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# A proof of a conjecture of Degtyarev on non-torus plane sextics

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**Abstract.**

A $\mathbb{D}_{10}$-sextic is an irreducible sextic $C \subset \mathbb{CP}^2$ with simple singularities such that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to the dihedral group $\mathbb{D}_{10}$. A $\mathbb{D}_{10}$-sextic is not of torus type. In this paper, we show that if $C$ is a $\mathbb{D}_{10}$-sextic with the set of singularities $4\mathbf{A}_4$ or $4\mathbf{A}_4 \oplus \mathbf{A}_1$, then $\pi_1(\mathbb{CP}^2 \setminus C)$ is isomorphic to $\mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$. This positively answers a conjecture by Degtyarev.

## §1. Introduction

A sextic $F(X,Y,Z) = 0$ in $\mathbb{CP}^2$ is said to be of *torus type* if there is an expression of the form $F(X,Y,Z) = F_2(X,Y,Z)^3 + F_3(X,Y,Z)^2$, where $F_2$ and $F_3$ are homogeneous polynomials of degree 2 and 3 respectively. A conjecture by the second author says that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ of the complement of an irreducible sextic $C$ with simple singularities and which is *not* of torus type is abelian. In [4] we checked this for a number of configurations of singularities, but early in the year 2007, Degtyarev [1] observed that this conjecture is false in general. Especially, Degtyarev proved that there exist 8 equisingular deformation families of irreducible non-torus sextics $C$ with simple singularities such that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to the dihedral group $\mathbb{D}_{10}$, one family for each of the following sets of singularities: $4\mathbf{A}_4$, $4\mathbf{A}_4 \oplus \mathbf{A}_1$, $4\mathbf{A}_4 \oplus 2\mathbf{A}_1$, $4\mathbf{A}_4 \oplus \mathbf{A}_2$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4$, $\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_1$,

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$\mathbf{A}_9 \oplus 2\mathbf{A}_4 \oplus \mathbf{A}_2$ and $2\mathbf{A}_9$.[1] Furthermore, in the special case where the set of singularities is $4\mathbf{A}_4$, he conjectured that $\pi_1(\mathbb{CP}^2 \setminus C) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$ (cf. [1, Conjecture 1.2.1]). The aim of this paper is to prove this conjecture.

Hereafter, we use the term $\mathbb{D}_{10}$-*sextic* for an irreducible sextic $C \subset \mathbb{CP}^2$ with simple singularities such that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to $\mathbb{D}_{10}$ (cf. [2]). By [1, 5, 8], a $\mathbb{D}_{10}$-sextic is not of torus type.

**Theorem 1.** *If $C$ is a $\mathbb{D}_{10}$-sextic with the set of singularities $4\mathbf{A}_4$ (respectively $4\mathbf{A}_4 \oplus \mathbf{A}_1$), then the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ is isomorphic to $\mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$.*

According to [1], there is only one equisingular deformation family of $\mathbb{D}_{10}$-sextics with the set of singularities $4\mathbf{A}_4$ (respectively $4\mathbf{A}_4 \oplus \mathbf{A}_1$). Therefore, to prove the theorem, it suffices to construct a $\mathbb{D}_{10}$-sextic $C_1$ with four $\mathbf{A}_4$-singularities (respectively a $\mathbb{D}_{10}$-sextic $C_2$ with four $\mathbf{A}_4$-singularities and one $\mathbf{A}_1$-singularity) — notice that in [1] only the existence of $\mathbb{D}_{10}$-sextics is proved — and show the isomorphism $\pi_1(\mathbb{CP}^2 \setminus C_i) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$ for $i = 1$ (respectively $i = 2$). This is done in sections 2 and 3 respectively.

Note that when this paper was being written, Degtyarev independently found the fundamental groups $\pi_1(\mathbb{CP}^2 \setminus C)$ for all $\mathbb{D}_{10}$-sextics $C$ (cf. [2]). Let us also mention that in addition to the statement about $\mathbb{D}_{10}$-sextics with four $\mathbf{A}_4$-singularities, Degtyarev's Conjecture 1.2.1 in [1] also says that $\pi_1(\mathbb{CP}^2 \setminus C) \simeq \mathbb{D}_{14} \times \mathbb{Z}/3\mathbb{Z}$ for any $\mathbb{D}_{14}$-sextic $C$ with three $\mathbf{A}_6$-singularities (a $\mathbb{D}_{14}$-sextic $C$ is just an irreducible sextic with simple singularities such that $\pi_1(\mathbb{CP}^2 \setminus C)$ factors to the dihedral group $\mathbb{D}_{14}$). This second point of the conjecture is proved in [3].

## §2. An example of a $\mathbb{D}_{10}$-sextic with the set of singularities $4\mathbf{A}_4$ and the fundamental group of its complement

Let $(X\colon Y\colon Z)$ be homogeneous coordinates on $\mathbb{CP}^2$ and $(x, y)$ the affine coordinates defined by $x := X/Z$ and $y := Y/Z$ on $\mathbb{CP}^2 \setminus \{Z = 0\}$, as usual. We consider the following one-parameter family of curves $C(u) : f(x, y, u) = 0$, $u \in \mathbb{C}$, where $f(x, y, u)$ is a polynomial given as $f(x, y, u) = g(x, y^2, u)$, with

$$g(x, y, u) := c_3\, y^3 + c_2\, y^2 + c_1\, y + c_0,$$

[1]We recall that a point $P$ in a curve $C$ is said to be an $\mathbf{A}_n$-singularity ($n \geq 1$) if the germs $(C, P)$ and $(\{x^2 + y^{n+1} = 0\}, O)$ are topologically equivalent as embedded germs, where $O$ is the origin in $\mathbb{C}^2$.

and the coefficients $c_3, \ldots, c_0$ are defined as follows:

$$\begin{aligned} c_3 &:= -64\,u^3 + 96\,u^2 + 16 + 16\,u^4 - 64\,u, \\ c_2 &:= 196\,u - 4\,x^2u^6 - 36\,xu^4 + 144\,xu^3 - 226\,xu^2 - 164\,x^2u + \\ &\quad 12\,x^2u^4 + 192\,u^3 - 289\,u^2 + 223\,x^2u^2 - 40\,x + 16\,x^2u^5 - \\ &\quad 128\,x^2u^3 - 52 + 160\,xu - 48\,u^4 + 44\,x^2, \end{aligned}$$

$$\begin{aligned} c_1 &:= 56 + 88\,x - 200\,u + 8\,x^2u^6 + 72\,xu^4 - 288\,xu^3 + 454\,xu^2 + \\ &\quad 208\,x^2u + 2\,x^2u^4 - 276\,x^2u^2 + 152\,x^2u^3 - 328\,xu - 192\,u^3 + \\ &\quad 48\,u^4 + 290\,u^2 - 64\,x^2 - 72\,x^3 + 40\,x^4 + 264\,x^3u - 32\,x^2u^5 + \\ &\quad 16\,x^4u^5 + 4\,x^3u^6 + 166\,x^4u^2 - 16\,x^3u^5 - 338\,x^3u^2 - 136\,x^4u + \\ &\quad 184\,x^3u^3 - 4\,x^4u^6 - 80\,x^4u^3 - 2\,x^4u^4 - 24\,x^3u^4, \\ c_0 &:= -20 - 48\,x + 68\,u - x^6u^6 + 144\,xu^3 - 36\,x^6u + 3\,x^4u^6 + \\ &\quad 56\,x^5u^3 + 52\,x^2u^2 - 40\,x^2u - 120\,x^5u^2 + 104\,x^5u - 44\,x^4u^2 - \\ &\quad 4\,x^3u^6 - 2\,x^6u^4 - 4\,x^2u^6 + 2\,x^5u^6 - 8\,x^5u^5 + 298\,x^3u^2 - \\ &\quad 24\,x^2u^3 - 240\,x^3u + 40\,x^4u + 18\,x^3u^4 + 39\,x^6u^2 + 16\,x^4u^3 - \\ &\quad 2\,x^5u^4 - 14\,x^2u^4 - 12\,x^4u^5 + 4\,x^6u^5 - 32\,x^5 + 72\,x^3 + 12\,x^6 - \\ &\quad 16\,x^4 + 16\,x^2 + 64\,u^3 - 16\,u^4 - 97\,u^2 - 160\,x^3u^3 + 16\,x^2u^5 + \\ &\quad 12\,x^4u^4 - 228\,xu^2 + 16\,x^3u^5 - 16\,x^6u^3 - 36\,xu^4 + 168\,xu. \end{aligned}$$

All the curves $C(u)$ in that family are symmetric with respect to the $x$-axis. All of them have four $\mathbf{A}_4$-singularities located at $(0, \pm 1)$ and $(1, \pm 1)$, except the curves $C(\frac{9 \pm \sqrt{33}}{6})$ which obtain, in addition, an $\mathbf{A}_1$-singularity at $(-1, 0)$, and the curve $C(1)$ which is a non-reduced cubic (union of a smooth conic and a line). All the curves are irreducible except $C(1)$. All of them are non-torus curves.

As a test curve with four $\mathbf{A}_4$-singularities, we take the curve $C_1 := C(11/5)$ defined by the equation $f_1(x, y) := f(x, y, 11/5) = 0$, where

$$\begin{aligned} a_0 \cdot f_1(x, y) &:= 518400\,y^6 + (808511\,x^2 - 1435150\,x - 1555825)\,y^4 + \\ &\quad (259536\,x^4 - 1580686\,x^3 - 297122\,x^2 + 2871550\,x + \\ &\quad 1556450)\,y^2 - 45216\,x^6 - 313968\,x^5 + 503423\,x^4 + \\ &\quad 1177536\,x^3 - 512014\,x^2 - 1436400\,x - 519025, \end{aligned}$$

with $a_0 := 15625$. In Fig. 1, we show its real plane section, that is, the set $\{(x, y) \in \mathbb{R}^2\,;\, f_1(x, y) = 0\}$. (In the figures we do not respect the numerical scale.)

**Theorem 2.** $\pi_1(\mathbb{CP}^2 \setminus C_1) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$.

*Proof.* We use the classical Zariski–van Kampen theorem (cf. [10] and [9]) with the pencil given by the vertical lines $L_\eta \colon x = \eta$, $\eta \in \mathbb{C}$. We always take the point $(0:1:0)$ as the base point for the fundamental groups. This point is nothing but the axis of the pencil, which is also the point at infinity of the lines $L_\eta$. Observe that it does not belong to $C_1$.

The discriminant $\Delta_y(f_1)$ of $f_1$ as a polynomial in $y$, which describes the singular lines of the pencil (notice that the line at infinity $Z = 0$ is not singular), is the polynomial in $x$ given by

$$\Delta_y(f_1)(x) = b_0\,(x+1)\,x^{10}\,(408839\,x^2 + 219050\,x - 625)^2\,(x-1)^{10}$$
$$(45216\,x^5 + 268752\,x^4 - 772175\,x^3 - 405361\,x^2 + 917375\,x + 519025),$$

where $b_0 \in \mathbb{Q} \setminus \{0\}$. This polynomial has exactly 10 distinct roots which are all real numbers: $\eta_1 = -7.9192\ldots$, $\eta_2 = -1$, $\eta_3 = -0.7182\ldots$, $\eta_4 = -0.7005\ldots$, $\eta_5 = -0.5386\ldots$, $\eta_6 = 0$, $\eta_7 = 0.0028\ldots$, $\eta_8 = 1$, $\eta_9 = 1.6969\ldots$, and $\eta_{10} = 1.6974\ldots$ The singular lines of the pencil are the lines $L_{\eta_i}$ $(1 \le i \le 10)$ corresponding to these 10 roots. The lines $L_{\eta_6}$ and $L_{\eta_8}$ pass through the singular points of the curve. All the other singular lines are tangent to $C_1$. See Fig. 1.

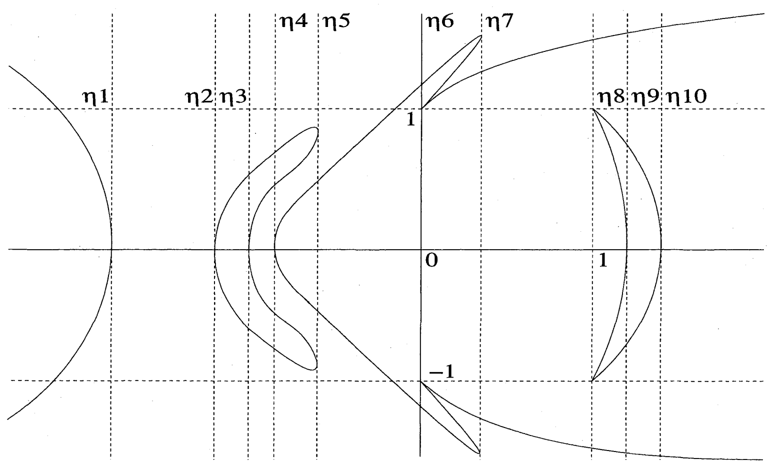


Fig. 1. Real plane section of $C_1$

We consider the generic line $L_{\eta_6-\varepsilon}$ and choose generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_6-\varepsilon} \setminus C_1)$ as in Fig. 2, where $\varepsilon > 0$ is small enough. The $\xi_j$'s are (the homotopy classes of) lassos oriented counter-clockwise (see [7] for the definition) around the intersection points of

$L_{\eta_6-\varepsilon}$ with $C_1$. In the figures, a lasso oriented counter-clockwise is always represented by a path ending with a bullet, as in Fig. 3. The Zariski–van Kampen theorem says $\pi_1(\mathbb{CP}^2 \setminus C_1) \simeq \pi_1(L_{\eta_6-\varepsilon} \setminus C_1)/G_1$, where $G_1$ is the normal subgroup of $\pi_1(L_{\eta_6-\varepsilon} \setminus C_1)$ generated by the monodromy relations associated with the singular lines of the pencil. To determine these relations, we fix a system of generators $\sigma_1, \ldots, \sigma_{10}$ for the fundamental group $\pi_1(\mathbb{C} \setminus \{\eta_1, \ldots, \eta_{10}\})$ as follows: each $\sigma_i$ is (the homotopy class of) a lasso oriented counter-clockwise around $\eta_i$ with base point $\eta_6 - \varepsilon$. Its tail is a union of real segments and half-circles around the exceptional parameters $\eta_j$ $(j \neq i)$ located between the base point $\eta_6 - \varepsilon$ and $\eta_i$. Its head is a circle around $\eta_i$. For example, for $i = 4$, the lasso $\sigma_4$ is obtained when the variable $x$ moves on the real axis from $x := \eta_6 - \varepsilon$ to $x := \eta_5 + \varepsilon$ (in short, from $x := \eta_6 - \varepsilon \to \eta_5 + \varepsilon$), makes half-turn counter-clockwise on the circle $|x - \eta_5| = \varepsilon$, moves on the real axis from $x := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$, runs once counter-clockwise on the circle $|x - \eta_4| = \varepsilon$, then comes back on the real axis from $x := \eta_4 + \varepsilon \to \eta_5 - \varepsilon$, makes half-turn clockwise on the circle $|x - \eta_5| = \varepsilon$, and moves on the real axis from $x := \eta_5 + \varepsilon \to \eta_6 - \varepsilon$ (cf. Fig. 4). For $i = 6$, we get $\sigma_6$ just by moving $x$ once counter-clockwise on the circle $|x - \eta_6| = \varepsilon$. The monodromy relations around the singular line $L_{\eta_i}$ are obtained by moving the generic fibre $F \simeq L_{\eta_6-\varepsilon} \setminus C_1$ isotopically 'above' the loop $\sigma_i$ so defined, and by identifying the generators $\xi_j$ $(1 \leq j \leq 6)$ with their own images by the terminal homeomorphism of this isotopy. For details see [10, 9]. Most of the remaining of the proof is to determine these relations.

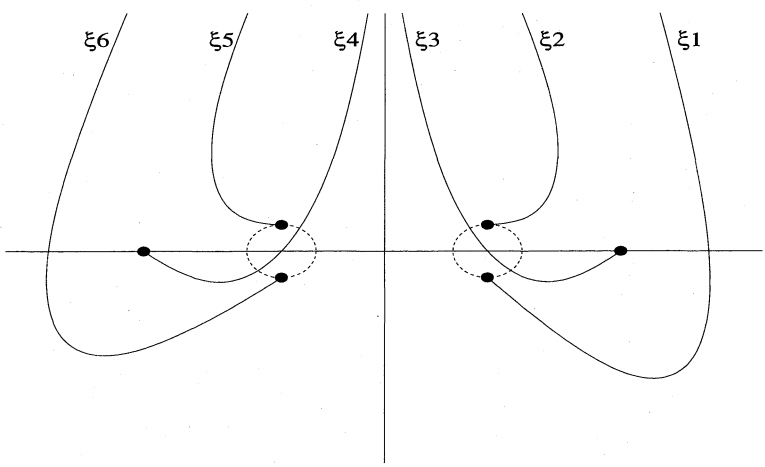


Fig. 2. Generators at $x = \eta_6 - \varepsilon$

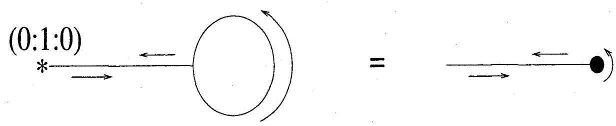


Fig. 3. Lasso oriented counter-clockwise

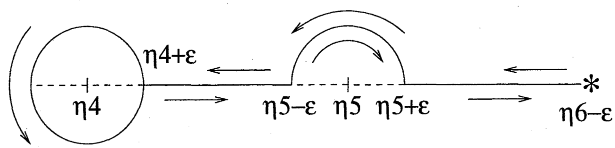


Fig. 4. Lasso $\sigma_4$

**Monodromy relations at $x = \eta_5$**

In Fig. 5, we show how the generators at $x = \eta_6 - \varepsilon$ (cf. Fig. 2) are deformed when $x$ moves on the real axis from $x := \eta_6 - \varepsilon \to \eta_5 + \varepsilon$. The line $L_{\eta_5}$ is tangent to the curve at two distinct simple points (i.e., non-singular points) $P_- = (\eta_5, -0.6132\ldots)$ and $P_+ = (\eta_5, +0.6132\ldots)$, and the intersection multiplicity of this line with the curve at these points is 2. Therefore, by the implicit function theorem, the germ $(C_1, P_\pm)$ is given by

$$x - \eta_5 = \alpha_\pm \cdot (y \mp 0.6132\ldots)^2 + \text{higher terms},$$

where $\alpha_\pm \neq 0$. So, when $x$ runs once counter-clokwise on the circle $|x - \eta_5| = \varepsilon$, the variable $y$ makes half-turn on the dotted circle around $\pm 0.6132\ldots$ (cf. Fig. 5), and therefore the monodromy relations at $x = \eta_5$ are given by

$$\xi_6 = \xi_5 \quad \text{and} \quad \xi_2 = \xi_1. \tag{1}$$

**Monodromy relations at $x = \eta_4$**

In Fig. 6, we show how the generators at $x = \eta_5 + \varepsilon$ (cf. Fig. 5) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_5| = \varepsilon$, then moves on the real axis from $x := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$. The singular line $L_{\eta_4}$ is tangent to the curve at one simple point $P$ and the intersection multiplicity of this line with the curve at $P$ is 2. Then, as above, the monodromy relation at $x = \eta_4$ is simply given by

$$\xi_4 = \xi_3. \tag{2}$$

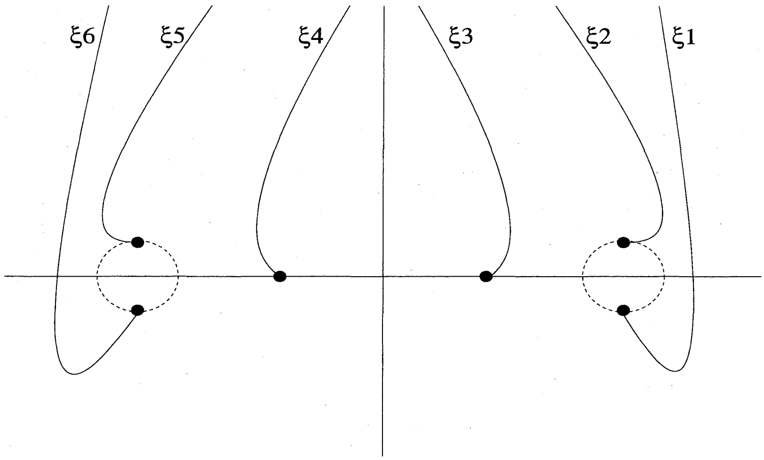


Fig. 5. Generators at $x = \eta_5 + \varepsilon$

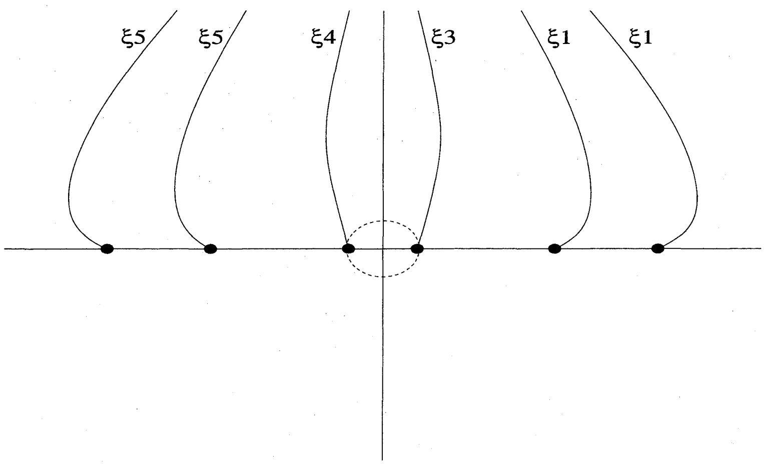


Fig. 6. Generators at $x = \eta_4 + \varepsilon$

**Monodromy relations at $x = \eta_3$**

In Fig. 7, we show how the generators at $x = \eta_4 + \varepsilon$ (cf. Fig. 6) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_4| = \varepsilon$, then moves on the real axis from $x := \eta_4 - \varepsilon \to \eta_3 + \varepsilon$. The line $L_{\eta_3}$ is also tangent to the curve at one simple point with intersection multiplicity 2, and the monodromy relation we are looking for is given by

$$\xi_5 = \xi_3\xi_1\xi_3^{-1}. \tag{3}$$

The monodromy relations around the singular lines $L_{\eta_2}$ and $L_{\eta_1}$ do not give any new information. The movement of the 6 complex roots of the equation $f_1(\eta, y) = 0$ for $\eta_1 \leq \eta \leq \eta_2$ can be chased easily using the real plane section of $g(x, y, 11/5) = 0$ (cf. Fig. 8). Indeed, from this picture, one gets the movement of the 3 real roots of the equation $g(\eta, y, 11/5) = 0$ for $\eta_1 \leq \eta \leq \eta_2$. The movement of the 6 complex roots of $f_1(\eta, y) = 0$, $\eta_1 \leq \eta \leq \eta_2$, can be then easily deduced (we recall that $f_1(\eta, y) = g(\eta, y^2, 11/5)$). For details see [6].

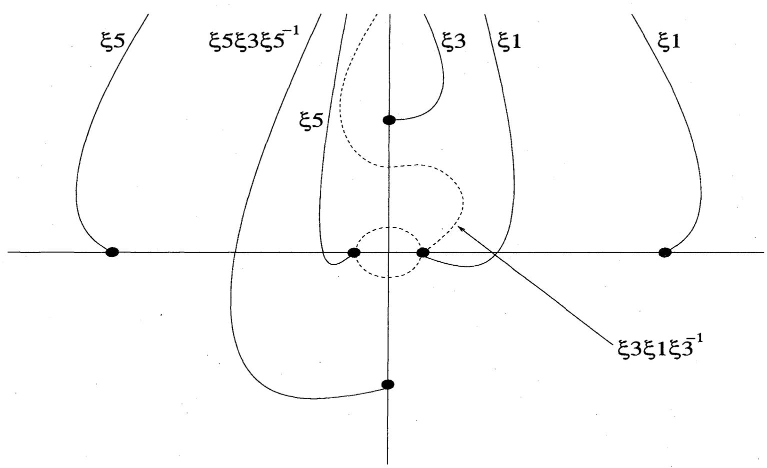


Fig. 7. Generators at $x = \eta_3 + \varepsilon$

**Monodromy relations at $x = \eta_6$**

By (1), (2) and (3), Fig. 2 (which shows the generators at $x = \eta_6 - \varepsilon$) is the same as Fig. 9, where

$$\zeta_1 := \xi_3\xi_1 \cdot \xi_3 \cdot (\xi_3\xi_1)^{-1}.$$

The line $L_{\eta_6}$ passes through the singular points $(0, 1)$ and $(0, -1)$ which are both $\mathbf{A}_4$-singularities. Puiseux parametrizations of the curve at these points are given by

$$x = t^2, \quad y = 1 + \frac{1}{2}t^2 + \frac{359}{200}t^4 + \frac{726}{125}\sqrt{22}\,t^5 + \text{higher terms} \tag{4}$$

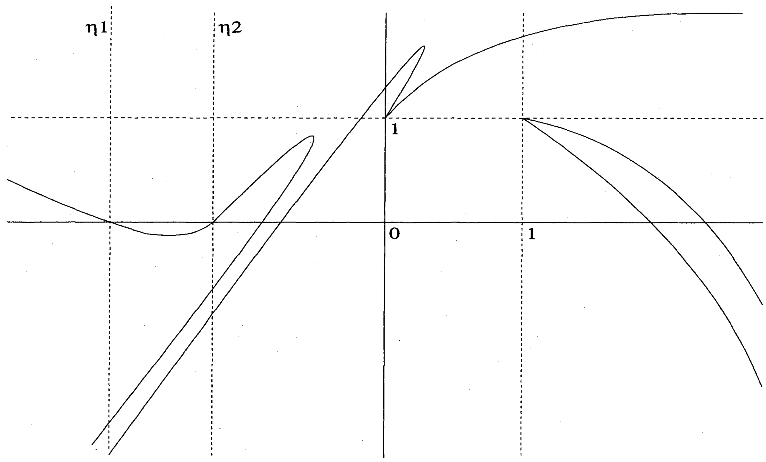


Fig. 8. Real plane section of $g(x, y, 11/5) = 0$

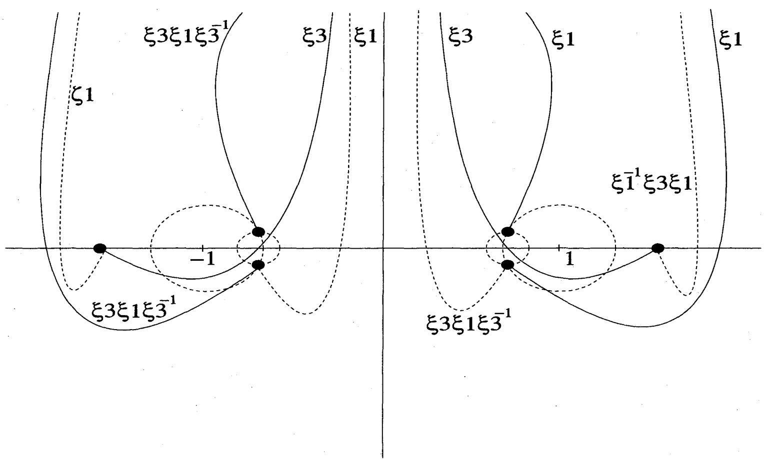


Fig. 9. Generators at $x = \eta_6 - \varepsilon$

and

$$(5) \qquad x = t^2, \quad y = -1 - \frac{1}{2}\,t^2 - \frac{359}{200}\,t^4 - \frac{726}{125}\,\sqrt{22}\,t^5 + \text{higher terms}$$

respectively. Equations (4) show that when $x = \varepsilon \exp(i\theta)$ moves once counter-clockwise on the circle $|x - \eta_6| = \varepsilon$, the topological behavior of the two points near 1 in Fig. 9 looks like the movement of two satellites (corresponding to $t = \sqrt{\varepsilon}\exp(i\nu)$, $\nu = \theta/2,\ \theta/2 + \pi$) accompanying a planet. The movement of the planet is described by the term $t^2/2$. It runs once counter-clockwise around 1 (this movement can be ignored in our case). The movement of the satellites around the planet is described by the term $\frac{726}{125}\sqrt{22}\,t^5$. Each of them makes (5/2)-turn counter-clockwise around the planet. Therefore the monodromy relation at $x = \eta_6$ that comes from the singular point $(0, 1)$ is given by

$$(6)\quad \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}.$$

Similarly, equations (5) show that the monodromy relation at $x = \eta_6$ that comes from the singular point $(0, -1)$ is also given by (6).

**Monodromy relations at $x = \eta_7$**

In Fig. 10, we show how the generators at $x = \eta_6 - \varepsilon$ (cf. Fig. 9) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_6| = \varepsilon$, then moves on the real axis from $x := \eta_6 + \varepsilon \to \eta_7 - \varepsilon$, where

$$\omega := \xi_3\xi_1\xi_3^{-1}\ (= \xi_5 = \xi_6).$$

The line $L_{\eta_7}$ is tangent to $C_1$ at two simple points, in both cases with intersection multiplicity 2, and the monodromy relations at $x = \eta_7$ reduce to the following single relation:

$$(7)\qquad \xi_1\xi_3\xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}.$$

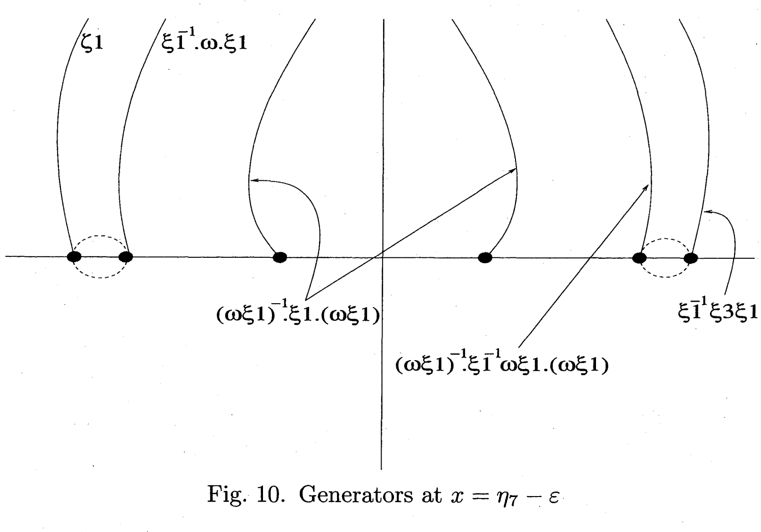


Fig. 10. Generators at $x = \eta_7 - \varepsilon$

**Monodromy relations at $x = \eta_8$**

In Fig. 11, we show how the generators at $x = \eta_7 - \varepsilon$ (cf. Fig. 10) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_7| = \varepsilon$, then moves on the real axis from $x := \eta_7 + \varepsilon \to \eta_8 - \varepsilon$, where

$$\begin{aligned}
\zeta_1 &:= (\xi_3\xi_1) \cdot \xi_3 \cdot (\xi_3\xi_1)^{-1}, \\
\zeta_2 &:= \xi_1^{-1} \cdot \zeta_1 \cdot \xi_1, \\
\zeta_3 &:= \xi_1^{-1} \cdot \omega \cdot \xi_1, \\
\zeta_4 &:= (\omega\xi_1)^{-1} \cdot \xi_1 \cdot (\omega\xi_1), \\
\zeta_5 &:= (\omega\xi_1)^{-1} \cdot \xi_1^{-1}\omega\xi_1 \cdot (\omega\xi_1) = \xi_1^{-1}\xi_3\xi_1 \text{ (by (7))}, \\
\zeta_6 &:= (\xi_3\xi_1\xi_1)^{-1} \cdot \xi_1 \cdot (\xi_3\xi_1\xi_1).
\end{aligned}$$

(To determine dotted lassos, we use the relation (7).) The singular line $L_{\eta_8}$ passes through the singular points $(1, 1)$ and $(1, -1)$ which are both $\mathbf{A}_4$-singularities, and Puiseux parametrizations of $C_1$ at these points are given by

$$x = 1 + t^2, \quad y = 1 - \frac{61}{144}t^2 - \frac{7063}{13824}t^4 - \frac{125}{684288}\sqrt{22}\,t^5 + \text{higher terms}$$

and

$$x = 1 + t^2, \quad y = -1 + \frac{61}{144}t^2 + \frac{7063}{13824}t^4 + \frac{125}{684288}\sqrt{22}\,t^5 + \text{higher terms}$$

respectively. As above, these equations show that the monodromy relation at $x = \eta_8$ is written as

$$\xi_1\xi_3\xi_1\xi_3\xi_1 = \xi_3\xi_1\xi_3\xi_1\xi_3. \tag{8}$$

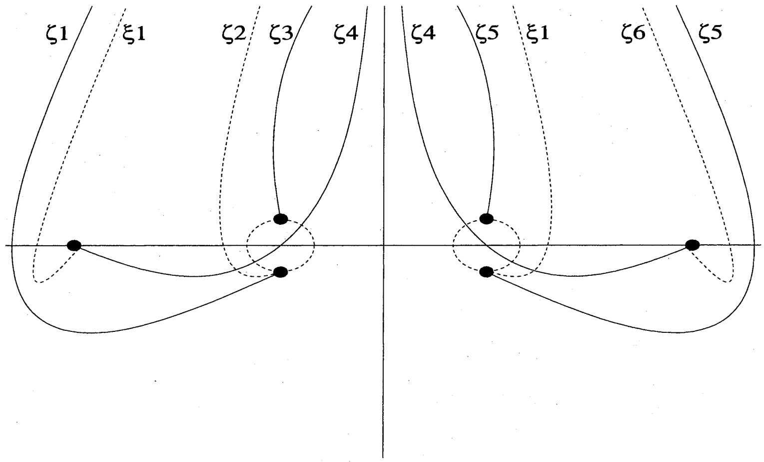


Fig. 11. Generators at $x = \eta_8 - \varepsilon$

The monodromy relations around the singular lines $L_{\eta_9}$ and $L_{\eta_{10}}$ do not give any new information (details are left to the reader).

Now, by the previous relations, it is easy to check that the vanishing relation at infinity $\xi_6\xi_5\xi_4\xi_3\xi_2\xi_1 = e$, where $e$ is the unit element, is written as

(9) $$\xi_3\xi_1\xi_1 \cdot \xi_3\xi_1\xi_1 = e.$$

This relation, combined with (7), shows that (6) is equivalent to

(10) $$\xi_1\xi_3\xi_1 \cdot \xi_1\xi_3\xi_1 = e.$$

Finally, we have proved that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is presented by the generators $\xi_1$ and $\xi_3$ and the relations (7), (8), (9) and (10).

**Simplification of the presentation**

By (10), the relation (8) can be written as

$$\xi_3\xi_1 = \xi_1\xi_3\xi_1 \cdot \xi_3\xi_1\xi_3\xi_1\xi_3,$$

that is,

(11) $$\xi_3\xi_1 = (\xi_1\xi_3)^4.$$

In addition, the relation (7) can be written as

$$\xi_1\xi_3\xi_1 \cdot \xi_3 \cdot (\xi_1\xi_3\xi_1)^{-1} = \xi_3\xi_1\xi_3^{-1}.$$

Combined with (10), this gives

$$\xi_1\xi_3\xi_1 \cdot \xi_3 \cdot (\xi_1\xi_3\xi_1) = \xi_3\xi_1\xi_3^{-1},$$

which is nothing but (11). Since the vanishing relation at infinity (9) is trivially equivalent to (10), it follows that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is presented by the generators $\xi_1$ and $\xi_3$ and the relations (10) and (11). Hence, after the change $a := \xi_1\xi_3\xi_1$ and $b := \xi_1\xi_3$, the presentation is given by

$$\pi_1(\mathbb{CP}^2 \setminus C_1) \simeq \langle a, b \mid a^2 = e, aba = b^4 \rangle.$$

Now, we observe that $b^{15} = e$ and $b^5$ is in the centre of $\pi_1(\mathbb{CP}^2 \setminus C_1)$. Indeed, since $a^2 = e$, the relation $aba = b^4$ gives $b^{16} = ab^4a = b$, that is, $b^{15} = e$ as desired. To show that $b^5$ is in the centre of $\pi_1(\mathbb{CP}^2 \setminus C_1)$ we write:

$$\begin{aligned} b^5ab^{-5}a^{-1} &= b \cdot b^4 \cdot ab^{-5}a^{-1} = b \cdot aba \cdot ab^{-5}a^{-1} = \\ &ba \cdot b^{-4} \cdot a^{-1} = ba \cdot a^{-1}b^{-1}a^{-1} \cdot a^{-1} = e. \end{aligned}$$

Hence $\pi_1(\mathbb{CP}^2 \setminus C_1)$ is also presented as:

$$\begin{aligned} \pi_1(\mathbb{CP}^2 \setminus C_1) &\simeq \langle a, b \mid a^2 = e,\ aba = b^4,\ b^{15} = e,\ b^5a = ab^5 \rangle \\ &\simeq \langle a, b, c, d \mid a^2 = b^{15} = e,\ aba = b^4,\ b^5a = ab^5, \\ &\qquad c = b^6,\ d = b^5,\ da = ad,\ db = bd,\ dc = cd \rangle \\ &\simeq \langle a, b, c, d \mid a^2 = b^{15} = e,\ aba = b^4,\ c = b^6,\ d = b^5, \\ &\qquad b = cd^{-1},\ da = ad,\ db = bd,\ dc = cd \rangle \\ &\simeq \langle a, c, d \mid a^2 = c^5 = d^3 = e,\ acd^{-1}a = c^4d^{-1}, \\ &\qquad da = ad,\ dc = cd \rangle \\ &\simeq \langle a, c, d \mid a^2 = c^5 = d^3 = e,\ aca = c^4,\ da = ad, \\ &\qquad dc = cd \rangle \\ &\simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

This completes the proof of Theorem 2. Q.E.D.

## §3. An example of a $\mathbb{D}_{10}$-sextic with the set of singularities $4\mathbf{A}_4 \oplus \mathbf{A}_1$ and the fundamental group of its complement

In this section, we consider the curve $C_2 := C(\frac{9+\sqrt{33}}{6})$ defined by the equation $f_2(x,y) := f(x,y,\frac{9+\sqrt{33}}{6}) = 0$, where

$$\begin{aligned} d_0 \cdot f_2(x,y) \quad := \quad & 3867 - 6\,x^3\,y^2\,\sqrt{33} + 6480\,x + 54\,y^2\,x\,\sqrt{33} + \\ & 219\,x^2\,y^4\,\sqrt{33} - 933\,x^4\,\sqrt{33} + 960\,x^3\,\sqrt{33} - 405\,\sqrt{33} - \\ & 9270\,y^2 + 2896\,x^5 + 3723\,x^4 - 8000\,x^3 - 4838\,x^2 - \\ & 1376\,x^6 - 432\,x^5\,\sqrt{33} + 810\,y^2\,\sqrt{33} + 1146\,x^2\,\sqrt{33} - \\ & 432\,x\,\sqrt{33} + 288\,x^6\,\sqrt{33} - 1770\,x^2\,y^2\,\sqrt{33} + 6939\,y^4 - \\ & 1536\,y^6 + 10102\,x^2\,y^2 - 8298\,y^2\,x - 3056\,x^4\,y^2 - \\ & 405\,y^4\,\sqrt{33} - 2933\,x^2\,y^4 + 1818\,y^4\,x + 3482\,x^3\,y^2 + \\ & 528\,x^4\,y^2\,\sqrt{33} + 378\,y^4\,x\,\sqrt{33}, \end{aligned}$$

with $d_0 := (3867 - 405\sqrt{33})/(-\frac{677}{18} - \frac{109}{18}\sqrt{33})$ (cf. section 2). We recall that this curve has four $\mathbf{A}_4$-singularities located at $(0,\pm 1)$ and $(1,\pm 1)$, and one $\mathbf{A}_1$-singularity situated at $(-1,0)$. In Fig. 12, we show its real plane section. Near the singular point $(-1,0)$, the equation of $C_2$ has the following form:

$$\frac{4}{9}\left(4\sqrt{33}+39\right)(x+1)^2 + \left(\frac{8}{3}+\frac{8}{9}\sqrt{33}\right)y^2 + \text{higher terms} = 0.$$

As the leading term $\frac{4}{9}\,(4\sqrt{33}+39)\,(x+1)^2 + (\frac{8}{3}+\frac{8}{9}\sqrt{33})\,y^2$ has no real factorization, the point $(-1,0)$ is an isolated point of the real plane section of the curve.

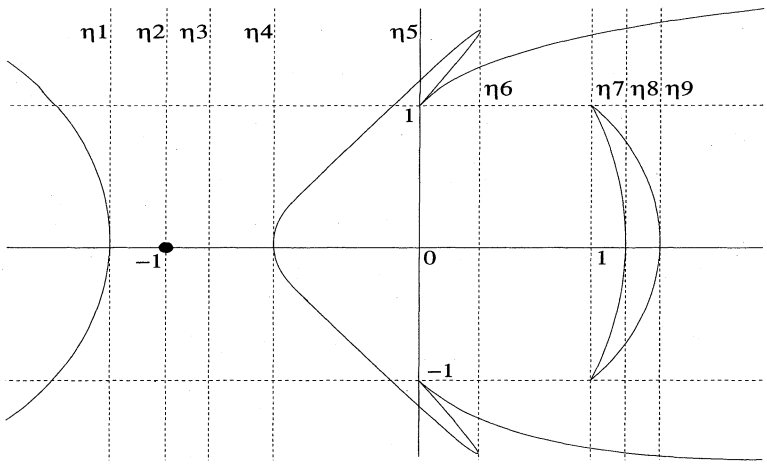


Fig. 12. Real plane section of $C_2$

**Theorem 3.** $\pi_1(\mathbb{CP}^2 \setminus C_2) \simeq \mathbb{D}_{10} \times \mathbb{Z}/3\mathbb{Z}$.

*Proof.* We use again the Zariski–van Kampen theorem with the pencil given by the vertical lines $L_\eta\colon x = \eta$, $\eta \in \mathbb{C}$. Observe that the axis of the pencil $(0 : 1 : 0)$ does not belong to $C_2$. The discriminant $\Delta_y(f_2)$ of $f_2$ as a polynomial in $y$ is the polynomial in $x$ given by

$$\Delta_y(f_2)(x) = e_0\,(6592\,x^4 - 14128\,x^3 + 1872\,x^3\sqrt{33} - 7589\,x^2 -$$
$$5397\,x^2\sqrt{33} + 14586\,x + 1242\,x\sqrt{33} + 11499 + 4347\sqrt{33})\,(x+1)^2$$
$$(x-1)^{10}\,x^{10}\,(16069\,x^2 + 10680\,x + 774\,x\sqrt{33} - 10917 + 1890\sqrt{33})^2,$$

where $e_0 \in \mathbb{R} \setminus \{0\}$. This polynomial has exactly 9 roots which are all real numbers: $\eta_1 = -2.2525\ldots$, $\eta_2 = -1$, $\eta_3 = -0.9452\ldots$, $\eta_4 = -0.7814\ldots$, $\eta_5 = 0$, $\eta_6 = 0.0039\ldots$, $\eta_7 = 1$, $\eta_8 = 1.7717\ldots$, and $\eta_9 = 1.7740\ldots$ The singular lines of the pencil are the lines $L_{\eta_i}$ $(1 \le i \le 9)$ corresponding to these 9 roots (notice that the line at infinity is not singular). The lines $L_{\eta_i}$, for $i = 2, 5, 7$, pass through the singular points of the curve. All the other singular lines are tangent to $C_2$. See Fig. 12. The line $L_{\eta_3}$ intersects the curve at 4 distinct non-real points. It is tangent to $C_2$ at $(\eta_3, \pm 0.2270\ldots i)$ and the intersection multiplicity of $L_{\eta_3}$ with $C_2$ at these two points is 2.

We consider the generic line $L_{\eta_5-\varepsilon}$ and choose generators $\xi_1, \ldots, \xi_6$ of the fundamental group $\pi_1(L_{\eta_5-\varepsilon} \setminus C_2)$ as in Fig. 13. The Zariski–van Kampen theorem says that $\pi_1(\mathbb{CP}^2 \setminus C_2) \simeq \pi_1(L_{\eta_5-\varepsilon} \setminus C_2)/G_2$, where $G_2$ is the normal subgroup of $\pi_1(L_{\eta_5-\varepsilon} \setminus C_2)$ generated by the monodromy relations around the singular lines $L_{\eta_i}$ $(1 \leq i \leq 9)$. The latter are given as follows.

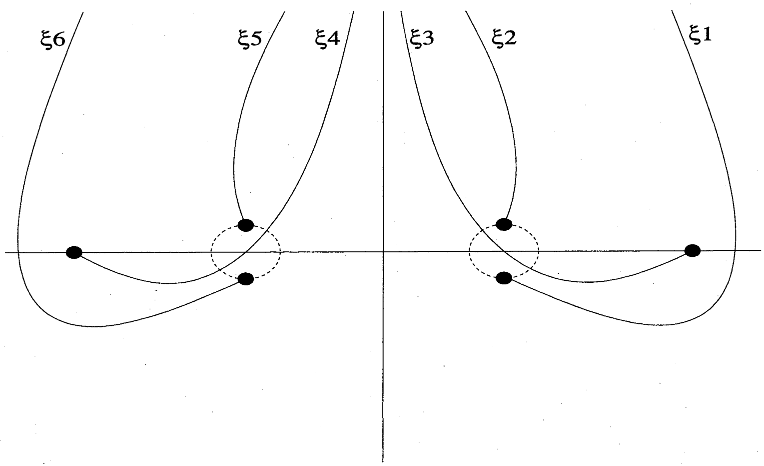


Fig. 13. Generators at $x = \eta_5 - \varepsilon$

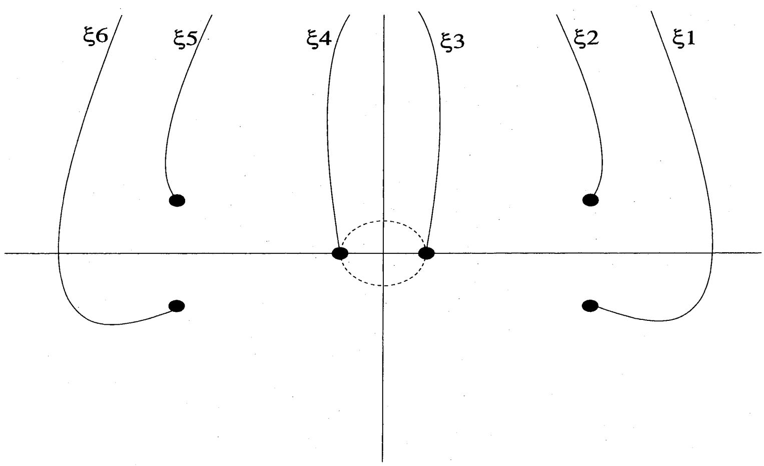


Fig. 14. Generators at $x = \eta_4 + \varepsilon$

**Monodromy relations at $x = \eta_4$**

In Fig. 14, we show how the generators at $x = \eta_5 - \varepsilon$ (cf. Fig. 13) are deformed when $x$ moves on the real axis from $x := \eta_5 - \varepsilon \to \eta_4 + \varepsilon$.

The line $L_{\eta_4}$ is tangent to the curve at one simple point with intersection multiplicity 2. Therefore, as above, the monodromy relation around this line is given by

$$\xi_4 = \xi_3. \tag{12}$$

**Monodromy relations at $x = \eta_3$**

In Fig. 15, we show how the generators at $x = \eta_4 + \varepsilon$ (cf. Fig. 14) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_4| = \varepsilon$, then moves on the real axis from $x := \eta_4 - \varepsilon \to \eta_3 + \varepsilon$. The singular line $L_{\eta_3}$ is tangent to $C_2$ at two non-real simple points, in both cases with intersection multiplicity 2, and therefore the monodromy relations we are looking for are given by

$$\xi_5 = \xi_3\xi_2\xi_3^{-1} \quad \text{and} \quad \xi_6 = (\xi_5\xi_3\xi_2) \cdot \xi_1 \cdot (\xi_5\xi_3\xi_2)^{-1}.$$

Equivalently,

$$\xi_5 = \xi_3\xi_2\xi_3^{-1} \quad \text{and} \quad \xi_6 = (\xi_3\xi_2\xi_2) \cdot \xi_1 \cdot (\xi_3\xi_2\xi_2)^{-1}. \tag{13}$$

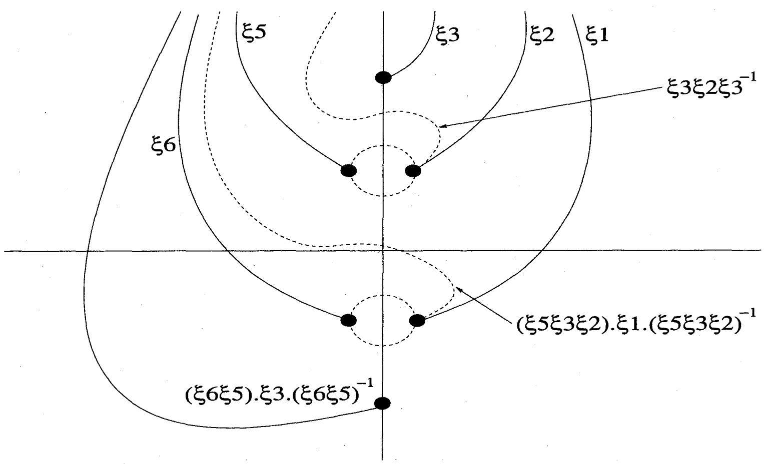


Fig. 15. Generators at $x = \eta_3 + \varepsilon$

**Monodromy relations at $x = \eta_2$**

In Fig. 16, we show how the generators at $x = \eta_3 + \varepsilon$ (cf. Fig. 15) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_3| = \varepsilon$, then moves on the real axis from $x := \eta_3 - \varepsilon \to \eta_2 + \varepsilon$.

The line $L_{\eta_2}$ passes throuh the singular point $(-1,0)$ which is an $\mathbf{A}_1$-singularity. At this point, the curve has two branches $K_1$ and $K_2$ given by

$$K_1: \qquad x = -1 + \frac{1}{331}\sqrt{3310 - 5958\sqrt{33}\,y} + \text{higher terms},$$
$$K_2: \qquad x = -1 - \frac{1}{331}\sqrt{3310 - 5958\sqrt{33}\,y} + \text{higher terms}.$$

These equations show up that when $x$ runs once counter-clockwise on the circle $|x - \eta_2| = \varepsilon$, the points near the origin in Fig. 16 run once counter-clockwise around it. So the monodromy relation at $x = \eta_2$ is given by

$$\xi_3\xi_2\xi_3^{-1} = \xi_6 \cdot \xi_3\xi_2\xi_3^{-1} \cdot \xi_6^{-1},$$

which can also be written, by (13), as

$$\xi_2\xi_1 = \xi_1\xi_2.$$

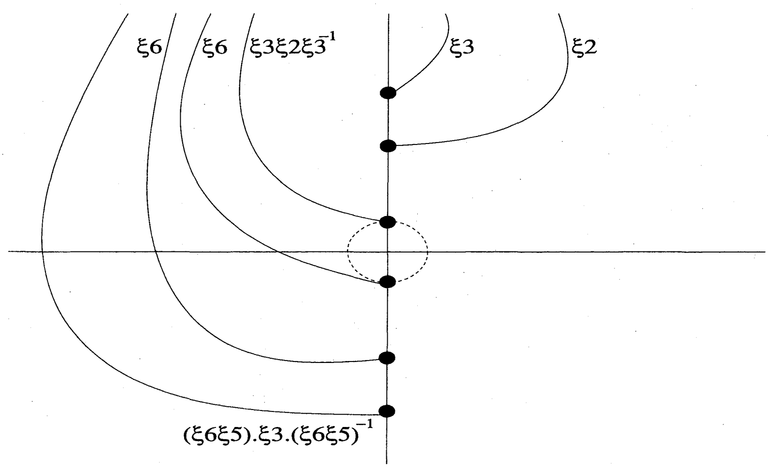


Fig. 16. Generators at $x = \eta_2 + \varepsilon$

### Monodromy relations at $x = \eta_1$

In Fig. 17, we show how the generators at $x = \eta_2 + \varepsilon$ (cf. Fig. 16) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_2| = \varepsilon$, then moves on the real axis from $x := \eta_2 - \varepsilon \to \eta_1 + \varepsilon$.

The line $L_{\eta_1}$ is tangent to $C_2$ at one simple point, with intersection multiplicity 2, and the monodromy relation at $x = \eta_1$ is given by

$$(\xi_3\xi_2) \cdot \xi_1 \cdot (\xi_3\xi_2)^{-1} = \xi_3\xi_2\xi_3^{-1},$$

that is,

$$\xi_1 = \xi_2. \tag{14}$$

In particular, by (13), it implies

$$\xi_5 = \xi_6. \tag{15}$$

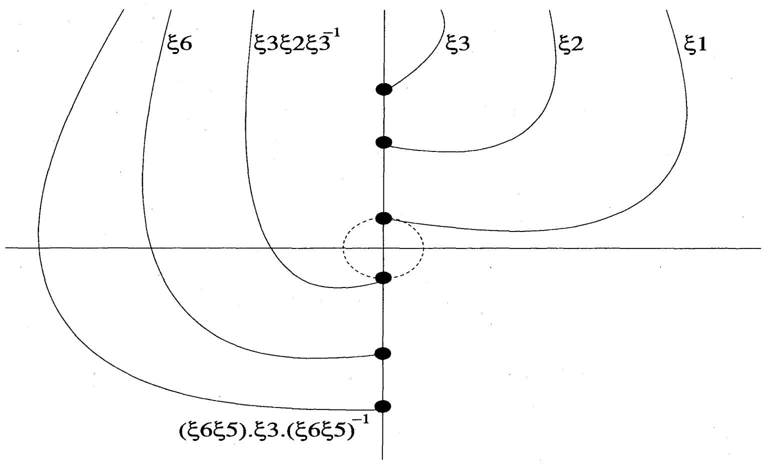


Fig. 17. Generators at $x = \eta_1 + \varepsilon$

**Monodromy relations at $x = \eta_5$**

By (12), (13), (14) and (15), Fig. 13 (which gives the generators at $x = \eta_5 - \varepsilon$) is the same as Fig. 18, where

$$\omega := \xi_3\xi_1\xi_3^{-1} \; (= \xi_5 = \xi_6).$$

The line $L_{\eta_5}$ passes through the singular points $(0,1)$ and $(0,-1)$ which are both $\mathbf{A}_4$-singularities. Puiseux parametrizations of $C_2$ at these points are given by

$$x = t^2, \quad y = 1 + \frac{1}{2}\,t^2 + \beta_4\,t^4 + \beta_5\,t^5 + \text{higher terms}$$

and

$$x = t^2, \quad y = -1 - \frac{1}{2}\,t^2 - \beta_4\,t^4 - \beta_5\,t^5 + \text{higher terms}$$

respectively, where $\beta_4, \beta_5 \in \mathbb{R} \setminus \{0\}$. We deduce that the monodromy relation at $x = \eta_5$ is given by

(16) $\xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}.$

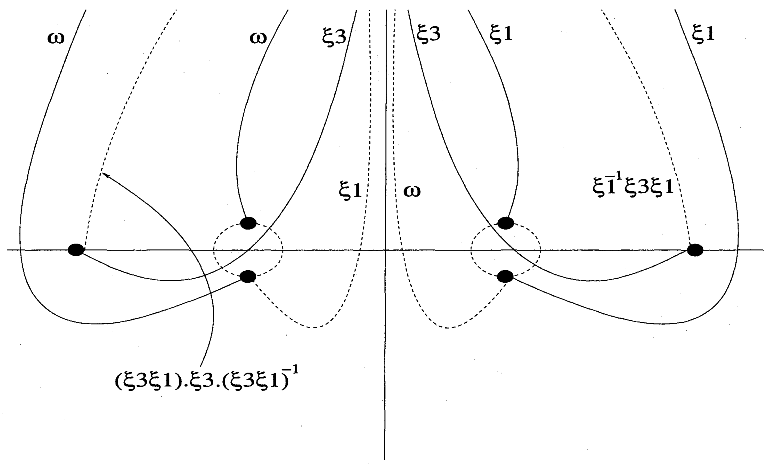


Fig. 18. Generators at $x = \eta_5 - \varepsilon$

**Monodromy relations at $x = \eta_6$**

In Fig. 19, we show how the generators at $x = \eta_5 - \varepsilon$ (cf. Fig. 18) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_5| = \varepsilon$, then moves on the real axis from $x := \eta_5 + \varepsilon \to \eta_6 - \varepsilon$, where

$$\begin{aligned}
\zeta_1 &:= \xi_1^{-1}\omega\xi_1, \\
\zeta_2 &:= (\omega\xi_1)^{-1} \cdot \xi_1 \cdot (\omega\xi_1), \\
\zeta_3 &:= (\omega\xi_1)^{-1} \cdot \xi_1^{-1}\omega\xi_1 \cdot (\omega\xi_1).
\end{aligned}$$

The line $L_{\eta_6}$ is tangent to the curve at two simple points, in both cases with intersection multiplicity 2. So, once more, the monodromy relation around this tangent line is simply given by

(17) $$\xi_1\xi_3\xi_1 = \xi_3\xi_1\xi_3^{-1} \cdot \xi_1 \cdot \xi_3\xi_1\xi_3^{-1}.$$

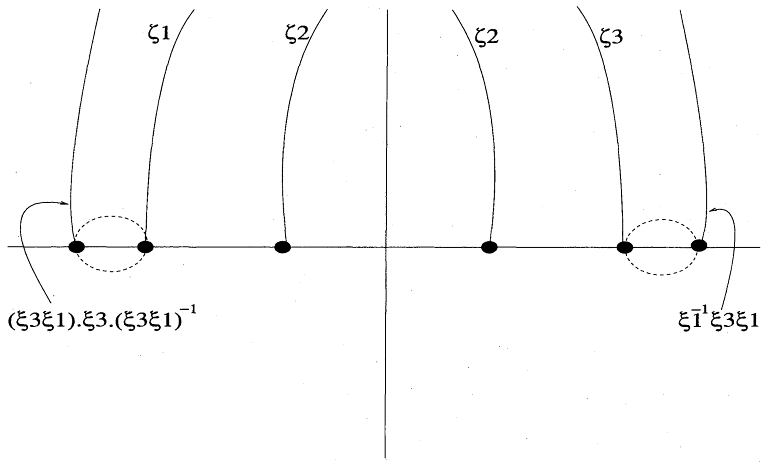


Fig. 19. Generators at $x = \eta_6 - \varepsilon$

**Monodromy relations at $x = \eta_7$**

In Fig. 20, we show how the generators at $x = \eta_6 - \varepsilon$ (cf. Fig. 19) are deformed when $x$ makes half-turn counter-clockwise on the circle $|x - \eta_6| = \varepsilon$, then moves on the real axis from $x := \eta_6 + \varepsilon \to \eta_7 - \varepsilon$ (use the relation (17) to determine all the lassos). The line $L_{\eta_7}$ passes through the singular points $(1, 1)$ and $(1, -1)$ which are both $\mathbf{A}_4$-singularities, and Puiseux parametrizations of the curve at these points are given by

$$x = 1 + t^2, \quad y = 1 + \gamma_2\, t^2 + \gamma_4\, t^4 + \gamma_5\, t^5 + \text{higher terms}$$

and

$$x = 1 + t^2, \quad y = -1 - \gamma_2\, t^2 - \gamma_4\, t^4 - \gamma_5\, t^5 + \text{higher terms}$$

respectively, where $\gamma_2, \gamma_4, \gamma_5 \in \mathbb{R} \setminus \{0\}$. Hence the monodromy relation at $x = \eta_7$ is given by

(18) $$\xi_3 \xi_1 \xi_3 \xi_1 \xi_3 = \xi_1 \xi_3 \xi_1 \xi_3 \xi_1.$$

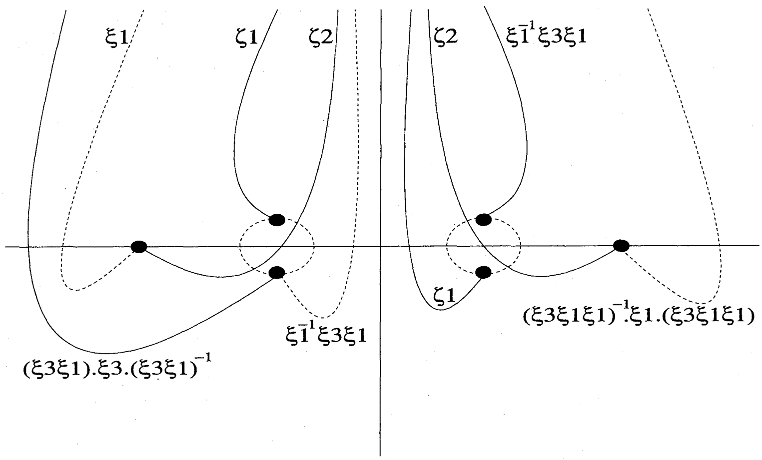


Fig. 20. Generators at $x = \eta_7 - \varepsilon$

The monodromy relations around the singular lines $L_{\eta_8}$ and $L_{\eta_9}$ do not give any new information (details are left to the reader).

Now, by the previous relations, it is easy to check that the vanishing relation at infinity is written as

(19) $$\xi_3\xi_1\xi_1 \cdot \xi_3\xi_1\xi_1 = e.$$

This relation, combined with (17), shows that (16) is equivalent to

(20) $$\xi_1\xi_3\xi_1 \cdot \xi_1\xi_3\xi_1 = e.$$

Finally, we have proved that the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C_2)$ is presented by the generators $\xi_1$ and $\xi_3$ and the relations (17), (18), (19) and (20). We conclude exactly as in section 2. Q.E.D.

## References

[1] A. Degtyarev, Oka's conjecture on irreducible plane sextics, J. London Math. Soc. (2), **78** (2008), 329–351.

[2] A. Degtyarev, On irreducible sextics with non-abelian fundamental group, in this volume.

[3] A. Degtyarev and M. Oka, A plane sextic with finite fundamental group, in this volume.

[4] C. Eyral and M. Oka, On the fundamental groups of the complements of plane singular sextics, J. Math. Soc. Japan, **57** (2005), 37–54.

[5] D. T. Pho, Classification of singularities on torus curves of type $(2,3)$, Kodai Math. J., **24** (2001), 259–284.
[6] M. Oka, Symmetric plane curves with nodes and cups, J. Math. Soc. Japan, **44** (1992), 375–414.
[7] M. Oka, Two transforms of plane curves and their fundamental groups, J. Math. Sci. Univ. Tokyo, **3** (1996), 399–443.
[8] M. Oka and D. T. Pho, Classification of sextics of torus type, Tokyo J. Math., **25** (2002), 399–433.
[9] E. R. van Kampen, On the fundamental group of an algebraic curve, Amer. J. Math., **55** (1933), 255–260.
[10] O. Zariski, On the problem of existence of algebraic functions of two variables possessing a given branch curve, Amer. J. Math., **51** (1929), 305–328.

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