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Vanishing theorem on the pointwise defect of a rational iteration sequence for moving targets

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§1. Introduction

Let f be a rational map, i.e., a holomorphic endomorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, of degree $d > 1$. The k times iteration of f is denoted by f^k for $k \in \mathbb{N}$.

The Nevanlinna theory for sequences was first studied in [19], [2], [8] and [10], and recently, motivated by complex dynamics, studied in [18], [16] and [15], where the sequence of rational maps correspond to a transcendental meromorphic function. Hence the following definition is natural:

Definition 1.1 (Picard exceptional value). The point $a \in \hat{\mathbb{C}}$ is called a *Picard exceptional value* of $\{f^k\}$ if

$$\# \bigcup_{k \in \mathbb{N}} f^{-k}(a) < \infty.$$

The point $a \in \hat{\mathbb{C}}$ is a Picard exceptional value if and only if it is periodic of period at most two and a and $f(a)$ are critical of order $d-1$. In particular, there exist at most two such values (cf. [9]), which is an analogue of the Picard theorem for transcendental meromorphic functions.

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Notation 1.1. The spherical area measure on $\hat{\mathbb{C}}$, which is normalized as $\sigma(\hat{\mathbb{C}}) = 1$, is denoted by σ, and the chordal distance between $z, w \in \hat{\mathbb{C}}$, which is normalized as $[0, \infty] = 1$, by $[z, w]$. Put $\mathbb{D}(x, r) := \{z \in \hat{\mathbb{C}}; [z, x] < r\}$ for $x \in \hat{\mathbb{C}}$ and $r > 0$.

One of the main aims of the Nevanlinna theory is to generalize the Picard theorem quantitatively by the *defects*, which are defined not only for each constant values but also for moving targets. See [14], Chapter 4 and also the recent significant result by Yamanoi [20].

Clearly, the degree $d = \int_{\hat{\mathbb{C}}} f^*(\mathrm{d}\sigma)$ of f is an analogue of the order (or characteristic) function of a transcendental meromorphic function.

Definition 1.2 (proximities and defects). For a rational map g, the *pointwise proximity function* of f is defined as

$$w(g, f) := \log \frac{1}{[g(\cdot), f(\cdot)]} : \hat{\mathbb{C}} \to [0, +\infty],$$

the *mean proximity* of f as

$$m(g, f) := \int_{\hat{\mathbb{C}}} w(g, f) \mathrm{d}\sigma,$$

and the *Valiron defect* of $\{f^k\}$ as

$$\delta_V(g; \{f^k\}) := \limsup_{k \to \infty} \frac{m(g, f^k)}{d^k}.$$

Convention 1.1. Each point $a \in \hat{\mathbb{C}}$ is identified with the constant map $g \equiv a$.

A point $a \in \hat{\mathbb{C}}$ is called a *Valiron exceptional value* of $\{f^k\}$ if $\delta_V(a; \{f^k\}) > 0$. It is easy to see that every Picard exceptional value of $\{f^k\}$ is a Valiron one. It seems surprising that the converse is true:

Theorem 1.1 (Valiron agrees with Picard, [12] and [13]). *Let f be a rational map of degree > 1. For a point $a \in \hat{\mathbb{C}}$,*

$$\delta_V(a; \{f^k\}) = 0$$

if and only if a is not a Picard exceptional value of $\{f^k\}$.

In [11], the following generalization of Theorem 1.1 below was shown and crucially used to obtain a new Diophantine condition for the nonlinearizability of f at its irrationally indifferent cycle.

Definition 1.3. The *Fatou set* $F(f)$ is the set of all the points in $\hat{\mathbb{C}}$ where $\{f^k\}$ is normal, and the *Julia set* $J(f)$ is $\hat{\mathbb{C}} - F(f)$.

Theorem 1.2 (vanishing theorem on the Valiron defects for moving targets). *Let f be a rational map of degree > 1 such that $F(f) \neq \emptyset$. Then for every non-constant rational map g,*

$$\delta_V(g; \{f^k\}) = 0. \tag{1}$$

In [11] we asked whether it is possible to remove the assumption $F(f) \neq \emptyset$. In the rest of this notes, we will answer affirmatively the following *pointwise* version of this problem:

Theorem 1 (vanishing theorem on the pointwise defect). *Let f be a rational map of degree $d > 1$. Then for every rational map g,*

$$\lim_{k \to \infty} \frac{w(g, f^k)}{d^k} = 0 \tag{2}$$

μ_f-almost everywhere on $\hat{\mathbb{C}}$. Here the measure μ_f appears in Theorem 2.1 in §2.

§2. The maximal entropy measures of rational maps

In this section, we gather some useful ergodic properties of rational maps which will be used in §3.

Let f be a rational map of degree $d > 1$.

Theorem 2.1 ([6] and [5]). *There exists the unique maximal entropy measure μ_f for f, and $h_{\mu_f}(f) = \log d$, which is the topological entropy of f.*

Moreover, the probability measure μ_f is exponentially mixing. More quantitatively, the following holds:

Theorem 2.2 (exponential decay of correlation [3]. See also [4]). *For every $\epsilon_0 > 0$, there exists $C = C(\epsilon_0) > 0$ such that for every $\psi \in L^\infty(\mu_f)$, every Lipschitz function ϕ on $\hat{\mathbb{C}}$, for which $\|\phi\|_{\mathrm{Lip}} := \sup_{z,w \in \hat{\mathbb{C}}, z \neq w} |\phi(z) - \phi(w)|/[z, w]$, and every $k \in \mathbb{N}$,*

(3)
$$\left| \int (\psi \circ f^k) \cdot \phi \mathrm{d}\mu_f - \int \psi \mathrm{d}\mu_f \int \phi \mathrm{d}\mu_f \right| \leq C \|\psi\|_\infty \|\phi\|_{\mathrm{Lip}} \left(\frac{1 + \epsilon_0}{d} \right)^{\frac{k}{2}}.$$

Let us also recall several properties of μ_f proved by Mañé:

Theorem 2.3 (Mañé [7], Theorem A). *Let μ be an f-ergodic probability measure on $\hat{\mathbb{C}}$ with the entropy $h_\mu(f) > 0$, then*

$$\int \log |f'| \mathrm{d}\mu > 0, \tag{4}$$

and for μ-a.e. $x \in \hat{\mathbb{C}}$,

$$\lim_{r \to 0} \frac{\log \mu(\mathbb{D}(x,r))}{\log r} = \frac{h_\mu(f)}{\int \log |f'| \mathrm{d}\mu} =: D(\mu). \tag{5}$$

Since $h_{\mu_f}(f) = \log d > 0$, Theorem 2.3 can be applied to μ_f.

Remark 2.1. The quantity in (4) is called the *Lyapunov exponent* of f, which is independent of an f-ergodic probability measure μ on $\hat{\mathbb{C}}$. The left hand side of (5) is called the *pointwise Hausdorff dimension* of μ at x. By the observation of Young [21], it holds that

$$D(\mu) = \inf\{\mathrm{HD}(X); X \subset \hat{\mathbb{C}}, \mu(X) = 1\},$$

where $\mathrm{HD}(X)$ is the Hausdorff dimension of X.

Theorem 2.4 (cf. Mañé [7], Lemma II.1). *There exist $\rho \in (0, 1]$ and $\gamma > 0$ such that for every $r \in (0, \rho)$ and every $x \in \hat{\mathbb{C}}$,*

$$\mu_f(\mathbb{D}(x,r)) \le r^\gamma. \tag{6}$$

§3. The long fly property of a rational map

Let f be a rational map of degree $d > 1$. The following is a refinement of Saussol's *long fly property* ([17]) of $(\hat{\mathbb{C}}, f, \mu_f)$ and proves Theorem 1:

Theorem 2. *For every rational map g, the following holds: for μ_f-almost every $z \in \hat{\mathbb{C}}$,*

$$\log \frac{1}{[f^k(z), g(z)]} = O(\log k) \tag{7}$$

as $k \to \infty$.

Proof. We extend the argument in the proof of [17], Lemma 9.

Let $\epsilon_0 \in (0, d-1)$, $C = C(\epsilon_0), D(\mu_f), \rho, \gamma$ be the constants in Theorems 2.2, 2.3 and 2.4. Fix $\delta \in (0, \gamma/2)$, $\epsilon_1 > 0$ and $\epsilon_2 \in (0, \gamma - 2\delta)$. For each $r_0 \in (0, \rho)$, let $G(r_0)$ be the set of all such $x \in \hat{\mathbb{C}}$ that for every $r \in (0, r_0)$,

$$\frac{\log \mu_f(\mathbb{D}(x,r))}{\log r} \le D(\mu_f) + \epsilon_1, \text{ and} \tag{8}$$

$$\mu_f(\mathbb{D}(x,4r)) \le \mu_f(\mathbb{D}(x,r)) r^{-\epsilon_2}. \tag{9}$$

By Theorem 2.3 and the *weak diametrical regularity* of μ_f (cf. Barreira and Saussol [1], p452), $G(r_0)$ is increasing as $r_0 \to 0$ and

$$\mu_f\Big(\bigcup_{r_0\in(0,\rho)} G(r_0)\Big) = 1,$$

by which, it is enough to show that for every sufficiently small r_0, (7) holds μ_f-almost everywhere on $G(r_0)$

For each $m \in \mathbb{N}$, put

$$A_\delta(m;g) := \{y \in \hat{\mathbb{C}};\ \inf_{k\in[e^{m\delta},e^{(m+1)\delta}]}[f^k(y), g(y)] < e^{-m}\}.$$

Then for every $x \in \hat{\mathbb{C}}$ and every $m \in \mathbb{N}$,

$$A_\delta(m;g)\cap\mathbb{D}(x,e^{-m}) \subset \bigcup_{k\in[e^{m\delta},e^{(m+1)\delta}]} \mathbb{D}(x,e^{-m})\cap f^{-k}(\mathbb{D}(g(x),(K+1)e^{-m})),$$

where $K > 0$ is a constant such that g is K-Lipschitz on $\hat{\mathbb{C}}$.

Put $\phi_{x,r}(y) := \eta_r([x,y])$, where $\eta_r : [0,\infty) \to \mathbb{R}$ is an $1/r$-Lipschitz function such that $1_{[0,r]} \le \eta_r \le 1_{[0,2r]}$. Then $\phi_{x,r}$ is $1/r$-Lipschitz on $\hat{\mathbb{C}}$ and $1_{\mathbb{D}(x,r)} \le \phi_{x,r} \le 1_{\mathbb{D}(x,2r)}$.

For every $r_0 \in (0,\rho)$ and every $r \in (0,r_0)$, from (3),

$$\begin{aligned}
&\mu_f\left(\mathbb{D}(x,r)\cap f^{-k}(\mathbb{D}(g(x),(K+1)r))\right)\\
\le&\int (1_{\mathbb{D}(g(x),(K+1)r))}\circ f^k)\cdot\phi_{x,r}\mathrm{d}\mu_f\\
\le& C\cdot 1\cdot\frac{1}{r}\left(\frac{1+\epsilon_0}{d}\right)^{k/2} + \mu_f(\mathbb{D}(g(x),(K+1)r))\cdot\mu_f(\mathbb{D}(x,2r)),
\end{aligned}$$

and by (6) and (9),

$$\begin{aligned}
&\mu_f(\mathbb{D}(g(x),(K+1)r))\cdot\mu_f(\mathbb{D}(x,2r))\\
\le&((k+1)r)^\gamma\cdot\mu_f(\mathbb{D}(x,r/2))(r/2)^{-\epsilon_2} \le \mu_f(\mathbb{D}(x,r/2))\cdot 2^{\epsilon_2}(K+1)^\gamma\cdot r^{\gamma-\epsilon_2}.
\end{aligned}$$

There exists so small $\rho' \in (0,\rho)$ that for every $r_0 \in (0,\rho')$, every $x \in G(r_0)$ and every $m > \log(1/r_0)$,

$$\begin{aligned}
&\mu_f\left(A_\delta(m;g)\cap\mathbb{D}(x,e^{-m})\right)\\
\le& C\cdot e^m\frac{\left(\frac{1+\epsilon_0}{d}\right)^{e^{m\delta}/2}}{1-\left(\frac{1+\epsilon_0}{d}\right)^{1/2}} + e^{(m+1)\delta}\cdot\mu_f(\mathbb{D}(x,e^{-m}/2))\cdot 2^{\epsilon_2}(K+1)^\gamma e^{-m(\gamma-\epsilon_2)}\\
\le&(e^{-m}/2)^{D(\mu_f)+\epsilon_1}\cdot e^{-m(\gamma-\epsilon_2-2\delta)} + \mu_f(\mathbb{D}(x,e^{-m}/2))\cdot e^{-m(\gamma-\epsilon_2-2\delta)}\\
\le&\mu_f(\mathbb{D}(x,e^{-m}/2))\cdot 2e^{-m(\gamma-\epsilon_2-2\delta)} \text{ (by (8))},
\end{aligned}$$

and hence for every $m > \log(1/r_0)$,

$$\mu_f\left(A_\delta(m;g) \cap G(r_0)\right) \le \sum_{x \in S_m} \mu_f\left(A_\delta(m;g) \cap \mathbb{D}(x, e^{-m})\right)$$
$$\le 2e^{-m(\gamma-\epsilon_2-2\delta)}\mu_f\Big(\bigcup_{x \in S_m} \mathbb{D}(x, e^{-m}/2)\Big) \le 2e^{-m(\gamma-\epsilon_2-2\delta)},$$

where S_m is a finite and *maximal* e^{-m}*-separated* set for $G(r_0)$, i.e., $G(r_0) \subset \cup_{x \in S_m} \mathbb{D}(x, e^{-m})$ and $\mathbb{D}(x, e^{-m}) \cap S_m = \{x\}$ for each $x \in S_m$, and finally $\sum_{m \in \mathbb{N}} \mu_f\left(A_\delta(m;g) \cap G(r_0)\right) < \infty$.

Hence by the first Borel-Cantelli lemma, $\mu_f(\limsup_{m \to \infty} A_\delta(m;g) \cap G(r_0)) = 0$, that is, for μ_f-almost every $z \in G(r_0)$, there exists $m(z) \in \mathbb{N}$ such that for every $m > m(z)$,

$$\inf_{k \in [e^{m\delta}, e^{(m+1)\delta}]} [f^k(z), g(z)] \ge e^{-m},$$

which proves (7). □

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