap E - V$, G has two orbits on triangles of Δ, so each triangle in Δ is fused under G_0 into θ.

Let $\mathcal{U} = (G, \Delta, P_3, \theta)$ and $\mathcal{U}_0 = (G_0, P_3^0, \Delta_0, \theta_0)$. As G_0 is simple, Δ_0 is simply connected, and each triangle in Δ_0 is fused into θ_0, so to show $G \cong G_0$ it suffices by Exercise 13.1 in [SG] to show that $\mathcal{U}$ and $\mathcal{U}_0$ are equivalent uniqueness systems.

It is trivial that $\mathcal{U}$ and $\mathcal{U}_0$ are uniqueness systems, given 8.6. The maps α, ζ define a similarity of $\mathcal{U}$ and $\mathcal{U}_0$ in the sense of section 37 of [SG]. To complete the proof we appeal to Exercise 13.3.3 in [SG]. For this we need geometries Γ and Γ_0 for G and G_0 respectively. Define $\Gamma = \Gamma(G, \mathcal{F})$ to be the coset geometry of $\mathcal{F} = (H, P_2, P_3)$ and define Γ_0 similarly. Hypothesis ($\Gamma 0$) of section 38 of [SG] can be seen to be satisfied by Γ and Γ_0 by checking the conditions at the top of page 205 of [SG]. Observe Γ is isomorphic to the geometry with point set z^G, line set V^G, and plane set V_3^G, with incidence defined by inclusion. A similar remark holds for Γ_0. Thus Δ and Δ_0 are isomorphic to the collinearity graphs of Γ and Γ_0, respectively, via the map $z^x \mapsto Hx$. Using these isomorphisms, Hypotheses (Γi), $1 \le i \le 5$, of section 38 of [SG] are easy to check as are the remaining conditions of Exercise 13.3.3 of [SG].

Q.E.D.

§9. Groups of type $\mathbf{Z}_2/{}^2E_6(2)$

Define a group $\hat{G}$ to be of *type* $\mathbf{Z}_2/{}^2E_6(2)$ if $\hat{G}$ possesses an involution z such that $\hat{H} = C_{\hat{G}}(z)$ satisfies $Q = F^*(\hat{H}) \cong 2^{1+20}$ and $\hat{H}$ has a subgroup H of index 2 with $H/Q \cong U_6(2)$, and z is not weakly closed in Q with respect to $\hat{G}$.

Throughout this section assume $\hat{G}$ is of type $\mathbf{Z}_2/{}^2E_6(2)$ and let z be an involution in $\hat{G}$ such that $\hat{H} = C_{\hat{G}}(z)$ and $Q = F^*(\hat{H})$ satisfy our hypotheses. We will show that $\hat{G}$ has a subgroup G of index 2 such that $H = C_G(z)$. Hence G is of type ${}^2E_6(2)$ and hence by Theorem 8.7:

Theorem 9.1. *If $\hat{G}$ is of type $\mathbf{Z}_2/{}^2E_6(2)$ then $F^*(\hat{G})$ is of index*

2 in $\hat{G}$ and isomorphic to ${}^2E_6(2)$.

Much of the initial analysis is the same as that for groups of type ${}^2E_6(2)$, so rather than repeat all details we only indicate where more needs to be said. Adopt the notation of section 6. In particular let $t = z^g \in Q - \{z\}$ and $E = Q \cap Q^g$. We observe first that

(9.2) (1) *$\hat{H}/\hat{Q}$ is the extension of $H^* \cong U_6(2)$ by an involutory outer automorphism τ.*

(2) *Lemma* 6.1 *holds in $\hat{G}$ with $N_{\hat{H}}(R^*)$ the split extension of $R^* \cong E_{2^9}$ by $L_3(4)$ extended by a field automorphism. This time $\hat{P}_2 = N_{\hat{G}}(V) = XC_{\hat{H}}(V)$ with*

$$R = O_2(N_{\hat{G}}(V)) = C_X(V),$$

$\hat{P}_2/R = X/R \times C_{\hat{G}}(V)/R$, $X/R \cong S_3$, and $C_{\hat{G}}(V)/R$ the extension of $L_3(4)$ by a field automorphism.

Proof. As $F^*(\hat{H}) = Q$ and H is of index 2 in $\hat{H}$, $F^*(\hat{H}/Q) = H^* \cong U_6(2)$ and hence (1) holds. The proof of Lemma 6.1 then goes through virtually unchanged once we observe that if $R \le \hat{T} \in Syl_2(\hat{H})$ and $T = \hat{T} \cap H$, then $J(\hat{T}/Q) = J(T^*) \cong E_{2^9}$. This follows from the fact that $N_{H^*}(J(T^*))$ is the parabolic described in 6.1.2 and $N_{\hat{H}/Q}(J(T^*))$ is the split extension of $J(T^*)$ by $L_3(4)$ extended by a field automorphism τ. Then as $m(J(T^*)/C_{J(T^*)}(\tau)) = 3$ while $C_{J(T^*)}(\tau)$ is not centralized by a complement $L_3(2)$ in $N_{H^*}(J(T^*)) \cap C_{H^*}(\tau)$, we conclude $J(T^*) = J(\hat{T})$ as claimed. Q.E.D.

Now with the analogue of 6.1 established, Lemma 6.2 also holds in $\hat{G}$ since its proof goes through verbatim. Similarly the analogue of Lemma 8.1 holds. Indeed if we let $\hat{G}_0$ be the extension of $G_0 = {}^2E_6(2)$ by the graph-field automorphism σ of Lemma 7.10, then $\hat{G}_0$ is of type $\mathbf{Z}_2/{}^2E_6(2)$ with $\hat{H}_0 = H_0\langle\sigma\rangle$. By 8.1, $\tilde{H}_0 \cong \tilde{H}$, and hence by 7.14, we have an isomorphism $\varphi : \hat{H}_0/\langle z_0\rangle \to \hat{H}/\langle z\rangle$. Let $\tilde{L}_0$ be then image in $\tilde{H}_0$ of a σ-invariant Levi factor of H_0 and $\tilde{L} = \varphi(\tilde{L}_0)$. Finally let $u \in \hat{H}$ with $\tilde{u} = \varphi(\sigma)$. Then by 7.10:

(9.3) (1) *$C_H(u)/C_Q(u) \cong Sp_6(2)$, $C_Q(u) = D_1D_2$ where $D_1 \cap D_2 = \langle z\rangle$, $\tilde{D}_1$ is the natural module for $C_H(u)/C_Q(u)$, and $C_Q(u)/D_1$ is the spin module.*

(9.4) *u is an involution.*

Proof. As $\tilde{u}$ is an involution, $u^2 = 1$ or z, so it remains to show $u^2 \neq z$. To see this we consider the local subgroup $\hat{P}_2$ of 9.2. Let $\bar{P}_2 = \hat{P}_2/V$. The isomorphism φ induces an isomorphism $\varphi : N_{\hat{H}_0}(V_0)/V_0 \to N_{\hat{H}}(V)/V$ which extends to an isomorphism $\psi : \bar{P}_{2,0} = P_{2,0}/V_0 \to \bar{P}_2$ by 21.12 in [3T] and 9.2. Hence by 7.10, $\bar{u}$ centralizes a subgroup $\bar{I} \cong S_3$ faithful on V. Then $I \cong S_4$ and $\langle u\rangle V \trianglelefteq I\langle u\rangle$, so it follows that $u^2 \neq z$, and hence indeed u is an involution. Q.E.D.

(9.5) (1) *All involutions in H are fused under $\hat{G}$ into Q.*

Proof. Let $j \in H$ be an involution. We wish to show $j^{\hat{G}} \cap Q \neq \varnothing$, so we may assume $j^* \neq 1$. Then by 7.7 and as $\varphi : \tilde{H}_0 \to \tilde{H}$ is an isomorphism, we may take $j^* \in R^*$ and j^* of type j_1, j_2 or j_3. Then by 7.4, $m([j, \tilde{Q}]) = 6, 8, 10$ in the respective case. Further by 7.4.2, if j^* is of type j_3 then Q is transitive on the involutions in jQ, so as $Q^g \cap H$ contains an involution in jQ, each involution j with j^* of type j_3 is fused into Q under $\hat{G}$.

In the remaining cases if $i \in jQ$ is an involution then $i = jx$ for some $\tilde{x} \in C_{\tilde{Q}}(j)$ and if $\tilde{x} \in [j, \tilde{Q}]$ then i is fused to j or jz under Q. From the proof of 7.4 and recalling that $\tilde{H} \cong \tilde{H}_0$, $\tilde{L}$ contains a subgroup $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$ with $\tilde{M}_1 \cong S_3$, $\tilde{M}_2 \cong U_4(2)$, and $\tilde{Q} = (\tilde{Q}_1 \oplus \tilde{Q}_2) \perp \tilde{Q}_3$ corresponding to the decomposition described in the proof of 7.4.

Suppose j^* is of type j_1. Then as we saw during the proof of 7.4, we may choose $\tilde{j} \in \tilde{M}_1$, so that $\tilde{M}_2 \leq C_{\tilde{L}}(j)$, $\tilde{Q}_1 = [\tilde{Q}, j]$, $C_{\tilde{Q}}(j) = \tilde{Q}_1 \oplus \tilde{Q}_3$, and $C_{\tilde{Q}}(j) = [C_{\tilde{Q}}(j), M_2]$. Then as $C_{\tilde{L}}(j) = O^2(C_{\tilde{L}}(j))$, also $C_{\tilde{H}}(j) = O^2(C_{\tilde{H}}(j))$, and hence $C_{\tilde{H}}(j) = C_H(j)/\langle z\rangle$. Thus if jx is an involution then x is an involution, so as $\tilde{M}_2$ is transitive on singular vectors of $\tilde{Q}_3$, each involution in jQ is conjugate under $C_H(j)$ to j, jz, jx, or jxz, for some fixed $\tilde{x} \in \tilde{Q}_3$ singular. Then as we may choose $x \in E$ and $j \in Q^g \cap H$, each involution $j \in H$ with j^* of type j_1 is fused into Q under $\hat{G}$.

Finally the case j^* of type j_2 is quite similar. Namely from the proof of 7.4, we may take $\tilde{j} \in \tilde{M}_2$ and $C_{\tilde{Q}}(j) = [\tilde{Q}, j] \oplus \tilde{Q}_4$ with $\tilde{Q}_4 \leq \tilde{Q}_1 \oplus \tilde{Q}_2$ a nondegenerate 4-dimensional space of sign $+1$ and a Sylow 3-subgroup of $C_L(j)$ is transitive on the singular vectors of $\tilde{Q}_4$ and one such is contained in E. So we can repeat the argument of the previous paragraph.

Q.E.D.

(9.6) $u^{\hat{G}} \cap H = \varnothing$.

Proof. Assume otherwise. Then by 9.5, $u^{\hat{G}} \cap Q \neq \varnothing$. Suppose first that $u = z^y$ for some $y \in \hat{G}$. Then as H^* has no $Sp_6(2)$-sections in parabolics, $z \in C_Q(u) = [C_Q(u), C_H(u)] \leq Q^y$, so $u \in Q$, a contradiction.

Therefore $u \notin z^{\hat{G}}$. Let $S \in Syl_2(C_{\hat{H}}(u))$ and $S \leq T_1 \in Syl_2(C_{\hat{G}}(u))$. By 4.3, $Z(T_1) = \langle z^y, u \rangle$ with $u \in Q^y$. Then $Z(T_1) \leq C_{T_1}(z) \leq S$, so $Z(T_1) \leq Z(S) = \langle z, a, u \rangle \cong E_8$ with $\langle z, a \rangle \leq Q$ by 7.10.4. In particular $1 \neq Z(T_1) \cap \langle z, a \rangle$.

Suppose $z^y \in Q$. Then $u \in Q^y \cap \hat{H} \leq H$, a contradiction. Therefore $uz^y \in Q$. Next $uz^y \in u^{Q^y}$ and $u^{\hat{G}} \neq z^{\hat{G}}$, so $uz^y \neq z$. Now $\tilde{a} \in [\tilde{Q}, u]$, so ua or $uaz \in u^Q$, and without loss $ua \in u^Q$. Thus $ua \neq z^y$, so $uz^y \neq a$. This leaves $uz^y = az$, so $z^y = uaz \in (uz)^Q$. Thus $uz \in z^{\hat{G}}$, so we have a contradiction by symmetry between u and uz. Q.E.D.

We are now in a position to complete the proof of Theorem 9.1. By 9.6 and a standard transfer argument such as 37.4 in [FGT], $\hat{G}$ has a subgroup G of index 2 with $u \notin G$. Then as H is the unique subgroup of $\hat{H}$ of index 2, $H = G \cap \hat{H}$. Therefore G is of type ${}^2E_6(2)$, so Theorem 8.7 completes the proof of Theorem 9.1.

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