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Global behavior of a two-dimensional monotone difference system

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Abstract.

We investigate global behavior of solutions of a nonlinear difference system

$$x_{n+1} = px_n(1 + y_n), \quad y_{n+1} = qy_n(1 + x_n), \quad n = 0, 1, 2, \ldots,$$

where parameters p, q and initial values x_0, y_0 are positive. We give sufficient conditions for every solution of the system to be unbounded and sufficient conditions for the global stable manifold of the positive equilibrium to exist, which is a unbounded separatrix for the system. Some related conjectures are also given.

§1. Introduction and preliminaries

Consider a nonlinear difference system

$$(1) \qquad u_{n+1} = \frac{au_nv_n}{b + cv_n}, \quad v_{n+1} = \frac{du_nv_n}{e + fu_n}, \quad n = 0, 1, 2, \ldots,$$

where all parameters a, b, c, d, e, f are positive and initial values u_0, v_0 are nonnegative. The system (1) may be regarded as a cooperative system (see [4, 5]). By the change of variables $x_n = e/(fu_n)$ and $y_n = b/(cv_n)$, the system (1) can be transformed into a *monotone* difference system

$$(2) \qquad x_{n+1} = px_n(1 + y_n), \quad y_{n+1} = qy_n(1 + x_n), \quad n = 0, 1, 2, \ldots,$$

where $p = c/a > 0$ and $q = f/d > 0$. For general study of monotone dynamical systems, one can refer to [3, 6, 7] and the references contained therein.

The purpose of this paper is to classify global behavior of solutions of the system (2) with positive initial values x_0, y_0. Our results are

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closely related to results in [5] due to Kulenović and Nurkanović for the system (1) with $b = c = e = f = 1$.

The equilibria of (2) are $(0,0)$ for any p and q, and $(1/q-1, 1/p-1)$ for $0 < p < 1$ and $0 < q < 1$. In addition, if $p = 1$, then every point on the x-axis is an equilibrium point, and if $q = 1$, then every point on the y-axis is an equilibrium point. The map $T : \mathbf{R}_+^2 \to \mathbf{R}_+^2$ of (2) is given by

$$T(x, y) = (px(1+y), qy(1+x)), \tag{3}$$

where $\mathbf{R}_+^2 = \{(x,y) \mid x \geq 0, y \geq 0\}$. Then, for any $(x,y) \in \mathbf{R}_+^2$, the Jacobian matrix for T is given by

$$J_T(x,y) = \begin{pmatrix} p(1+y) & px \\ qy & q(1+x) \end{pmatrix}$$

and the characteristic equation of the Jacobian evaluted at $E_0 = (0,0)$ is

$$(\lambda - p)(\lambda - q) = 0.$$

As is well known (see, e.g. [1, 2]), the equilibrium E_0 of (2) is locally asymptotically stable if $0 < p < 1$ and $0 < q < 1$ and it is unstable if $p > 1$ or $q > 1$. On the other hand, the characteristic equation of the Jacobian evaluted at $E_{p,q} = (1/q - 1, 1/p - 1)$ is

$$\lambda^2 - 2\lambda + 1 - (1-p)(1-q) = 0$$

with roots

$$\lambda_\pm = 1 \pm \sqrt{(1-p)(1-q)}.$$

Obviously, $|\lambda_+| > 1$ and $|\lambda_-| < 1$ if $0 < p < 1$ and $0 < q < 1$, which implies that the equilibrium $E_{p,q}$ of (2) is a saddle point. Therefore, we summarize local stability properties of the equilibria of (2) as follows:

Cases	Equilibria of (2)
$p > 1$ or $q > 1$	E_0 (unstable)
$0 < p < 1$ and $0 < q < 1$	E_0 (locally AS) and $E_{p,q}$ (unstable)
$p = q = 1$	x-axis and y-axis
$p = 1$ and $0 < q < 1$	x-axis
$0 < p < 1$ and $q = 1$	y-axis

Table 1. Linearized stability analysis of (2)

In the following, we will give some notations to state a basic property on the map T given by (3). For $v = (v_1, v_2)$, $w = (w_1, w_2) \in \mathbf{R}_+^2$, we say that $v \leq w$ if $v_1 \leq w_1$ and $v_2 \leq w_2$. Two points v, $w \in \mathbf{R}_+^2$ are said to be *related* if $v \leq w$ or $w \leq v$. Also, a strict inequality between points is defined as $v < w$ if $v \leq w$ and $v \neq w$. A strong inequality is defined as $v \ll w$ if $v_1 < w_1$ and $v_2 < w_2$.

Proposition 1. *Let v, $w \in \operatorname{Int} \mathbf{R}_+^2$. If $v < w$, then $T^n(v) \ll T^n(w)$ for $n = 1, 2, \ldots$.*

Proof. Let $v = (v_1, v_2)$, $w = (w_1, w_2) \in \operatorname{Int} \mathbf{R}_+^2$. If $v < w$, then

$$v_1(1 + v_2) < w_1(1 + w_2), \quad v_2(1 + v_1) < w_2(1 + w_1),$$

which implies $T(v) \ll T(w)$. By induction, we have $T^n(v) \ll T^n(w)$ for $n = 1, 2, \ldots$. Q.E.D.

Proposition 1 shows that if two points in $\operatorname{Int} \mathbf{R}_+^2$ are related, then all iterates of these points are related.

§2. Global results

2.1. The case $p > 1$ or $q > 1$

In this case, there exists a unique equilibrium E_0 which is unstable. Then we have the following result on global behavior of solutions of (2).

Theorem 1. *Assume that $p > 1$ or $q > 1$. Then every solution (x_n, y_n) of (2) satisfies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

Proof. We consider the case $p > 1$. In case $q > 1$, the proof is similar and will be omitted. By (2), we have

$$x_{n+1} > p x_n, \quad n = 0, 1, 2, \ldots,$$

which implies that $x_n > p^n x_0 \to \infty$ as $n \to \infty$, and so $\lim_{n\to\infty} x_n = \infty$. There are two possible cases to consider.

Case (i): $p > 1$ and $q > 1$. An argument similar to that above yields $\lim_{n\to\infty} y_n = \infty$.

Case (ii): $p > 1$ and $q \leq 1$. Since $\lim_{n\to\infty} x_n = \infty$, there exists a positive integer N such that

$$x_n > \frac{1}{q} - 1 \quad \text{for } n \geq N.$$

Then we have

$$\frac{y_{n+1}}{y_n} = q(1+x_n) > 1 \quad \text{for } n \geq N,$$

which implies that y_n is an increasing sequence for $n \geq N$. Suppose that y_n is bounded above. Then there exists a positive number β such that $\lim_{n\to\infty} y_n = \beta$, and hence

$$q(1+x_n) = \frac{y_{n+1}}{y_n} \to 1 \quad \text{as} \quad n \to \infty.$$

This is a contradiction to the fact that the left-hand side tends to ∞ as $n \to \infty$, and so $\lim_{n\to\infty} y_n = \infty$. This completes the proof. Q.E.D.

2.2. The case $0 < p < 1$ and $0 < q < 1$

In this case, there exist two equilibria E_0 which is locally asymptotically stable and $E_{p,q}$ which is a saddle point. For each $v = (v_1, v_2) \in \operatorname{Int} \mathbf{R}^2_+$, we define $Q_i(v)$ for $i = 1, \ldots, 4$ to be the usual four quadrants centered at v and numbered in a counterclockwise direction, e.g., $Q_1(v) = \{(x,y) \in \operatorname{Int} \mathbf{R}^2_+ \mid v_1 \leq x, v_2 \leq y\}$.

The first lemma deals with behavior of solutions of (2) in $Q_1(E_{p,q})$ or $Q_3(E_{p,q})$.

Lemma 1. *Assume that $0 < p < 1$ and $0 < q < 1$. Let (x_n, y_n) be a solution of* (2). *Then the following statements hold:*

(a) *If $(x_0, y_0) \in Q_1(E_{p,q}) \setminus E_{p,q}$, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

(b) *If $(x_0, y_0) \in Q_3(E_{p,q}) \setminus E_{p,q}$, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.*

Proof. We will prove the statement (a). The proof of the statement (b) is similar and will be omitted. Let $(x_0, y_0) \in Q_1(E_{p,q}) \setminus E_{p,q}$, that is, $x_0 > 1/q - 1$ and $y_0 > 1/p - 1$. Then

$$\frac{x_1}{x_0} = p(1+y_0) > 1, \quad \frac{y_1}{y_0} = q(1+x_0) > 1,$$

which, by induction, implies

$$x_{n+1} > x_n > \frac{1}{q} - 1, \quad y_{n+1} > y_n > \frac{1}{p} - 1, \quad n = 0, 1, 2, \ldots.$$

Therefore x_n and y_n are increasing sequences. Suppose that x_n and y_n are bounded above. Then there exist positive numbers α, β such that $\lim_{n\to\infty} x_n = \alpha > 1/q - 1$, $\lim_{n\to\infty} y_n = \beta > 1/p - 1$, and thus, the point (α, β) is another equilibrium in $Q_1(E_{p,q})$. This is a contradiction to the fact that (2) has only two equilibria E_0 and $E_{p,q}$. Suppose that y_n (resp. x_n) is bounded above and x_n (resp. y_n) tends to ∞ as $n \to \infty$. Then a contradiction also arises by a similar fashion in the proof of Theorem 1. Consequently, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$. The proof is complete. Q.E.D.

Lemma 1 shows that a stable eigenvector of $E_{p,q}$ can be chosen to have components of opposite signs. Thus there exists a monotonically decreasing function $h(x)$ such that the local stable manifold W^s_{loc} of $E_{p,q}$ is the graph of h. In fact, the following lemma holds.

Lemma 2. *Assume that $0 < p < 1$ and $0 < q < 1$. Then the global stable manifold W^s of $E_{p,q}$ is a graph of a decreasing function of x.*

Proof. It follows from Lemma 1 that $W^s \subset Q_2(E_{p,q}) \cup Q_4(E_{p,q})$. Hence we will show that W^s contains no related points, then the proof will be complete. To this end, we have only to verify that W^s_{loc} contains no related points by Proposition 1. Suppose that there exist related points $v = (v_1, v_2)$, $w = (w_1, w_2) \in W^s_{\mathrm{loc}}$ such that $v < w$. Then $h(v_1) = v_2 \le w_2 = h(w_1)$, which contradicts the fact that h is monotonically decreasing, and thus, W^s_{loc} contains no related points. This completes the proof. Q.E.D.

Next we will observe behavior of solutions of (2) in $Q_2(E_{p,q})$ or $Q_4(E_{p,q})$.

Lemma 3. *Assume that $0 < p < 1$ and $0 < q < 1$. Let (x_n, y_n) be a solution of (2). Then the following statements hold:*

(a) *If $(x_0, y_0) \in Q_2(E_{p,q}) \setminus E_{p,q}$ and $x_1 > 1/q - 1$, then $(x_1, y_1) \in Q_1(E_{p,q})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

(b) *If $(x_0, y_0) \in Q_4(E_{p,q}) \setminus E_{p,q}$ and $y_1 > 1/p - 1$, then $(x_1, y_1) \in Q_1(E_{p,q})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

(c) *If $(x_0, y_0) \in Q_2(E_{p,q}) \setminus E_{p,q}$ and $y_1 < 1/p - 1$, then $(x_1, y_1) \in Q_3(E_{p,q})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.*

(d) *If $(x_0, y_0) \in Q_4(E_{p,q}) \setminus E_{p,q}$ and $x_1 < 1/q - 1$, then $(x_1, y_1) \in Q_3(E_{p,q})$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.*

Proof. We will prove the statement (a). The proof of the statements (b), (c) and (d) are similar and will be omitted. Let $(x_0, y_0) \in Q_2(E_{p,q}) \setminus E_{p,q}$ and $x_1 = px_0(1 + y_0) > 1/q - 1$. Then we have

$$1 + x_0 > 1 + \frac{1}{p(1 + y_0)} \left(\frac{1}{q} - 1 \right),$$

which implies

$$y_1 = qy_0(1+x_0) > qy_0 + \frac{(1-q)y_0}{p(1+y_0)}. \tag{4}$$

Then it follows from $y_0 > 1/p - 1$ that

$$\begin{aligned}
&qy_0 + \frac{(1-q)y_0}{p(1+y_0)} - \left(\frac{1}{p} - 1\right) \\
&= \frac{1}{1+y_0}\left\{qy_0(1+y_0) + \frac{1-q}{p}y_0 - \left(\frac{1}{p} - 1\right)(1+y_0)\right\} \\
&= \frac{1}{1+y_0}\left\{qy_0^2 + \left(q + 1 - \frac{q}{p}\right)y_0 - \frac{1}{p} + 1\right\} \\
&= \frac{(qy_0+1)(y_0 - 1/p + 1)}{1+y_0} > 0.
\end{aligned}$$

This, together with (4), yields $y_1 = qy_0(1+x_0) > 1/p - 1$, that is, $(x_1, y_1) \in Q_1(E_{p,q})$. By virtue of Lemma 1, we therefore conclude that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ and the statement (a) holds. The proof is complete. Q.E.D.

Lemma 4. *Assume that* $0 < p < 1$ *and* $0 < q < 1$. *Then the global stable manifold* W^s *of* $E_{p,q}$ *is unbounded and separates* $\operatorname{Int}\mathbf{R}_+^2$ *into two invariant regions.*

Proof. Let W_1^s denote the connected component of W^s which contains $E_{p,q}$. Lemma 2 shows that $W_1^s \subset Q_2(E_{p,q}) \cup Q_4(E_{p,q})$ and W_1^s is a graph of a decreasing function of x. Each of the components $Q_2(E_{p,q})$ and $Q_4(E_{p,q})$ may contain only one limiting point of W_1^s which does not belong to W_1^s. Suppose that $v_2 \in Q_2(E_{p,q})$ and $v_4 \in Q_4(E_{p,q})$ denote these limiting points. Since W_1^s is invariant under T, it follows that the set $\{v_2, v_4\}$ is invariant. Hence $\{v_2, v_4\}$ is a period-two solution or each point is fixed. Lemma 3, however, implies that if $v_4 \in Q_4(E_{p,q})$ and $T(v_4)$ belongs to the region $\{(x,y) \in \operatorname{Int}\mathbf{R}_+^2 \mid y > 1/p - 1\}$, then $T(v_4) \in Q_1(E_{p,q})$; if $v_2 \in Q_2(E_{p,q})$ and $T(v_2)$ belongs to the region $\{(x,y) \in \operatorname{Int}\mathbf{R}_+^2 \mid y < 1/p - 1\}$, then $T(v_2) \in Q_3(E_{p,q})$. Thus $\{v_2, v_4\}$ is not a period-two solution. Also v_2 and v_4 are not fixed points because the map T has only two fixed points E_0 and $E_{p,q}$. Therefore, each of the components $Q_2(E_{p,q})$ and $Q_4(E_{p,q})$ contain no limiting points of W_1^s except for $E_{p,q}$. This means that W^s is unbounded and separates $\operatorname{Int}\mathbf{R}_+^2$ into two invariant regions and the proof is complete. Q.E.D.

Now the following theorem is the main result of this subsection.

Theorem 2. *Assume that* $0 < p < 1$ *and* $0 < q < 1$. *Then the following statements hold:*

(a) *Every solution* (x_n, y_n) *of* (2) *starting above* W^s *remains above* W^s *and satisfies* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.

(b) *Every solution* (x_n, y_n) *of* (2) *starting below* W^s *remains below* W^s *and satisfies* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Proof. We will prove the statement (a). The proof of the statement (b) is similar and will be omitted. Let v denote the positive initial point (x_0, y_0) above W^s. By virtue of Lemma 1, we have only to consider the case where $v \in Q_2(E_{p,q}) \cup Q_4(E_{p,q})$. Let w be the point where the vertical line through v intersects W^s. Then $w < v$, and so, Proposition 1 shows that $T^n(w) \ll T^n(v)$ for $n = 0, 1, 2, \ldots$. This, together with Lemma 2, implies that $T^n(v)$ remains above W^s. Suppose that $T^n(v)$ remains in $Q_2(E_{p,q})$ or $Q_4(E_{p,q})$. Then $T^n(v)$ tends to $E_{p,q}$ as $n \to \infty$ because $T^n(w) \ll T^n(v)$ and $T^n(w)$ tends to $E_{p,q}$ as $n \to \infty$. This means that $v \in W^s$, which contradicts the definition of v. Therefore, there exists a positive integer N such that $T^N(v) \in Q_3(E_{p,q})$. By virtue of Lemma 3, we thus conclude that $T^n(v)$ tends to (∞, ∞) as $n \to \infty$. This completes the proof. Q.E.D.

2.3. The case $p = q = 1$

In this case, every point on each coordinate axis is an equilibrium point.

Theorem 3. *Assume that* $p = q = 1$. *Then every solution* (x_n, y_n) *of* (2) *satisfies* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.

Proof. By (2), we have $x_{n+1} > x_n$, $y_{n+1} > y_n$ for $n = 0, 1, 2, \ldots$, which implies that x_n and y_n are increasing sequences. Suppose that x_n and y_n are bounded above. Then there exist positive numbers α, β such that $\lim_{n\to\infty} x_n = \alpha > x_0$, $\lim_{n\to\infty} y_n = \beta > y_0$, and hence, the point (α, β) is another equilibrium in $\operatorname{Int} \mathbf{R}^2_+$. This contradicts the fact that the equilibria of (2) only exist on each coordinate axis. Suppose that y_n (resp. x_n) is bounded above and x_n (resp. y_n) tends to ∞ as $n \to \infty$. Then a contradiction also arises by a similar fashion in the proof of Theorem 1. Consequently, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ and the proof is complete. Q.E.D.

2.4. The case $p = 1$ and $0 < q < 1$

In this case, every point on the x-axis is an equilibrium point. For simplicity, we divide $\mathrm{Int}\,\mathbf{R}_+^2$ into the following two regions:

$$D_1 = \left\{(x, y) \mid x \geq \frac{1}{q}-1,\ y > 0\right\}, \quad D_2 = \left\{(x, y) \mid 0 < x < \frac{1}{q}-1,\ y > 0\right\}.$$

Let (x_n, y_n) be a solution of (2) with $(x_0, y_0) \in D_1$. Then we have $x_{n+1} > x_n > x_0 \geq 1/q - 1$ for $n = 1, 2, \ldots$, which implies that $y_{n+1} > y_n > 0$ for $n = 1, 2, \ldots$. Hence it is easily seen that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ by a similar manner in the proof of Lemma 1 (a), and therefore, the following theorem holds.

Theorem 4. *Assume that $p = 1$ and $0 < q < 1$. Then every solution (x_n, y_n) of (2) starting from D_1 satisfies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

A computer simulation suggests that the following conjecture is true.

Conjecture 1. *Assume that $p = 1$ and $0 < q < 1$. Then there exists a decreasing function $\varphi(x)$ such that $\varphi(1/q - 1) = 0$ and the graph of φ lies in D_2 and has the y-axis as the asymptote, and the following statements hold:*

(a) *Every solution (x_n, y_n) of (2) starting above the graph of φ enters D_1 and satisfies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

(b) *Every solution (x_n, y_n) of (2) starting below the graph of φ remains below this curve and satisfies $\lim_{n\to\infty} x_n = x^* < 1/q - 1$ and $\lim_{n\to\infty} y_n = 0$ where x^* is some positive number depending on (x_0, y_0).*

2.5. The case $0 < p < 1$ and $q = 1$

In this case, every point on the y-axis is an equilibrium point. For simplicity, we divide $\mathrm{Int}\,\mathbf{R}_+^2$ into the following two regions:

$$\Delta_1 = \left\{(x, y) \mid x > 0,\ y \geq \frac{1}{p}-1\right\}, \quad \Delta_2 = \left\{(x, y) \mid x > 0,\ 0 < y < \frac{1}{p}-1\right\}.$$

Theorem 5. *Assume that $0 < p < 1$ and $q = 1$. Then every solution (x_n, y_n) of (2) starting from Δ_1 satisfies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

Conjecture 2. *Assume that $0 < p < 1$ and $q = 1$. Then there exists a decreasing function $\psi(x)$ such that $\psi(0) = 1/p - 1$ and the graph* *of ψ lies in Δ_2 and has the x-axis as the asymptote, and the following statements hold:*

(a) *Every solution (x_n, y_n) of (2) starting above the graph of ψ enters Δ_1 and satisfies $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$.*

(b) *Every solution (x_n, y_n) of (2) starting below the graph of ψ remains below this curve and satisfies $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = y^* < 1/p - 1$ where y^* is some positive number depending on (x_0, y_0).*

Finally, to illustrate our global results for (2), we consider the following cases:

(i) $p = 2, q = 0.5, (x_0, y_0) = (0.005, 1), (0.02, 1), (0.06, 1), (0.12, 1), (0.2, 1), (0.4, 1), (0.7, 1)$;
(ii) $p = q = 0.5, (x_0, y_0) = (0.03, 5), (0.07, 5), (0.13, 5), (0.3, 5), (0.6, 5), (5, 0.03), (5, 0.07), (5, 0.13), (5, 0.3), (5, 0.6), (1, 1)$;
(iii) $p = q = 1, (x_0, y_0) = (0.2, 0.2), (0.2, 1), (0.2, 2), (0.2, 3), (0.2, 4), (1, 0.2), (2, 0.2), (3, 0.2), (4, 0.2)$;
(iv) $p = 1, q = 0.5, (x_0, y_0) = (0.06, 1), (0.1, 1), (0.18, 1), (0.35, 1), (0.6, 1), (2, 0.05), (3, 0.05), (4, 0.05)$.

We give some portraits of solution orbits of (2) drawn by a computer. Figures 1–4 correspond to the cases (i)–(iv), respectively. The dotted curve in Figure 2 is the global stable manifold of the positive equilibrium.

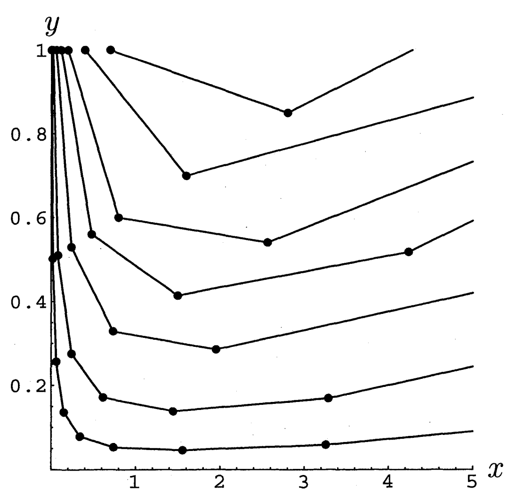

Figure 1. $p = 2,\ q = 0.5$

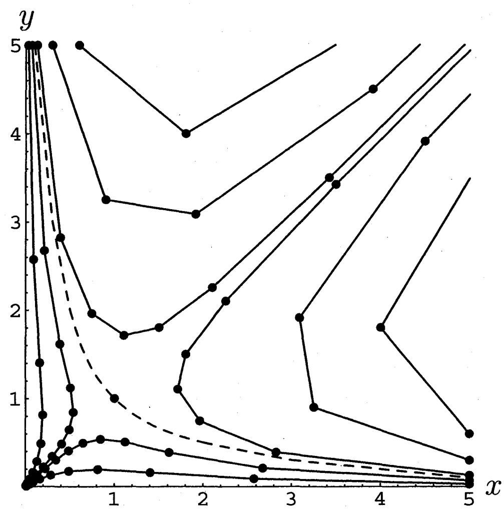

Figure 2. $p = q = 0.5$

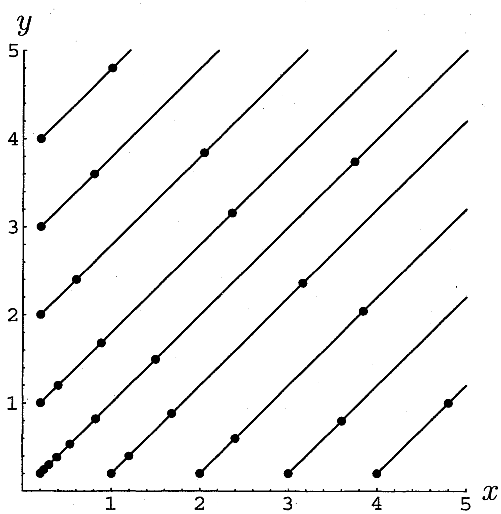

Figure 3. $p = q = 1$

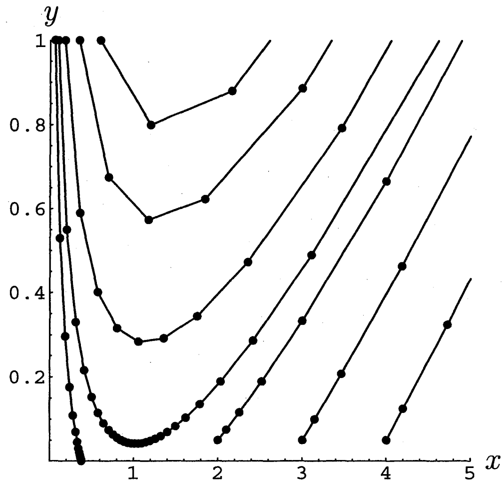

Figure 4. $p = 1,\ q = 0.5$

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