

## Uniform decay estimates for the wave equation in an exterior domain

Hideo Kubo

### Abstract.

The aim of this article is to establish a uniform pointwise decay estimate for the solution of the mixed problem for the linear wave equation in three space dimensions. We prove such an estimate by using the “cut-off method” as in the work of Shibata and Tsutsumi [23]. As an application, we treat the mixed problem for quadratically quasilinear wave equations exterior to a non-trapping obstacle.

### §1. Introduction

Let  $\mathcal{O}$  be a bounded domain with smooth boundary in  $\mathbf{R}^n$  with  $n \geq 3$  and put  $\Omega := \mathbf{R}^n \setminus \overline{\mathcal{O}}$ . We consider the mixed problem:

$$(1.1) \quad (\partial_t^2 - \Delta)u = f, \quad (t, x) \in (0, T) \times \Omega,$$

$$(1.2) \quad u(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\Omega,$$

$$(1.3) \quad u(0, x) = u_0(x), \quad (\partial_t u)(0, x) = u_1(x), \quad x \in \Omega,$$

where  $\Delta = \sum_{j=1}^n \partial_j^2$  and  $\partial_t = \partial_0 = \partial/\partial t$ ,  $\partial_j = \partial/\partial x_j$  ( $j = 1, \dots, n$ ). The aim of this article is to establish a uniform pointwise decay estimate for the solution of the above problem. Following Shibata and Tsutsumi [23], we shall use the so-called “cut-off method” based on the local energy decay estimate. In order to guarantee the local energy decay, we need to restrict the shape of the obstacle  $\mathcal{O}$ . Specifically, we assume that the obstacle is non-trapping. Once we obtain a pointwise decay estimate of type (2.8) below, then we are able to handle the mixed problem for quadratically nonlinear wave equations in three space dimensions. In fact, (2.8) gives us the standard  $O(t^{-1})$  decay with an additional  $O(|x|^{-1})$  decay or  $O((t - |x|)^{-1})$  decay according to the region.

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In [23]  $L^p$ - $L^q$  time decay estimates for the mixed problem (1.1)–(1.3) was derived from that for the Cauchy problem via the cut-off method developed by Shibata [22]. Moreover, global solvability in time of the following nonlinear problem for small initial data was studied in [23]:

$$(1.4) \quad (\partial_t^2 - \Delta)u = F(\partial u, \nabla_x \partial u), \quad (t, x) \in (0, \infty) \times \Omega,$$

$$(1.5) \quad u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega,$$

$$(1.6) \quad u(0, x) = \varepsilon f(x), \quad \partial_t u(0, x) = \varepsilon g(x), \quad x \in \Omega,$$

where  $\varepsilon > 0$ ,  $f, g \in C_0^\infty(\overline{\Omega})$ ,  $\partial = (\partial_t, \nabla_x)$ ,  $\nabla_x = (\partial_1, \dots, \partial_n)$  and

$$F(\partial u, \nabla_x \partial u) = Q(\partial u, \nabla_x \partial u) + O(|\partial u|^3 + |\nabla_x \partial u|^3)$$

near  $(\partial u, \nabla_x \partial u) = 0$ . Here  $Q(\partial u, \nabla_x \partial u)$  is the quadratic part, that is,

$$\begin{aligned} & Q(\partial u, \nabla_x \partial u) \\ &= \sum_{a,b,c=0}^n A_{a,b,c}(\partial_a u)(\partial_b \partial_c u) + \sum_{a,b=0}^n B_{a,b}(\partial_a u)(\partial_b u) \end{aligned}$$

with real constants  $A_{a,b,c}$  and  $B_{a,b}$  satisfying  $A_{a,0,0} = 0$  and  $A_{a,b,c} = A_{a,c,b}$  for all  $a, b$  and  $c$ . It was shown in [23] that the problem admits a unique global small amplitude solution, provided either  $n \geq 6$  or the quadratic part vanishes. Since the dispersive property becomes weaker in the lower space dimensions, it seems difficult to handle the problem by  $L^p$ - $L^q$  time decay estimates when  $3 \leq n \leq 5$  and the quadratic part does not vanish. In spite of the fact, there are already many contributions to that case (see e.g. [3, 4], [5], [12, 13, 14], [19], [20], [21] and the references cited therein). Here we focus on the work of Keel, Smith and Sogge [13, 14] in which an *almost global existence theorem* for the problem (1.4)–(1.6) with  $n = 3$  was shown, provided either the constants  $A_{a,b,c} = 0$  for all  $a, b$  and  $c$ , or the obstacle is star-shaped. Here “almost global existence theorem” means that the lifespan  $T_\varepsilon$  of the solution satisfies  $T_\varepsilon \geq \exp(C/\varepsilon)$  for some positive constant  $C$ .

However, it is not clear from their proof of the almost global existence theorem if the leading part of the solution is localized near the light cone  $t = |x|$ , because their proof relies on the weighted space-time  $L^2$  estimate. On the one hand, it is well-known that the solution of the corresponding Cauchy problem does concentrate close to the light cone in the sense that there is a positive constant  $C$  such that

$$(1.7) \quad |\partial u(t, x)| \leq C(1 + |x|)^{-1}(1 + |t - |x||)^{-\kappa}$$

for  $(t, x) \in [0, T_\varepsilon) \times \mathbf{R}^3$  and suitably chosen  $\kappa > 0$ . We remark that such estimates with hyperbolic weight as (1.7) play an important role in the

Cauchy problem, since it compensates the deficiency of the dispersive property. This idea goes back to the pioneering work of John [7]. Therefore, it seems to be worth while posing the question whether a similar estimate for the solution of the mixed problem can be established or not.

In the Cauchy problem we can make use of the invariance of the wave operator  $\partial_t^2 - \Delta$  under the translations, spatial rotations, scaling and Lorentz boosts. In fact, by introducing the vector fields  $Z$  associated with the invariance, that is,

$$Z = \{\partial_t, \nabla_x, x_i \partial_j - x_j \partial_i, t \partial_t + x \cdot \nabla_x, t \partial_j + x_j \partial_t\},$$

the generalized energy approach was developed by Klainerman [15, 16] in combination with Klainerman's inequality :

$$(1.8) \quad \begin{aligned} & |v(t, x)| (1 + t + |x|)^{(n-1)/2} (1 + |t - |x||)^{1/2} \\ & \leq C \sum_{|\alpha| \leq [n/2] + 1} \|Z^\alpha v(t) : L^2(\mathbf{R}^n)\| \end{aligned}$$

for  $(t, x) \in [0, \infty) \times \mathbf{R}^n$ . This estimate holds for any function  $v(t, x)$  as long as its right-hand side is finite.

On the contrary, if we pose the Dirichlet boundary condition (1.5), then it seems difficult to use the Lorentz boost fields  $L_j = t \partial_j + x_j \partial_t$  ( $j = 1, \dots, n$ ), because they have normal components to the boundary of size  $t$ . Consequently, it would not be possible to have an analogue to (1.8) for the mixed problem. We notice that similar difficulty appears when we consider the system of wave equations with multiple speeds. In that case one can overcome the difficulty by establishing uniform pointwise decay estimates like (1.7) for solutions to the linear problem (see e.g. [17, 1, 6, 18, 24, 9, 11]). Therefore, one possibility to study the mixed problem in the case of quadratic nonlinearity is to derive the corresponding estimate for the linear problem (1.1)–(1.3). This approach is rather similar to that of [23]. The only difference is the fact that our estimate (2.8) below involves the hyperbolic weight as in (1.7) at the cost of the use of spatial rotation fields in addition to translation fields.

This paper is organized as follows. In the next section we state our main results concerning uniform pointwise decay estimates. In the section 3 we give some preliminaries needed for the proof of the main results. The section 4 is devoted to the proof of Theorems 2.1 and 2.2. Making use of the estimates (2.7) and (2.8), we shall give an alternative proof of the almost global existence theorem given by [13, 14] in the section 5.

We conclude this section by introducing the notion of the compatibility condition.

**Definition 1.1.** Let  $m$  be a non-negative integer and let

$$(1.9) \quad \begin{aligned} u_0 &\in H^m(\Omega), \quad u_1 \in H^{m-1}(\Omega), \\ f(t) &\in \bigcap_{j=0}^{m-1} C^j([0, T]; H^{m-1-j}(\Omega)). \end{aligned}$$

For  $2 \leq j \leq m-1$  we put

$$u_j(x) \equiv \Delta u_{j-2}(x) + (\partial_t^{j-2} f)(0, x) \quad \text{a.e. } x \in \Omega.$$

We say that  $u_0$ ,  $u_1$  and  $f$  satisfy the compatibility condition of order  $(m-1)$  for the d'Alembertian equation in  $\Omega$ , if

$$u_j = 0 \quad \text{on} \quad \partial\Omega$$

holds for all  $j$  with  $0 \leq j \leq m-1$ . In addition, when  $u_0$ ,  $u_1$  and  $f$  satisfy the compatibility condition of order  $(m-1)$  for all  $m$ , then we say that  $u_0$ ,  $u_1$  and  $f$  satisfy the compatibility condition to infinite order for the d'Alembertian equation in  $\Omega$ .

## §2. Main results

In order to state our results, we first prepare several notation. Let us put  $\vec{u}_0 := (u_0, u_1)$  and we denote by  $K[\vec{u}_0](t, x)$  the solution of the mixed problem (1.1)–(1.3) with  $f \equiv 0$ . Similarly, we denote by  $L[f](t, x)$  the solution of the problem with  $\vec{u}_0 \equiv 0$ .

Next we introduce vector fields:

$$\partial_0 = \partial_t, \quad \partial_j \quad (j = 1, 2, 3), \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i \quad (1 \leq i < j \leq 3)$$

and denote them by  $\Gamma_j$  ( $j = 0, 1, \dots, 6$ ). Notice that

$$[\Gamma_i, \Gamma_j] = \sum_{k=0}^6 c_{ij}^k \Gamma_k \quad (i, j = 0, 1, \dots, 6),$$

where  $c_{ij}^k$  is a suitable constant and  $[A, B] := AB - BA$ . In addition, we have

$$(2.1) \quad [\Gamma_i, \partial_t^2 - \Delta] = 0 \quad (i = 0, 1, \dots, 6).$$

Denoting  $\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \dots \Gamma_6^{\alpha_6}$  with a multi-index  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_6)$ , we set

$$|v(t, x)|_m = \sum_{|\alpha| \leq m} |\Gamma^\alpha v(t, x)|$$

for a smooth function  $v(t, x)$  and a non-negative integer  $m$ .

Next for  $\nu, \kappa \in \mathbf{R}$  we define weight functions as follows:

$$(2.2) \quad \tilde{\Phi}_\nu(t, x) = \begin{cases} \langle t + |x| \rangle^\nu & \text{if } \nu < 0, \\ \log^{-1} \left( 2 + \frac{\langle t + |x| \rangle}{\langle t - |x| \rangle} \right) & \text{if } \nu = 0, \\ \langle t - |x| \rangle^\nu & \text{if } \nu > 0, \end{cases}$$

$$(2.3) \quad \Phi_\kappa(t) = \begin{cases} \log(2 + t) & \text{if } \kappa = 1, \\ 1 & \text{if } \kappa \neq 1 \end{cases}$$

and

$$(2.4) \quad W(t, x) = \begin{cases} \langle t - |x| \rangle & \text{if } (t, |x|) \in \Lambda, \\ \langle x \rangle & \text{if } (t, |x|) \in [0, \infty)^2 \setminus \Lambda, \end{cases}$$

where  $\Lambda := \{(t, r) \in [0, \infty)^2 \mid r/2 \leq t \leq 2r\}$  and  $\langle y \rangle = \sqrt{1 + |y|^2}$  for  $y \in \mathbf{R}^n$ .

Now we are in a position to state our main results concerning the uniform pointwise decay estimates. Theorem 2.1 is the result for the homogeneous wave equation, while Theorem 2.2 is for the inhomogeneous wave equation.

**Theorem 2.1.** *Suppose that  $\mathcal{O}$  is a non-trapping obstacle. Let  $\tilde{u}_0 \in (C_0^\infty(\bar{\Omega}))^2$  satisfy the compatibility condition to infinite order for the d'Alembertian equation in  $\Omega$  and let  $k$  be a non-negative integer. Then there exists a constant  $C > 0$  such that*

$$(2.5) \quad |K[\tilde{u}_0](t, x)|_k \leq C \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1}$$

for all  $(t, x) \in [0, \infty) \times \Omega$ .

**Theorem 2.2.** *Suppose that  $\mathcal{O}$  is a non-trapping obstacle. Let  $f \in C^\infty([0, T) \times \Omega)$  satisfy the compatibility condition to infinite order for the d'Alembertian equation in  $\Omega$ . For  $\nu, \kappa \geq 0$  and a non-negative integer  $k$ , we put*

$$(2.6) \quad \begin{aligned} & \|f(t) : N_k(\nu, \kappa)\| \\ &= \sup_{(s, x) \in [0, t] \times \Omega} |x| \langle s + |x| \rangle^\nu W(s, x)^\kappa |f(s, x)|_k. \end{aligned}$$

(i) *If  $0 < \nu \leq 2$  and  $\kappa \geq 1$ , then there exists a constant  $C > 0$  such that*

$$(2.7) \quad \begin{aligned} & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L[f](t, x)|_k \\ & \leq C(\Phi_\kappa(t) \|f(t) : N_{k+3}(\nu, \kappa)\| + \|f(0) : N_{k+2}(\nu, \kappa)\|) \end{aligned}$$

for all  $(t, x) \in [0, T) \times \Omega$ .

(ii) If  $1 \leq \nu \leq 2$  and  $\kappa \geq 1$ , then there exists a constant  $C > 0$  such that

$$(2.8) \quad \begin{aligned} & \langle x \rangle \langle t - |x| \rangle^\rho |\partial_{t,x} L[f](t, x)|_k \\ & \leq C(\Phi_\rho(t) \|f(t) : N_{k+4}(\nu, \kappa)\| + \|f(0) : N_{k+3}(\nu + 1, \kappa)\|) \end{aligned}$$

for all  $(t, x) \in [0, T) \times \Omega$ . Here we put  $\rho = \min(\nu, \kappa)$ .

**Remark 2.3.** It is well-known that similar estimates to the above hold for the solution to the Cauchy problem :

$$(2.9) \quad (\partial_t^2 - \Delta)v = g, \quad (t, x) \in (0, T) \times \mathbf{R}^3,$$

$$(2.10) \quad v(0, x) = v_0(x), \quad (\partial_t v)(0, x) = v_1(x), \quad x \in \mathbf{R}^3.$$

We shall give concrete statements in Lemma 3.3, Lemma 3.4 and Corollary 3.5 below, since our main results are deduced from these estimates in combination with the local energy decay estimate.

### §3. Preliminaries

For  $a \geq 1$ , let  $\psi_a$  be a smooth radially symmetric function on  $\mathbf{R}^3$  satisfying

$$\psi_a(x) = 0 \quad (|x| \leq a), \quad \psi_a(x) = 1 \quad (|x| \geq a + 1).$$

We set

$$\Omega_r = \Omega \cap B_r(0),$$

where  $B_r(x)$  stands for an open ball with radius  $r$  centered at  $x \in \mathbf{R}^3$ . Besides, putting  $\vec{v}_0 := (v_0, v_1)$ , we denote by  $K_0[\vec{v}_0](t, x)$  and  $L_0[g](t, x)$  the solution of the Cauchy problem (2.9)–(2.10) with  $g \equiv 0$  and  $\vec{v}_0 \equiv 0$ , respectively.

First we derive identities (3.1) and (3.2) below.

**Lemma 3.1.** Let  $a > 0$  and let  $\mathcal{O}$  be  $\overline{\mathcal{O}} \subset B_a(0)$ . Suppose that  $\vec{u}_0 \in (C^\infty(\Omega))^2$  and  $f \in C^\infty([0, T) \times \Omega)$  satisfy

$$\text{supp } u_j \subset \overline{\Omega_a} \quad (j = 0, 1), \quad \text{supp } f(t, \cdot) \subset \overline{\Omega_{t+a}} \quad (t \geq 0)$$

and the compatibility condition to infinite order for the d'Alembertian equation in  $\Omega$ . Then we have

$$(3.1) \quad K[\vec{u}_0](t, x) = \psi_1(x) K_0[\psi_2 \vec{u}_0](t, x) + \sum_{i=1}^4 K_i[\vec{u}_0](t, x),$$

$$(3.2) \quad L[f](t, x) = \psi_1(x) L_0[\psi_2 f](t, x) + \sum_{i=1}^4 L_i[f](t, x)$$

for all  $(t, x) \in [0, T) \times \Omega$ . Here we set

$$(3.3) \quad K_1[\vec{u}_0](t, x) = (1 - \psi_2(x))L[\psi_1, -\Delta]K_0[\psi_2\vec{u}_0](t, x),$$

$$(3.4) \quad K_2[\vec{u}_0](t, x) = -L_0[\psi_2, -\Delta]L[\psi_1, -\Delta]K_0[\psi_2\vec{u}_0](t, x),$$

$$(3.5) \quad K_3[\vec{u}_0](t, x) = (1 - \psi_3(x))K[(1 - \psi_2)\vec{u}_0](t, x),$$

$$(3.6) \quad K_4[\vec{u}_0](t, x) = -L_0[\psi_3, -\Delta]K[(1 - \psi_2)\vec{u}_0](t, x)$$

and

$$(3.7) \quad L_1[f](t, x) = (1 - \psi_2(x))L[\psi_1, -\Delta]L_0[\psi_2f](t, x),$$

$$(3.8) \quad L_2[f](t, x) = -L_0[\psi_2, -\Delta]L[\psi_1, -\Delta]L_0[\psi_2f](t, x),$$

$$(3.9) \quad L_3[f](t, x) = (1 - \psi_3(x))L[(1 - \psi_2)f](t, x),$$

$$(3.10) \quad L_4[f](t, x) = -L_0[\psi_3, -\Delta]L[(1 - \psi_2)f](t, x).$$

*Proof.* We give a proof of (3.1) only, since (3.2) can be shown similarly. First we show

$$(3.11) \quad K_1[\vec{u}_0] + K_2[\vec{u}_0] = L[\psi_1, -\Delta]K_0[\psi_2\vec{u}_0] \quad \text{in } (0, T) \times \Omega,$$

$$(3.12) \quad K_3[\vec{u}_0] + K_4[\vec{u}_0] = K[(1 - \psi_2)\vec{u}_0] \quad \text{in } (0, T) \times \Omega.$$

Observe that  $[\psi_2, -\Delta] = [\psi_2, \partial_t^2 - \Delta]$ . Therefore, it is easy to see from (3.3) and (3.4) that (3.11) follows from

$$(3.13) \quad \begin{aligned} & \psi_2 L[\psi_1, -\Delta]K_0[\psi_2\vec{u}_0] \\ &= L_0[(\partial_t^2 - \Delta)(\psi_2 L[\psi_1, -\Delta]K_0[\psi_2\vec{u}_0])] \end{aligned}$$

in  $(0, T) \times \mathbf{R}^3$ . In order to verify this identity, it suffices to observe that the left-hand side satisfies the inhomogeneous wave equation in the whole space and the zero initial data. Besides, (3.12) can be deduced from the following identity which is shown similarly to (3.13):

$$\psi_3 K[(1 - \psi_2)\vec{u}_0] = L_0[(\partial_t^2 - \Delta)(\psi_3 K[(1 - \psi_2)\vec{u}_0])]$$

in  $(0, T) \times \mathbf{R}^3$ .

Now, (3.1) follows from (3.11) and (3.12), once we check

$$(3.14) \quad \psi_1 K_0[\psi_2\vec{u}_0] + L[\psi_1, -\Delta]K_0[\psi_2\vec{u}_0] = K[\psi_2\vec{u}_0]$$

in  $(0, T) \times \Omega$ . Since the left-hand side satisfies the homogeneous wave equation in  $(0, T) \times \Omega$  together with the boundary condition (1.2) and it has the initial data  $\psi_2\vec{u}_0$ , we find from the uniqueness of the classical solution that (3.14) holds good. Thus we have shown (3.1). Q.E.D.

Observe that the first terms on the right-hands side of (3.1), (3.2) can be evaluated by applying the known estimates for the whole space

case. While, in  $K_j[\vec{u}_0]$  and  $L_j[f]$  ( $j = 1, \dots, 4$ ), we always have some localized factor in front of the operators  $K$ ,  $L$  and behind of them as well. Therefore the local energy decay estimate works well in estimating  $K_j[\vec{u}_0]$  and  $L_j[f]$ . The following type of local energy decay was established by [23, Lemmas 4.3 and Ap. 4].

**Lemma 3.2.** *Let  $d > 0$  and let  $\mathcal{O}$  be a non-trapping obstacle such that  $\overline{\mathcal{O}} \subset B_d(0)$ . Suppose that  $\vec{u}_0$  and  $f$  satisfy (1.9), the compatibility condition of order  $(m - 1)$  for the d'Alembertian equation in  $\Omega$ , and*

$$\text{supp } u_j \subset \overline{\Omega_a} \quad (j = 0, 1), \quad \text{supp } f(t, \cdot) \subset \overline{\Omega_a} \quad (t \geq 0).$$

*If  $0 < \gamma \leq 2$ ,  $a, b > d$  and  $m \geq 2$ , then there exists a positive constant  $C = C(\gamma, a, b, m, \Omega)$  such that the solution  $u(t)$  of the problem (1.1)–(1.3) satisfies*

$$(3.15) \sum_{|\alpha| \leq m} \|\partial_{t,x}^\alpha u(t, \cdot) : L^2(\Omega_b)\| \leq C(1+t)^{-\gamma} \left( \|\vec{u}_0 : H^m(\Omega) \times H^{m-1}(\Omega)\| \right. \\ \left. + \sup_{0 \leq s \leq t} (1+s)^\gamma \sum_{|\alpha| \leq m-1} \|\partial_{s,x}^\alpha f(s, \cdot) : L^2(\Omega)\| \right)$$

for  $t \in [0, T)$ .

On the one hand, we also need to prepare the known estimates for the Cauchy problem. The first one is the decay estimate for solutions of the homogeneous wave equation due to Asakura [2, Proposition 1.1].

**Lemma 3.3.** *Let  $\tilde{\Phi}_\nu(t, x)$  be the function defined by (2.2). Then for  $\vec{v}_0 \in (C_0^\infty(\mathbf{R}^3))^2$  and  $\nu > 0$ , there is a positive constant  $C = C(\nu)$  such that*

$$(3.16) \quad \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |K_0[\vec{v}_0](t, x)| \\ \leq C \left( \sum_{|\alpha| \leq 1} \| |\cdot| \langle \cdot \rangle^\nu \partial^\alpha v_0 : L^\infty(\mathbf{R}^3) \| + \| |\cdot| \langle \cdot \rangle^\nu v_1 : L^\infty(\mathbf{R}^3) \| \right)$$

for  $(t, x) \in [0, T) \times \mathbf{R}^3$ .

The second one is the decay estimates for solutions of the inhomogeneous wave equation due to Yokoyama [24, Proposition 3.1]. We define

$$(3.17) \quad \|g(t) : M_k(\nu, \kappa)\| \\ = \sup_{(s,x) \in [0,t] \times \mathbf{R}^3} |x| \langle s + |x| \rangle^\nu W(s, x)^\kappa |g(s, x)|_k$$



with  $W(t, x)$  defined by (2.4), and

$$(3.18) \quad \begin{aligned} & \|g(t):M_k(\nu, \kappa; c)\| \\ &= \sup_{(s,x) \in [0,t] \times \mathbf{R}^3} |x| \langle s + |x| \rangle^\nu \langle cs - |x| \rangle^\kappa |g(s, x)|_k, \end{aligned}$$

where  $\nu, \kappa, c \geq 0$  and  $k$  is a non-negative integer.

**Lemma 3.4.** *Let  $\Phi_\kappa(t)$  be the function defined by (2.3).*

(i) *If  $c \geq 0$ ,  $\nu > 0$  and  $\kappa \geq 1$ , then there exists a positive constant  $C = C(c, \nu, \kappa)$  such that*

$$(3.19) \quad \begin{aligned} & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_0[g](t, x)|_k \\ & \leq C(\Phi_\kappa(t) \|g(t):M_k(\nu, \kappa; c)\| + \|g(0):M_{k-1}(\nu, 0; c)\|) \end{aligned}$$

for  $(t, x) \in [0, T] \times \mathbf{R}^3$ . Here the second term on the right-hand side vanishes when  $k = 0$ .

(ii) *Let  $c = 1$ . If  $\nu, \kappa \geq 1$ , then we have*

$$\begin{aligned} & \langle x \rangle \langle t - |x| \rangle^\rho |\partial_{t,x} L_0[g](t, x)|_k \\ & \leq C(\Phi_\rho(t) \|g(t):M_{k+1}(\nu, \kappa; 1)\| + \|g(0):M_k(\nu + 1, 0; 1)\|) \end{aligned}$$

for  $(t, x) \in [0, T] \times \mathbf{R}^3$ . Here  $\rho = \min(\nu, \kappa)$  and the second term on the right-hand side vanishes when  $k = 0$ .

(iii) *Let  $c \neq 1$ . If  $\nu > 0$ ,  $\kappa \geq 1$ , then we have*

$$(3.20) \quad \begin{aligned} & \langle x \rangle \langle t - |x| \rangle^\nu |\partial_{t,x} L_0[g](t, x)|_k \\ & \leq C(\Phi_\kappa(t) \|g(t):M_{k+1}(\nu, \kappa; c)\| + \|g(0):M_k(\nu + 1, 0; c)\|) \end{aligned}$$

for  $(t, x) \in [0, T] \times \mathbf{R}^3$ . Here the second term on the right-hand side vanishes when  $k = 0$ .

The following decay estimates for solutions of the inhomogeneous wave equation are deduced from Lemma 3.4. They are useful in the application for the nonlinear problem.

**Corollary 3.5.** (i) *If  $\nu > 0$ ,  $\kappa \geq 1$ , then we have*

$$(3.21) \quad \begin{aligned} & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_0[g](t, x)|_k \\ & \leq C(\Phi_\kappa(t) \|g(t):M_k(\nu, \kappa)\| + \|g(0):M_{k-1}(\nu, 0)\|) \end{aligned}$$

for  $(t, x) \in [0, T] \times \mathbf{R}^3$ . Here the second term on the right-hand side vanishes when  $k = 0$ .

(ii) If  $\nu, \kappa \geq 1$ , then we have

$$(3.22) \quad \begin{aligned} & \langle x \rangle \langle t - |x| \rangle^\rho |\partial_{t,x} L_0[g](t, x)|_k \\ & \leq C(\Phi_\rho(t) \|g(t) : M_{k+1}(\nu, \kappa)\| + \|g(0) : M_k(\nu + 1, 0)\|) \end{aligned}$$

for  $(t, x) \in [0, T) \times \mathbf{R}^3$ . Here we put  $\rho = \min(\nu, \kappa)$  and the second term on the right-hand side vanishes when  $k = 0$ .

*Proof.* We prove (3.21) only. As for (3.22), we refer to [24] (see also [11]). It follows from (2.1) that

$$\Gamma^\alpha L_0[g] = L_0[\Gamma^\alpha g] + K_0[(\phi_\alpha, \psi_\alpha)],$$

where we put

$$\phi_\alpha(x) = (\Gamma^\alpha L_0[g])(0, x), \quad \psi_\alpha(x) = (\partial_t \Gamma^\alpha L_0[g])(0, x).$$

From the equation (1.1) we get

$$\phi_\alpha(x) = \sum_{|\beta| \leq |\alpha| - 2} C_\beta(\Gamma^\beta g)(0, x), \quad \psi_\alpha(x) = \sum_{|\beta| \leq |\alpha| - 1} C'_\beta(\Gamma^\beta g)(0, x)$$

with suitable constants  $C_\beta, C'_\beta$ . Therefore, it is enough to show

$$(3.23) \quad \begin{aligned} & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_0[\Gamma^\alpha g](t, x)| \\ & \leq C \Phi_\kappa(t) \|g(t) : M_k(\nu, \kappa)\| \end{aligned}$$

for  $(t, x) \in [0, T) \times \mathbf{R}^3$ , because of (3.16).

Recalling (3.17) and using the fact that

$$L_0[w + w^*] \leq L_0[w] + L_0[w^*]$$

for any non-negative functions  $w$  and  $w^*$ , we see from (3.19) with  $c = 1$  and  $c = 0$  that (3.23) follows, hence (3.21) is valid. Q.E.D.

Finally, we introduce the following Sobolev type inequality.

**Lemma 3.6.** *Let  $v \in C_0^2(\bar{\Omega})$ . Then we have*

$$(3.24) \quad \sup_{x \in \Omega} |x| |v(x)| \leq C \sum_{|\alpha| \leq 2} \|\Gamma^\alpha v : L^2(\Omega)\|.$$

*Proof.* It is well-known that for  $w \in C_0^2(\mathbf{R}^3)$  we have

$$\sup_{x \in \mathbf{R}^3} |x| |w(x)| \leq C \sum_{|\alpha| \leq 2} \|\Gamma^\alpha w : L^2(\mathbf{R}^3)\|$$

(for the proof, see e.g. [15]). Now, if we rewrite  $v$  as  $v = \psi_1 v + (1 - \psi_1)v$ , then we see that the left-hand side on (3.24) is evaluated by

$$\begin{aligned} & C \sum_{|\alpha| \leq 2} \|\Gamma^\alpha(\psi_1 v): L^2(\mathbf{R}^3)\| + C|v(x)| \\ & \leq C \sum_{|\alpha| \leq 2} \|\Gamma^\alpha v: L^2(\Omega)\| + C \sum_{|\alpha| \leq 2} \|\partial_x^\alpha v: L^2(\Omega)\|, \end{aligned}$$

hence we obtain (3.24). This completes the proof. Q.E.D.

#### §4. Proof of the decay estimates

In this section we shall carry out the proof of Theorems 2.1 and 2.2.

*Proof of Theorem 2.1.* First, observe that for  $\vec{v}_0 \in (C_0^\infty(\mathbf{R}^3))^2$  we have

$$(4.1) \quad \sum_{|\beta| \leq m} |\Gamma^\beta K_0[\vec{v}_0](t, x)| \leq C \langle t + |x| \rangle^{-1} \langle t - |x| \rangle^{-1}$$

for  $(t, x) \in [0, T) \times \mathbf{R}^3$ . In fact, when  $m = 0$ , we get this estimate from (3.16) with  $\nu = 2$ . Besides, we see from (2.1) that the general case  $m \geq 1$  can be reduced to the case  $m = 0$ . Therefore, the first term on the right-hand side of (3.1) has the desired bound.

Next we consider  $K_1[\vec{u}_0]$ . For this end, we first show

$$(4.2) \quad \sum_{|\beta| \leq m} \|\partial^\beta L[[\psi_1, -\Delta]K_0[\psi_2 \vec{u}_0]](t): L^2(\Omega_4)\| \leq C \langle t \rangle^{-2}$$

for  $t \in (0, T)$ , where  $\partial = (\partial_t, \nabla_x)$ . It is easy to check that

$$[\psi_a, -\Delta]u(t, x) = u(t, x)\Delta\psi_a(x) + 2\nabla_x u(t, x) \cdot \nabla_x \psi_a(x)$$

for  $(t, x) \in (0, T) \times \mathbf{R}^3$ , and

$$\sum_{|\alpha| \leq m} \|\Gamma^\alpha[\psi_a, -\Delta]u(t): L^2(\Omega)\| \leq C \sum_{|\alpha| \leq m+1} \|\partial^\alpha u(t): L^2(\Omega_{a+1})\|$$

for  $t \in (0, T)$ . Now, using (3.15) as  $\vec{u}_0 = 0$ ,  $\gamma = 2$ , the left-hand side of (4.2) is evaluated by

$$\begin{aligned} & C \langle t \rangle^{-2} \sup_{0 \leq s \leq t} (1+s)^2 \sum_{|\alpha| \leq m-1} \|\partial^\alpha [\psi_1, -\Delta]K_0[\psi_2 \vec{u}_0](s): L^2(\Omega)\| \\ & \leq C \langle t \rangle^{-2} \sup_{0 \leq s \leq t} (1+s)^2 \sum_{|\alpha| \leq m} \|\partial^\alpha K_0[\psi_2 \vec{u}_0](s): L^2(\Omega_2)\|. \end{aligned}$$

By (4.1) the last quantity is bounded by  $C\langle t \rangle^{-2}$ , hence (4.2) is valid.

We turn to the estimation of  $K_1[\vec{u}_0]$ . Since  $\text{supp}(1 - \psi_2) \subset \overline{\Omega_3}$ , we see from (3.3) that

$$|K_1[\vec{u}_0](t, x)|_k \leq C \sum_{|\alpha| \leq k} (1 - \psi_3(x)) |\partial^\alpha L[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]](t, x)|.$$

By the Sobolev inequality and (4.2) we get

$$\begin{aligned} |K_1[\vec{u}_0](t, x)|_k &\leq C \sum_{|\alpha| \leq k+2} \|\partial^\alpha L[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]](t) : L^2(\Omega_4)\| \\ &\leq C \langle t \rangle^{-2}. \end{aligned}$$

Noting that  $\text{supp} K_1[\vec{u}_0](t, \cdot) \subset \overline{\Omega_3}$  again, we see that  $K_1[\vec{u}_0]$  has the desired bound.

Next we evaluate  $K_2[\vec{u}_0]$ . Using (3.19) as  $c = 0$ ,  $\nu = 2$  and  $\kappa > 1$ , we have from (3.4)

$$\begin{aligned} &\langle t + |x| \rangle \langle t - |x| \rangle |K_2[\vec{u}_0](t, x)|_k \\ &\leq C \|\psi_2, -\Delta\| L[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]](t) : M_k(2, \kappa; 0)\| \\ &\quad + C \|\psi_2, -\Delta\| L[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]](0) : M_{k-1}(2, 0; 0)\|. \end{aligned}$$

Recalling (3.18), the first term is estimated by

$$\begin{aligned} &C \sup_{(s, x) \in [0, t] \times \mathbf{R}^3} \langle s \rangle^2 |\psi_2, -\Delta\| L[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]](s, x)|_k \\ &\leq C \sup_{s \in [0, t]} \langle s \rangle^2 \sum_{|\beta| \leq k+3} \|\partial^\beta L[\psi_1, -\Delta] K_0[\psi_2 \vec{u}_0]](s) : L^2(\Omega_4)\|, \end{aligned}$$

which is bounded due to (4.2). Besides, we can evaluate the second term in a similar fashion. Hence we see that  $K_2[\vec{u}_0]$  has the desired bound.

Next we consider  $K_3[\vec{u}_0]$ . Using (3.15) as  $f(t) = 0$ ,  $\gamma = 2$ , we get

$$(4.3) \quad \sum_{|\beta| \leq m} \|\partial^\beta K[(1 - \psi_2) \vec{u}_0](t) : L^2(\Omega_4)\| \leq C \langle t \rangle^{-2}$$

for  $t \in [0, T)$ . Therefore, recalling (3.5) and proceeding as in the estimation of  $K_1[\vec{u}_0]$ , we find that  $K_3[\vec{u}_0]$  has the desired bound.

Finally, we evaluate  $K_4[\vec{u}_0]$ . Using (3.19) as  $c = 0$ ,  $\nu = 2$  and  $\kappa > 1$ , we have

$$\begin{aligned} &\langle t + |x| \rangle \langle t - |x| \rangle |K_4[\vec{u}_0](t, x)|_k \\ &\leq C \|\psi_3, -\Delta\| K[(1 - \psi_2) \vec{u}_0]](t) : M_k(2, \kappa; 0)\| \\ &\quad + C \|\psi_3, -\Delta\| K[(1 - \psi_2) \vec{u}_0]](0) : M_{k-1}(2, 0; 0)\|. \end{aligned}$$

Therefore, recalling (3.6) and proceeding as in the estimation of  $K_2[\vec{u}_0]$ , we can conclude from (4.3) that  $K_4[\vec{u}_0]$  has the desired bound, as well. This completes the proof. Q.E.D.

*Proof of Theorem 2.2.* Without loss of generality, we may assume that  $0 \in \mathcal{O}$  by the translation. First we prove (2.7). It is easy to see from (3.21) that the first term on the right-hand side of (3.2) has the desired bound, if we observe that  $\text{supp}(\psi_2 f)(t, \cdot) \subset \Omega$ .

Next we consider  $L_1[f]$ . First we deduce

$$(4.4) \quad \begin{aligned} & \langle t \rangle^\nu \sum_{|\beta| \leq m} \|\partial^\beta L[\psi_1, -\Delta] L_0[\psi_2 f](t) : L^2(\Omega_3)\| \\ & \leq C(\Phi_\kappa(t) \|f(t) : N_m(\nu, \kappa)\| + \|f(0) : N_{m-1}(\nu, 0)\|) \end{aligned}$$

for  $t \in [0, T]$ . Using (3.15) as  $\vec{u}_0 = 0$ ,  $\gamma = \nu$ , the left-hand side of (4.4) is evaluated by

$$\begin{aligned} & C \sup_{0 \leq s \leq t} (1+s)^\nu \sum_{|\alpha| \leq m-1} \|\partial^\alpha [\psi_1, -\Delta] L_0[\psi_2 f](s) : L^2(\Omega)\| \\ & \leq C \sup_{0 \leq s \leq t} (1+s)^\nu \sum_{|\alpha| \leq m} \|\partial^\alpha L_0[\psi_2 f](s) : L^2(\Omega_2)\|. \end{aligned}$$

Noting the fact that  $\tilde{\Phi}_{\nu-1}(s, x)$  is equivalent to  $\langle s \rangle^{\nu-1}$  when  $x \in \Omega_2$ , we see from (3.21) and (2.6) that (4.4) holds.

Since  $\text{supp} L_1[f](t, \cdot) \subset \overline{\Omega}_3$ , by the Sobolev inequality we get from (3.7) and (4.4)

$$(4.5) \quad \begin{aligned} & \langle t \rangle^\nu |L_1[f](t, x)|_k \\ & \leq C(\Phi_\kappa(t) \|f(t) : N_{k+2}(\nu, \kappa)\| + \|f(0) : N_{k+1}(\nu, 0)\|). \end{aligned}$$

Thus we find that  $L_1[f]$  has the desired bound.

Next we evaluate  $L_2[f]$ . Using (3.19) as  $c = 0$ ,  $0 < \nu \leq 2$  and  $\kappa > 1$ , we have from (3.8)

$$\begin{aligned} & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_2[f](t, x)|_k \\ & \leq C \|\psi_2, -\Delta\| L[\psi_1, -\Delta] L_0[\psi_2 f](t) : M_k(\nu, \kappa; 0)\| \\ & \quad + C \|\psi_2, -\Delta\| L[\psi_1, -\Delta] L_0[\psi_2 f](0) : M_{k-1}(\nu, 0; 0)\|. \end{aligned}$$

Recalling (3.18), we see that the first term is estimated by

$$\begin{aligned}
& C \sup_{(s,x) \in [0,t) \times \mathbf{R}^3} \langle s \rangle^\nu |[\psi_2, -\Delta]L[[\psi_1, -\Delta]L_0[\psi_2 f]](s, x)|_k \\
& \leq C \sup_{s \in [0,t)} \langle s \rangle^\nu \sum_{|\beta| \leq k+3} \|\partial^\beta L[[\psi_1, -\Delta]L_0[\psi_2 f]](s) : L^2(\Omega_3)\| \\
& \leq C(\Phi_\kappa(t) \|f(t) : N_{k+3}(\nu, \kappa)\| + \|f(0) : N_{k+2}(\nu, 0)\|),
\end{aligned}$$

thanks to (4.4). Since the second term can be handled similarly, we get

$$\begin{aligned}
(4.6) \quad & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_2[f](t, x)|_k \\
& \leq C(\Phi_\kappa(t) \|f(t) : N_{k+3}(\nu, \kappa)\| + \|f(0) : N_{k+2}(\nu, \kappa)\|).
\end{aligned}$$

Next we consider  $L_3[f]$ . First we derive

$$\begin{aligned}
(4.7) \quad & \langle t \rangle^\nu \sum_{|\beta| \leq m} \|\partial^\beta L[(1 - \psi_2)f](t) : L^2(\Omega_4)\| \\
& \leq C \|f(t) : N_{m-1}(\nu, \kappa)\|.
\end{aligned}$$

Using (3.15) as  $\vec{u}_0 = 0$ ,  $\gamma = \nu$ , the left-hand side is estimated by

$$\begin{aligned}
& C \sup_{0 \leq s \leq t} (1+s)^\nu \sum_{|\alpha| \leq m-1} \|\partial^\alpha ((1 - \psi_2)f)(s) : L^2(\Omega)\| \\
& \leq C \sup_{0 \leq s \leq t} (1+s)^\nu \sup_{x \in \Omega_3} |f(s, x)|_{m-1},
\end{aligned}$$

which implies (4.7), since  $|x|$  is strictly positive by the assumption  $0 \in \mathcal{O}$ .

By the Sobolev inequality we get from (3.9) and (4.7)

$$(4.8) \quad \langle t \rangle^\nu |L_3[f](t, x)|_k \leq C \|f(t) : N_{k+1}(\nu, \kappa)\|.$$

Hence  $L_3[f]$  has the desired bound, since  $\text{supp } L_3[f](t, \cdot) \subset \overline{\Omega_4}$ .

Next we estimate  $L_4[f]$ . Using (3.19) as  $c = 0$ ,  $0 < \nu \leq 2$  and  $\kappa > 1$ , we have from (3.10)

$$\begin{aligned}
& \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_4[f](t, x)|_k \\
& \leq C \|[\psi_3, -\Delta]L[(1 - \psi_2)f](t) : M_k(\nu, \kappa; 0)\| \\
& \quad + C \|[\psi_3, -\Delta]L[(1 - \psi_2)f](0) : M_{k-1}(\nu, 0; 0)\|.
\end{aligned}$$

Since the first term can be estimated by

$$C \sup_{s \in [0,t)} \langle s \rangle^\nu \sum_{|\beta| \leq k+3} \|\partial^\beta L[(1 - \psi_2)f](s) : L^2(\Omega_4)\|$$

and the second term is also estimated similarly, we get from (4.7)

$$(4.9) \quad \begin{aligned} & \langle t + |x| \rangle \tilde{\Phi}_{\nu-1}(t, x) |L_4[f](t, x)|_k \\ & \leq C(\|f(t) : N_{k+2}(\nu, \kappa)\| + \|f(0) : N_{k+1}(\nu, \kappa)\|). \end{aligned}$$

Thus (2.7) follows from (4.5), (4.6), (4.8) and (4.9).

Secondly we prove (2.8). It is easy to see from (3.22) that the first term on the right-hand side of (3.2) has the desired bound. Moreover, it follows from (4.5), (4.8) that  $L_1[f]$  and  $L_3[f]$  have the desired bound, since  $\text{supp } L_j[f](t, \cdot)$  are bounded for  $j = 1, 3$  and  $\Phi_\kappa(t) \leq \Phi_\rho(t)$ .

Next we consider  $L_2[f]$ . Using (3.20) as  $c = 0$ ,  $0 < \nu \leq 2$  and  $\kappa > 1$ , we get

$$\begin{aligned} & \langle x \rangle \langle t - |x| \rangle^\nu |\partial L_2[f](t, x)|_k \\ & \leq C\|[\psi_2, -\Delta]L[\psi_1, -\Delta]L_0[\psi_2 f]](t) : M_{k+1}(\nu, \kappa; 0)\| \\ & \quad + C\|[\psi_2, -\Delta]L[\psi_1, -\Delta]L_0[\psi_2 f]](0) : M_k(\nu + 1, 0; 0)\|. \end{aligned}$$

Therefore, we see from (4.4) that  $L_2[f]$  has the desired bound, as before.

Finally, we estimate  $L_4[f]$ . Using (3.20) as  $c = 0$ ,  $0 < \nu \leq 2$  and  $\kappa > 1$ , we get

$$\begin{aligned} & \langle x \rangle \langle t - |x| \rangle^\nu |\partial L_4[f](t, x)|_k \\ & \leq C\|[\psi_3, -\Delta]L[(1 - \psi_2)f](t) : M_{k+1}(\nu, \kappa; 0)\| \\ & \quad + C\|[\psi_3, -\Delta]L[(1 - \psi_2)f](0) : M_k(\nu + 1, 0; 0)\|. \end{aligned}$$

By virtue of (4.7), we find that  $L_4[f]$  has the desired bound, as before. Thus we have shown (2.8). This completes the proof. Q.E.D.

## §5. Application

The aim of this section is to apply the uniform pointwise decay estimates given by Theorems 2.1 and 2.2 for the mixed problem to the quasilinear wave equation :

$$(5.1) \quad (\partial_t^2 - \Delta)u = F(\partial u, \nabla_x \partial u), \quad (t, x) \in (0, \infty) \times \Omega,$$

$$(5.2) \quad u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \partial\Omega,$$

$$(5.3) \quad u(0, x) = \varepsilon \phi(x), \quad \partial_t u(0, x) = \varepsilon \psi(x), \quad x \in \Omega,$$

where  $\varepsilon$  is a positive parameter,  $\phi, \psi \in C_0^\infty(\bar{\Omega})$  and

$$(5.4) \quad \begin{aligned} & F(\partial u, \nabla_x \partial u) \\ & = \sum_{a,b,c=0}^3 A_{a,b,c}(\partial_a u)(\partial_b \partial_c u) + \sum_{a,b=0}^3 B_{a,b}(\partial_a u)(\partial_b u) \end{aligned}$$

with real constants  $A_{a,b,c}$  and  $B_{a,b}$  satisfying  $A_{a,0,0} = 0$  and  $A_{a,b,c} = A_{a,c,b}$  for all  $a, b$  and  $c$ .

When we consider the corresponding Cauchy problem, we can evaluate the generalized derivatives  $\Gamma^\alpha u$  in the energy norm without any essential difficulty, thanks to the commutator relations (2.1). On the contrary, it is not so simple to do that any more in the mixed problem, because we have a boundary term in the integration by parts argument which may cause some loss of the derivatives. For this reason, we estimate  $\partial^\alpha u$  and  $\Gamma^\alpha u$  in the energy norm separately and improve the estimate for  $\Gamma^\alpha u$  step by step (recall that  $\Gamma$  contains spatial rotation fields  $\Omega_{ij}$  which does not preserve the boundary condition (5.2)). More precisely, we shall try to evaluate the quantity  $e(T)$  defined by (5.6) below.

In this way, we obtain the following result.

**Theorem 5.1.** *Let  $F(\partial u, \nabla_x \partial u)$  take the form of (5.4) and let  $\phi, \psi \in C_0^\infty(\bar{\Omega})$  satisfy the compatibility condition to infinite order for the d'Alembertian equation in  $\Omega$ . Then there exist positive constants  $\varepsilon_0, C$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  the mixed problem (5.1)–(5.3) admits a unique solution  $u \in C^\infty([0, T_\varepsilon) \times \Omega)$  and its lifespan  $T_\varepsilon$  satisfies*

$$T_\varepsilon \geq \exp(C\varepsilon^{-1}).$$

Moreover, for  $(t, x) \in [0, \exp(C\varepsilon^{-1})) \times \Omega$  we have

$$(5.5) \quad |\partial u(t, x)| \leq C\varepsilon \langle x \rangle^{-1} \langle t - |x| \rangle^{-1}.$$

**Remark 5.2.** *As is already mentioned in the introduction, this result was proved by [13] for the semilinear case and by [14] for the star-shaped obstacles. We underline that our argument below gives a unified proof for these results without using the scaling vector fields  $S = t\partial_t + x \cdot \nabla_x$  and that the pointwise estimate (5.5) shows that the derivatives of the solution have the  $O(t^{-1})$  decay with an additional  $O((t - |x|)^{-1})$  decay or  $O(|x|^{-1})$  decay according to the region. Besides, our approach is also applicable to the system of wave equations with multiple speeds as are the works of [13, 14], since both arguments do not involve the Lorentz boost fields.*

We remark that in order to get a global existence theorem, we need algebraic condition on the nonlinearity called “null condition” in general, due to John [8]. Assuming the “null condition”, Metcalfe, Nakamura and Sogge [20] proved the existence result for multiple speed systems of quadratic, quasilinear wave equations in exterior domains. Recently, we find an alternative proof of their result based on the approach below with suitable modification. This type of result will appear elsewhere ([10]).



*Proof of Theorem 5.1.* Since the existence of the local solution for the mixed problem (5.1)–(5.3) has been shown by [23], what we need to do for showing the large time existence of the solution is to derive suitable a-priori estimates. For simplicity, we consider only the semilinear case, namely  $A_{a,b,c} = 0$  for all  $a, b$  and  $c$  in (5.4), since the general case can be treated analogously. Besides, without loss of generality, we may assume that  $\bar{O} \subset B_1(0)$  by the scaling. We denote the generalized Sobolev norm by  $\|v: H_*^m(\Omega)\| = \sum_{|\alpha| \leq m} \|\Gamma^\alpha v: L^2(\Omega)\|$  and define

$$(5.6) \quad e(T) \equiv \sup_{(t,x) \in [0,T) \times \Omega} \langle x \rangle \langle t - |x| \rangle |\partial u(t, x)|_N \\ + \sup_{t \in [0,T)} \left( \sum_{|\alpha| \leq 2N} \|\partial^\alpha \partial u(t): L^2(\Omega)\| + \langle t \rangle^{-1/2} \|\partial u(t): H_*^{2N-1}(\Omega)\| \right. \\ \left. + \log^{-1/2}(2+t) \|\partial u(t): H_*^{2N-8}(\Omega)\| + \|\partial u(t): H_*^{2N-15}(\Omega)\| \right).$$

Our goal is to show that for  $N \geq 21$ ,  $0 < \varepsilon \leq 1$

$$(5.7) \quad e(T) \leq C_0(\varepsilon + D(T)),$$

where  $C_0$  is a universal constant, independent of  $T$ , and we put

$$(5.8) \quad D(T) = (\log(2+T)e(T)^3)^{1/2} + \log(2+T)e(T)^2.$$

**1st Step.** On the energy estimate for the derivatives in time.

First we set

$$E(u; t) = \frac{1}{2} \int_{\Omega} \{ |\partial_t u(t, x)|^2 + |\nabla_x u(t, x)|^2 \} dx.$$

By the boundary condition (5.2) we have  $\partial_t^j u(t, x) = 0$  ( $j = 0, 1, \dots$ ) for all  $(t, x) \in (0, T) \times \partial\Omega$ . Therefore we find

$$\frac{d}{dt} E(\partial_t^j u; t) = \int_{\Omega} \partial_t^j F(\partial u)(t, x) \partial_t^{j+1} u(t, x) dx.$$

Now, using  $|\partial u(t, x)|_N \leq C \langle t \rangle^{-1} e(T)$ , we get

$$\begin{aligned} \frac{d}{dt} E(\partial_t^j u; t) &\leq C \langle t \rangle^{-1} e(T) \sum_{k=0}^j \int_{\Omega} |\partial_t^k \partial u(t, x)| |\partial_t^{j+1} u(t, x)| dx \\ &\leq C \langle t \rangle^{-1} e(T) \sum_{k=0}^{2N} \|\partial_t^k \partial u(t): L^2(\Omega)\|^2 \\ &\leq C \langle t \rangle^{-1} e(T)^3 \end{aligned}$$

for all  $j$  with  $j = 0, 1, \dots, 2N$ . Thus we obtain

$$(5.9) \quad \sum_{j=0}^{2N} \|\partial_t^j \partial u(t): L^2(\Omega)\| \leq C(\varepsilon + (\log(2+T)e(T)^3)^{1/2})$$

for  $t \in [0, T]$ .

**2nd Step.** On the energy estimate for the space-time derivatives.

Since the spatial derivatives do not preserve the the boundary condition (5.2), we make use of the following elliptic regularity: Let  $m$  be an integer with  $m \geq 2$  and  $v \in H^m(\Omega) \cap H_{\nabla}(\Omega)$ . Then we have

$$(5.10) \quad \|\partial_x^\alpha v: L^2(\Omega)\| \leq C(\|\Delta v: H^{m-2}(\Omega)\| + \|\nabla_x v: L^2(\Omega)\|)$$

for  $|\alpha| = m$ . Here  $H_{\nabla}(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  with respect to the Dirichlet norm  $\|\nabla_x v: L^2(\Omega)\|$ .

Based on the above estimate, we shall show

$$(5.11) \quad \begin{aligned} & \|\partial_x^\alpha \partial_t^j u(t): L^2(\Omega)\| \\ & \leq C(\varepsilon + (\log(2+T)e(T)^3)^{1/2} + e(T)^2) \end{aligned}$$

for all  $(j, \alpha)$  such that  $1 \leq j + |\alpha| \leq 2N + 1$ , by the inductive argument. It is clear that (5.11) follows from (5.9) when  $j + |\alpha| = 1$ .

Next we let  $l$  be an integer with  $l \leq 2N$  and suppose that (5.11) holds for  $1 \leq j + |\alpha| \leq l$ . Let  $j + |\alpha| = l + 1$ . When  $(j, |\alpha|) = (l + 1, 0)$ ,  $(l, 1)$ , (5.11) follows from (5.9). While, when  $j = l + 1 - m$ ,  $|\alpha| = m$  ( $2 \leq m \leq l + 1$ ), (5.10) yields

$$\begin{aligned} & \|\partial_x^\alpha \partial_t^j u(t): L^2(\Omega)\| \\ & \leq C(\|\Delta \partial_t^j u(t): H^{m-2}(\Omega)\| + \|\nabla_x \partial_t^j u(t): L^2(\Omega)\|). \end{aligned}$$

Since  $0 \leq j \leq l - 1 \leq 2N - 1$ , we see from (5.9) that the second term has the desired bound. On the other hand, using (5.1), the first term is estimated by

$$C(\|\partial_t^{l+3-m} u(t): H^{m-2}(\Omega)\| + \|\partial_t^{l+1-m} F(\partial u)(t): H^{m-2}(\Omega)\|),$$

since  $j = l + 1 - m$ . Moreover, the second term is evaluated by

$$C|\partial u(t, x)|_N \sum_{|\alpha| \leq 2N-1} \|\partial^\alpha \partial u(t): L^2(\Omega)\| \leq Ce(T)^2,$$

since  $(m - 2) + (l + 1 - m) = l - 1 \leq 2N - 1$ . (Notice that we obtain the same estimate even if  $F$  contains the second derivatives.) In conclusion,

we have

$$\begin{aligned} \|\partial_x^\alpha \partial_t^{l+1-m} u(t) : L^2(\Omega)\| &\leq C \|\partial_t^{l+3-m} u(t) : H^{m-2}(\Omega)\| \\ &\quad + C(\varepsilon + (\log(2+T) e(T)^3)^{1/2} + e(T)^2) \end{aligned}$$

for  $|\alpha| = m$  and  $2 \leq m \leq l+1$ . Now, if we vary  $m$  from 2 to  $l+1$ , then we can evaluate the first term on the right-hand side by using the assumption of the induction step by step. Indeed, for example, the case  $m = 2$  is reduced to the case  $(j, |\alpha|) = (l+1, 0)$ , the case  $m = 3$  is to  $(j, |\alpha|) = (l, 0)$ ,  $(l, 1)$ , and the case  $m = 4$  is to  $(j, |\alpha|) = (l-1, 0)$ ,  $(l-1, 1)$ ,  $(l-1, 2)$ . Therefore, (5.11) is valid for all  $(j, \alpha)$  such that  $1 \leq j + |\alpha| \leq 2N+1$ .

Thus we find from (5.11) that

$$(5.12) \quad \sum_{|\alpha| \leq 2N} \|\partial^\alpha \partial u(t) : L^2(\Omega)\| \leq C(\varepsilon + D(T))$$

holds for  $t \in [0, T)$ . Here  $D(T)$  is given by (5.8).

**3rd Step.** On the energy estimate for the generalized derivatives.

It follows from (2.1) that

$$\begin{aligned} \frac{d}{dt} E(\Gamma^\alpha u; t) &= \int_{\Omega} \Gamma^\alpha F(\partial u)(t, x) \partial_t \Gamma^\alpha u(t, x) dx \\ &\quad + \int_{\partial\Omega} \nu \cdot \nabla_x \Gamma^\alpha u(t, x) \partial_t \Gamma^\alpha u(t, x) dS, \end{aligned}$$

where  $\nu = \nu(x)$  is the unit outer normal vector at  $x \in \partial\Omega$  and  $dS$  is the surface measure on  $\partial\Omega$ . Since  $|\partial u(t, x)|_N \leq C\langle t \rangle^{-1} e(T)$ , the first term on the right-hand side is estimated by

$$C\langle t \rangle^{-1} e(T) \sum_{|\beta| \leq |\alpha|} \|\Gamma^\beta \partial u(t) : L^2(\Omega)\|^2$$

when  $|\alpha| \leq 2N-1$ . On the other hand, since  $\partial\Omega \subset B_1(0)$ , we have  $|\Gamma^\alpha u(t, x)| \leq C \sum_{|\beta| \leq |\alpha|} |\partial^\beta u(t, x)|$  for all  $(t, x) \in (0, T) \times \partial\Omega$ . Moreover, by the trace theorem, we see that the second term is evaluated by

$$C \sum_{|\beta| \leq |\alpha|+1} \|\partial^\beta \partial u(t) : L^2(\Omega_2)\|^2.$$

Therefore we get

$$\begin{aligned} (5.13) \quad \frac{d}{dt} E(\Gamma^\alpha u; t) &\leq C\langle t \rangle^{-1} e(T) \|\partial u(t) : H_*^{|\alpha|}(\Omega)\|^2 \\ &\quad + C \sum_{|\beta| \leq |\alpha|+1} \|\partial^\beta \partial u(t) : L^2(\Omega_2)\|^2. \end{aligned}$$

First we shall show

$$(5.14) \quad \langle t \rangle^{-1/2} \|\partial u(t) : H_*^{2N-1}(\Omega)\| \leq C(\varepsilon + D(T))$$

for  $t \in [0, T)$ . Since  $\|\partial u(t) : H_*^{2N-1}(\Omega)\| \leq \langle t \rangle^{1/2} e(T)$ , by using (5.13) and (5.12), we get

$$\frac{d}{dt} E(\Gamma^\alpha u; t) \leq C e(T)^3 + C(\varepsilon + D(T))^2$$

for all  $\alpha$  with  $|\alpha| \leq 2N - 1$ . Hence

$$\begin{aligned} E(\Gamma^\alpha u; t) &\leq C\varepsilon^2 + C\langle t \rangle (e(T)^3 + (\varepsilon + D(T))^2) \\ &\leq C\langle t \rangle (\varepsilon^2 + D(T)^2), \end{aligned}$$

which implies (5.14).

Next we deduce

$$(5.15) \quad \log^{-1/2}(2+t) \|\partial u(t) : H_*^{2N-8}(\Omega)\| \leq C(\varepsilon + D(T))$$

for  $t \in [0, T)$ . To this end, we first prove

$$(5.16) \quad \sum_{|\beta| \leq 2N-7} \|\partial^\beta \partial u(t) : L^2(\Omega_2)\| \leq C\langle t \rangle^{-1/2} (\varepsilon + D(T)).$$

Observe that  $u(t, x)$  is decomposed as

$$(5.17) \quad u = \varepsilon K[\phi, \psi] + L[F(\partial u)] \quad \text{in } (0, T) \times \Omega$$

and that (3.15) yields

$$(5.18) \quad \sum_{|\beta| \leq m} \|\partial^\beta \partial K[\phi, \psi](t) : L^2(\Omega_2)\| \leq C\langle t \rangle^{-\nu}$$

for all  $\nu > 0$  and non-negative integer  $m$ . Therefore, in order to show (5.16), it suffices to prove

$$\begin{aligned} (5.19) \quad &\sum_{|\alpha| \leq 2N-6} \langle t \rangle^{1/2} \|\partial^\alpha L[F(\partial u)](t) : L^2(\Omega_2)\| \\ &\leq C(\varepsilon^2 + \log(2+t) e(T)^2), \end{aligned}$$

since  $0 < \varepsilon \leq 1$ . We see from (2.7) that (5.19) follows from

$$(5.20) \quad \|F(\partial u)(t) : N_{2N-3}(1/2, 1)\| \leq C e(T)^2,$$

since  $\tilde{\Phi}_{\nu-1}(t, x)$  is equivalent to  $\langle t \rangle^{\nu-1}$  when  $x \in \Omega_2$ . By (3.24) we get

$$(5.21) \quad \sum_{|\beta| \leq 2N-3} |x| |\Gamma^\beta \partial u(t, x)| \leq C \|\partial u(t): H_*^{2N-1}(\Omega)\| \\ \leq C \langle t \rangle^{1/2} e(T).$$

While, from (2.4) we have

$$(5.22) \quad |\partial u(t, x)|_N \leq C \langle t+r \rangle^{-1} W(t, r)^{-1} e(T).$$

Since (5.21), (5.22) imply (5.20), we thus obtain (5.16).

We now prove (5.15). Since  $\|\partial u(t): H_*^{2N-8}(\Omega)\| \leq \log^{1/2}(2+t) e(T)$ , by using (5.16) and (5.13), we get

$$\frac{d}{dt} E(\Gamma^\alpha u; t) \leq C \langle t \rangle^{-1} \log(2+t) e(T)^3 + C \langle t \rangle^{-1} (\varepsilon + D(T))^2$$

for all  $\alpha$  with  $|\alpha| \leq 2N-8$ . Hence

$$E(\Gamma^\alpha u; t) \leq C\varepsilon^2 + C \log^2(2+t) e(T)^3 + C \log(2+t) (\varepsilon + D(T))^2 \\ \leq C \log(2+t) (\varepsilon^2 + \log(2+t) e(T)^3 + D(T)^2) \\ \leq C \log(2+t) (\varepsilon^2 + D(T)^2),$$

which implies (5.15).

Next we deduce

$$(5.23) \quad \|\partial u(t): H_*^{2N-15}(\Omega)\| \leq C(\varepsilon + D(T))$$

for  $t \in [0, T)$ . To this end, it suffices to prove

$$(5.24) \quad \sum_{|\beta| \leq 2N-14} \|\partial^\beta \partial u(t): L^2(\Omega_2)\| \leq C \langle t \rangle^{-\nu} (\varepsilon + D(T))$$

for  $\nu$  with  $1/2 < \nu < 1$ . In fact, assuming this estimate and using  $\|\partial u(t): H_*^{2N-15}(\Omega)\| \leq e(T)$  and (5.13), we obtain

$$\frac{d}{dt} E(\Gamma^\alpha u; t) \leq C \langle t \rangle^{-1} e(T)^3 + C \langle t \rangle^{-2\nu} (\varepsilon + D(T))^2$$

for all  $\alpha$  with  $|\alpha| \leq 2N-15$ . Hence, by  $2\nu > 1$  we find

$$E(\Gamma^\alpha u; t) \leq C\varepsilon^2 + C \log(2+t) e(T)^3 + C(\varepsilon + D(T))^2 \\ \leq C(\varepsilon^2 + D(T)^2),$$

which implies (5.23).

Finally, we show (5.24). By (5.18) it is enough to prove

$$\sum_{|\alpha| \leq 2N-13} \langle t \rangle^\nu \|\partial^\alpha L[F(\partial u)](t): L^2(\Omega_2)\| \leq C(\varepsilon^2 + D(T)).$$

We see from (2.7) that this estimate follows from

$$(5.25) \quad \|F(\partial u)(t): N_{2N-10}(\nu, 1)\| \leq Ce(T)^2,$$

as before. It follows from (3.24) that

$$\begin{aligned} \sum_{|\beta| \leq 2N-10} |x| |\Gamma^\beta \partial u(t, x)| &\leq C \|\partial u(t): H_*^{2N-8}(\Omega)\| \\ &\leq C \log^{1/2}(2+t) e(T). \end{aligned}$$

Combining this with (5.22), we get (5.25) for  $\nu < 1$ . Thus we find (5.24), hence (5.23).

**4th Step.** On the pointwise decay estimates.

We shall show

$$(5.26) \quad \langle x \rangle \langle t - |x| \rangle |\partial u(t, x)|_N \leq C(\varepsilon + D(T))$$

for  $(t, x) \in [0, T] \times \Omega$ . In order to get this estimate, it suffices to prove

$$\langle x \rangle \langle t - |x| \rangle |\partial L[F(\partial u)](t, x)|_N \leq C(\varepsilon^2 + D(T)),$$

thanks to (5.17) and (2.5). We see from (2.8) that the above estimate follows from

$$(5.27) \quad \|F(\partial u)(t): N_{N+4}(1, 1)\| \leq Ce(T)^2.$$

Observe that when  $N \geq 21$ , we have  $[(N+4)/2] \leq N$ ,  $N+6 \leq 2N-15$ . Therefore, by (5.22) we get

$$|\partial u(t, x)|_{[(N+4)/2]} \leq C \langle t+r \rangle^{-1} W(t, r)^{-1} e(T).$$

While, by (3.24) we obtain

$$|x| |\partial u(t, x)|_{N+4} \leq C \|\partial u(t): H_*^{N+6}(\Omega)\| \leq Ce(T).$$

From these estimates we arrive at (5.27), hence (5.26).

**Final Step.** End of the proof of Theorem 5.1.

It follows from (5.12), (5.14), (5.15), (5.23) and (5.26) that (5.7) holds for  $N \geq 21$  and  $0 < \varepsilon \leq 1$ . Now, we take a positive number  $M$

to be  $M \geq 3C_0$ ,  $e(0) \leq M\varepsilon/2$ . Suppose that  $e(T) \leq M\varepsilon$ . Then we find from (5.7) that

$$(5.28) \quad e(T) \leq C_0\varepsilon + (C_0(M\varepsilon \log(2+T)))^{1/2} + C_0M\varepsilon \log(2+T)) M\varepsilon.$$

Without loss of generality, we may assume  $C_0 \geq 1$ . Then, as long as  $T$  satisfies

$$(5.29) \quad C_0^2 M\varepsilon \log(2+T) \leq 1/9$$

for given  $\varepsilon$ , we see from (5.28) that

$$e(T) \leq \frac{M}{3}\varepsilon + \left(\frac{1}{3} + \frac{1}{9}\right)M\varepsilon = \frac{7}{9}M\varepsilon < M\varepsilon.$$

This means that the solution of the problem (5.1)–(5.3) can be continued, provided (5.29) holds. Thus we have shown the theorem. Q.E.D.

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*Department of Mathematics  
Graduate School of Science  
Osaka University  
Toyonaka, Osaka 560-0043, Japan  
E-mail address: kubo@math.sci.osaka-u.ac.jp*