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A quantization of the sixth Painlevé equation

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Abstract.

The sixth Painlevé equation has the affine Weyl group symmetry of type $D_4^{(1)}$ as a group of Bäcklund transformations and is written as a Hamiltonian system. We propose a quantization of the sixth Painlevé equation with the extended affine Weyl group symmetry of type $D_4^{(1)}$.

§1. Introduction

The Painlevé equations $\mathrm{P_J}$ (J=I,. . . ,VI) were discovered by Painlevé and Gambier around the beginning of the twentieth century, as a result of seeking new special functions as solutions of second-order nonlinear ordinary differential equations without movable singular points [11], [3]. After the discovery of the Painlevé equations, the sixth Painlevé equations was derived from the monodromy preserving deformation by R. Fuchs [2].

As for the problem of quantization of the monodromy preserving deformation, it was noticed by N. Resehtikhin [12] that a quantization of the Schlesinger equations which are deformation equations that preserve the monodromy of the rational connection with regular singularities can be viewd as the Knizhnik–Zamolodchikov equations in the conformal field theory. As the argument in his paper, one hope that one would obtain "a quantization of the sixth Painlevé equation" from the Knizhik–Zamolochikov equation in the special case and no one suceeds to obtain it so far.

On the other hand, the Painlevé equations are integrable systems in some sense. For example, they are closely related with the soliton equations (for example, see [6] and reference therein). Quantization of

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the soliton equations has been studied since 90's (for example, see [1] and reference therein), however, quantization of (continuous) Painlevé equations had not been studied.

We attack the problem of quantization from the symmetry. K. Okamoto showed that the Painlevé equations, except the first Painlevé equation, have affine Weyl group symmetries as a group of Bäcklund transformations [10]. The sixth Painlevé equation, which we deal with in this article, has the affine Weyl group symmetry of type $D_4^{(1)}$. The Painlevé equations are written as Hamiltonian systems and their hamiltonians are polynomials in the canonical coordinates. We consider the following problem: does there exist a quantization of Painlevé equations with affine Weyl group symmetry? What we mean by quantization is the canonical quantization, that is, a Poisson bracket is replaced with a commutator.

In [7], we constructed a quantization of differential systems with affine Weyl group symmeties of type $A_l^{(1)}$ [8] which includes the quantum second, the quantum fourth and the quantum fifth Painlevé equation, which has the affine Weyl group symmetry of type $A_1^{(1)}$, $A_2^{(1)}$ and $A_3^{(1)}$, respectively. The first Painlevé equation does not have an affine Weyl group symmetry and its quantization is uniquely determined because its hamiltonian does not include unseparated terms between the canonical coordinate p and q.

In this article, we construct a quantization of the sixth Painlevé equation with the extended affine Weyl group symmetry of type $D_4^{(1)}$. We hope that this is a first step to understand the quantum Hamiltonian reduction of the Knizhnik–Zamolodchikov equations or the relation between quantum soliton equations and quantum Painlevé equaions.

§2. The sixth Painlevé equation

Let us recall the Hamiltonian and the Bäcklund transformations of the sixth Painlevé equation. We follow the notation from [9]. The Hamiltonian of the sixth Painlevé equation is given by

$$(1)\quad \begin{aligned} H =& \frac{1}{t(t-1)}\left[p^2q(q-1)(q-t) - p\left\{(\alpha_0-1)q(q-1)+\alpha_3 q(q-t)\right.\right.\\ &\left.\left.+\alpha_4(q-1)(q-t)\right\} + \alpha_2(\alpha_1+\alpha_2)(q-t)\right], \end{aligned}$$

where q, p are functions of t and α_i $(i=0,1,2,3,4)$ are parameters in $\mathbb{C}$ such that $\alpha_0+\alpha_1+2\alpha_2+\alpha_3+\alpha_4=1$. The sixth Painlevé equation is expressed as the Hamiltonian system

$$\frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial p},\tag{2}$$

and we have

$$\begin{aligned} t(t-1)\frac{dq}{dt} =&2pq(q-1)(q-t)-\{\alpha_4(q-1)(q-t)\\ &+\alpha_3 q(q-t)+(\alpha_0-1)q(q-1)\},\end{aligned}\tag{3}$$

$$\begin{aligned} t(t-1)\frac{dp}{dt} =&-p^2(3q^2-2(1+t)q+t)+p\{2(\alpha_0+\alpha_3+\alpha_4-1)q\\ &-\alpha_4(1+t)-\alpha_3 t-\alpha_0+1\}-\alpha_2(\alpha_1+\alpha_2).\end{aligned}\tag{4}$$

It is known that the sixth Painlevé equation has the extended affine Weyl group symmetry of type $D_4^{(1)}$. To illustrate this, let us consider the field of rational functions

$$K=\mathbb{C}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,q,p,t).\tag{5}$$

Let K be equipped with Poisson bracket defined by

$$\{p,q\}=1,\tag{6}$$

$$\{p,\alpha_i\}=\{q,\alpha_i\}=\{p,t\}=\{q,t\}=\{t,\alpha_i\}=0\quad(1\le i\le 4).\tag{7}$$

Moreover, we set

$$\varphi_0=q-t,\quad \varphi_1=1,\quad \varphi_2=-p,\quad \varphi_3=q-1,\quad \varphi_4=q.\tag{8}$$

We define automorphisms s_i $(0\le i\le 4)$ and r_1, r_3, r_4 on K as follows:

$$s_i(\alpha_j)=\alpha_j-\alpha_i a_{ij},\quad s_i(\varphi_j)=\varphi_j+\{\varphi_i,\varphi_j\}\frac{\alpha_i}{\varphi_i}\quad(0\le i,j\le 4),\tag{9}$$

where $A=(a_{ij})$ is the Cartan matrix of type $D_4^{(1)}$:

$$A=\begin{bmatrix} 2 & 0 & -1 & 0 & 0\\ 0 & 2 & -1 & 0 & 0\\ -1 & -1 & 2 & -1 & -1\\ 0 & 0 & -1 & 2 & 0\\ 0 & 0 & -1 & 0 & 2\end{bmatrix},\tag{10}$$

and

$$r_1(\alpha_0) = \alpha_1, \quad r_1(\alpha_1) = \alpha_0, \quad r_1(\alpha_2) = \alpha_2, \tag{11}$$

$$r_1(\alpha_3) = \alpha_4, \quad r_1(\alpha_4) = \alpha_3, \tag{12}$$

$$r_1(q) = \frac{t(q-1)}{q-t}, \quad r_1(p) = -\frac{(q-t)\left((q-t)p+\alpha_2\right)}{t(t-1)}, \tag{13}$$

$$r_3(\alpha_0) = \alpha_3, \quad r_3(\alpha_1) = \alpha_4, \quad r_3(\alpha_2) = \alpha_2, \tag{14}$$

$$r_3(\alpha_3) = \alpha_0, \quad r_3(\alpha_4) = \alpha_1, \tag{15}$$

$$r_3(q) = \frac{t}{q}, \quad r_3(p) = -\frac{q\,(qp+\alpha_2)}{t}, \tag{16}$$

$$r_4(\alpha_0) = \alpha_4, \quad r_4(\alpha_1) = \alpha_3, \quad r_4(\alpha_2) = \alpha_2, \tag{17}$$

$$r_4(\alpha_3) = \alpha_1, \quad r_4(\alpha_4) = \alpha_0, \tag{18}$$

$$r_4(q) = \frac{q-t}{q-1}, \quad r_4(p) = -\frac{(q-1)\left((q-1)p+\alpha_2\right)}{t-1}. \tag{19}$$

Then automorphisms s_i $(0 \le i \le 4)$ and r_1, r_3, r_4 give a representation of the extended affine Weyl group of type $D_4^{(1)}$, and commute with $\frac{d}{dt}$. This describes the extended affine Weyl group symmetry of type $D_4^{(1)}$ on the sixth Painlevé equation.

§3. Quantization of the sixth Painlevé equation

Let $\mathcal{K}$ be the skew field over $\mathbb{C}$ with generators $\hat{q}$, $\hat{p}$, α_1, α_2, α_3, α_4, t and the commutation relations defined by

$$[\hat{p}, \hat{q}] = h, \tag{20}$$

$$[\hat{p}, \alpha_i] = [\hat{q}, \alpha_i] = [\hat{p}, t] = [\hat{q}, t] = [t, \alpha_i] = 0 \quad (1 \le i \le 4). \tag{21}$$

This commutation relations correspond to the Poisson bracket on the rational function field K.

Let an element $\widehat{H}$ in $\mathcal{K}$ be defined by

(22)

$$\begin{aligned} t(t-1)\widehat{H} =& \frac{1}{6}\left[\hat{q}\hat{p}(\hat{q}-1)\hat{p}(\hat{q}-t) + (\hat{q}-1)\hat{p}(\hat{q}-t)\hat{p}\hat{q} + (\hat{q}-t)\hat{p}\hat{q}\hat{p}(\hat{q}-1)\right. \\ & \left. +(\hat{q}-t)\hat{p}(\hat{q}-1)\hat{p}\hat{q} + (\hat{q}-1)\hat{p}\hat{q}\hat{p}(\hat{q}-t) + \hat{q}\hat{p}(\hat{q}-t)\hat{p}(\hat{q}-1)\right] \\ & - \frac{1}{2}\left[(\alpha_0-1)(\hat{q}\hat{p}(\hat{q}-1) + (\hat{q}-1)\hat{p}\hat{q}) + \alpha_3(\hat{q}\hat{p}(\hat{q}-t)\right. \\ & \left. +(\hat{q}-t)\hat{p}\hat{q}) + \alpha_4((\hat{q}-1)\hat{p}(\hat{q}-t) + (\hat{q}-t)\hat{p}(\hat{q}-1))\right] \\ & + \alpha_2(\alpha_1+\alpha_2)(\hat{q}-t), \end{aligned}$$

where $\alpha_0 = 1 - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4$. This element $\widehat{H}$ is a canonical quantization of the Hamiltonian of the sixth Painlevé equation. We define transformations s_i $(0 \le i \le 4)$ and r_1, r_3, r_4 for the generators of $\mathcal{K}$ as follows:

$$(23)\quad s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad s_i(\varphi_j) = \hat{\varphi}_j + [\hat{\varphi}_i, \hat{\varphi}_j]\frac{\alpha_i}{h\hat{\varphi}_i} \quad (0 \le i, j \le 4),$$

where $A = (a_{ij})_{0 \le i,j \le 4}$ is the Cartan matrix of type $D_4^{(1)}$ (10) and where $\hat{\varphi}_i$ are defined by

$$(24)\qquad \hat{\varphi}_0 = \hat{q} - t, \quad \hat{\varphi}_1 = 1, \quad \hat{\varphi}_2 = -\hat{p}, \quad \hat{\varphi}_3 = \hat{q} - 1, \quad \hat{\varphi}_4 = \hat{q},$$

and

$$(25)\qquad r_1(\alpha_0) = \alpha_1, \quad r_1(\alpha_1) = \alpha_0, \quad r_1(\alpha_2) = \alpha_2,$$

$$(26)\qquad r_1(\alpha_3) = \alpha_4, \quad r_1(\alpha_4) = \alpha_3,$$

$$(27)\qquad r_1(\hat{q}) = \frac{t(\hat{q}-1)}{\hat{q}-t}, \quad r_1(\hat{p}) = -\frac{(\hat{q}-t)\hat{p}(\hat{q}-t) + \alpha_2(\hat{q}-t)}{t(t-1)},$$

$$(28)\qquad r_3(\alpha_0) = \alpha_3, \quad r_3(\alpha_1) = \alpha_4, \quad r_3(\alpha_2) = \alpha_2,$$

$$(29)\qquad r_3(\alpha_3) = \alpha_0, \quad r_3(\alpha_4) = \alpha_1,$$

$$(30)\qquad r_3(\hat{q}) = \frac{t}{\hat{q}}, \quad r_3(\hat{p}) = -\frac{\hat{q}\hat{p}\hat{q} + \alpha_2\hat{q}}{t},$$

$$(31)\qquad r_4(\alpha_0) = \alpha_4, \quad r_4(\alpha_1) = \alpha_3, \quad r_4(\alpha_2) = \alpha_2,$$

$$(32)\qquad r_4(\alpha_3) = \alpha_1, \quad r_4(\alpha_4) = \alpha_0,$$

$$(33)\qquad r_4(\hat{q}) = \frac{\hat{q}-t}{\hat{q}-1}, \quad r_4(\hat{p}) = -\frac{(\hat{q}-1)\hat{p}(\hat{q}-1) + \alpha_2(\hat{q}-1)}{t-1}.$$

Proposition 1. *Transformations s_i $(0 \le i \le 4)$ and r_1, r_3, r_4 preserve the fundamental commutation relations (20), namely, they become automorphisms on $\mathcal{K}$.*

Proof. We can check that by straightforward calculations. For example, we check that r_1 preserves (20):

$$\begin{aligned}[r_1(\hat{p}), r_1(\hat{q})] &= \left[-\frac{(\hat{q}-t)\hat{p}(\hat{q}-t)}{t(t-1)}, \frac{t(\hat{q}-1)}{\hat{q}-t}\right] \\ &= \frac{\hat{q}-t}{t-1}\left[-\hat{p}, \frac{\hat{q}-1}{\hat{q}-t}\right](\hat{q}-t) \\ &= \frac{h}{t-1}\left(\hat{q}-1-(\hat{q}-t)\right) = h.\end{aligned}$$

Q.E.D.

Lemma 1. *The automorphisms s_i $(0 \le i \le 4)$ act $\widehat{H}$ as follows:*

(34)
$$s_0(\widehat{H}) = \widehat{H} + \frac{\alpha_0}{t(t-1)}\left(\hat{q} - \alpha_0\frac{\hat{q}(\hat{q}-1)}{\hat{q}-t} - \alpha_4 - (\alpha_0+\alpha_1+2\alpha_2)t\right),$$
(35)
$$s_1(\widehat{H}) = \widehat{H},$$
(36)
$$s_2(\widehat{H}) = \widehat{H} + \frac{1}{t(t-1)}((1-\alpha_0+\alpha_1)\alpha_2 t - \alpha_2(\alpha_1+\alpha_2+\alpha_3)),$$
(37)
$$s_3(\widehat{H}) = \widehat{H} + \frac{(\alpha_0-1)\alpha_3}{(t-1)},$$
(38)
$$s_4(\widehat{H}) = \widehat{H} + \frac{(\alpha_0-1)\alpha_4}{t}.$$

Proof. These relations immediately follows from the definitions of the Hamiltonian $\widehat{H}$ and automorphisms s_i. For example, we check (37):

$$\begin{aligned}
&t(t-1)s_3(\widehat{H})\\
&= t(t-1)\widehat{H} + \frac{\alpha_3}{6}\left\{-4qp(q-t) - 4(q-t)pq\right.\\
&\left.-2(q-1)p\frac{q(q-t)}{q-1} - 2\frac{q(q-t)}{q-1}p(q-1) + 6\frac{\alpha_3 q(q-t)}{q-1}\right\}\\
&-\alpha_3\left\{-(\alpha_0-1)q - qp(q-t) - (q-t)pq + \frac{\alpha_3 q(q-t)}{q-1} - \alpha_4(q-t)\right\}\\
&+(\alpha_2\alpha_3 + \alpha_3(\alpha_1+\alpha_2+\alpha_3))(q-t)\\
&= t(t-1)\widehat{H} + (\alpha_0-1)\alpha_3 t.
\end{aligned}$$

Q.E.D.

Let δ be the $\mathbb{C}$-derivation on $\mathcal{K}$ defined by

$$\delta(\varphi) = \frac{1}{h}[\widehat{H},\varphi] + \frac{\partial\varphi}{\partial t} \quad (\varphi \in \mathcal{K}). \tag{39}$$

We write down $\delta(\hat{q})$ and $\delta(\hat{p})$ in the following.

(40)
$$
\begin{aligned}
t(t-1)\delta(\hat{q}) =& \hat{q}(\hat{q}-1)\hat{p}(\hat{q}-t) + (\hat{q}-t)\hat{p}\hat{q}(\hat{q}-1) \\
& - \{\alpha_4(\hat{q}-1)(\hat{q}-t) + \alpha_3\hat{q}(\hat{q}-t) + (\alpha_0-1)\hat{q}(\hat{q}-1)\},
\end{aligned}
$$

(41)
$$
\begin{aligned}
t(t-1)\delta(\hat{p}) =& -\left(\hat{p}\hat{q}\hat{p}\hat{q} + \hat{q}\hat{p}^2\hat{q} + \hat{q}\hat{p}\hat{q}\hat{p}\right) + 2(1+t)\hat{p}\hat{q}\hat{p} - t\hat{p}^2 \\
& + (\alpha_0+\alpha_3+\alpha_4-1)(\hat{p}\hat{q}+\hat{q}\hat{p}) \\
& + \hat{p}\{-\alpha_4(1+t) - \alpha_3 t - \alpha_0 + 1\} - \alpha_2(\alpha_1+\alpha_2).
\end{aligned}
$$

Theorem 1. *The automorphisms s_i $(0 \le i \le 4)$ and r_1, r_3, r_4 define a representation of the extended affine Weyl group of type $D_4^{(1)}$ and commute with the $\mathbb{C}$-derivation δ.*

Proof. We can check by straightforward calculations that s_i $(0 \le i \le 4)$ and r_1, r_3, r_4 satisfy the following relations:

$$s_i^2 = 1, \quad s_i s_j = s_j s_i \quad (i, j \neq 2), \tag{42}$$

$$s_i s_2 s_i = s_2 s_i s_2 \quad (i = 0, 1, 3, 4), \tag{43}$$

$$r_i^2 = 1 \quad (i = 1, 3, 4), \quad r_k r_l = r_m \quad (\{k, l, m\} = \{1, 3, 4\}), \tag{44}$$

$$r_i s_j = s_{\sigma_i(j)} r_i \quad (i = 1, 3, 4;\ j = 0, 1, 2, 3, 4), \tag{45}$$

where σ_i $(i = 1, 3, 4)$ are the permutations defined by

$$\sigma_1 = (01)(34), \quad \sigma_3 = (03)(04), \quad \sigma_4 = (04)(13). \tag{46}$$

This proves the first statement.

For the second statement, it is enough that we show $\delta(s_i(\hat{\varphi}_i)) = s_i(\delta(\hat{\varphi}_i))$ for $i = 0, 1, 2, 3, 4$, and $\delta(r_i(\hat{p})) = r_i(\delta(\hat{p}))$ and $\delta(r_i(\hat{q})) = r_i(\delta(\hat{q}))$ for $(i = 1, 3, 4)$. From Lemma 1, $s_i(\widehat{H})$ are equivalent to the Hamiltonian $\widehat{H}$ plus commutative parts. Then, the computation to confirm commutativity between δ and s_i is the same as the classical case. As for commutativity between δ and r_i, also we can check that by direct computations.

Q.E.D.

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