Advanced Studies in Pure Mathematics 51, 2008
Surveys on Geometry and Integrable Systems
pp. 315–337

Isoparametric geometry and related fields

Reiko Miyaoka

§1. Introduction

The purpose of this paper is to give a perspective in the theory of isoparametric hypersurfaces and related topics. Starting with a brief introduction of the subject, we explain what is now going on, describing important results and remaining problems as well as new aspects.

The classification problem is now being solved in the cases $g = 4, 6$, where g is the number of principal curvatures. In §3, we give a simple proof of Cartan's theorem on the classification for $g = 3$, and generalize the strategy to any g. For this we use the relation between the curvature and the Lax equation, as well as Singer's strongly curvature-homogeneous theorem. On the other hand, the behavior of the kernel of the differential of the Gauss map of the focal submanifolds is important. In §4, we treat the degeneracy of the Gauss map, which seems related to the homogeneity.

Some isoparametric hypersurfaces and all the focal submanifolds provide us with many examples of austere submanifolds, whose twisted normal cones are special Lagrangian submanifolds in $\mathbb{C}^n$. In §5, we introduce this topic. We explain how to obtain explicit solutions of the special Lagrangian equation from isoparametric functions (§5.3). Special Lagrangian submanifolds are volume minimizing, and the topology of stable minimal submanifolds is restricted. We discuss the topology of austere submanifolds in §6, using Morse theory as in the proof of Lefschetz' theorem. In §7, we discuss hypersurface geometry from the viewpoint of Hamiltonian systems of hydrodynamic type.

Received February 8, 2002.
Revised March 3, 2007 (updated).
2000 *Mathematics Subject Classification.* Primary 53C42.

Except for some new results obtained mainly in §3, §5 and §6, we give complete references rather than proofs.

§2. Isoparametric hypersurfaces

First, we refer readers to the excellent survey on isoparametric geometry given by Thorbergsson [Th3], who proved that isoparametric submanifolds in S^n with codimension greater than one are homogeneous [Th2],[O].

By isoparametric hypersurfaces, we mean hypersurfaces with constant principal curvatures. A rich class of isoparametric hypersurfaces exists in the sphere, including all homogeneous hypersurfaces, which are classified as principal orbits of the linear isotropy representation of rank two symmetric spaces [HL]. Moreover, there are infinitely many non-homogeneous isoparametric hypersurfaces with four distinct principal curvatures, constructed by using Clifford algebras [OT],[FKM].

The fundamental facts on isoparametric hypersurfaces were obtained by E. Cartan and by E. Münzner [C1]~[C4],[Mu1],[Mu2]. Isoparametric hypersurfaces in S^n exist as families of parallel hypersurfaces; such a family includes two lower dimensional submanifolds called the focal submanifolds. All of these are level sets of the isoparametric function f, which is the restriction of the so-called Cartan-Münzner polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ to S^n. This polynomial is homogeneous and of degree equal to the number g of distinct principal curvatures $\lambda_1 > \lambda_2 > \cdots > \lambda_g$, and it satisfies

$$\begin{cases} |\nabla F|^2 = g^2 r^{2g-2}, & r(x) = |x|, \\ \triangle F = c r^{g-2}, & c = \dfrac{m_1 - m_2}{2} g^2, \end{cases}$$

where m_i is the multiplicity of λ_i [Mu1]. This implies that the hypersurfaces are algebraic, in particular, compact. Moreover, g is known to be limited to 1,2,3,4 and 6.

Isoparametric hypersurfaces with $g = 1$ are hyperspheres, $g = 2$ the Clifford hypersurfaces $S^k \times S^{n-1-k}$, and $g = 3$ the Cartan hypersurfaces given by tubes over standardly embedded Veronese surfaces corresponding to the four division algebras. The last result is known as Cartan's thoerem. Thus all isoparametric hypersurfaces with $g \leq 3$ are homogeneous. Two homogeneous examples for $g = 6$ are given as principal orbits of the isotropy representation of the symmetric spaces $G_2/SO(4)$

and $G_2 \times G_2/G_2$. Dorfmeister and Neher conjecture that these are all for $g = 6$. This has been confirmed in the case of principal curvatures of multiplicity one [DN],[M4], and is still open in the multiplicity two case.

For $g = 4$, H. Ozeki and M. Takeuchi [OT] were the first to construct non-homogeneous examples. In order to distinguish them from the homogeneous ones, they used the classification in [HL], but posed the question of how to distinguish them more naturally [OTT]. The number of principal curvatures and their multiplicities do not answer to this question, because there are homogeneous and non-homogeneous isoparametric families with $g = 4$ both having multiplicities $(m_1, m_2) = (4, 3)$.

Homogenous hypersurfaces in the sphere correspond to integrable Hamiltonian systems of hydrodynamic type in low dimensional cases [F2]. In fact, starting from the Hamiltonian density given by the isoparametric functions, we can deform the associated Hamiltonian system into the N-wave system. This argument uses the explicit forms of the isoparametric functions hence depends also on the classification of homogeneous hypersurfaces. The proof needs calculation for each case, which restricts the proof to lower dimensional cases, though it seems likely to be true for all homogeneous cases.

All this suggests the importance of characterizing homogeneity without using the classification. Ferus, Karcher and Münzner distinguished non-homogeneous hypersurfaces among examples of FKM-type, without using the classification, but using shape operators of the focal submanifolds. In particular, their kernel plays an important role. Incidentally, the author has characterized the homogeneity of isoparametric hypersurfaces with $g = 6$ by using the kernel structure of the focal submanifolds [M2],[M4].

While we were writing this article, a remarkable result on the classification in the case $g = 4$ was reported by Cecil, Chi and Jensen [CCJ]. They show that if the multiplicities satisfy $m_2 \geq 2m_1 - 1$, all isoparametric hypersurfaces are of FKM-type. Together with the known result by Takagi [T] and Ozeki-Takeuchi [OT], the remaining gaps in the classification are the following 4 cases: $(m_1, m_2) = (3, 4), (4, 5), (6, 9), (7, 8)$[1]. In the proof, they use special frame fields of the focal submanifolds, and using the structure equations and the symmetry satisfied by the structure coefficients, they construct Clifford algebras. Here an argument involving ideals of a polynomial ring plays an important role. The proof, therefore, depends on an algebraic argument, as in the proof of the

[1] Immervoll [I] obtained the same result.

Dorfmeister-Neher theorem for $(g, m) = (6, 1)$. For the latter, the author gives a simple geometric proof in [M4], and the case $(g, m) = (6, 2)$ is now in progress [M6]; we give some details in the next section.

§3. Homogeneity

In this section, we propose a strategy to prove the homogeneity of hypersurfaces.

3.1. Gauss equation and Cartan's theorem

Let M be an isoparametric hypersurface in S^n. Let $\lambda_1, \ldots, \lambda_g$ be the principal curvatures, and choose an orthonormal frame $\{e_i\}$ consisting of principal vectors $\{e_{a_1}, \ldots e_{a_m}\}$ of each λ_a, where m is the multiplicity (which depends on λ_a). With respect to this frame, we have the following fundamental formulae. Put

$$\tilde{\nabla}_{e_\alpha} e_\beta = \Lambda^\gamma_{\alpha\beta} e_\gamma + \lambda_\alpha \delta_{\alpha\beta}\xi, \quad \Lambda^\gamma_{\alpha\beta} = -\Lambda^\beta_{\alpha\gamma},$$

where $\tilde{\nabla}$ is the Levi-Civita connection of S^n, $1 \leq \alpha, \beta, k \leq n-1$, and we use the Einstein convention. The curvature tensor $R_{\alpha\beta\gamma\delta}$ of M is given by

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= (1 + \lambda_\alpha\lambda_\beta)(\delta_{\beta\gamma}\delta_{\alpha\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}) \\ &= e_\alpha(\Lambda^\delta_{\beta\gamma}) - e_\beta(\Lambda^\delta_{\alpha\gamma}) + \Lambda^k_{\beta\gamma}\Lambda^\delta_{\alpha k} - \Lambda^k_{\alpha\gamma}\Lambda^\delta_{\beta k} \\ &\quad -\Lambda^k_{\alpha\beta}\Lambda^\delta_{k\gamma} + \Lambda^k_{\beta\alpha}\Lambda^\delta_{k\gamma}, \end{aligned} \tag{1}$$

which we call the Gauss equation. We have the following [M2]:

$$\Lambda^\gamma_{\alpha\beta}(\lambda_\beta - \lambda_\gamma) = \Lambda^\beta_{\gamma\alpha}(\lambda_\alpha - \lambda_\beta) = \Lambda^\alpha_{\beta\gamma}(\lambda_\gamma - \lambda_\alpha), \ (\text{distinct } \lambda_\alpha, \lambda_\beta, \lambda_\gamma) \tag{2}$$

$$\Lambda^\gamma_{ab} = 0, \Lambda^\gamma_{aa} = \Lambda^\gamma_{bb}, \ (\lambda_a = \lambda_b \neq \lambda_\gamma \text{ and } a \neq b). \tag{3}$$

In particular for isoparametric hypersurfaces, we have

$$\Lambda^\gamma_{\alpha\alpha} = 0 \quad \text{if} \quad \lambda_\gamma \neq \lambda_\alpha. \tag{4}$$

Using the Gauss equation, we obtain Cartan's theorem easily. This argument may be applied to the proof of the homogeneity for $g \geq 4$ (see Theorem 3.4).

Proposition 3.1. (Cartan's Theorem [C2]) *Isoparametric hypersurfaces with $g = 3$ are homogeneous.*

Proof: From the Gauss equation,

(5)

$$\begin{aligned} R_{aiaj} &= -(1+\lambda_a\lambda_i)\delta_{ij} \\ &= e_a(\Lambda^j_{ia}) - e_i(\Lambda^j_{aa}) + \Lambda^k_{ia}\Lambda^j_{ak} - \Lambda^k_{aa}\Lambda^j_{ik} - \Lambda^k_{ai}\Lambda^j_{ka} + \Lambda^k_{ia}\Lambda^j_{ka} \\ &= e_a(\Lambda^j_{ia}) + \Lambda^k_{ia}\Lambda^j_{ak} - \Lambda^k_{ai}\Lambda^j_{ka} + \Lambda^k_{ia}\Lambda^j_{ka}, \end{aligned}$$

we know that if λ_a, λ_i, λ_j are distinct, the left hand side vanishes. Moreover, the latter three terms in the last line vanish by (3), because in the summation in k, λ_k must be one of λ_a, λ_i, λ_j. This means that Λ^j_{ia} is constant along e_a. Since Λ^j_{ia} is non-zero only when $\lambda_a, \lambda_i, \lambda_j$ are distinct, and since we know $e_a(\Lambda^j_{ia}) = 0$, we conclude that Λ^j_{ia} is constant in all directions (see (2)). Now recall

Definition ([KN], p.357). A Riemannian manifold M is strongly curvature-homogeneous if, for any points x, y of M, there is a linear isomorphism of T_xM onto T_yM which maps g_x (the metric at x) and $(\nabla^k R)_x$ (higher covariant derivatives of the curvature tensor R), $k = 0, 1, 2, \ldots$ upon g_y and $(\nabla^k R)_y$, $k = 0, 1, 2, \ldots$.

Theorem 3.2. (Singer [S], Nomizu [N]) *If a connected Riemannian manifold M is strongly curvature-homogeneous, then it is locally homogeneous. If moreover, M is complete and simply connected, then it is homogeneous.*

Applying this, we know that M is locally intrinsically homogeneous. Then using the rigidity theorem on hypersurfaces with type number greater than 2 (which we apply to a non-minimal isoparametric hypersurface), we conclude that M is homogeneous. Q.E.D.

Cartan proved the theorem by determining all isoparametric functions explicitly, which made it possible to classify the hypersurfaces with $g = 3$.

3.2. Focal submanifolds.

In order to relate the Gauss equation to the Lax equation, we introduce the focal submanifolds of isoparametric hypersurfaces. As is well known, the principal curvatures are given by $\lambda_i = \cot\theta_i$, $\theta_i = \omega + \frac{(i-1)\pi}{g}$, $1 \le i \le g$, $0 < \omega < \frac{\pi}{g}$. Denote each curvature distribution by $D_i = D_{\lambda_i}$. Define a focal map $f_a \colon M \to S^n$ by

$$f_a(p) = \cos\theta_a p + \sin\theta_a \xi_p,$$

and let $\bar{p} = f_a(p)$. Then we have

$$df_a(e_{\hat{j}}) = \sin\theta_a(\lambda_a - \lambda_j)e_{\hat{j}},$$

where $e_{\hat{j}}$ is a principal vector with respect to λ_j, and the right hand side is considered as a vector in $T_{\bar{p}}S^n$ by a parallel translation in $\mathbb{R}^{n+1}$. We always use such identifications in what follows. The rank of f_a is constant and we obtain the focal submanifold M_a of M corresponding to $\lambda_a = \cot\theta_a$: $M_a = \{\cos\theta_a p + \sin\theta_a \xi_p \mid p \in M\}$. We have $T_{\bar{p}}M_a = \oplus_{j\neq a} D_j(q)$, for any $q \in {f_a}^{-1}(\bar{p})$. An orthonormal basis of the normal space of M_a at $\bar{p}$ is given by η_p and $e_{\hat{a}}(p)$, where

$$\eta_p = -\sin\theta_a p + \cos\theta_a \xi_p. \tag{6}$$

Lemma 3.3. [M2],[M4] *When we identify $T_{\bar{p}}M_a$ with $\oplus_{j=1}^{g-1} D_{a+j}(p)$ where the indices are modulo g, the second fundamental tensors B_{η_p} and B_{ζ_p} at $\bar{p}$, where $\zeta_p = e_a(p)$, are given respectively by the symmetric matrices*

$$B_{\eta_p} = \begin{pmatrix} \mu_1 I_{m_1} & 0 & 0 & 0 \\ 0 & \mu_2 I_{m_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mu_{g-1} I_{m_{g-1}} \end{pmatrix},$$

where

$$\mu_i = \frac{1 + \lambda_i \lambda_a}{\lambda_a - \lambda_i}, \tag{7}$$

and

$$B_{\zeta_p} = \begin{pmatrix} 0 & B_{i+1\,i+2} & B_{i+1\,i+3} & \cdots & B_{i+1\,i+b} \\ B_{i+1\,i+2} & 0 & B_{i+2\,i+3} & \cdots & B_{i+2\,i+b} \\ B_{i+1\,i+3} & B_{i+2\,i+3} & 0 & & \vdots \\ \vdots & \vdots & & \ddots & B_{i+4\,i+b} \\ B_{i+1\,i+b} & B_{i+2\,i+b} & \cdots & B_{i+4\,i+b} & 0 \end{pmatrix},$$

where $b = g - 1$ and

$$B_{jk} = \frac{1}{\sin\theta_a} \frac{\Lambda^{k_t}_{j_s a}}{\lambda_j - \lambda_a}, \quad \lambda_{j_s} = \lambda_j, \lambda_{k_t} = \lambda_k,$$

is an $m_j \times m_k$ matrix.

In the following, we denote $\frac{\Lambda^{k_t}_{j_s a}}{\lambda_j - \lambda_a}$ by $\frac{\Lambda^{\hat{k}}_{\hat{j} a}}{\lambda_j - \lambda_a}$ for simplicity.

3.3. Isospectrality

Because any normal vector is expressed as η_q for some $q \in L_a$ as in (6), all the shape operators have eigenvalues μ_i, i.e., they are isospectral. For a unit normal vector $n_t = \cos t\eta + \sin t\zeta$, the shape operator $L(t) = B_{n_t}$ is given by

$$L(t) = \cos tB_\eta + \sin tB_\zeta. \tag{8}$$

Since $L(t)$ is isospectral and symmetric, we have orthogonal matrices $U(t) \in O(n-1-m_a)$ such that

$$B_{n_t} = L(t) = U(t)L(0)U(t)^{-1} = U(t)B_\eta U(t)^{-1}. \tag{9}$$

Define $H(t) \in \mathfrak{o}(n-1-m_a)$ by

$$H(t) = \frac{d}{dt}U(t)\,U(t)^{-1} \tag{10}$$

then L satisfies the Lax equation

$$\frac{d}{dt}L(t) = [H(t), L(t)], \tag{11}$$

i.e.

$$-\sin tB_\eta + \cos tB_\zeta = \cos t[H(t), B_\eta] + \sin t[H(t), B_\zeta]. \tag{12}$$

In particular, we obtain $B_\zeta = [H(0), B_\eta]$. Let $U(t)$ be the solution of (10) with $U(0) = I$. Now the eigenvector $e_i(t)$ of $L(t)$ with respect to the eigenvalue λ_i is given by $e_i(t) = U(t)e_i(0)$, where

$$\frac{d}{dt}e_i(t) = H(t)e_i(t).$$

If we use a moving frame $\{e_i(t)\}$ consisting of the principal vectors along the geodesic $p(t) \subset L_a(p)$ joining p and q such that $\eta = \eta_p$ and $\zeta = \eta_q$, we have by Lemma 3.3,

$$L(t) = \begin{pmatrix} \mu_1 I_{m_1} & 0 & 0 & 0 \\ 0 & \mu_2 I_{m_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mu_{g-1} I_{m_{g-1}} \end{pmatrix},$$

and

$$L(t+\pi/2) = \begin{pmatrix} 0 & & B_{ij} \\ & \ddots & \\ {}^tB_{ij} & & 0 \end{pmatrix},$$

where the $m_i \times m_j$ matrix B_{ij}, $i, j \neq a$ is given by

$$B_{ij} = (b_{\hat{i}\hat{j}}), \quad b_{\hat{i}\hat{j}} = \frac{\Lambda^{\hat{j}}_{\hat{i}a}(p(t))}{\sin\theta_a(\lambda_i - \lambda_a)}.$$

The following is crucial [M6].

Theorem 3.4. *An isoparametric hypersurface is homogeneous if and only if there exists a local frame so that $e_a(\Lambda^j_{ia}) = 0$ holds for all a, i, j such that λ_a, λ_i and λ_j are distinct.*

The philosophy of the proof is the same as the proof of Cartan's theorem in §3.1, though we need more arguments for larger g, because there are many choices of the frame when $m_i > 1$. We omit the details, but instead give the following lemma, which is rather easy to prove.

Lemma 3.5. *$e_a(\Lambda^j_{ia}) = 0$ holds for all a, i, j such that λ_a, λ_i and λ_j are distinct if and only if $H(t) - H(0)$ commutes with $L(t)$.*

Proof: Noting that $\dfrac{d}{dt} = \sin\theta_a \nabla_{e_a}$, we have

$$\begin{aligned} \tfrac{d}{dt}L(t)_{ij} &= \sin\theta_a \nabla_{e_a}(\mu_i\delta_{ij}) \\ &= \sin\theta_a\{e_a(\mu_i\delta_{ij}) - \mu_k\delta_{kj}\Lambda^k_{ai}(t) - \mu_i\delta_{ik}\Lambda^k_{aj}(t)\} \\ &= [H(t), L(t)]_{ij}, \end{aligned} \tag{13}$$

where $H(t) = (\sin\theta_a\Lambda^i_{ak}(t))$, and i denotes the index of the rows. On the other hand, we have

(14)

$$\begin{aligned} \frac{d}{dt}L(t+\pi/2)_{ij} &= \sin\theta_a\nabla_{e_a}\left(\frac{1}{\sin\theta_a}\frac{\Lambda^j_{ia}(t)}{\lambda_i - \lambda_a}\right) \\ &= \sin\theta_a\{e_a(b_{ij}(t)) - b_{kj}(t)\Lambda^k_{ai}(t) - b_{ik}(t)\Lambda^k_{aj}(t)\} \\ &= e_a\left(\frac{\Lambda^j_{ia}(t)}{\lambda_i - \lambda_a}\right) + [H(t), L(t+\pi/2)]_{ij}. \end{aligned}$$

Because we also have $\frac{d}{dt}L(t+\pi/2)_{ij} = [H(t+\pi/2), L(t+\pi/2)]_{ij}$ by (13), $e_a(\Lambda^j_{ia}) = 0$ holds if and only if $H(t) - H(t+\pi/2)$ commutes with $L(t+\pi/2)$. In particular, when $H(0) - H(\pi/2)$ commutes with $L(\pi/2)$, noting that $\frac{d}{dt}L(t) = -\sin tL(0) + \cos tL(\pi/2) = L(t+\pi/2) = [H(t), L(t)]$, we obtain

$$\begin{aligned} \frac{d}{dt}L(t) &= -\sin tL(0) + \cos tL(\pi/2) = \sin tL(\pi) + \cos tL(\pi/2) \\ &= \sin t[H(\pi/2), L(\pi/2)] + \cos t[H(0), L(0)] \\ &= [H(0), L(t)], \end{aligned}$$

hence $H(t) - H(0)$ commutes with $L(t)$. Q.E.D.

These play an important role in the investigation of the case $g = 6$ in [M6].

Next, we demonstrate a relation between the Lax equation and the curvature.

Lemma 3.6. *The Lax equation satisfied by the shape operators of the focal submanifolds is a part of the Gauss equation of M.*

Proof: From (7), we have

$$\begin{aligned} \mu_i\delta_{ij} &= -\frac{(1+\lambda_a\lambda_i)\delta_{ij}}{\lambda_i-\lambda_a} = \frac{R_{aiaj}}{\lambda_i-\lambda_a} \\ &= \frac{e_a(\Lambda^j_{ia})}{\lambda_i-\lambda_a} + \frac{\Lambda^k_{ia}}{\lambda_i-\lambda_a}\Lambda^j_{ak} - \frac{\Lambda^k_{ai}}{\lambda_i-\lambda_a}\Lambda^j_{ka} + \frac{\Lambda^k_{ia}}{\lambda_i-\lambda_a}\Lambda^j_{ka} \\ &= \sin\theta_a\{e_a(b_{ij}) - b_{ik}\Lambda^k_{aj} - b_{jk}\Lambda^k_{ai}\}, \end{aligned} \tag{15}$$

which coincides with (14), where we use (2) to obtain

$$\begin{aligned} -\frac{\Lambda^k_{ai}}{\lambda_i-\lambda_a}\Lambda^j_{ka} + \frac{\Lambda^k_{ia}}{\lambda_i-\lambda_a}\Lambda^j_{ka} &= -\frac{\Lambda^j_{ka}\Lambda^k_{ai}}{\lambda_i-\lambda_a}\left(1-\frac{\lambda_i-\lambda_k}{\lambda_a-\lambda_k}\right) \\ &= \frac{\Lambda^j_{ka}\Lambda^k_{ai}}{\lambda_a-\lambda_k} = -\sin\theta_a b_{jk}\Lambda^k_{ai}. \end{aligned}$$

Q.E.D.

§4. Submanifolds with degenerate Gauss mapping

In [IKM], we discuss submanifolds in the sphere whose generalized Gauss map degenerates, in the sense that the rank of the Gauss map is less than the dimension of the submanifold. The Ferus number has the property that if the Gauss map has rank less than the Ferus number, then the submanifold must be totally geodesic. Here we consider submanifolds whose Gauss map degenerates with rank equal to the Ferus number. We find many examples by using isoparametric hypersurfaces.

For a submanifold M^p in S^n, by the Gauss map $\gamma : M \to \mathrm{Gr}(p+1, n+1)$, we mean the map assigning each point of M to the linear subspace spanned by the tangent space and the position vector. The Gauss map degenerates when the intersection of the kernels of the shape operators contains non-zero elements. In particular, for the focal submanifolds of isoparametric hypersurfaces, we can prove that if the kernel of the shape operator does not depend on the normal direction (hence the Gauss map degenerates), they are homogeneous in all cases but $g = 4$. Thus it is interesting to know what happens when $g = 4$, and how it relates with the homogeneity. In this case, the known examples consist of

(1) Homogeneous ones of Clifford type :
$(m_1, m_2) = (1, k)$, $(2, 2k - 1)$, $(4, 4k - 1)$, (9,6)

(2) Homogeneous ones of non-Clifford type :
$(m_1, m_2) = (2, 2), (4, 5)$

(3) Non-homogeneous ones of Clifford type :
$(m_1, m_2) = (3, 4k), (7, 8k)$.

Investigating all homogeneous examples (excluding, for a technical reason[2] the last case in (1)) we obtain the following :

Theorem 4.1. [IKM] *Let M be a homogeneous hypersurface with four principal curvatures. Let $M_\pm$ be the focal submanifolds of M. Then we have:*

(1) When $(m_1, m_2) = (1, k - 2)$ and if $k \geq 3$, M_+ has degenerate Gauss mapping while M_- does not.

(2) When $(m_1, m_2) = (2, 2k - 3), (4, 4k - 5)$ and $k \geq 2$, M_- has degenerate Gauss mapping while M_+ does not.

(3) When $(m_1, m_2) = (2, 2)$, M_+ has degenerate Gauss mapping while M_- does not.

(4) When $(m_1, m_2) = (4, 5)$, M_- has degenerate Gauss mapping while M_+ does not.

For M_- in (2) with $(m_1, m_2) = (2, 2^p - 3)$, $p \geq 2$ and $(4, 2^q - 5)$, $q \geq 3$, and M_+ in (3), the degenerate Gauss map has rank equal to the Ferus number.

Here, $M_\pm$ are defined as in [FKM]. Note that M_+ is characterized in [W],[PT] as a Clifford-Stiefel submanifold. It is curious that neither all M_+ nor all M_- have degenerate Gauss map. Thus when $g = 4$, the homogeneity does not imply the invariance of the kernel of the Gauss map of both $M_\pm$. But it is still interesting because one of $M_\pm$ has this property. In fact, in [FKM], non-homogenous examples are found

[2]My student F. Kaneda investigated this case and obtained a similar result [K].

when they have kernels whose intersection has non-constant dimension. Conversely, it seems that if the intersection of the kernels of each focal submanifold has a constant dimension, then the hypersurfaces are homogeneous.

§5. Special Lagrangian submanifolds

Special Lagrangian geometry is the intersection of minimal submanifold theory with symplectic geometry. Special Lagrangian submanifolds in $\mathbb{C}^n$ consist of volume minimizing submanifolds among submanifolds with the same homology. They are necessarily non-compact, and may have singularities. Neverthress, they are interesting enough from the viewpoints of minimal submanifold theory and symplectic geometry, related with the stability of minimal graphs, and with integrability of finite and infinite dimensional Hamiltonian systems, as is shown below.

Rich families of examples of special Lagrangian submanifolds in $\mathbb{C}^n$ are obtained from isoparametric hypersurfaces and their focal submanifolds. They are nice examples because

—infinitely many homogeneous and non-homogeneous examples exist,

—they are given as a graph of explicit function, hence provide explicit solutions to the special Lagrangian equation,

—they have clear topology, and

—they are related with Hamiltonian system of hydrodynamic type.

In the following, we investigate these examples.

5.1. Definitions and known results

Let X be a Riemannian manifold with a closed p-form φ which satisfies

$$|\varphi(\zeta)| \leq \mathrm{vol}(\zeta)$$

for any oriented p-plane ζ of T_xX. Then (X, φ) is called a calibrated manifold, and φ a calibration. When a submanifold M of X satisfies $\varphi|_M = \mathrm{vol}$, M is called a φ manifold.

Fact 5.1. If M is a φ manifold of a calibrated manifold (X, φ), then M minimizes volume among manifolds in the same homology class.

Complex submanifolds of a Kähler manifold are examples of ω^p-manifolds, where ω is the Kähler form and p is the complex dimension of M.

When $X = \mathbb{C}^n$, using the coordinates $z_1, \dots, z_n$, we can see that

$$\alpha^\theta = \Re\{e^{i\theta} dz\}, \quad dz = dz_1 \wedge \cdots \wedge dz_n$$

is a calibration, and $(\mathbb{C}^n, \alpha^\theta)$ is a calibrated manifold. Let J be the standard complex structure of $\mathbb{C}^n$, and let $\omega = \frac{i}{\sqrt{2}} \sum dz_j \wedge d\bar{z}_j$ be the Kähler form of $\mathbb{C}^n$. An oriented real n-plane ζ is called Lagrangian if $\omega|_\zeta = 0$, or equivalently, $Ju \perp \zeta$ holds for all $u \in \zeta$. Moreover, ζ is special Lagrangian if there exists $A \in SU(n)$ such that $\zeta = A\zeta_0$ where $\zeta_0 = \mathbb{R}^n = \{(x,0) \in \mathbb{R}^n \oplus \mathbb{R}^n \cong \mathbb{C}^n\}$. A submanifold M in $\mathbb{C}^n$ is called *special Lagrangian* if all the tangent spaces of M are special Lagrangian.

Fact 5.2. [HaL] (1) M is special Lagrangian in $\mathbb{C}^n$ if and only if M is an α^θ-manifold for some θ.

(2) M is special Lagrangian in $\mathbb{C}^n$ if and only if M is minimal and Lagrangian.

5.2. Austere submanifolds and twisted normal cones

A submanifold of a Riemannian manifold is called *austere* [HaL] when all the shape operators have eigenvalues in pairs $\pm\lambda$ for each eigenvalue λ (including 0). All minimal surfaces are austere. A submanifold M of $S^{n-1} \subset \mathbb{R}^n$ is austere if and only if the cone CM is austere in $\mathbb{R}^n$. Austere 3-folds in $\mathbb{R}^n$ have been classified by Bryant [B].

Identifying $T_x^*\mathbb{R}^n = \mathbb{C}^n$, we have $J(X,Y) = (-Y, X)$ for $(X,Y) \in T\mathbb{C}^n$. Thus if M is a submanifold of $\mathbb{R}^n$, then the canonically embedded normal bundle

$$NM = \{(x, \nu(x)) \mid x \in M, \nu(x) \in NM\} \subset T\mathbb{R}^n \cong T^*\mathbb{R}^n$$

is Lagrangian.

Fact 5.3. [HaL] (1) When M is a submanifold of $\mathbb{R}^n$, the canonically embedded normal bundle NM is a special Lagrangian submanifold in $\mathbb{C}^n$ if and only if M is austere.

(2) When M is a submanifold of $S^{n-1} \subset \mathbb{R}^n$, the twisted normal cone

$$NCM = \{(tx, s\nu(x)) \mid (x, \nu(x)) \in T^\perp M, s, t \in \mathbb{R}\}$$

is a special Lagrangian submanifold in $\mathbb{C}^n$ if and only if M is austere.

Corollary 5.1. *Let M be a compact austere submanifold in S^{n-1}. Then the twisted normal cone is an n-dimensional cone of least mass in $\mathbb{R}^{2n}$.*

Examples of cones of least mass are given in [HaL] as twisted normal cones over compact minimal surfaces in S^3, Veronese surface in S^4 and Clifford hypersurfaces $S^k \times S^k$ in S^{2k+1}. The latter two are included in the class of submanifolds related to isoparametric hypersurfaces.

Recall that the principal curvatures $\lambda_i(t)$ of the level set $M_t = F^{-1}(t) \cap S^{n-1}$ of the isoparametric function $f = F|_{S^{n-1}}$ are given by

$$\lambda_i = \cot\left(\frac{\pi}{2g}(1-t) + \frac{(i-1)\pi}{g}\right), \quad i = 1, \ldots, g, \quad t \in (-1, 1),$$

where M_0 as well as $M_\pm = M_{\pm 1}$ are minimal.

Proposition 5.2. *Let M be a minimal isoparametric hypersurfaces in S^{n-1}. If each principal curvature has the same multiplicity, then M is austere. Thus the twisted normal cone over M is special Lagrangian, and volume minimizing in $\mathbb{C}^n$.*

Proof: The case $g = 1$ is trivial. When $g = 2$, we have two principal curvatures λ and μ, hence it is austere if and only if $\mu = -\lambda = 1$ and $m_1 = m_2$. When $g = 3$ and 6, the multiplicities are known to coincide, and austere means minimal with principal curvatures $\pm\sqrt{3}, 0$ for $g = 3$, and $\pm(2+\sqrt{3}), \pm 1, \pm(2-\sqrt{3})$, for $g = 6$. When $g = 4$, the principal curvatures $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$ have multiplicities m_1, m_2, m_1, m_2, respectively. Thus it is austere if and only if $\lambda_4 = -\lambda_1 = \sqrt{2}+1, \lambda_3 = -\lambda_2 = \sqrt{2}-1$ and $m_1 = m_2 = 1$, or 2 [St]. Q.E.D.

All known examples of isoparametric hypersurfaces with principal curvatures having the same multiplicities are homogeneous. That is, all the cases for $g \leq 3$, and $m = m_1 = m_2 = 1, 2$ for $g = 4, 6$. Thus the twisted normal cones obtained in Proposition 5.2 are all homogeneous up to Dorfmeister-Neher's conjecture. However, the non-homogeneous case occurs as well.

Proposition 5.3. *The focal submanifolds of any isoparametric hypersurface in S^{n-1} are austere. The twisted normal cones over them are special Lagrangian, and volume minimizing in $\mathbb{C}^n$.*

Proof: When $g = 1$, each focal submanifold is a point manifold, and the twisted normal cone is $\mathbb{R}^n$. When $g = 2$, each focal submanifold is a great subsphere of S^{n-1} and the twisted normal cone is also $\mathbb{R}^n$. The eigenvalues of the shape operators $M_\pm$ are given by (7) as $\mu_i = \frac{1+\lambda\lambda_i}{\lambda_i-\lambda}$ where λ is the maximal or minimal principal curvature and $\lambda \neq \lambda_i$. When $g = 3$, they are given by $\pm\frac{1}{\sqrt{3}}$ with the same multiplicity. When $g = 4$, they are ± 1 with a common multiplicity, and 0. When $g = 6$, they are $\pm\sqrt{3}, \pm\frac{1}{\sqrt{3}}$ and 0 with the same multiplicity. Hence $M_\pm$ are austere. Q.E.D.

Corollary 5.4. *There are infinitely many non-homogeneous compact austere submanifolds of dimension greater than two in the sphere. Thus there are infinitely many non-homogeneous special Lagrangian cones in $\mathbb{C}^n$.*

We have many other compact austere submanifolds in the sphere [IKM].

5.3. Expression as a graph and the SL equation

A Lagrangian submanifold of $\mathbb{C}^n$ is known to be expressable locally as a graph of the gradient of a function [HaL]. The converse is evident: let $G : U \to \mathbb{R}$ be a smooth function on an open domain U of $\mathbb{R}^n$. Then $\Gamma_G = \{(x, \nabla G) \in \mathbb{R}^n \oplus \mathbb{R}^n\}$ is Lagrangian. Moreover this is special Lagrangian with respect to α^θ if and only if Hess G satisfies

$$\Im\{e^{i\theta}\det_{\mathbb{C}}(I + i\mathrm{Hess}\ G)\} = 0. \tag{16}$$

Though the Dirichlet problem for this equation is known to have solutions [HaL], not many explicit solutions are known. Here, we express the twisted normal cone obtained in Proposition 5.2 as a graph, and give the corresponding solutions of (16) explicitly by using the isoparametric functions.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be a Cartan-Münzner function. The minimal isoparametric hypersurface in the unit sphere S^{n-1} of $\mathbb{R}^n$ is given by $M = F^{-1}(0) \cap S^{n-1}$, and its cone by $CM = F^{-1}(0)$, because of the homogeneity of F. This means that we have a local function

$$x_n = f(x_1, \ldots, x_{n-1})$$

such that $F((x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) = 0$.

More generally, consider a hypersurface M given by a graph

$$(x_1, \ldots, x_{n-1}, f(x_1, \ldots, x_{n-1})) \in \mathbb{R}^n.$$

of a certain function f on $\mathbb{R}^{n-1}$. The canonically embedded normal bundle of M is given by

$$(17)\quad NM = \{((x_1,\dots,x_{n-1},f),(-tf_1,\dots,-tf_{n-1},t)) \in \mathbb{R}^n \oplus \mathbb{R}^n\},$$

where $f_i = \dfrac{\partial f}{\partial x_i}$, since the tangent vectors of M are given by $v_i = (0,\dots,1,\dots,f_i)$. Now define a function $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x_1,\dots,x_{n-1},t) = -tf(x_1,\dots,x_{n-1}).$$

Then we have $\nabla G = (-tf_1,\dots,-tf_{n-1},-f)$, and

$$(18)\quad \Gamma_G = \{((x_1,\dots,x_{n-1},t),(-tf_1,\dots,-tf_{n-1},-f))\} \in \mathbb{R}^n \oplus \mathbb{R}^n$$

is a Lagrangian submanifold of $\mathbb{C}^n$. Let us identify $\mathbb{C}^n$ with $\mathbb{R}^n \oplus \mathbb{R}^n$ by $(z_1,\dots,z_n) = (x_1,\dots,x_n,y_1,\dots,y_n)$, $z_j = x_j + iy_j$. Comparing (17) with (18), we see that $f + it$ in (17) is replaced by $t - if$ in (18). This means that Γ_G is the image of NM under $U = \operatorname{diag}\begin{pmatrix}1 & \cdots & 1 & -i\end{pmatrix} \in U(n)$ acting on $\mathbb{C}^n$. Thus NM is special Lagrangian with respect to a calibration α^θ if and only if Γ_G is so with respect to $\alpha^{\theta - n\pi/2}$.

Any hypersurface can be expressed locally as a graph. Thus by the above procedure, we obtain a solution of the special Lagrangian equation from an austere hypersurface given as a graph. For instance, consider austere isoparametric hypersurfaces in S^{n-1}, i.e. isoparametric hypersurfaces satisfying the condition of Proposition 5.2. We give f and G explicitly in the first non-trivial case, $g = 3$. ([OT], pp.23-4). For $x \in \mathbb{F}$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or the real Cayley algebra $\mathcal{C}$, define

$$t(x) = x + \bar{x}, \quad n(x) = x\bar{x}.$$

Then the isoparametric functions on $\mathbb{F}^3 \times \mathbb{R}^2$ are given by $F(u) =$

$$\tfrac{3\sqrt{3}}{2}\{-\xi_1\xi_2(\xi_1+\xi_2) - \xi_1(n(x_1)-n(x_3)) - \xi_2(n(x_2)-n(x_3)) + t(x_1x_2x_3)\},$$

for $u = (x_1,x_2,x_3,\xi_1,\xi_2) \in \mathbb{F}^3 \times \mathbb{R}^2$. Solving $F(u) = 0$, we get $f = \xi_2$ as a function of x_1, x_2, x_3, ξ_1 on a suitable domain in $\mathbb{F}^3 \times \mathbb{R}$:

$$\begin{aligned} f &= \frac{-(\xi_1^2 + (n(x_2) - n(x_3))}{2\xi_1} \\ &\quad \pm \frac{\sqrt{(\xi_1^2 + (n(x_2) - n(x_3))^2 - 4\xi_1(\xi_1(n(x_1) - n(x_3)) - t(x_1x_2x_3))}}{2\xi_1}, \end{aligned}$$

and $G = -tf$.

For $g = 4$ and $m_1 = m_2 = 1, 2$, the polynomial is of degree 4 and we can solve it with respect to some variables [OT], pp.27. For $g = 6$, see [OT], pp.27–29.

§6. Topological aspects

The canonically embedded normal bundle over an austere submanifold M in $\mathbb{R}^n$ is volume minimizing, hence is stable. The stability of minimal submanifolds implies some topological restrictions [Pa],[M1]. We give a topological characterization of complete austere submanifolds in $\mathbb{R}^n$, as well as their normal bundles.

Theorem 6.1. *A complete proper austere submanifold M in $\mathbb{R}^n$ has the homotopy type of a CW complex of dimension not greater than $\frac{1}{2}\dim M$.*

Proof: This can be shown by using Morse theory in analogy with the proof of Lefschetz' theorem on algebraic varieties in [AF],[Mi]. First, since M is real analytic and proper, we can apply Morse theory to the squared distance function L_p^2, $p \in \mathbb{R}^n$. On each normal line of M, focal points are located symmetrically on both sides, with the same index at the corresponding focal points. Since the shape operator in this normal direction has at most dim M eigenvalues, which correspond to the focal points in a one to one way (counting multiplicities), there are at most $\frac{1}{2}\dim M$ focal points on each side. If M has a cycle of dimension greater than $\frac{1}{2}\dim M$, a function L_p^2 for some $p \in \mathbb{R}^n$ has a critical point of index greater than $\frac{1}{2}\dim M$, a contradiction. Q.E.D.

Corollary 6.2. *The normal bundle over a complete proper austere submanifold M in $\mathbb{R}^n$ has the homotopy type of a CW complex of dimension not greater than $\frac{1}{2}\dim M$*

More generally, it should be interesting to investigate the topology of special Lagrangian submanifolds, from the viewpoint of stability.

The cone over a minimal submanifold in S^{n-1} has a singularity at the origin, hence Theorem 6.1 does not apply in this case. Moreover, compact minimal submanifolds in S^{n-1} are not stable [S], and we cannot expect a topological restriction. In fact, the topology of an isoparametric hypersurface M can be described as follows: The cases $g = 1, 2$ are obvious. When $g = 3, 4, 6$, the non-trivial $\mathbb{Z}_2$ coefficient homology groups are $H_0(M) = H_{n-1}(M) = \mathbb{Z}_2$ and

$$\begin{array}{ll} g=3 & H_{mi}(M) = 2\mathbb{Z}_2, \quad i=1,2\,(m=1,2,4,8) \\ g=4 & H_{m_i}(M) = H_{n-1-m_i}(M) = \mathbb{Z}_2,\ H_{m_1+m_2}(M) = 2\mathbb{Z}_2,\ i=1,2 \\ g=6 & H_{mi}(M) = 2\mathbb{Z}_2, \quad i=1,2,3,4,5\,(m=1,2) \end{array}$$

hence the sum of the Betti numbers is $2g$. The non-trivial homology groups of the focal submanifolds $M_\pm$ are

$$\begin{array}{ll} g=3 & H_{mi}(\bar{M}) = \mathbb{Z}_2, \quad i=0,1,2\,(m=1,2,4,8) \\ g=4 & H_0(\bar{M}) = H_{m_i}(\bar{M}) = H_{m_1+m_2}(\bar{M}) = H_{n-1-m_i}(\bar{M}) = \mathbb{Z}_2,\ i=1,2 \\ g=6 & H_{mi}(\bar{M}) = \mathbb{Z}_2, \quad i=0,1,2,3,4,5\,(m=1,2) \end{array}$$

hence the sum of the Betti numbers is g. In relation to Morse theory, isoparametric hypersurfaces, and more generally, compact embedded Dupin hypersurfaces, are taut [Th1]. For more details, see [Ce],[CR],[M5].

§7. Relations to integrable systems

D. Joyce has proposed the problem of clarifying the relation between special Lagrangian submanifolds and some integrable systems [J1],[J2]. For instance, minimal surfaces in S^n, $\mathbb{C}P^n$ are given by conformal harmonic maps, whose equations are an integrable system known as the Toda equations [BPW],[M3]. Therefore the twisted normal cones over them are related to integrable systems. Limiting the discussion to examples obtained from isoparametric hypersurfaces, we consider this problem in higher dimensions.

Hamiltonian systems of hydrodynamic type [DuN1],[DuN2], correspond hypersurfaces in space forms [F1]. Roughly speaking, a certain Hamiltonian system on the space of curves (an infinite dimensional space with a certain symplectic structure) corresponds to the Weingarten equation satisfied by the shape operators of a hypersurface in $\mathbb{R}^n$ or in S^n. When these hypersurfaces are homogeneous, the Hamiltonian systems seem to be integrable [F2]. In view of Proposition 5.2, the twisted normal cone over an isoparametric hypersurface is special Lagrangian if and only if the hypersurface is homogeneous up to Dorfmeister-Neher's conjecture, hence corresponds to an integrable Hamiltonian system of hydrodynamic type. However, as for twisted normal cones over focal submanifolds, we have both homogeneous and non-homogeneous examples, and so we can say nothing about relations between the homogeneity and integrability in an easy way. Still we feel there should exist some relation between isoparametric geometry and integrable systems.

We shall give a brief explanation of how hypersurfaces correspond to Hamiltonian systems. Let $\mathcal{C} = \{u = u(x) : \mathbb{R} \to \mathbb{R}^n\}$ be a space of curves in $\mathbb{R}^n$ with a suitable decay condition so that $\mathcal{C}$ becomes a topological linear space. For functionals

$$F(u) = \int f(u, u_x, \dots)dx, \quad G(u) = \int g(u, u_x, \dots)dx$$

where f and g are polynomials in u and its derivatives, we consider a bracket

$$\{F, G\} = \int \frac{\delta F}{\delta u^i}\omega^{ij}\frac{\delta G}{\delta u^j}dx, \tag{19}$$

where δ is the variational derivative [P]

$$\frac{\delta F}{\delta u^i} = \frac{\partial f}{\partial u^i} - (\frac{\partial f}{\partial u^i_x})_x + (\frac{\partial f}{\partial u^i_{xx}})_{xx} - \cdots .$$

In order for $\{\, ,\, \}$ to be a Poisson bracket, ω^{ij} should satisfy certain conditions. When this is of hydrodynamic type i.e., $\omega^{ij} = g^{ij}(u)d + b^{ij}_k(u)u^k_x$, we have:

Fact 7.1 [DuN1] (19) defines a Poisson bracket if and only if

(1) $\det g^{ij} \neq 0$

(2) $g^{ij} = g^{ji}$

(3) $\Gamma^j_{sk}(u)$ defined by $b^{ij}_k = -g^{is}(u)\Gamma^j_{sk}(u)$ is a torsion free flat connection ∇.

(4) $\nabla g = 0$.

When $h(u)$ is independent of the derivatives of u, the Hamiltonian equation with respect to the Hamiltonian function $H = \int h(u)dx$, namely

$$u^i_t = \omega^{ij}(u)\frac{\delta H}{\delta u^j} = v^i_j(u)u^j_x, \quad v^i_j(u) = \nabla^i\nabla_j h, \tag{20}$$

is called a Hamiltonian system of hydrodynamic type [DuN1]. Conversely, when there exists a flat metric g_{ij} such that

$$g_{ik}v^k_j = g_{jk}v^k_i, \quad \nabla_k v^i_j = \nabla_j v^i_k, \tag{21}$$

the system $u^i_t = v^i_j(u)u^j_x$ is of hydrodynamic type [Ts], and v is called local. With respect to flat coordinates, we can express it as

$$u^i_t = h_{ij}u^j_x, \quad h_{ij} = \frac{\partial^2 h}{\partial u^i \partial u^j} \tag{22}$$

Similarly for $\omega^{ij} = g^{ij}d - g^{is}\Gamma^j_{sk}u^k_x + cu^i_x d^{-1}u^j_x$, where c is a constant and d^{-1} is an integral operator, $\{\, , \}$ is a Poisson bracket if and only if $g^{ij} = g^{ji}$, $\nabla g = 0$ where the connection ∇ is defined as above and satisfies $\Gamma^j_{sk} = \Gamma^j_{ks}$, $R^{ij}_{kl} = c(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k)$ for a constant c. Then (20) can be written as

$$u^i_t = (g^{is}\nabla_s\nabla_j h + c\delta^i_j h)u^j_x.$$

Conversely, in the system $u^i_t = v^i_j(u)u^j_x$, v^i_j is of the form $\nabla^i\nabla_j h + c\delta^i_j h$ if there exists a non-degenerate curvature 1 metric g_{ij}, for which v^i_j satisfies (21). In this case, v is called non-local.

Fact 7.2. [FM] The quadratic forms

$$\mathrm{I} = v^k_i g_{kl} v^l_j du^i du^j, \quad \mathrm{II} = v^k_i g_{kj} du^i du^j, \quad \mathrm{III} = g_{ij}du^i du^j,$$

satisfy the Gauss-Coddazi equations if and only if g_{ij} is of curvature 1 and v^i_j satisfies (21).

Therefore, a non-local hydrodynamic system corresponds to a hypersurface in $\mathbb{R}^{n+1}$, and through a certain transformation, a hypersurface in S^{n+1}. In fact when (20) is given, we can change the parameters t, x by the so-called reciprocal transformation [F1]

$$\begin{cases} dX = Bdx + Adt = \frac{1}{2}((u^i)^2 + 1)dx + (h_j u^j - h)dt \\ dT = Ndx + Mdt = dt, \end{cases}$$

then (20) changes into

$$u^i_t = V^i_j u^j_X = \{h_{ij}B - \delta^i_j A\}u^j_X,$$

which is non-local with respect to a new Hamiltonian given by $\tilde{h}(u) = h(u)/(\sum_k \frac{(u^k)^2}{2} + \frac{1}{2})$, hence corresponds to a hypersurface by Fact 7.2. In fact, putting

$$\begin{aligned} r &= (h_1 - u^1\frac{A}{B}, \ldots, h_n - u^n\frac{A}{B}, -\frac{A}{B}), \\ n &= (\frac{u^1}{B}, \ldots, \frac{u^n}{B}, \frac{1}{B} - 1), \end{aligned} \tag{23}$$

we obtain a hypersurface $r : \mathbb{R}^n \to \mathbb{R}^{n+1}$ with unit normal vector n satisfying the Weingarten equation

$$\frac{\partial n}{\partial u^j} = (V^{-1})^i_j(u)\frac{\partial r}{\partial u^i}.$$

Moreover, if we transform

$$\begin{cases} dX = Bdx + Adt = \frac{1}{2}(\sum(u^i)^2 + 1)dx + (h_j u^j - h)dt \\ dT = Ndx + Mdt = hdx + \frac{1}{2}(\sum(h_i)^2 + 1)dt, \end{cases}$$

then (20) changes into

$$u^i_T = \left\{\frac{1}{2}(\sum(u^k)^2 + 1)h_{ij} - (h_k u^k - h)\delta^i_j)\right\} u^j_X$$

and

$$(24) \qquad \begin{cases} r &= \dfrac{1}{AN - BM}(u^i A - h_i B, A, B + AN - BM) \\ n &= \dfrac{1}{BM - AN}(u^i M - h_i N, M - BM + AN, N) \end{cases}$$

defines a hypersurface in S^n, with unit normal vector n satisfying the Weingarten equation

$$(25) \qquad n_T = r_X,$$

which comes from the original Hamiltonian equation. Thus from a given Hamiltonian system of hydrodynamic type, we can construct locally a hypersurface in $\mathbb{R}^{n+1}$ and in S^{n+1}, whose shape operators satisfy (25) for r, n given by (23), and (24), respectively. Note that they correspond via stereographic projection. In particular when h is an isoparametric function on $\mathbb{R}^n$ for $n \leq 10$, (19) can be deformed into the N-wave system which is known to be integrable [F2]. Hence, homogeneous hypersurfaces seem to correspond to integrable systems. We are, however, not sure if the integrability of this system for homogeneous hypersurafaces can be shown directly.

There are many interesting geometrical objects such as Dupin hypersurfaces, tight and taut immersions, and Lie sphere geometry, which have a close relation to this Hamiltonian system. For more details, see [Ce],[CR],[F2],[M5].

References

[A] U. Abresch, Isoparametric hypersurfaces with four or six distinct principal curvatures. Necessary conditions on the multiplicities, Math. Ann., **264** (1983), 283–302.

[AF] A. Andreotti and T. Frankel, The Lefschetz theorem on hyperplane sections, Ann. of Math., **69** (1959), 713–717.

[B] R. L. Bryant, Some remarks on the geometry of austere submanifolds, Boletim da Sociedade Brasileira de Matematica (Nova Sie), **21** (1991), 133–157.

[BPW] J. Bolton, F. Pedit and L. Woodward, Minimal surfaces and the affine Toda field model, J. Reine Angew. Math., **459** (1995), 119–150.

[C1] E. Cartan, Familles de surfaces isoparamétriques dan les espaces à courbure constante, Ann. di Mat., **17** (1938), 177–191.

[C2] E. Cartan, Sur des familles remarkables d'hypersurfaces isoparamétriques dan les espaces sphérique, Math. Z., **45** (1939), 335–367.

[C3] E. Cartan, Sur quelques familles remarquables d'hypersurfaces, C. R. Congrès Math. Liège (1939), 30–41.

[C4] E. Cartan, Sur des familles d'hypersurfaces isoparamétriques des espaces sphèriques à 5 et à 9 dimensions, Revista Univ. Tucumán, **1** (1940), 5–22.

[Ce] T. Cecil, Lie sphere geometry, Universitext, Springer-Verlag, New York etc. (1992).

[CCJ] T. Cecil, Q. S. Chi and G. Jensen, Isoparametric hypersurfaces with four principal curvartures, to appear in Ann. of Math.

[CR] T. Cecil and P. Ryan, Tight and taut immersions of manifolds, Research Notes in Mathematics, **107**, Pitman, Boston, London, Melbourne (1985),

[DN] J. Dorfmeister and E. Neher, Isoparametric hypersufaces, case $g = 6, m = 1$, Comm. in Alg., **13** (1985), 2299–2368.

[DuN1] B. A. Dubrovin and S. P. Novikov, Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method, Dokl. Akad. Nauk SSSR, **270** (1983), 781–785.

[DuN2] B. A. Dubrovin and S. P. Novikov, Hydrodynamics of weakly deformed soliton lattices, Differential geometry and Hamiltonian theory, Russian Math. Surveys, **44** (1989), no. 6, 35–124.

[F1] E. V. Ferapontov, Dupin hypersurfaces and integrable hamiltonian systems of hydrodynamic type, which do not possess Riemann invariants, Diff. Geom. and its Appl., **5** (1995), 121–152.

[F2] E. V. Ferapontov, Isoparametric hypersurfaces in spheres, integrable nondiagonalizable systems of hydrodynamic type, and N-wave systems, Diff. Geom. and its Appl., **5** (1995), 335–369.

[FM] E. Ferapontov and O. Mokhov, Nonlocal Hamiltonian operators of hydrodynamic type that are connected with metrics of constant curvature, Russian Math. Surveys, **45** (1990), 218–219.

[FKM] D. Ferus, H. Karcher and H. F. Münzner, Cliffordalgebren und neue isoparametrische Hyperflächen, Math. Z., **177** (1981), 479–502.

[HaL] R. Harvey and H. B. Lawson, Calibrated geometries, Acta Math., **148** (1982), 47–157.

[HL] W. Y. Hsiang and H. B. Lawson, Minimal submanifolds of low cohomogeneity, J. Diff. Geom., **5** (1971), 1–38.

[I] S. Immervoll, The classification of isoparametric hypersurfaces with four distinct principal curvatures in spheres, to appear in Ann. of Math.

[IKM] G. Ishikawa, M. Kimura and R. Miyaoka, Submanifolds with degenerate Gauss mappings in the spheres, Advanced Stud. in Pure Math., **37** (2002), 115–149.

[J1] D. Joyce, Special Lagrangian 3-folds and integrable systems, this volume.

[J2] D. Joyce, Lectures on special Lagrangian geometry, math.DG/0111111 (2001).

[K] F. Kaneda, Isotropy orbits of the Hermitian symmetric spaces of EIII type, Master's thesis, Sophia Univ., 2003.

[KN] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry II, Interscience Publ., 1963.

[Mi] J. W. Milnor, Morse theory, Ann. of Math. Studies, Princeton U.P., **51** (1963).

[M1] R. Miyaoka, L^2-harmonic 1-forms on a complete stable minimal hypersurface, In: Geometry and Global Analysis, Report of the First MSJ International Research Institute, (eds. by T. Kotake, S. Nishikawa and R. Schoen), 1993, pp. 289–294.

[M2] R. Miyaoka, The linear isotropy group of $G_2/SO(4)$, the Hopf fibering and isoparametric hypersurfaces, Osaka J. of Math., **30** (1993), 179–202.

[M3] R. Miyaoka, The family of isometric superconformal harmonic maps and the affine Toda equations, J. Reine Angew. Math., **481** (1996), 1–25.

[M4] R. Miyaoka, The Dorfmeister-Neher theorem on isoparametric hypersurfaces, math.DG/0602519.

[M5] R. Miyaoka, Isoparametric hypersurfaces, revisited, Sugaku Exposition, **17** (2004), 185–202,

[M6] R. Miyaoka, Geometry of G_2 orbits and isoparametric hypersurfaces, preprint, 2008.

[Mu1] H. F. Münzner, Isoparametrische Hyperflächen in Sphären I, Math. Ann., **251**, (1980), 57–71.

[Mu2] M. F. Münzner, Isoparametrische Hyperflächen in Sphären II, Math. Ann., **256** (1981), 215–232.

[N] K. Nomizu, Sur les algèbres de Lie de générateuer de Killing et l'homogéneité d'une variété riemannienne, Osaka J. Math., **14** (1962), 45–51.

[O] C. Olmos, Isoparametric submanifolds and their homogeneous structures, J. Diff. Geom., **38** (1993), 225–234.

[OT] H. Ozeki and M. Takeuchi, On some types of isoparametric hypersurfaces in spheres I, II, Tohoku Math. J., **27** (1975), 515–559, **28** (1976), 7–55.

[OTT] H. Ozeki, R. Takagi and M. Takeuchi, On isoparametric hypersurfaces (Japanese), Sugaku, **30** (1978), 23–32.

[P] R. Palais, Symmetries of solitons, Bull. Amer. Math. Soc., **34** (1997), 339–403.

[Pa] B. Palmer, Stability of minimal hypersurfaces, Comment Math. Helvetici, **66** (1991), 185–188.

[PT] U. Pinkall and G. Thorbergsson, Deformations of Dupin hypersurfaces, Proc. Amer. Math. Soc., **107** (1989), 1037–1043.

[S] J. Simons, Minimal varieties in Riemannian varieties, Ann. Math., **88** (1968), 62–105.

[Si] I. M. Singer, Infinitesimally homogeneous spaces, Comm. Pure Appl. Math., **13** (1960), 685–697.

[St] S. Stolz, Multiplicities of Dupin hypersurfaces, Invent. Math., **138** (1999), 253–279.

[T] R. Takagi, A class of hypersurfaces with constant principal curvatures in a shpere, J. Diff. Geom., **11** (1976), 225–233.

[TeT] C. L. Terng and G. Thorbergsson, Submanifold geometry in symmetric spaces, J. Diff. Geom., **42** (1995), 665–718.

[Th1] G. Thorbergsson, Dupin hypersurfaces, Bull. London Math. Soc., **15** (1983), 493–498.

[Th2] G. Thorbergsson, Isoparametric foliations and their buildings, Ann. of Math., **133** (1991), 429–446.

[Th3] G. Thorbergsson, A survey on isoparametric hypersurfaces and their generalizations, In: Handbook of Differential Geometry Vol. I, North-Holland, Amsterdam, 2000, pp. 963–995.

[Ts] S.P. Tsarëv, Poisson brackets and one-dimensional Hamiltonian system of hydrodynamic type, Dokl. Akad. Nauk SSSR, **282** (1985), 534–537.

[TT] R. Takagi and T. Takahashi, On the principal curvatures of homogeneous hypersurfaces in a sphere, In: Differential Geometry, in honor of K. Yano, Kinokuniya, 1972, 469–481.

[W] Q. M. Wang, On the topology of Clifford isoparametric hypersurfaces, J. Diff. Geom., **27** (1988), 55–66.

Mathematical Institute
Graduate School of Science
Tohoku University
Aramaki, Aoba-ku
Sendai 980-8578
JAPAN

E-mail address:
`r-miyaok@math.tohoku.ac.jp`