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Relative weight filtrations on completions of mapping class groups

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§1. Introduction

One of the primary themes of Morita's work over the last 20 years has been the study of the structure of mapping class groups via their actions on nilpotent quotients of surface groups. A secondary theme has been the relation of this work to the study of invariants of 3-manifolds and homology spheres. The goal of this paper is to introduce to topologists a new tool that may be useful in these pursuits.

The tool is the relative weight filtration of the relative completion of a mapping class group of a surface that is associated to a system of simple closed curves on the surface. Establishing the existence of relative

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weight filtrations on completions of mapping class groups is non-trivial and was established with Makoto Matsumoto using Galois actions in [20], and with Gregory Pearlstein and Tomohide Terasoma using Hodge theory in [22]. The existence of relative weight filtrations on completions of mapping class groups follows from general results about fundamental groups of smooth varieties and because all mapping class groups occur as fundamental groups of moduli spaces of curves. The general theories of the Hodge and Galois theory of fundamental groups of algebraic varieties imply that relative weight filtrations have strong exactness properties.

The paper begins with an exposition for topologists of the theory of relative completion of discrete groups. It is illustrated by the examples of mapping class groups and automorphism groups of free groups. The paper continues with an exposition of the weight filtration associated to a nilpotent endomorphism of a vector space, and its generalization, the relative weight filtration associated to a nilpotent endomorphism of a *filtered* vector space, which is due to Deligne [11] and was further developed by Steenbrink and Zucker [48], and Kashiwara [31]. Since the generic nilpotent endomorphism of a filtered vector space does not have a relative weight filtration, the existence of a relative weight filtration of a nilpotent endomorphism of a filtered vector space imposes non-trivial restrictions on the endomorphism. (Cf. [48].)

Relative weight filtrations appear in the study of mapping class groups in the following context. Suppose that S is a compact oriented surface of genus g which, for simplicity, we suppose to be ≥ 3. Denote the relative completion of its mapping class group Γ_S by $\mathcal{G}_S$. This is a proalgebraic group (defined over $\mathbb{Q}$) that is an extension

$$1 \to \mathcal{U}_S \to \mathcal{G}_S \to \mathrm{Sp}(H_1(S)) \to 1$$

of the symplectic group that is associated to the first homology of S and its intersection form by a prounipotent group $\mathcal{U}_S$, which is essentially (but not quite) the unipotent completion of the Torelli group T_S of S. (Cf. [18].) There is a natural Zariski dense homomorphism $\Gamma_S \to \mathcal{G}_S$; the image of the Torelli group T_S is Zariski dense in $\mathcal{U}_S$. The Lie algebra of $\mathcal{G}_S$ is an extension

$$0 \to \mathfrak{u}_S \to \mathfrak{g}_S \to \mathfrak{sp}(H_1(S)) \to 0$$

of the symplectic Lie algebra associated to $H_1(S)$ by $\mathfrak{u}_S$, which is a pronilpotent Lie algebra. It has a natural weight filtration which is defined by

$$W_0\mathfrak{g}_S = \mathfrak{g}_S,\ W_{-1}\mathfrak{g}_S = \mathfrak{u}_S,\ W_{-m}\mathfrak{g}_S = L^m\mathfrak{u}_S\ (m \geq 1),$$

where $L^m\mathfrak{u}_S$ denotes the mth term of the lower central series of $\mathfrak{u}_S$. A system $\gamma = \{c_0, \dots, c_m\}$ of disjoint simple closed curves on S determines commuting Dehn twists $\tau_0, \dots, \tau_m$. Their product τ_γ lies in a prounipotent subgroup of $\mathcal{G}_S$ and has a unique logarithm $N_\gamma := \log \tau_\gamma \in \mathfrak{g}_S$ whose adjoint action

$$\mathrm{ad}(N_\gamma) : \mathfrak{g}_S \to \mathfrak{g}_S$$

preserves the weight filtration $W_\bullet$. It therefore induces an inverse system of nilpotent endomorphisms

$$\mathrm{ad}(N_\gamma) : \mathfrak{g}_S/W_{-n}\mathfrak{g}_S \to \mathfrak{g}_S/W_{-n}\mathfrak{g}_S$$

each of which preserves the induced weight filtration $W_\bullet$. General results in Hodge and Galois theory imply the existence of the relative weight filtration $M_\bullet^\gamma$ of $\mathfrak{g}_S$ associated to the curve system γ. This filtration is compatible with the bracket in the sense that the bracket induces a map

$$M_r^\gamma \mathfrak{g}_S \otimes M_s^\gamma \mathfrak{g}_S \to M_{r+s}^\gamma \mathfrak{g}_S.$$

This filtration and the weight filtration $W_\bullet$ have very strong exactness and naturality properties. For example, there is a natural (though not canonical) isomorphism of pronilpotent Lie algebras

$$\mathfrak{g}_S \cong \prod_{m,k} \mathrm{Gr}_k^{M^\gamma} \mathrm{Gr}_m^W \mathfrak{g}_S.$$

There are similar results when S has decorations, such as points and boundary components. The Lie algebra $\mathfrak{p}(S,x)$ of the unipotent completion of $\pi_1(S,x)$ also has a relative weight filtration associated to each curve system γ of (S,x). The natural action $\mathfrak{g}_S \to \mathrm{Der}\,\mathfrak{p}(S,x)$ preserves the relative weight filtration $M_\bullet^\gamma$ as well as the weight filtration $W_\bullet$. This map has strong exactness properties with respect to both of these filtrations. In particular, it is compatible with their identifications with their associated bigraded objects.

Since the bracket preserves the relative weight filtration, $M_k^\gamma\mathfrak{g}_S$ is a subalgebra of $\mathfrak{g}_S$ whenever $k \le 0$. These correspond to subgroups $M_k^\gamma\mathcal{G}_S$ ($k \le 0$) of $\mathcal{G}_S$. The subalgebras $M_0^\gamma\mathfrak{g}_S$, parametrized by curve systems γ on S are natural analogues of parabolic subalgebras of semi-simple Lie algebras as they are parametrized by the boundary components of the corresponding moduli space of curves and they equal their normalizers in $\mathfrak{g}_S$, as we establish in Proposition 8.5.

A particularly interesting case occurs when the curve system γ is maximal. Maximal curve systems correspond to pants decompositions. Since each oriented "pair of pants" naturally bounds a ball, a pants decomposition of S determines a handlebody U with boundary S in which each $c_j \in \gamma$ bounds an imbedded disk in U. The handlebody U is unique in the sense that if V is another such handlebody, then there is a diffeomorphism $U \to V$ which restricts to the identity on S. In Section 9 we use Morse Theory to show that the relative weight filtration corresponding to a pants decomposition depends only on the handlebody U that it determines. We denote it by $M_\bullet^U$. In Section 10 we use the exactness properties of the relative weight filtration and results of Griffiths, Luft and Pitsch [14, 35, 43] to show that the relative weight filtrations $M_\bullet^U$ and $M_\bullet^V$ of $\mathfrak{g}_S$ associated to two handlebodies U and V are equal if and only if there is a diffeomorphism $U \to V$ whose restriction to their boundaries is the identity $S \to S$.

Each handlebody U with boundary S determines a subgroup Λ_U of Γ_S which consists of those elements of Γ_S that extend to an isotopy class of diffeomorphisms of U. In the pointed case, we combine results of Griffiths, Luft and Pitsch [14, 35, 43] with the exactness properties of $M_\bullet^U$ to prove that

$$\Lambda_{U,x} = \Gamma_{S,x} \cap M_0^U \mathcal{G}_{S,x}, \ \Lambda_{U,x} \cap M_{-2}^U \mathcal{G}_{S,x} = \ker\{\Lambda_{U,x} \to \operatorname{Aut} \pi_1(U,x)\}.$$

These results provide an upper bound on the size of $\Lambda_{U,x}$ in $\Gamma_{S,x}$. They also imply that there is an injection

$$\operatorname{Aut} \pi_1(U,x) \to \operatorname{Gr}_0^{M_U} \mathcal{G}_{S,x}.$$

This induces a homomorphism $\mathcal{A}_g \to \operatorname{Gr}_0^{M_U} \mathcal{G}_{S,x}$ from the relative completion $\mathcal{A}_g$ of $\operatorname{Aut} \pi_1(U,x)$. In Section 10 we show that this homomorphism is not surjective, which implies the unexpected result that Λ_U is not Zariski dense in $M_0^U \mathcal{G}_S$. This implies that the relative weight filtration of $\mathcal{G}_S$ is not obtained simply by taking the Zariski closure in $\mathcal{G}_S$ of a filtration of Γ_S.

We give an application to the problem of determining which elements of Γ_S extend to a handlebody. Very similar results have been obtained independently by Jamie Jorgensen [30] by different methods. If S bounds the handlebody U, then the elements of Γ_S that extend to some handlebody is

$$C = \bigcup_{\phi \in \Gamma_S} \phi \Lambda_U \phi^{-1}.$$

In Section 11, we use properties of $M_\bullet^U$ to show that the Zariski closure of the intersection of C with the mth term of the lower central series of $\mathcal{U}_S$ is a proper (i.e., $\subsetneq$) closed subvariety of the mth term of the lower central series of $\mathcal{U}_S$ for all $m \geq 1$ when $g \geq 7$ and slightly restricted ranges when $3 \leq g < 7$.

A more substantial potential application should be to finite type invariants of 3-manifolds and homology 3-spheres. The set of all genus g Heegaard decompositions of 3-manifolds and homology 3-spheres are the double coset spaces

$$\Lambda_U \backslash \Gamma_S / \Lambda_U \text{ and } T\Lambda_V \backslash T_S / T\Lambda_U,$$

where $S^3 = U \cup V$ is a Heegaard decomposition of the 3-sphere, $\partial U = \partial V = S$, and $T\Lambda_U := T_S \cap \Lambda_U$. The sets of all 3-manifolds and homology 3-manifolds are obtained from these by a suitable stabilization where the genus of S goes to infinity.[1] It is thus natural to consider the double coset spaces

$$\mathcal{L}_U \backslash \mathcal{G}_S / \mathcal{L}_U \text{ and } W_{-1}\mathcal{L}_V \backslash \mathcal{U}_S / W_{-1}\mathcal{L}_U,$$

where $\mathcal{L}_U$ denotes the Zariski closure of Λ_U in $\mathcal{G}_S$, as well as their stabilizations as $g(S) \to \infty$. There are natural mappings

$$\Lambda_U \backslash \Gamma_S / \Lambda_U \to \mathcal{L}_U \backslash \mathcal{G}_S / \mathcal{L}_U$$

and

$$T\Lambda_V \backslash T_S / T\Lambda_U \to W_{-1}\mathcal{L}_V \backslash \mathcal{U}_S / W_{-1}\mathcal{L}_U.$$

and their stabilizations as $g(S) \to \infty$. Functions on $\mathcal{L}_U \backslash \mathcal{G}_S / \mathcal{L}_U$ should yield finite type invariants of Heegaard decompositions and functions on the stabilization should yield finite type invariants of 3-manifolds — and similarly for homology 3-spheres. However, there is one major difficulty in carrying out this program. One needs to take the quotients

$$\mathcal{L}_U \backslash \mathcal{G}_S / \mathcal{L}_U$$

using geometric invariant theory (GIT). But since the group $\mathcal{L}_U$ is not reductive, the GIT problems are more difficult and less likely to be well behaved. (Cf. [12].) Amassa Fauntleroy and I are attempting to use the strictness properties of $M_\bullet^U$ to construct and study the GIT quotients above.

The reader should be aware that, in an attempt to make this material more accessible to non-experts, the Hodge and Galois theoretic aspects of the theory have been suppressed. This choice comes at the expense of giving the basic properties of relative weight filtrations a false

[1]The stabilization is by taking the connected sum with the standard genus one Heegaard decomposition of the 3-sphere, cf. [6]. To construct the stabilization maps, one needs to use the mapping class group $\Gamma_{S,D}$ associated to a surface with one boundary component and the corresponding handlebody subgroup $\Lambda_{S,D}$.

aura of mystery. Readers wanting more background should consult the papers of Deligne [11], Steenbrink-Zucker [48] and Kashiwara [31].

Convention 1.1. Throughout the paper the default coefficient group in all homology and cohomology groups is $\mathbb{Q}$, the rational numbers. So, for example, $H_\bullet(S)$ denotes the rational homology of S and $H^\bullet(\Gamma)$ denotes the rational cohomology of the group Γ. All other coefficients will be made explicit.

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§2. Filtrations

Since filtrations play a central role in this paper, it is wise to first lay out the general conventions used in this paper. Deligne's conventions on filtrations [9] are used systematically as they work well and as they are used in Hodge theory and the study of Galois actions.

Suppose that V is a vector space over a field F of characteristic zero. An increasing filtration $G_\bullet$ of V is a sequence of subspaces

$$\cdots \subseteq F_{m-1}V \subseteq F_m V \subseteq F_{m+1}V \subseteq \cdots$$

where $m \in \mathbb{Z}$. When V is finite dimensional, we require that the intersection of the $F_m V$ be trivial and that their union be all of V. The infinite dimensional case is more subtle and is discussed below.

The mth graded quotient $F_m V/F_{m-1}V$ of $F_\bullet$ will be denoted by $\mathrm{Gr}^F_m V$. The associated graded vector space will be denoted by $\mathrm{Gr}^F_\bullet V$.

Decreasing filtrations of V will be denoted with an upper index:

$$\cdots \supseteq F^{m-1}V \supseteq F^m V \supseteq F^{m+1}V \supseteq \cdots$$

The mth graded quotient $F^m V/F^{m+1}V$ will be denoted by $\mathrm{Gr}_F^\bullet V$. We require that the intersection of the $F^m V$ be trivial and that their union be all of V.

An increasing filtration $F_\bullet$ of V can be regarded as a decreasing filtration by "raising indices": $F^mV := F_{-m}V$. For this reason, we will discuss the remaining properties only for increasing filtrations.

If $(V, F_\bullet)$ and $(W, F_\bullet)$ are filtered vector spaces, then $V \otimes W$ and $\mathrm{Hom}(V, W)$ inherit natural filtrations:

$$F_m(V \otimes W) := \sum_{j+k=m} F_jV \otimes F_kW$$

and

$$F_m \mathrm{Hom}(V, W) := \{\phi : V \to W : \phi(F_kV) \subseteq F_{m+k}W \text{ for all } k \in \mathbb{Z}\}.$$

In particular, the dual V^* of V has a natural filtration

$$F_mV^* = \{\phi \in V^* : \phi(F_{-m-1}V) = 0\}.$$

With these definitions, there are natural isomorphisms

$$\begin{aligned} \mathrm{Gr}^F_m(V \otimes W) &\cong \bigoplus_{j+k=m} \mathrm{Gr}^F_j V \otimes \mathrm{Gr}^F_k V \\ \mathrm{Gr}^F_m \mathrm{Hom}(V, W) &\cong \bigoplus_k \mathrm{Hom}(\mathrm{Gr}^F_k V, \mathrm{Gr}^F_{m+k} W) \\ \mathrm{Gr}^F_m (V^*) &\cong (\mathrm{Gr}^F_{-m} V)^*. \end{aligned}$$

A filtration $F_\bullet$ of V naturally induces one on every subspace W and every quotient $p : V \to V/U$ by

$$F_mW := W \cap F_mV \text{ and } F_m(V/U) := p(F_mV).$$

There are thus two ways of inducing a filtration on a subquotient $q : W \to p(W)$. One way is to restrict the quotient filtration on V/U to the subspace $p(W)$; the other is to give $p(W)$ the image of the filtration induced by $F_\bullet$ on W. These are easily seen to agree. (Cf. [9]).

In particular, if a vector space V has two filtrations $F_\bullet$ and $G_\bullet$, then the filtration $F_\bullet$ induces a natural filtration (also denoted by $F_\bullet$) on each $G_\bullet$-graded quotient $\mathrm{Gr}^G_m V$ of V. There are then natural isomorphisms

$$\mathrm{Gr}^F_m \mathrm{Gr}^G_n V \cong \mathrm{Gr}^G_n \mathrm{Gr}^F_m V.$$

2.1. The infinite dimensional case

Unless otherwise noted, all infinite dimensional vector spaces considered will be either ind- or pro- objects of the category $\mathsf{Vec}_F^{\mathrm{fin}}$ of finite dimensional F-vector spaces. The dual of a pro-object of $\mathsf{Vec}_F^{\mathrm{fin}}$ is an ind-object of $\mathsf{Vec}_F^{\mathrm{fin}}$, and vice-versa. If V is an ind-object of $\mathsf{Vec}_F^{\mathrm{fin}}$, then it is naturally isomorphic to its double dual. Similarly, an ind-object of $\mathsf{Vec}_F^{\mathrm{fin}}$ is naturally isomorphic to its double dual. A filtration of a pro-object (resp. ind-object) of $\mathsf{Vec}_F^{\mathrm{fin}}$ is simply a filtration of it by pro-objects (resp. ind-objects) of $\mathsf{Vec}_F^{\mathrm{fin}}$.

§3. Relative Completion of Discrete Groups

Here we summarize the theory of relative unipotent completion of discrete groups. Some of the statements are stronger than results in the literature. Full proofs will appear in [21]. Versions of many of these results for the related notion of weighted completion can be found in [19].

3.1. Unipotent and Prounipotent Groups

Suppose that F is a field of characteristic zero. Recall that a unipotent algebraic group over F is a subgroup U, for some n, of the group of the $n \times n$ unipotent upper triangular matrices

$$\{X \in \mathrm{GL}_n(F) : X - I \text{ is strictly upper triangular}\}$$

that is defined by polynomial equations. The Baker-Campbell-Hausdorff formula implies that

$$\mathfrak{u} = \{\log u \in \mathfrak{gl}_n(F) : u \in U\}$$

is a Lie algebra with bracket $[x, y] = xy - yx$. The exponential mapping $\exp : \mathfrak{u} \to U$ is a polynomial bijection. The algebraic subgroups of U correspond bijectively to the Lie subalgebras of $\mathfrak{u}$ via the exponential mapping.

A *pronilpotent* Lie algebra is, by definition, the inverse limit of finite dimensional nilpotent Lie algebras:

$$\mathfrak{u} = \varprojlim_{\alpha} \mathfrak{u}_\alpha$$

It has a natural topology; a base of neighbourhoods of 0 consists of the kernels of the projections of $\mathfrak{u}$ to each of the $\mathfrak{u}_\alpha$.

A *prounipotent group* $\mathcal{U}$ is the inverse limit

$$\mathcal{U} = \varprojlim_{\alpha} U_\alpha$$

of an inverse system of unipotent groups. The Lie algebra $\mathfrak{u}$ of $\mathcal{U}$ is the inverse limit of the Lie algebras of the U_α. It is a pronilpotent Lie algebra The exponential mapping $\exp : \mathcal{U} \to \mathfrak{u}$ is an isomorphism of proalgebraic varieties.

Pronilpotent Lie algebras have nice presentations. Suppose that $\mathfrak{u}$ is a pronilpotent Lie algebra. Define $H_1(\mathfrak{u})$ to be the abelianization of $\mathfrak{u}$. It is a topological vector space — if $\mathfrak{u}$ is the inverse limit of the finite dimensional nilpotent Lie algebras $\mathfrak{u}_\alpha$, then $H_1(\mathfrak{u})$ is the inverse limit of the $H_1(\mathfrak{u}_\alpha)$. A continuous section $s : H_1(\mathfrak{u}) \to \mathfrak{u}$ of the natural projection $\mathfrak{u} \to H_1(\mathfrak{u})$ induces a continuous Lie algebra homomorphism

$$s_* : \mathbb{L}(H_1(\mathfrak{u}))^\frown \to \mathfrak{u}$$

from the free completed Lie algebra generated by $H_1(\mathfrak{u})$ to $\mathfrak{u}$. This induces an isomorphism on abelianizations. Since $\mathfrak{u}$ is pronilpotent, s_* is surjective. It follows that $\mathfrak{u}$ has a presentation of the form

$$\mathfrak{u} \cong \mathbb{L}(H_1(\mathfrak{u}))^\frown/\mathfrak{r}$$

where $\mathfrak{r} = \ker s_*$ is a closed ideal contained in the commutator subalgebra of $\mathbb{L}(H_1(\mathfrak{u}))^\frown$. Such presentations are said to be *minimal.*

Define the continuous cohomology $H^\bullet(\mathfrak{u})$ of a pronilpotent Lie algebra $\mathfrak{u}$ that is the inverse limit of the finite dimensional nilpotent Lie algebras $\mathfrak{u}_\alpha$ by

$$H^\bullet(\mathfrak{u}) := \varinjlim_\alpha H^\bullet(\mathfrak{u}_\alpha).$$

This will be regarded an ind-object of the category of finite dimensional F-vector spaces. It is easy to check that for all $k \geq 0$,

$$H^k(\mathfrak{u}) = \mathrm{Hom}_{\mathrm{cts}}(H_k(\mathfrak{u}), F) \text{ and } H_k(\mathfrak{u}) = \mathrm{Hom}(H^k(\mathfrak{u}), F).$$

This generalizes to pronilpotent coefficients (i.e., projective systems of nilpotent coefficients): If $V = \varprojlim V_\alpha$ where V_α is a nilpotent $\mathfrak{u}_\alpha$-module, then

$$H_k(\mathfrak{u}, V) := \varprojlim_\alpha H_k(\mathfrak{u}_\alpha, V_\alpha) \text{ and } H^k(\mathfrak{u}, V^*) := \varinjlim_\alpha H^k(\mathfrak{u}_\alpha, V_\alpha^*).$$

There are natural isomorphisms

$$H^k(\mathfrak{u}, V^*) = \mathrm{Hom}_{\mathrm{cts}}(H_k(\mathfrak{u}, V), F) \text{ and } H_k(\mathfrak{u}, V) = \mathrm{Hom}(H^k(\mathfrak{u}, V^*), F)$$

If $\mathfrak{f}$ is a free pronilpotent Lie algebra, then $H^k(\mathfrak{f}; V) = 0$ for all $k > 1$ and all nilpotent $\mathfrak{f}$-modules V.

A complete proof of the following analogue of Hopf's Theorem will appear in [21].

Proposition 3.1. *If $\mathfrak{r}$ is a closed ideal in the free pronilpotent Lie algebra $\mathfrak{f}$ that is contained in $[\mathfrak{f},\mathfrak{f}]$, then there is a natural isomorphism*

$$H^2(\mathfrak{f}/\mathfrak{r}) \cong \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{r}/[\mathfrak{r},\mathfrak{f}], F)$$

of ind-vector spaces. Moreover, if $\theta : \mathfrak{r}/[\mathfrak{r},\mathfrak{f}] \to \mathfrak{r}$ is a continuous section of the quotient mapping, then $\mathfrak{r}$ is generated as a closed ideal by $\operatorname{im}\theta$.

Proof. The fact that every subalgebra of a free Lie algebra is free [45], implies that every subalgebra of a free pronilpotent Lie algebra is also a free pronilpotent Lie algebra. Consequently, $\mathfrak{r}$ is a free pronilpotent Lie algebra and has vanishing cohomology in degrees > 1. Using standard cochains, one can show that there is a spectral sequence

$$E_2^{s,t} = H^s(\mathfrak{f}/\mathfrak{r}, H^t(\mathfrak{r})) \Rightarrow H^{s+t}(\mathfrak{f}).$$

Since $\mathfrak{r} \subseteq [\mathfrak{f},\mathfrak{f}]$, $H^1(\mathfrak{f}) = H^1(\mathfrak{f}/\mathfrak{r})$. Since $H^1(\mathfrak{r}) = \mathrm{Hom}_{\mathrm{cts}}(H_1(\mathfrak{r}), F)$ and since

$$H_0(\mathfrak{f}/\mathfrak{r}, H_1(\mathfrak{r})) = H_0(\mathfrak{f}, H_1(\mathfrak{r})) \cong \mathfrak{r}/[\mathfrak{r},\mathfrak{f}],$$

the vanishing of the higher cohomology of $\mathfrak{f}$ and $\mathfrak{r}$ imply (when plugged into the spectral sequence) that

$$H^2(\mathfrak{f}/\mathfrak{r}) = H^0(\mathfrak{f}, H^1(\mathfrak{r})) = \mathrm{Hom}_{\mathrm{cts}}(H_0(\mathfrak{f}, H_1(\mathfrak{r})), F) = \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{f}/[\mathfrak{r},\mathfrak{f}], F).$$

Q.E.D.

An immediate corollary is an analogue of Stallings' result [47]. A detailed proof will appear in [21].

Corollary 3.2. *A homomorphism $\phi : \mathfrak{u}_1 \to \mathfrak{u}_2$ of pronilpotent Lie algebras is an isomorphism if and only if it induces an isomorphism $H^1(\mathfrak{u}_2) \to H^1(\mathfrak{u}_1)$ and a monomorphism $H^2(\mathfrak{u}_2) \to H^2(\mathfrak{u}_1)$ of ind-vector spaces.*

Sketch of Proof. The only if assertion is trivially true. Suppose that $H^k(\mathfrak{u}_2) \to H^k(\mathfrak{u}_1)$ is an isomorphism when $k = 1$ and a monomorphism when $k = 2$. Since $\mathfrak{u}_1$ and $\mathfrak{u}_2$ are pronilpotent, the isomorphism on H^1 implies that ϕ is a quotient map in the category of pronilpotent Lie algebras. Choose a minimal presentation $\mathfrak{u}_1 = \mathfrak{f}/\mathfrak{r}_1$. Let $\mathfrak{r}_2$ be the kernel of $\mathfrak{f} \to \mathfrak{u}_1 \to \mathfrak{u}_2$. Then $\mathfrak{u}_2 = \mathfrak{f}/\mathfrak{r}_2$ and ϕ is an isomorphism if and only if the inclusion $\phi : \mathfrak{r}_1 \to \mathfrak{r}_2$ is a quotient mapping. But this holds if and only if

$$H^2(\mathfrak{u}_2) = \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{r}_2/[\mathfrak{r}_2,\mathfrak{f}], F) \to \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{r}_1/[\mathfrak{r}_1,\mathfrak{f}], F) = H^2(\mathfrak{u}_1)$$

is a monomorphism. Q.E.D.

Corollary 3.3. *A pronilpotent Lie algebra* $\mathfrak{u}$ *is trivial if and only if* $H^1(\mathfrak{u}) = 0$ *and free if and only if* $H^2(\mathfrak{u}) = 0$. $\square$

3.2. Relative Unipotent Completion

The data for relative completion are:

(i) a discrete group Γ;
(ii) a field F of characteristic zero;
(iii) a reductive algebraic group R over F, such as $\mathrm{GL}_n(F)$, $\mathrm{SL}_n(F)$ or $\mathrm{Sp}_n(F)$;
(iv) a Zariski dense homomorphism $\rho : \Gamma \to R$.[2]

The completion of Γ with respect to ρ consists of a proalgebraic group (i.e., an inverse limit of algebraic groups) $\mathcal{G}$ over F that is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1 \tag{1}$$

where $\mathcal{U}$ is prounipotent and a homomorphism $\hat{\rho} : \Gamma \to \mathcal{G}$ whose composition with $\mathcal{G} \to R$ is ρ. It is characterized by the following universal mapping property:

If G is an affine (pro)algebraic group over F that is an extension

$$1 \to U \to G \to R \to 1$$

of R by a (pro)unipotent group U, and if $\tilde{\rho} : \Gamma \to G$ is a homomorphism whose composition with $G \to R$ is ρ, then there is a unique homomorphism $\phi : \mathcal{G} \to G$ of (pro)algebraic F-groups such that

$$\begin{tikzcd} \Gamma \arrow[r, "\hat{\rho}"] \arrow[d, "\tilde{\rho}"'] & \mathcal{G} \arrow[d] \arrow[dl, "\phi"'] \\ G \arrow[r] & R \end{tikzcd}$$

commutes.

The universal mapping property implies that the homomorphism $\hat{\rho} : \Gamma \to \mathcal{G}$ is Zariski dense — that is, if $\mathcal{G}'$ is a proalgebraic subgroup of $\mathcal{G}$ defined over F that contains the image of $\hat{\rho}$, then $\mathcal{G}' = \mathcal{G}$. The point being that the Zariski closure of $\operatorname{im} \hat{\rho}$ in $\mathcal{G}$ has the same universal mapping property as $\mathcal{G}$.

Suppose that K is an extension field of F. Every (pro)algebraic group G over F gives rise to a (pro)algebraic group $G \otimes_F K$ over K

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[2] In this paper I will not distinguish between an algebraic group G over F and its group of F- rational points $G(F)$.

by extension of scalars. The universal mapping property of the relative completion $\mathcal{G}_K$ of Γ over K with respect to $\rho : \Gamma \to R \otimes_F K$ implies that $\Gamma \to \mathcal{G} \otimes_F K$ induces a homomorphism $\mathcal{G}_K \to \mathcal{G} \otimes_F K$.

Theorem 3.4 (Hain-Matsumoto [21]). *The homomorphism $\mathcal{G}_K \to \mathcal{G} \otimes_F K$ is an isomorphism.*

In general we will work over the smallest field F possible, which is the smallest field over which both R and ρ are defined. In all principal examples in this paper, the field will be $\mathbb{Q}$.

3.3. Unipotent Completion

When R is the trivial group, relative completion reduces to classical unipotent completion, which is also known as Malcev completion and which can be computed by the methods of rational homotopy theory due to Quillen, Chen and Sullivan. We shall denote the unipotent completion of Γ over F by $\Gamma^{\mathrm{un}}_{/F}$. The default field will be $\mathbb{Q}$. We shall abbreviate $\Gamma^{\mathrm{un}}_{/\mathbb{Q}}$ to Γ^{un}. The unipotent completion of Γ over F is obtained from Γ^{un} by extension of scalars:

$$\Gamma^{\mathrm{un}}_{/F} = \Gamma^{\mathrm{un}} \otimes_{\mathbb{Q}} F.$$

If Γ is a free group $F_n = \langle x_1, \dots, x_n \rangle$ on n-generators, the Lie algebra of $F^{\mathrm{un}}_{n/F}$ is the completed free Lie algebra

$$\mathfrak{f}_n := \mathbb{L}(X_1, \dots, X_n)^{\wedge}$$

which is the closure of the free Lie algebra $\mathbb{L}(X_1, \dots, X_n)$ in the noncommutative power series ring $F\langle\langle X_1, \dots, X_n \rangle\rangle$. The prounipotent group $F^{\mathrm{un}}_{n/F}$ is

$$F^{\mathrm{un}}_{n/F} = \{\exp u \in F\langle\langle X_1, \dots, X_n \rangle\rangle : u \in \mathfrak{f}_n\}.$$

The natural homomorphism $\hat{\rho} : F_n \to F^{\mathrm{un}}_{n/F}$ is defined by $\hat{\rho}(x_j) = \exp X_j$. This can be proved using universal mapping properties. A theorem of Magnus [36] implies that the homomorphism $F_n \to F^{\mathrm{un}}_n$ is injective.

3.4. Completions of $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$

Denote the automorphism group of the free group F_n by $\mathrm{Aut}\, F_n$ and its quotient by inner automorphisms of F_n by $\mathrm{Out}\, F_n$. There are natural surjections

$$\mathrm{Aut}\, F_n \to \mathrm{GL}_n(\mathbb{Z}) \text{ and } \mathrm{Out}\, F_n \to \mathrm{GL}_n(\mathbb{Z}).$$

Denote their kernels by IA_n and IO_n, respectively. Let $\mathrm{Aut}^+ F_n$ be the index 2 subgroup of $\mathrm{Aut}\, F_n$ whose image in $\mathrm{GL}_n(\mathbb{Z})$ is $\mathrm{SL}_n(\mathbb{Z})$. Let $\mathrm{Out}^+ F_n$ be its quotient by the group of inner automorphisms.

Take $F = \mathbb{Q}$, $R = \mathrm{SL}_n(\mathbb{Q})$ and let

$$\rho : \mathrm{Aut}^+ F_n \to \mathrm{SL}_n(\mathbb{Q})$$

be the natural representation. This is Zariski dense. The completion of $\mathrm{Aut}^+ F_n$ with respect to ρ is an extension

$$1 \to \mathcal{IA}_n \to \mathcal{A}_n \to \mathrm{SL}_n(\mathbb{Q}) \to 1.$$

Similarly, we have the completion of $\mathrm{Out}^+ F_n$. It is an extension

$$1 \to \mathcal{IO}_n \to \mathcal{O}_n \to \mathrm{SL}_n(\mathbb{Q}) \to 1.$$

The corresponding sequences of Lie algebras

$$0 \to \mathfrak{ia}_n \to \mathfrak{a}_n \to \mathfrak{sl}_n \to 0 \text{ and } 0 \to \mathfrak{io}_n \to \mathfrak{o}_n \to \mathfrak{sl}_n \to 0$$

are exact. There are natural homomorphisms

$$\mathfrak{a}_n \to \mathrm{Der}\, \mathfrak{f}_n \text{ and } \mathfrak{o}_n \to \mathrm{OutDer}\, \mathfrak{f}_n,$$

where $\mathrm{OutDer}\, \mathfrak{f}_n$ denotes the Lie algebra of outer derivations of $\mathfrak{f}_n$.

The natural homomorphisms $IA_n \to \mathcal{IA}_n$ and $IO_n \to \mathcal{IO}_n$ induce homomorphisms $IA_n^{\mathrm{un}} \to \mathcal{IA}_n$ and $IO_n^{\mathrm{un}} \to \mathcal{IO}_n$. We shall see later (cf. Cor. 3.14) that these homomorphisms are isomorphisms when $n \geq 4$, surjective when $n = 3$ and are far from surjective when $n = 2$.

Proposition 3.5. *The natural homomorphism* $\hat{\rho} : \mathrm{Aut}^+ F_n \to \mathcal{A}_n$ *is injective.*

Proof. Since the unipotent completion $F_n \to F_n^{\mathrm{un}}$ is injective, it follows that the natural representation $\theta : \mathrm{Aut}\, F_n \to \mathrm{Aut}\, F_n^{\mathrm{un}}$ is injective. The Zariski closure of the image of $\mathrm{Aut}^+ F_n$ under θ is easily seen to be an extension of $\mathrm{SL}_n(\mathbb{Q})$ by a prounipotent group. The universal mapping property of relative completion induces a homomorphism $\psi : \mathcal{A}_n \to \mathrm{Aut}\, F_n^{\mathrm{un}}$ such that the diagram

$$\begin{tikzcd} \mathrm{Aut}^+ F_n \arrow[r, "\hat{\rho}"] \arrow[dr, "\theta"'] & \mathcal{A}_n \arrow[d, "\psi"] \\ & \mathrm{Aut}\, F_n^{\mathrm{un}} \end{tikzcd}$$

The injectivity of $\hat{\rho}$ follows from the injectivity of θ. Q.E.D.

The injectivity of $\mathrm{Out}^+ F_n \to \mathrm{Out}\, F_n^{\mathrm{un}}$ would follow if one could prove that $F_n^{\mathrm{un}} \cap \mathrm{Aut}\, F_n = F_n$ in $\mathrm{Aut}\, F_n^{\mathrm{un}}$, where F_n and F_n^{un} are regarded as subgroups of $\mathrm{Aut}\, F_n^{\mathrm{un}}$ via the inner action. This is not clear.

3.5. Properties of Relative Completion

Here we list some of the basic properties of relative completion that we shall need. The proofs of these are sprinkled throughout the literature and are sometimes proved for the related notion of *weighted completion* [19]. The notes [21] will give an efficient and uniform presentation of the theory relative and related completions of discrete and profinite groups.

Proposition 3.6 (Naturality). *Suppose that $\rho_j : \Gamma_j \to R_j$, $j = 1, 2$ are Zariski dense homomorphisms from discrete groups to reductive groups over F. Let $\Gamma_j \to \mathcal{G}_j$ be the completion of Γ_j with respect to ρ_j over F. If the diagram*

$$\begin{tikzcd} \Gamma_1 \arrow[r, "\rho_1"] \arrow[d, "\phi_\Gamma"'] & R_1 \arrow[d, "\phi_R"] \\ \Gamma_2 \arrow[r, "\rho_2"] & R_2 \end{tikzcd}$$

commutes where ϕ_R is a homomorphism of algebraic groups, then there is a unique homomorphism $\phi_\mathcal{G} : \mathcal{G}_1 \to \mathcal{G}_2$ such that the diagram

$$\begin{tikzcd} \Gamma_1 \arrow[r, "\hat\rho_1"] \arrow[d, "\phi_\Gamma"'] & \mathcal{G}_1 \arrow[r] \arrow[d, "\phi_\mathcal{G}"] & R_1 \arrow[d, "\phi_R"] \\ \Gamma_2 \arrow[r, "\hat\rho_2"] & \mathcal{G}_2 \arrow[r] & R_2 \end{tikzcd}$$

Completions are, in general, right exact. Here we state a useful special case.

Proposition 3.7 (Right exactness). *Suppose that $\rho : \Gamma \to R$ is a Zariski dense homomorphism from a discrete group to a reductive F-group. Denote the completion Γ with respect to ρ by $\mathcal{G}$ and the completion of $\operatorname{im}\rho$ with respect to the inclusion $\operatorname{im}\rho \hookrightarrow R$ by $\mathcal{R}$. Then the sequence*

$$\big(\ker\rho\big)^{\mathrm{un}}_{/F} \to \mathcal{G} \to \mathcal{R} \to 1$$

is exact.

A generalization of Levi's Theorem implies that, when Γ is finitely generated, the extension (1) is split, and that any two splitting are conjugate by an element of $\mathcal{U}$. It follows that $\mathfrak{u}$ is a Lie algebra in the category of pro-representations[3] of R and that there is an isomorphism

$$\mathcal{G} \cong R \ltimes \exp \mathfrak{u}$$

that is unique up to conjugation by an element of $\exp \mathfrak{u}$.

Relative completions are manageable and somewhat computable as they are quite tightly controlled by cohomology.

Suppose that $\overline{F}$ is an algebraic closure of F. An irreducible representation V of R is *absolutely irreducible* if $V \otimes_F \overline{F}$ is an irreducible representation of $R \otimes_F \overline{F}$.

Theorem 3.8. *For all finite dimensional R-modules V, there is a homomorphism*

$$\operatorname{Hom}_R(H_k(\mathfrak{u}), V) \cong \left(H^k(\mathfrak{u}) \otimes V\right)^R \to H^k(\Gamma; V)$$

that is natural with respect to the maps described in Proposition 3.6. It is an isomorphism when $k = 1$ and injective when $k = 2$. If every irreducible finite dimensional representation of R is absolutely irreducible, then there is a natural R-module isomorphism

$$H^1(\mathfrak{u}) \cong \bigoplus_\alpha H^1(\Gamma; V_\alpha) \otimes V_\alpha^*$$

and a natural R-module injection

$$H^2(\mathfrak{u}) \hookrightarrow \bigoplus_\alpha H^2(\Gamma; V_\alpha) \otimes V_\alpha^*$$

where $\{V_\alpha\}$ is a set of representatives of the isomorphism classes of irreducible finite dimensional R-modules and where each $H^1(\Gamma; V_\alpha)$ is regarded as a trivial R-module.

This theorem alone and in combination with the Base Change Theorem 3.4 can often be used to compute $\mathfrak{u}$. Combined with Corollary 3.3, it gives the following criterion for the vanishing of $\mathfrak{u}$.

Corollary 3.9. *The prounipotent radical of $\mathcal{G}$ vanishes if and only if $H^1(\Gamma; V) = 0$ for all irreducible finite dimensional R-modules.* □

Combining this with right exactness (Prop. 3.7) yields:

[3]That is, an inverse limit of finite dimensional R-modules. Since R is reductive, the pro-representations of R are direct products of finite dimensional R-modules.

Corollary 3.10. *If $H^1(\operatorname{im}\rho; V) = 0$ for all finite dimensional R-modules V, then*

$$\big(\ker\rho\big)^{\mathrm{un}}_{/F} \to \mathcal{G} \to R \to 1$$

is exact. That is, $\mathcal{U}$ is a quotient of $\big(\ker\rho\big)^{\mathrm{un}}_{/F}$.

Additional hypotheses give an upper bound on the kernel. The following result is a refined version of [16, Prop. 4.13].

Theorem 3.11. *Suppose that $\rho : \Gamma \to R$ is a Zariski dense homomorphism. Denote* $\ker\rho$ *by T. If the* $\operatorname{im}\rho$ *module $H_1(T; F)$ is finite dimensional and the restriction (via ρ) of a finite dimensional R-module, and if $H^1(\operatorname{im}\rho; V) = 0$ for all irreducible finite dimensional representations of R, then there is a natural exact sequence*

$$1 \to K \to T^{\mathrm{un}}_{/F} \to \mathcal{G} \to R \to 1$$

where K is contained in the center of $T^{\mathrm{un}}_{/F}$. Moreover, if $H^2(\operatorname{im}\rho; V)$ is finite dimensional for all irreducible R-modules V, then K is an R-submodule of the abelian prounipotent group

$$\prod_\alpha H^2(\operatorname{im}\rho; V_\alpha)^* \otimes V_\alpha.$$

where V_α ranges over representatives of the isomorphism classes of finite dimensional R-modules.

3.6. Examples

Equipped with the results of the previous section, we can approach the problem of computing the relative completions in natural examples.

Example 3.12 (Lattices). If Γ is an irreducible lattice in a semi-simple real Lie group G of rank ≥ 2, then Raghunathan's vanishing theorem [44] states that

$$H^1(\Gamma; V) = 0$$

for all irreducible representations V of G. Corollary 3.9 implies that the completion of Γ with respect to the inclusion $\Gamma \to G$ over $\mathbb{R}$ is G.

In particular, when $n \geq 3$, the completion of any finite index subgroup Γ of $\mathrm{SL}_n(\mathbb{Z})$ with respect to the inclusion $\Gamma \to \mathrm{SL}_n(\mathbb{Q})$ is $\mathrm{SL}_n(\mathbb{Q})$. When $g \geq 2$, the completion of any finite index subgroup of $\mathrm{Sp}_g(\mathbb{Z})$ with respect to the inclusion $\Gamma \to \mathrm{Sp}_g(\mathbb{Q})$ is $\mathrm{Sp}_g(\mathbb{Q})$.

The rank condition is necessary. The groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{Sp}_1(\mathbb{R})$ are isomorphic and have real rank 1. If we take Γ to be one of the isomorphic groups $\mathrm{SL}_2(\mathbb{Z})$, $\mathrm{Aut}^+ F_2$, $\Gamma_{S,x}$, where S is a genus 1 surface, then the prounipotent radical of the completion of Γ with respect to the inclusion $\Gamma \to \mathrm{SL}_2(\mathbb{Q})$ is a free prounipotent group whose abelianization is infinite dimensional. (Cf. [18, Rem. 3.9].) It is closely connected with classical modular forms and elliptic motives. (Cf. [22].)

Results of Borel [2] imply that if $\operatorname{im}\rho$ is arithmetic of sufficiently high rank, then $H^2(\operatorname{im}\rho; V)$ vanishes for all non-trivial R-modules and $H^2(\operatorname{im}\rho; \mathbb{Q})$ is isomorphic to the corresponding cohomology group of the compact dual of the symmetric space of $R \otimes \mathbb{R}$. In particular, Borel's formula implies the vanishing of $H^2(\mathrm{SL}_2(\mathbb{Z}); V)$ for *all* SL_n-modules V when $n \geq 4$. It also implies the vanishing of $H^2(\mathrm{Sp}_g(\mathbb{Z}), V)$ for all non-trivial irreducible Sp_g-modules when $g \geq 3$.

Example 3.13 (Universal Central Extensions)**.** Suppose that Γ is a non-zero multiple of the universal central extension of $\mathrm{Sp}_g(\mathbb{Z})$, where $g \geq 2$:

$$0 \to \mathbb{Z} \to \Gamma \to \mathrm{Sp}_g(\mathbb{Z}) \to 1.$$

Let $R = \mathrm{Sp}_g(\mathbb{Q})$ and $\rho : \Gamma \to \mathrm{Sp}_g(\mathbb{Q})$ be the obvious homomorphism. Denote the relative completion of Γ with respect to ρ by $\mathcal{G}$. By Example 3.12, the completion of $\mathrm{Sp}_g(\mathbb{Z})$ with respect to the inclusion $\mathrm{Sp}_g(\mathbb{Z}) \to \mathrm{Sp}_g(\mathbb{Q})$ is $\mathrm{Sp}_g(\mathbb{Q})$. Raghunathan's Theorem implies that $H^1(\mathrm{Sp}_g(\mathbb{Z}); V)$ vanishes for all finite dimensional representations V of Sp_g. An elementary spectral sequence argument implies that $H^1(\Gamma, V)$ also vanishes for all finite dimensional Sp_g-modules V. Cor. 3.9 then implies that $\mathcal{G} \to \mathrm{Sp}_g(\mathbb{Q})$ is an isomorphism.

This provides an interesting example of Theorem 3.11. Borel's vanishing theorem implies that, when $g \geq 3$, $H^2(\mathrm{Sp}_g(\mathbb{Z}), V)$ vanishes for all non-trivial irreducible $\mathrm{Sp}_g(\mathbb{Q})$-modules and that $H^2(\mathrm{Sp}_g(\mathbb{Z}), \mathbb{Q})$ is 1-dimensional. Since the unipotent completion of $\mathbb{Z}$ is $\mathbb{Q}$, Theorem 3.11 implies that we have an exact sequence

$$H^2(\mathrm{Sp}_g(\mathbb{Q}); \mathbb{Q})^* \to \mathbb{Q} \to \mathcal{G} \to \mathrm{Sp}_g(\mathbb{Q}) \to 1.$$

Since $\mathcal{G} \to \mathrm{Sp}_g(\mathbb{Q})$ is an isomorphism, it follows that $H^2(\mathrm{Sp}_g(\mathbb{Q}); \mathbb{Q})^* \to \mathbb{Q}$ is an isomorphism and that $\mathbb{Q} \to \mathcal{G}$ is trivial. $\square$

As remarked in Example 3.12, $IA_2^{\mathrm{un}} \to \mathcal{IA}_2$ and $IO_2^{\mathrm{un}} \to \mathcal{IO}_2$ are far from surjective. However, when $n \geq 3$, the situation improves.

Corollary 3.14. *If $n \geq 3$, then the natural homomorphisms $IA_n^{\mathrm{un}} \to \mathcal{IA}_n$ and $IO_n^{\mathrm{un}} \to \mathcal{IO}_n$ are surjective. If $n \geq 4$, they are isomorphisms.*

Proof. By results of Magnus [36] and Kawazumi [33], there are natural $\mathrm{GL}_n(\mathbb{Z})$-equivariant isomorphisms

$$H_1(IA_n) \cong \mathrm{Hom}(V, \Lambda^2 V) \text{ and } H_1(IO_n) \cong \mathrm{Hom}(V, \Lambda^2 V)/V,$$

where $V = H_1(F_n)$, from which it follows that the $\mathrm{SL}_n(\mathbb{Z})$-modules $H_1(IA_n)$ and $H_1(IO_n)$ are the restrictions of $\mathrm{SL}(V)$-modules. Surjectivity when $n \geq 3$ follows from Corollary 3.10 and Raghunathan's vanishing result. When $n \geq 4$, the result follows from Theorem 3.11 and Borel's vanishing result, stated above. Q.E.D.

Another situation in which left exactness holds, that we shall need later, is the following. Suppose that

$$\begin{tikzcd}
1 \arrow[r] & \Gamma \arrow[r] \arrow[d, "\rho_\Gamma"'] & \hat{\Gamma} \arrow[r] \arrow[d, "\rho_{\hat{\Gamma}}"'] & G \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & R \arrow[r] & \widehat{R} \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

is a commutative diagram of groups with exact rows where:

(i) Γ and $\hat{\Gamma}$ are discrete groups;
(ii) R and $\widehat{R}$ are reductive F-groups;
(iii) G is a finite group;
(iv) ρ_Γ is Zariski dense (which implies that $\rho_{\hat{\Gamma}}$ is also Zariski dense).

Denote the completion of Γ with respect to ρ_Γ by $\mathcal{G}$ and the completion of $\hat{\Gamma}$ with respect to $\rho_{\hat{\Gamma}}$ by $\widehat{\mathcal{G}}$. Naturality implies that there is a homomorphism $\mathcal{G} \to \widehat{\mathcal{G}}$ such that the diagram

$$\begin{tikzcd}
\Gamma \arrow[r] \arrow[d] & \mathcal{G} \arrow[d] \\
\hat{\Gamma} \arrow[r] & \widehat{\mathcal{G}}
\end{tikzcd}$$

commutes. Right exactness implies that the sequence $\mathcal{G} \to \widehat{\mathcal{G}} \to G \to 1$ is exact. Denote the prounipotent radicals of $\mathcal{G}$ and $\widehat{\mathcal{G}}$ by $\mathcal{U}$ and $\widehat{\mathcal{U}}$, respectively.

Proposition 3.15. *The natural homomorphism $\mathcal{G} \to \widehat{\mathcal{G}}$ is injective. Consequently, the induced mapping $\mathcal{U} \to \widehat{\mathcal{U}}$ of prounipotent radicals is an isomorphism.*

Proof. Stallings' criterion (Cor. 3.2) will be used to show that $\mathfrak{u} \to \widehat{\mathfrak{u}}$ is an isomorphism. To prove this we need the notion of an induced module. (This is sometimes called a co-induced module, cf. [3, p. 67].)

For an R-module V, define the representation induced from V to $\widehat{R}$ by

$$\mathrm{Ind}_R^{\widehat{R}} V = \mathrm{Fun}_R(\widehat{R}, V),$$

where Fun_R denotes the set of left R-invariant functions $\phi : \widehat{R} \to V$. This is a left $\widehat{R}$-module with respect to the action $(r\phi)(x) = \phi(xr)$, where $r, x \in \widehat{R}$. Since R has finite index in $\widehat{R}$, the induced representation is a rational representation of $\widehat{R}$ whenever V is a rational representation of R.

For all R-modules U and $\widehat{R}$-modules V, there is a natural isomorphism

$$\mathrm{Hom}_R(\mathrm{Res}_R^{\widehat{R}}\, U, V) \cong \mathrm{Hom}_{\widehat{R}}(U, \mathrm{Ind}_R^{\widehat{R}}\, V),$$

where $\mathrm{Res}_R^{\widehat{R}}\, U$ denotes the restriction of U to R.

Likewise, for any Γ module V, we can define $\mathrm{Ind}_\Gamma^{\hat{\Gamma}}\, V = \mathrm{Fun}_\Gamma(\hat{\Gamma}, V)$. If V is an R-module, viewed as a Γ-module via ρ_Γ, then the restriction mapping

$$\mathrm{Ind}_R^{\widehat{R}}\, V \xrightarrow{\simeq} \mathrm{Ind}_\Gamma^{\hat{\Gamma}}\, V,$$

is an isomorphism of $\hat{\Gamma}$-modules.

To apply Stallings' criterion, we need to show that $H^k(\widehat{\mathfrak{u}}) \to H^k(\mathfrak{u})$ is an isomorphism (resp., injection) when $k = 1$ (resp., $k = 2$). Since R is reductive, it suffices to show that the natural mapping

$$\phi_k : \mathrm{Hom}_R(\mathrm{Res}_R^{\widehat{R}}\, H_k(\widehat{\mathfrak{u}}), V) \to \mathrm{Hom}_R(H_k(\mathfrak{u}), V)$$

is an isomorphism (resp., injection) for all finite dimensional R-modules V when $k = 1$ (resp., $k = 2$). Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(\mathrm{Res}_R^{\widehat{R}}\, H_k(\widehat{\mathfrak{u}}), V) \arrow[r, "\simeq"] \arrow[d, "\phi_k"'] & \mathrm{Hom}_{\widehat{R}}(H_k(\widehat{\mathfrak{u}}), \mathrm{Ind}_R^{\widehat{R}}\, V) \arrow[r] & H^k(\hat{\Gamma}; \mathrm{Ind}_\Gamma^{\hat{\Gamma}}\, V) \arrow[d, equal] \\
\mathrm{Hom}_R(H_k(\mathfrak{u}), V) \arrow[rr] & & H^k(\Gamma; V)
\end{tikzcd}$$

The right hand vertical map is an isomorphism by Shapiro's Lemma [3, p. 73]. We apply Theorem 3.8. When $k = 1$, all horizontal mappings are isomorphisms, which implies that ϕ_1 is an isomorphism. When $k = 2$, all horizontal mappings are injective, which implies that ϕ_2 is injective. Q.E.D.

Example 3.16. Suppose that $n \geq 1$. Denote the subgroup of $\mathrm{GL}_n(R)$ that consists of matrices with determinant ± 1 by $\widehat{\mathrm{SL}}(R)$. Then

Proposition 3.15 implies that the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Aut}^+ F_n \arrow[r] \arrow[d, "\hat\rho^+"'] & \mathrm{Aut}\, F_n \arrow[r] \arrow[d, "\hat\rho"'] & C_2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathcal{A}_n \arrow[r, "\phi"] \arrow[d] & \widehat{\mathcal{A}}_n \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathrm{SL}_n(\mathbb{Q}) \arrow[r] & \widehat{\mathrm{SL}}_n(\mathbb{Q}) \arrow[r] & C_2 \arrow[r] & 1
\end{tikzcd}$$

has exact rows. It follows that $\mathcal{A}_n$ is the identity component of $\widehat{\mathcal{A}}_n$. There is a similar story for $\mathrm{Out}\, F_n$. It is for this reason that in Section 3.4 we considered only the completions of $\mathrm{Aut}^+ F_n$ and $\mathrm{Out}^+ F_n$.

§4. Mapping Class Groups and their Completions

Suppose that g, n, r are non-negative integers. A decorated surface of type (g, n, r) is a pair (S, D) where S is a compact oriented surface of genus g and $D = P \cup V$ is a set of decorations, where $P = \{x_1, \dots, x_n\}$ is a set of n points of S and $V = \{v_1, \dots, v_r\}$ is a set of r non-zero tangent vectors of S. If $v_j \in T_{y_j} S$, the points $x_1, \dots, x_n, y_1, \dots, y_r$ are required to be distinct. The decorated surface (S, D) is *stable* if the punctured surface $S'_D := S - \{x_1, \dots, x_n, y_1, \dots, y_r\}$ has negative Euler characteristic:

$$\chi(S'_D) = \chi(S) - |P| - |V| = 2 - 2g - (r + n) < 0.$$

The mapping class group $\hat\Gamma_{S,D}$ of a stable decorated surface (S, D) is the group of connected components of the group of orientation preserving diffeomorphisms of S that fix P and V set wise. There is a natural surjection

$$\hat\Gamma_{S,D} \to \mathrm{Aut}\, D := \mathrm{Aut}\, P \times \mathrm{Aut}\, V.$$

For a subgroup G of $\mathrm{Aut}\, D$ define $\Gamma^G_{S,D}$ to be the inverse image of G under this homomorphism. The classical mapping group of (S, D) corresponds to the trivial group $\mathbf{1}$:

$$\Gamma_{S,D} := \Gamma^{\mathbf{1}}_{S,D} = \pi_0\, \mathrm{Diff}^+(S, D).$$

There is a natural extension

$$1 \to \Gamma_{S,D} \to \Gamma^G_{S,D} \to G \to 1.$$

The classification of surfaces implies that $\Gamma^G_{S,D}$ depends only on (g,n,r) and the subgroup G of $S_n \times S_r$.

For a commutative ring R, set $H_R = H_1(S;R)$. The group of automorphisms of H that preserve the intersection pairing is an algebraic group over $\mathbb{Q}$ that we shall denote by $\mathrm{Sp}(H)$. There is a natural surjective homomorphism

$$\rho : \Gamma^G_{S,D} \to G \times \mathrm{Sp}(H_{\mathbb{Z}}).$$

Its kernel is, by definition, the Torelli group $T_{S,D}$.

4.1. Boundary Components versus Tangent Vectors

Tangent vectors are essentially interchangeable with marked boundary components. Because boundary components are less natural in algebraic geometry, we prefer to work with tangent vectors. A marked boundary component of a surface is a boundary component of the surface together with a point on the boundary component. Marked boundary components can be exchanged with tangent vectors as follows:

If $v \in T_y S$ is a non-zero tangent vector of a surface S, then one can replace (S,v) by a surface $\hat{S}$ with a marked boundary component. Here $\hat{S}$ is the real oriented blowup of S at y. This is the surface obtained from S by replacing y by the circle of rays in $T_y S$. The marked point on the boundary of $\hat{S}$ corresponds to the ray $\mathbb{R}^+ v$ in $T_y S$ determined by v. It will be denoted by $[v]$.

This process may be reversed by collapsing the boundary component to a point y and choosing any non-zero tangent vector at y that lies in the ray in $T_y S$ determined by the marked point. These identifications are well defined and mutually inverse up to isotopy.

The corresponding mapping class groups are isomorphic. For example, if S is compact, then the natural homomorphisms

$$\pi_0 \operatorname{Diff}^+(S,v) \xrightarrow{\simeq} \pi_0 \operatorname{Diff}^+(\hat{S},[v]) \xleftarrow{\simeq} \pi_0 \operatorname{Diff}^+(\hat{S},\partial\hat{S})$$

are isomorphisms.

4.2. Completions of Mapping Class Groups

The ground field F will be $\mathbb{Q}$ unless otherwise stated. Suppose that (S,D) is a stable decorated surface and that G is a subgroup of $\operatorname{Aut} D$, where $D = P \cup V$. The group $G \times \mathrm{Sp}(H)$ is a reductive algebraic group over $\mathbb{Q}$ and the representation $\rho : \Gamma^G_{S,D} \to G \times \mathrm{Sp}(H)$ is Zariski dense. Denote the completion of $\Gamma^G_{S,D}$ relative to ρ by $\mathcal{G}^G_{S,D}$. It is an extension

$$1 \to \mathcal{U}^G_{S,D} \to \mathcal{G}^G_{S,D} \to G \times \mathrm{Sp}(H) \to 1.$$

The next result follows directly from Proposition 3.15.

Proposition 4.1. *For all subgroups G of* $\operatorname{Aut} D$, *the sequence*

$$1 \to \mathcal{G}_{S,D} \to \mathcal{G}_{S,D}^G \to G \to 1$$

is exact.

The proposition implies that $\mathcal{G}_{S,D}$ is the connected component of the identity of $\mathcal{G}_{S,D}^G$.

Corollary 4.2. *For all subgroups G of* $\operatorname{Aut} D$, *the Lie algebra of* $\mathcal{G}_{S,D}^G$ *is* $\mathfrak{g}_{S,D}$.

Theorem 4.3. *If (S,D) is a stable decorated surface where $g(S) \geq 3$, then*

$$0 \to \mathbb{Q} \to T_{S,D}^{\mathrm{un}} \to \mathcal{G}_{S,D} \to \mathrm{Sp}(H) \to 1$$

is exact. When $g = 2$, the homomorphism $T_{S,D}^{\mathrm{un}} \to \mathcal{U}_{S,D}$ is surjective.

This result deserves some comment. Corollary 3.10 implies that $T_{S,D}^{\mathrm{un}} \to \mathcal{U}_{S,D}$ is surjective when $g \geq 2$. Theorem 3.11 implies that

$$\mathbb{Q} \to T_{S,D}^{\mathrm{un}} \to \mathcal{G}_{S,D} \to \mathrm{Sp}(H) \to 1$$

is exact when $g \geq 3$. The injectivity of $\mathbb{Q} \to T_{S,D}^{\mathrm{un}}$ is equivalent to the non-vanishing of a Chern class. A clumsy proof of the non-vanishing is given in [16]. However, the non-vanishing follows directly from an earlier result of Morita [39], as explained in [23].

4.3. Tautological Homomorphisms

Suppose that (S,D) is a decorated surface. A decoration $\widetilde{D} = \widetilde{P} \cup \widetilde{V}$ of S is a *refinement* of D if

$$D \subseteq \widetilde{D},\ P \subseteq \widetilde{P} \cup \widetilde{V} \text{ and } V \subseteq \widetilde{V},$$

where $D = P \cup V$ and $\widetilde{D} = \widetilde{P} \cup \widetilde{V}$. Thus, in passing from $\widetilde{D}$ to D, tangent vectors can become points, and points and tangent vectors can be forgotten.

Suppose that (S,D) is stable. This implies that $(S,\widetilde{D})$ is also stable. For each $G \subseteq \operatorname{Aut} D \cap \operatorname{Aut} \widetilde{D}$, there is a natural homomorphism $\Gamma_{S,\widetilde{D}}^G \to \Gamma_{S,D}^G$.

4.4. Natural Actions

The natural actions of mapping class groups on the fundamental groups of associated surfaces can be completed.

Suppose that (S, D) is a stable decorated surface where $D = P \cup V$. Recall that S'_D is the surface obtained from S by removing the support of D.

Definition 4.4. An *admissible base point* x of S'_D is either (1) a point x of S'_D or (2) a tangent vector $x \in V$. Let $\widetilde{D} = D \cup \{x\}$. This equals D when $x \in V$.

If x is an admissible base point of S'_D, then $\pi_1(S'_D, x)$ is defined.

Suppose that G is a subgroup of $\operatorname{Aut}\widetilde{D}$ that fixes x. It can also be viewed as a subgroup of $\operatorname{Aut} D$. Denote the Lie algebra of $\pi_1(S'_D, x)^{\mathrm{un}}$ by $\mathfrak{p}(S'_D, x)$. There are natural actions

$$\tilde{\theta}_x : \Gamma^G_{S,\widetilde{D}} \to \operatorname{Aut}\mathfrak{p}(S'_D, x) \text{ and } \theta : \Gamma^G_{S,D} \to \operatorname{Out}\mathfrak{p}(S'_D).$$

The Zariski closure of the image of each of these is an extension of $G \times \mathrm{Sp}(H)$ by a prounipotent group. The universal mapping property of relative completion implies that $\tilde{\theta}_x$ and θ induce homomorphisms

$$\tilde{\phi}_x : \mathcal{G}^G_{S,\widetilde{D}} \to \operatorname{Aut}\mathfrak{p}(S'_D, x) \text{ and } \phi : \mathcal{G}^G_{S,D} \to \operatorname{Out}\mathfrak{p}(S'_D).$$

These, in turn, induce Lie algebra homomorphisms

$$d\tilde{\phi}_x : \mathfrak{g}_{S,\widetilde{D}} \to \operatorname{Der}\mathfrak{p}(S'_D, x) \text{ and } d\phi : \mathfrak{g}_{S,D} \to \operatorname{OutDer}\mathfrak{p}(S'_D).$$

Proposition 4.5. *If D is non-empty, then $\hat{\rho} : \Gamma^G_{S,D} \to \mathcal{G}^G_{S,D}$ is injective.*

Proof. It suffices to prove that $T_{S,D}$ injects into $\mathcal{U}_{S,D}$. It also suffices to prove the case where D consists only of points. Write $D = D' \cup \{x\}$. Set $S' = S - D'$ and $\pi = \pi_1(S', x)$. Then the natural homomorphism $\Gamma_{S,D} \to \operatorname{Aut}\pi$ is injective. Since π is resdidually torsion free nilpotent, $\pi \to \pi^{\mathrm{un}}$ is injective. It follows that $\Gamma_{S,D} \to \operatorname{Aut}\mathfrak{p}$ is injective, where $\mathfrak{p}$ is the Lie algebra of π^{un}. The result follows as this homomorphism factors $\Gamma_{S,D} \to \mathcal{G}_{S,D} \to \operatorname{Aut}\mathfrak{p}$, which forces $\Gamma_{S,D} \to \mathcal{G}_{S,D}$ to be injective. Q.E.D.

Denote the Lie algebra of $T^{\mathrm{un}}_{S,D}$ by $\mathfrak{t}_{S,D}$. Since the natural representations $T^{\mathrm{un}}_{S,x} \to \operatorname{Aut}\mathfrak{p}(S, x)$ and $T^{\mathrm{un}}_S \to \operatorname{Out}\mathfrak{p}(S)$ factor through $\mathcal{G}_{S,x} \to \operatorname{Aut}\mathfrak{p}(S, x)$ and $\mathcal{G}_S \to \operatorname{Out}\mathfrak{p}(S)$, Theorem 4.3 implies:

Proposition 4.6. *When $g \geq 3$, the natural representations $T_{S,x}^{\mathrm{un}} \to \operatorname{Aut}\mathfrak{p}(S,x)$ and $T_S^{\mathrm{un}} \to \operatorname{Out}\mathfrak{p}(S)$ have non-trivial kernel. Equivalently, both $\mathfrak{t}_{S,x} \to \operatorname{Der}\mathfrak{p}(S,x)$ and $\mathfrak{t}_S \to \operatorname{OutDer}\mathfrak{p}(S)$ have non-trivial kernel.* □

When $g \geq 3$, the only known elements of the kernel of $\mathfrak{t}_{S,x} \to \operatorname{Der}\mathfrak{p}(S,x)$ are those in $\ker\{\mathfrak{t}_{S,x} \to \mathfrak{u}_{S,x}\}$. So it is natural (and interesting) to ask whether $\mathfrak{u}_{S,x} \to \operatorname{Der}\mathfrak{p}(S,x)$ is injective when $g \geq 3$. (This homomorphism fails to be injective when $g = 1, 2$.)

§5. Weight Filtrations on Homology and Cohomology

The rational cohomology[4] of a complex algebraic variety X carries a natural filtration

$$0 = W_0 H^m(X) \subseteq W_1 H^m(X) \subseteq \cdots \\ \cdots \subseteq W_{2m-1}H^m(X) \subseteq W_{2m}H^m(X) = H^m(X)$$

called the *weight filtration*, which was constructed by Deligne using Hodge theory in [9, 10]. Weight filtrations can be constructed Galois actions as well [11]. Algebraic maps between complex algebraic varieties induce weight filtration preserving maps of their cohomology [10]. In particular, $(H^\bullet(X), W_\bullet)$ is a filtered algebra. The weight filtration is a powerful tool for studying the topology of complex algebraic varieties due to its strong exactness properties. In this section we give a brief introduction to weight filtrations directed at topologists. Deligne's paper [7] provides a more complete exposition of the yoga of weight filtrations. Full details can be found in [9].

An integer k is a (non-trivial) weight of $H^m(X)$ if its kth weight graded quotient

$$\operatorname{Gr}_k^W H^m(X) := W_k H^m(X)/W_{k-1}H^m(X)$$

is non-zero. We say that $H^m(X)$ is *pure* of weight k if k is the only non-trivial weight of $H^m(X)$. The weights on $H^m(X)$ are $\geq m$ when X is smooth and $\leq m$ when X is compact. So if X is smooth and projective, then $H^m(X)$ is pure of weight m.

Since we are working with fundamental groups, it is more natural to work with weight filtrations on homology than on cohomology. The

[4]Recall Convention 1.1: all (co)homology is with rational coefficients unless otherwise noted.

weight filtration on $H_m(X)$ is defined by

$$W_{-k}H_m(X) = \mathrm{Hom}(H^m(X)/W_{k-1}, \mathbb{Q}).$$

When X is smooth, the weights on $H_m(X)$ lie in $\{-2m, \ldots, -m\}$.

Example 5.1. The weight filtration on the homology of a smooth complex algebraic curve is determined by the topology of the underlying surface. Suppose that S is a compact oriented surface and that D is a finite subset. Set $S' = S - D$. Then one has the exact sequence (the dual of the Gysin sequence):

$$0 \to \widetilde{H}_0(D) \to H_1(S') \to H_1(S) \to 0.$$

The weight filtration on $H_1(S')$ is given by

$$W_{-k}H_1(S') = \begin{cases} 0 & k \geq 3 \\ \widetilde{H}_0(D) & k = 2 \\ H_1(S') & k \leq 1. \end{cases}$$

Note that $\mathrm{Gr}^W_{-1} H_1(S') = H_1(S)$. The weight filtration on $H_0(S')$ and $H_2(S')$ are uninteresting.

Higher dimensional examples with non-trivial weight filtrations can be constructed by taking products of curves. The weight filtration on the product of two varieties is the tensor product of the two weight filtrations:

$$W_kH^n(X \times Y) = \bigoplus_{\ell+m=n} \sum_{i+j=k} W_iH^\ell(X) \otimes W_jH^m(Y).$$

This induces an isomorphism

$$\mathrm{Gr}^W_k H^n(X \times Y) \cong \bigoplus_{\ell+m=n} \sum_{i+j=k} Gr^W_i H^\ell(X) \otimes \mathrm{Gr}^W_j H^m(Y).$$

5.1. Strictness and Exactness Properties

Weight filtrations that arise from Hodge and/or Galois theory have strong exactness properties which make them a powerful tool in studying the topology of algebraic varieties and algebraic maps.

Definition 5.2. A morphism $f(V_1, W_\bullet) \to (V_2, W_\bullet)$ of filtered vector spaces is *strict* with respect to $W_\bullet$ if for all $m \in \mathbb{Z}$

$$W_mV_2 \cap f(V_1) = f(W_mV_1).$$

Suppose that V is a vector space and that A is a subspace of V and $q : V \to B$ is a quotient. A filtration $W_\bullet$ of V induces one on A and B by restriction and projection, respectively:

$$W_m A := A \cap W_m V \text{ and } W_m B = q(W_m V).$$

In particular, we can induce filtrations on the kernel and cokernel of a filtration preserving map $f : (V_1, W_\bullet) \to (V_2, W_\bullet)$.

It is easy to check that f is strict with respect to $W_\bullet$ if and only if

$$0 \to \operatorname{Gr}^W_m \ker f \to \operatorname{Gr}^W_m V_1 \to \operatorname{Gr}^W_m V_2 \to \operatorname{Gr}^W_m \operatorname{coker} f \to 0$$

is exact for all $m \in \mathbb{Z}$.

Another important property of weight filtrations on cohomology groups of algebraic varieties, established in [9], is that there are natural (but not canonical) isomorphisms

$$H^m(X;\mathbb{C}) \cong \bigoplus \operatorname{Gr}^W_k H^m(X;\mathbb{C})$$

that are preserved by algebraic maps and which are compatible with tensor products, cup products, the Künneth isomorphism, etc. Establishing the existence of natural splittings of the weight filtration is the essential ingredient in establishing the strictness and exactness properties stated above.

Many other invariants of algebraic varieties and maps (such as the Leray spectral sequence, Gysin sequences, long exact sequences of a pair) carry natural weight filtrations, and all of their internal maps (differentials, Gysin maps, connecting homomorphisms) and all maps induced between them by algebraic maps preserve the weight filtration (sometimes with a shift) and are strict. The following example of Deligne [10] illustrates the basic yoga of weights and how it can be used to prove a topological result.

Example 5.3 (Deligne). Suppose that G is a connected linear algebraic group over $\mathbb{C}$ and that X is a smooth complex projective variety. Suppose that $\mu : G \times X \to X$ is an algebraic action. Deligne [10] shows that the weights on $H^k(G)$ are strictly larger than k except when $k = 0$. Since X is smooth and projective, $H^k(X)$ is pure of weight k. The mapping

$$\mu^* : H^n(X) \to H^n(G \times X) \cong \bigoplus_{\ell+m=n} H^\ell(G) \otimes H^m(X)$$

is thus filtration preserving. Since $H^n(X) = W_n H^n(X)$, strictness implies that

$$\operatorname{im} \mu^* = \operatorname{im} \mu^* \cap W_n H^n(G \times X) = H^0(G) \otimes H^n(X)$$

from which it follows that μ^* is the inclusion

$$H^n(X) \cong H^0(G) \otimes H^n(X) \hookrightarrow H^n(G \times X).$$

That is, rational cohomology cannot distinguish μ from the trivial action.

§6. Weight Filtrations on Completed Mapping Class Groups

Completions of mapping class groups have natural weight filtrations that are preserved by the natural homomorphisms between them. They arise because mapping class groups occur as fundamental groups of smooth stacks (moduli spaces of curves) and are constructed using either Hodge theory [18] or Galois theory [20].

Denote the lower central series of a Lie algebra $\mathfrak{h}$ by

$$\mathfrak{h} = L^1\mathfrak{h} \supseteq L^2\mathfrak{h} \supseteq \mathfrak{h} \supseteq \cdots$$

where $L^{m+1}\mathfrak{h} := [\mathfrak{h}, L^m\mathfrak{h}]$.

Theorem 6.1 (Hain [18]). *If (S, D) is a stable decorated surface and G is a subgroup of* $\operatorname{Aut} D$, *then $\mathcal{O}(\mathcal{G}^G_{S,D})$ has a natural weight filtration with which the product, antipode and coproduct are strictly compatible. This corresponds to a weight filtration*

$$\cdots \subseteq W_{-2}\mathcal{G}^G_{S,D} \subseteq W_{-1}\mathcal{G}^G_{S,D} \subseteq W_0\mathcal{G}^G_{S,D} = \mathcal{G}^G_{S,D}$$

by subgroups, where $W_{-1}\mathcal{G}^G_{S,D} = \mathcal{U}_{S,D}$. It also induces a filtration of the Lie algebra $\mathfrak{g}_{S,D}$ of the identity component. It has the property that $\mathfrak{g}_{S,D} = W_0\mathfrak{g}_{S,D}$ and $\mathfrak{u}_{S,D} = W_{-1}\mathfrak{g}_{S,D}$. The adjoint action

$$\mathfrak{g}_{S,D} \otimes \mathcal{O}(\mathcal{G}^G_{S,D}) \to \mathcal{O}(\mathcal{G}^G_{S,D}),$$

the bracket $\mathfrak{g}_{S,D} \otimes \mathfrak{g}_{S,D} \to \mathfrak{g}_{S,D}$ and the natural homomorphisms $\mathfrak{g}_{S,\tilde{D}} \to \mathfrak{g}_{S,D}$ are all strictly compatible with the weight filtration. When $g \geq 3$ and $\#D = 1$, the weight filtration is related to the lower central series of $\mathfrak{u}_{S,D}$ by

$$W_{-m}\mathfrak{g}_{S,D} = L^m\mathfrak{u}_{S,D}.$$

When $g = 0$ and $m \geq 1$,

$$W_{-2m+1}\mathfrak{g}_{S,D} = W_{-2m}\mathfrak{g}_{S,D} = L^m\mathfrak{u}_{S,D}.$$

In particular, when $g = 0$, $\mathfrak{g}_{S,D} = W_{-2}\mathfrak{g}_{S,D}$ and all odd weight graded quotients of $\mathfrak{g}_{S,D}$ are trivial.

For each subgroup G of $\operatorname{Aut} D$, conjugation induces infinitesimal actions

$$\mathrm{ad} : \mathfrak{g}_{S,D} \to \operatorname{Der} \mathcal{O}(\mathcal{G}^G_{S,D}) \text{ and } \mathrm{ad} : \mathfrak{g}_{S,D} \to \operatorname{Der} \mathfrak{g}_{S,D}.$$

Since $\mathrm{Gr}^W_0 \mathfrak{g}_{S,D} = \mathfrak{g}_{S,D}/\mathfrak{u}_{S,D} \cong \mathfrak{sp}(H)$, we have:

Corollary 6.2. *Each $\mathrm{Gr}^W_m \mathcal{O}(\mathcal{G}^G_{S,D})$ is a direct sum of finite dimensional $\mathfrak{sp}(H)$-modules and each $\mathrm{Gr}^W_m \mathfrak{g}_{S,D}$ is a direct product of finite dimensional $\mathfrak{sp}(H)$-modules.* $\square$

These weight filtrations are compatible with those constructed (in [38, 15]) on fundamental groups of algebraic curves and their configuration spaces:

Theorem 6.3 (Morgan, Hain). *If (S,D) is a stable decorated surface, then the Lie algebra $\mathfrak{p}$ of the unipotent completion of the fundamental group of the configuration space of m ordered points in S'_D has a natural weight filtration that satisfies $\mathfrak{p} = W_{-1}\mathfrak{p}$. In particular, $\mathfrak{p}(S'_D, x)$ has a natural weight filtration that satisfies $\mathfrak{p}(S'_D,x) = W_{-1}\mathfrak{p}(S'_D,x)$. The bracket and the surjection $\mathfrak{p}(S'_D,x) \to H_1(S'_D)$ are strictly compatible with the weight filtration. When $\#D \leq 1$, the weight filtration of $\mathfrak{p}(S'_D)$ is given by its lower central series:*

$$W_{-m}\mathfrak{p}(S'_D,x) = L^m\mathfrak{p}(S'_D,x)$$

when $m \geq 1$.

The natural action of the $\mathfrak{g}_{S,D}$ on the $\mathfrak{p}(S'_D)$ is compatible with these weight filtrations.

Theorem 6.4 (Hain [18]). *If (S,D) is a stable decorated surface and x is an admissible base point of S'_D, then the natural homomorphisms*

$$\mathfrak{g}_{S,D\cup\{x\}} \to \operatorname{Der} \mathfrak{p}(S'_D,x) \text{ and } \mathfrak{g}_{S,D} \to \operatorname{OutDer} \mathfrak{p}(S'_D)$$

are strictly compatible with the natural weight filtrations.

For all stable decorated surfaces (S,D), the weight filtrations on

$$\mathfrak{g}_{S,D},\ \mathfrak{p}(S'_D,x),\ \operatorname{Der} \mathfrak{p}(S'_D,x),\ \operatorname{OutDer} \mathfrak{p}(S'_D)$$

all have natural splittings.[5] That is, if $\mathfrak{g}$ is such a Lie algebra, then there is a natural isomorphism of complete Lie algebras

$$\mathfrak{g} \cong \prod_m \mathrm{Gr}_m^W \mathfrak{g},$$

and if $\phi : \mathfrak{g} \to \mathfrak{h}$ is a natural homomorphism between two such Lie algebras, then the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\simeq"] \arrow[d, "\phi"'] & \prod_m \mathrm{Gr}_m^W \mathfrak{g} \arrow[d, "\mathrm{Gr}_\bullet^W \phi"] \\ \mathfrak{h} \arrow[r, "\simeq"] & \prod_m \mathrm{Gr}_m^W \mathfrak{h} \end{tikzcd}$$

commutes.[6]

The existence of natural splittings allows one to study, *without loss of information*, the infinitesimal actions

$$d\tilde{\phi}_x : \mathfrak{u}_{S,\tilde{D}} \to \mathrm{Der}\, \mathfrak{p}(S'_D, x) \text{ and } d\phi : \mathfrak{u}_{S,D} \to \mathrm{OutDer}\, \mathfrak{p}(S'_D)$$

using the associated graded actions

$$\mathrm{Gr}_\bullet^W \mathfrak{u}_{S,\tilde{D}} \to \mathrm{Der}\, \mathrm{Gr}_\bullet^W \mathfrak{p}(S'_D, x) \text{ and } \mathrm{Gr}_\bullet^W \mathfrak{u}_{S,D} \to \mathrm{OutDer}\, \mathrm{Gr}_\bullet^W \mathfrak{p}(S'_D).$$

It also allows us to construct presentations of $\mathfrak{u}_{S,D}$ by giving presentations to their associated weight graded quotients as was done in [18] for the $\mathfrak{u}_S$ when $g \geq 6$.

Remark 6.5. One might hope that there are natural weight filtrations on the Lie algebras $\mathfrak{f}_n$, $\mathfrak{a}_n$ and $\mathfrak{o}_n$ associated to $\mathrm{Aut}^+ F_n$ and $\mathrm{Out}^+ F_n$ with respect to which the natural actions $\mathfrak{a}_n \to \mathrm{Der}\, \mathfrak{f}_n$ and $\mathfrak{o}_n \to \mathrm{OutDer}\, \mathfrak{f}_n$ are strict and for which each Lie algebra is naturally isomorphic to its associated graded. This would simplify the problem of finding presentations of $\mathfrak{ia}_n$ and $\mathfrak{io}_n$.

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[5] In Hodge theory, one usually tensors with $\mathbb{C}$ first to construct these splittings. However, the machinery of tannakian categories implies the existence of such splittings over $\mathbb{Q}$. Cf. [9, 38]

[6] Examples of morphisms $\phi : \mathfrak{g} \to \mathfrak{h}$ that are strictly compatible with the weight filtration are those which are induced by morphisms of moduli spaces of curves or are associated with monodromy representations of fundamental groups of moduli spaces of curves associated to natural local systems over moduli spaces such as those associated to families of unipotent completions of fundamental groups of universal curves and other tautological bundles.

Such weight filtrations would probably exist if $\operatorname{Aut} F_n$ or $\operatorname{Out} F_n$ were the fundamental group of an algebraic variety or stack defined over a number field, or if one could construct an action of the Galois group of (say) $\mathbb{Q}(\boldsymbol{\mu}_n)$, where $\boldsymbol{\mu}_n$ denotes the nth roots of unity, on their profinite completions that was compatible with the action of the Galois group on the profinite completion of $\pi_1(\mathbb{A}^1 - \boldsymbol{\mu}_n, 0)$. It is not clear how to proceed, or if this could ever be true.

§7. The Relative Weight Filtration of a Nilpotent Endomorphism

This section is an exposition of the linear algebra of nilpotent endomorphisms of filtered vector spaces, which arises naturally in the study of degenerations of complex algebraic varieties. For example, suppose that

$$f : X \to \Delta$$

is a family of complex algebraic varieties over the unit disk that is topologically locally trivial over the punctured disk Δ^*. Denote the fiber of f over $t \in \Delta$ by X_t. Fix a base point $t_o \in \Delta^*$. Since the family is locally topologically trivial over Δ^*, there is a monodromy operator[7]

$$h : H^m(X_{t_o}) \to H^m(X_{t_o})$$

for each $m \in \mathbb{N}$. A general result of Griffiths-Landman-Grothendieck (Cf. [32, 34]) implies that the eigenvalues of h are roots of unity. So, by replacing the family by its pullback along a finite covering $\Delta \to \Delta, s \mapsto s^c$ if necessary, we may assume that h is unipotent (i.e., all of its eigenvalues are 1). In this case it is the exponential of a nilpotent matrix $N = \log h$.

Example 7.1. A classical and relevant example occurs when the fiber X_t over $t \in \Delta^*$ is a compact Riemann surface and the central fiber X_0 is obtained from $S = X_{t_o}$ by contracting a a finite set of disjoint simple closed curves (the *vanishing cycles*) $\{c_1, c_2, \dots, c_r\}$ in S. The geometric monodromy τ is the product of the Dehn twists about the c_j. The induced mapping

$$h : H_1(S) \to H_1(S)$$

[7]Recall Convention 1.1: all (co)homology is with rational coefficients unless otherwise noted.

is given by the *Picard-Lefschetz formula*:

$$h(x) = x + \sum_{j=1}^{r} \langle c_j, x \rangle c_j.$$

This is clearly unipotent. Its logarithm $N : H_1(S) \to H_1(S)$ is the operator

$$N : x \mapsto \sum_{j=1}^{r} \langle c_j, x \rangle c_j$$

which satisfies $N^2 = 0$. (This formula is independent of the orientations assigned to the c_j.) $\square$

7.1. The Weight Filtration of a Nilpotent Endomorphism

There is a natural weight filtration of a vector space associated to a nilpotent endomorphism N of it.

Proposition 7.2. *If N is a nilpotent endomorphism of a finite dimensional vector space V, then there is a unique filtration*

$$0 = W(N)_{-m-1} \subseteq W(N)_{-m} \subseteq W(N)_{-m+1} \subseteq \cdots$$
$$\cdots \subseteq W(N)_{m-1} \subseteq W(N)_m = V$$

of V such that

(i) *for all $n \in \mathbb{Z}$, $NW(N)_n \subseteq W(N)_{n-2}$;*
(ii) *for each $k \in \mathbb{Z}$,*

$$N^k : \operatorname{Gr}_k^{W(N)} V \to \operatorname{Gr}_{-k}^{W(N)} V$$

is an isomorphism.

The filtration $W(N)_\bullet$ of V is called the *weight filtration of N*.

Proof. To prove existence , it is enough to consider the case where N has a single Jordan block. There is a basis

$$e_{-m}, e_{-m+2}, e_{-m+4}, \ldots, e_{m-2}, e_m$$

of V such that $Ne_j = e_{j-2}$. Define

$$W(N)_j = \operatorname{span}\{e_k : k \leq j\}.$$

Uniqueness is proved by induction on the exponent of nilpotency of N. If $N = 0$, then uniqueness is clear. Suppose that $m > 0$ and that $N^{m+1} = 0$, but that $N^m \neq 0$. The vanishing of N^{m+1} implies that

$$W(N)_k = V \text{ and } W(N)_{-k-1} = 0 \text{ for all } k \geq m.$$

Since $N^m : \mathrm{Gr}_m V \to \mathrm{Gr}_{-m} V$ is an isomorphism, we must have

$$W(N)_{m-1} = \ker N^m \text{ and } W(N)_{-m} = \operatorname{im} N^m.$$

Since $N^m \neq 0$,

$$0 \neq W(N)_{-m} \subseteq W(N)_{m-1} \neq V.$$

The induced endomorphism $\bar{N}$ of $V' := W(N)_{m-1}/W(N)_{-m}$ satisfies $\bar{N}^m = 0$. By induction, the weight filtration of $W(\bar{N})$ of $\bar{N}$ is unique. All weight filtrations of V must satisfy

$$W(N)_k = \text{ inverse image of } W(\bar{N})_k \text{ whenever } -m < k < m.$$

Uniqueness follows. Q.E.D.

Note that $W(N)_\bullet$ is centered at 0. If $V = H^m(X)$, where X is a smooth projective variety, it is natural to reindex the weight filtration of a nilpotent endomorphism N of V so that it is centered at the weight m of V. The shifted filtration

$$M_k V := W(N)_{k-m}$$

is centered at m. The reindexed filtration $M_\bullet$ satisfies $NM_k \subseteq M_{k-2}$ and

$$N^k : \mathrm{Gr}^M_{m+k} V \xrightarrow{\simeq} \mathrm{Gr}^M_{m-k} V$$

is an isomorphism for all $k \in \mathbb{Z}$. We will call the shifted weight filtration $M_\bullet$ the monodromy weight filtration of $N : V \to V$.

Example 7.3. The monodromy weight filtration for nilpotent endomorphism N of the weight -1 vector space $H_1(S)$ in Example 7.1 is:

$$\begin{aligned} M_0 &= W(N)_1 = H_1(S) \\ M_{-1} &= W(N)_0 = \ker N \\ M_{-2} &= W(N)_{-1} = \operatorname{im} N = \operatorname{span}\{c_1, \ldots, c_r\} \\ M_{-3} &= W(N)_{-2} = 0. \end{aligned} \tag{2}$$

The existence of a weight filtration extends to arbitrary direct sums and direct products of nilpotent N-modules.

7.2. Curve Systems

A *curve system* on a stable decorated surface (S, D) is a set

$$\gamma = \{c_0, \dots, c_r\}$$

of disjoint simple closed curves such that each connected component of

$$S'_D - |\gamma| := S'_D - \bigcup_{j=1}^{r} c_j$$

has negative Euler characteristic. Equivalently, no two c_j are isotopic in S'_D and no c_j bounds a disk or punctured disk in S'_D. Two curve systems are considered to be equal if they are isotopic in S.

Denote the Dehn twist about c_j by τ_j. Since the c_j are disjoint, these commute. Set $\tau = \prod_j \tau_j$. The action

$$\tau_* : H_1(S'_D) \to H_1(S'_D)$$

of τ on $H_1(S'_D)$ is given by the *Picard-Lefschetz formula*:

$$\tau_*(x) = x + \sum_{j=0}^{r} \langle c_j, x \rangle c_j.$$

This is clearly unipotent. Its logarithm $N_\gamma : H_1(S'_D) \to H_1(S'_D)$ is the operator $\tau_* - \mathrm{id}$ which is given by

$$N_\gamma : x \mapsto \sum_{j=1}^{r} \langle c_j, x \rangle c_j.$$

It satisfies $N_\gamma^2 = 0$. Note that it preserves the weight filtration $W_\bullet$ defined on $H_1(S'_D)$ in Example 5.1 and acts trivially on $W_{-2}H_1(S'_D)$.

The following example will be used later in the paper.

Example 7.4. Suppose that $H = H_1(S)$ and $N = N_\gamma$, where H has weight -1. Denote the corresponding monodromy weight filtration by $M_\bullet$. Set

$$A = \mathrm{Gr}_0^M H, \ H_0 = \mathrm{Gr}_{-1}^M H, \ B = \mathrm{Gr}_{-2}^M H.$$

There is a natural isomorphism $\mathfrak{sp}(H) \cong S^2H$, which we consider to have weight 0 as it is a subspace of $\mathrm{End}(H)$, which has weight 0. Note that

$$\mathrm{Gr}_\bullet^M \mathfrak{sp}(H) = \mathfrak{sp}(\mathrm{Gr}_\bullet^M H)$$

and that there is a natural Lie algebra isomorphism

$$\operatorname{Gr}_0^M \mathfrak{sp}(H) \cong \mathfrak{gl}(A) \oplus \mathfrak{sp}(H_0).$$

Denote by ξ the element of $\mathfrak{sp}(\operatorname{Gr}_\bullet^M H)$ that corresponds to the identity element of $\mathfrak{gl}(A)$. Note that ξ acts as the identity on A, minus the identity on B and trivially on H_0. It follows that if we consider $H^{\otimes n}$ to have weight $-n$, then ξ acts on $\operatorname{Gr}_k^M H^{\otimes n}$ as multiplication by $k-n$. It follows that if V is an $\mathfrak{sp}(H)$-submodule of $H^{\otimes n}$, then ξ acts on $\operatorname{Gr}_k^M V$ as multiplication by $k-n$.

7.3. The Weight Filtration of a Nilpotent Endomorphism of a Filtered Vector Space

Now suppose that N is a nilpotent endomorphism of a filtered finite dimensional vector space V. That is, V has a filtration

$$0 \subseteq \cdots \subseteq W_{m-1}V \subseteq W_m V \subseteq W_{m+1}V \subseteq \cdots \subseteq V$$

which is stable under N. This is extended to the infinite dimensional case using the conventions of Section 2.1. Namely, infinite dimensional examples are either ind- or pro-objects of the category of finite dimensional filtered vector spaces; the nilpotent endomorphism N is replaced by a locally nilpotent endomorphism (i.e., a direct limit of nilpotent endomorphisms) in the ind case and a pronilpotent endomorphism (i.e., an inverse limit of nilpotent endomorphisms) in the pro case.

We will often call the filtration $W_\bullet$ of V the *weight filtration* of V and $\operatorname{Gr}_m^W V$ the mth weight graded quotient of V.

Natural examples of a filtered vector space $(V, W_\bullet)$ with a nilpotent endomorphism arise from degenerations of smooth (not necessarily compact) varieties. In this case $(V, W_\bullet)$ is $H^m(X_t)$ endowed with its natural weight filtration, and N is the logarithm of the unipotent part of the monodromy operator.

Since N preserves the weight filtration, it induces an endomorphism

$$N_m := \operatorname{Gr}_m^W N : \operatorname{Gr}_m^W V \to \operatorname{Gr}_m^W V.$$

of the mth weight graded quotient of V. Since, by assumption, $\operatorname{Gr}_m^W V$ is the product or sum of nilpotent N-modules, Proposition 7.2 implies that each graded quotient has a weight filtration $W(N_m)$. The reindexed filtration $W(N_m)[m]_\bullet$ is centered at m. Denote it by $M_\bullet^{(m)}$

Definition 7.5. A filtration $M_\bullet$ of V is called a *relative weight filtration* of $N : (V, W_\bullet) \to (V, W_\bullet)$ if

(i) for each $k \in \mathbb{Z}$, $NM_k \subseteq M_{k-2}$;

(ii) the filtration induced by $M_\bullet$ on $\mathrm{Gr}^W_m V$ is the reindexed weight filtration $M^{(m)}_\bullet$.

Relative weight filtrations, if they exist, are unique. (Cf. [48]).

Example 7.6. If $N : (V, W_\bullet) \to (V, W_\bullet)$ satisfies $N(W_m V) \subseteq W_{m-2}V$ for all $m \in \mathbb{Z}$, then each $N_m = 0$ and the relative weight flirtation $M_\bullet$ of N exists and equals the original weight filtration $W_\bullet$.

Example 7.7. Suppose that γ is a curve system on a stable decorated surface (S, D). Take $V = H_1(S'_D)$ with the weight filtration defined in Example 5.1 and N to be the nilpotent endomorphism N_γ associated to γ defined in Section 7.2. The non-trivial weight graded quotients of $H_1(S'_D)$ are

$$\mathrm{Gr}^W_{-1} H_1(S'_D) \cong H_1(S) \text{ and } \mathrm{Gr}^W_{-2} H_1(S'_D) \cong \widetilde{H}_0(D).$$

Note that $N_{-1} : H_1(S) \to H_1(S)$ is the operator given in Example 7.1. Consequently $M^{(-1)}_\bullet$ is given by Example 7.3:

$$M^{(-1)}_{-2}H_1(S) = \mathrm{im}\, N_{-1}, M^{(-1)}_{-1}H_1(S) = \ker N_{-1}, M^{(-1)}_0 H_1(S) = H_1(S).$$

Since $N_{-2} = 0$,

$$0 = M^{(-2)}_{-3} \subseteq M^{(-2)}_{-2} = \widetilde{H}_0(D).$$

The relative weight filtration of $N_\gamma : H_1(S'_D) \to H_1(S'_D)$ exists. It is defined by

$$M_{-3} = 0, M_{-2} = \mathrm{im}\, N_\gamma + \widetilde{H}_0(D), M_{-1} = \ker N_\gamma + \widetilde{H}_0(D), M_0 = H_1(S'_D).$$

Even though the weight filtration of a nilpotent endomorphism of a finite dimensional vector space always exists, the relative weight filtration of a nilpotent endomorphism of a *filtered* vector space $(V, W_\bullet)$ usually does not exist. Necessary and sufficient conditions for the existence of a relative weight filtration are given in [48].

The existence of a relative weight filtration on the rational cohomology of the general fiber of a degeneration of complex algebraic varieties was first established for degenerations of varieties by Deligne [11, (1.8)] using ℓ-adic methods, and for smooth varieties over the complex numbers using Hodge theory by Steenbrink and Zucker [48]. It provides non-trivial restrictions on the possible monodromy operators of degenerations of algebraic varieties. For example, the existence relative weight filtration is a strong enough invariant to show that a bounding pair (BP) map cannot be the geometric monodromy of a degeneration of complex algebraic curves:[8]

Example 7.8. Suppose that $g(S) \geq 1$ and that the curve system $\{c_0, c_1\}$ is a bounding pair of simple closed curves in S. (That is, $S - |\gamma|$ has two connected components.) Suppose that $P = \{x_0, x_1\}$ is a pair of points in $S - |\gamma|$, one in each component. Denote the Dehn twist about c_j by τ_j. The associated bounding pair map is $\tau = \tau_1 \tau_0^{-1}$. It acts non-trivially and unipotently on $H_1(S'_D)$. Its logarithm $N = \tau_* - \mathrm{id}$ acts trivially on both weight graded quotients of $H_1(S'_D)$. Because of this, the relative weight filtration $M_\bullet$, if it exists, must agree with the weight filtration $W_\bullet$. But since these satisfy

$$M_{-3}H_1(S'_D) = W_{-3}H_1(S'_D) = 0 \text{ and}$$
$$M_{-1}H_1(S'_D) = W_{-1}H_1(S'_D) = H_1(S'_D),$$

the condition $NM_{-1} \subseteq M_{-3}$ implies that $N = 0$. But this contradicts the non-triviality of τ_*. Consequently, the endomorphism N of $(H_1(S'_D), W_\bullet)$ has no relative weight filtration.

§8. Relative Weight Filtrations on Mapping Class Groups

An element σ of a proalgebraic group $\mathcal{G}$ is *prounipotent* if it lies in a prounipotent subgroup. An element N of a pro-Lie algebra $\mathfrak{g}$ is *pronilpotent* if it lies in a pronilpotent subalgebra.

Lemma 8.1. *Each prounipotent element of a proalgebraic group $\mathcal{G}$ can be written uniquely as the exponential of a pronilpotent element of $\mathfrak{g}$, the Lie algebra of $\mathcal{G}$.*

Proof. Suppose that τ is a prounipotent element of $\mathcal{G}$. The existence of a pronilpotent logarithm of τ is clear as it lies in a prounipotent subgroup. Since every algebraic subgroup of a prounipotent group is prounipotent, the intersection of two prounipotent subgroups of $\mathcal{G}$ is also prounipotent. If $\tau = \exp N_1 = \exp N_2$, then lies in the intersection of the two unipotent 1-parameter subgroups $\{\exp tN_j : t \in F\}$, $j = 1, 2$. If $\tau \neq 1$, this forces $N_1 = N_2$. If $\tau = 1$, the unique logarithm is $N = 0$. Q.E.D.

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[8]This can be proved by elementary and direct arguments. However, using the non-existence of the relative weight filtration to prove that a BP map cannot be the geometric monodromy of degeneration of curves illustrates the kinds of restrictions that the existence of relative weight filtrations places on the monodromy of degenerations of varieties in general.

Proposition 8.2. *If $\tau \in \Gamma^G_{S,D}$ is a Dehn twist, then $\hat{\rho}(\tau)$ is a prounipotent element of $\mathcal{G}_{S,D}$.*

Proof. The Picard-Lefschetz formula implies that $\rho(\tau)$ is a unipotent element of $\mathrm{Sp}(H_1(S))$ and that it is the exponential of $N = \rho(\tau) - \mathrm{id}$. The inverse image $\mathcal{H}$ of the unipotent subgroup $L := \{\exp tN : t \in \mathbb{Q}\}$ of $\mathrm{Sp}(H)$ in $\mathcal{G}_{S,D}$ is prounipotent as it is an extension

$$1 \to \mathcal{U}_{S,D} \to \mathcal{H} \to L \to 1$$

of a unipotent group by a prounipotent group. Since $\hat{\rho}(\tau) \in \mathcal{H}$, it is prounipotent and thus has a unique pronilpotent logarithm. Q.E.D.

Suppose that (S, D) is a stable decorated surface and that $\gamma = \{c_0, \dots, c_m\}$ is a curve system on (S, D). Denote the Dehn twist on c_j by τ_j. By Proposition 8.2 $\hat{\rho}(\tau_j)$ has a canonical logarithm $N_j \in \mathfrak{g}_{S,D}$.

Define the closed cone in $\mathfrak{g}_{S,D}$ associated to γ by

$$C(\gamma) = \Big\{ \sum_{j=0}^{m} r_j N_j : r_j \in \mathbb{Q} \text{ and } r_j \geq 0 \Big\}$$

and the open cone by

$$C^o(\gamma) = \Big\{ \sum_{j=0}^{m} r_j N_j : r_j \in \mathbb{Q} \text{ and } r_j > 0 \Big\}$$

Then $C(\gamma)$ is a simplicial cone in $\mathbb{Q}^\gamma$ whose faces correspond to the subsets σ of γ:

$$C(\gamma) = \coprod_{\sigma \subseteq \gamma} C^o(\sigma).$$

Suppose that $N \in C(\gamma)$ and that G is a subgroup of $\mathrm{Aut}\, D$. The infinitesimal actions

$$\mathrm{ad} : \mathfrak{g}_{S,D} \to \mathrm{Der}\, \mathcal{O}(\mathcal{G}^G_{S,D}) \text{ and } \mathrm{ad} : \mathfrak{g}_{S,D} \to \mathrm{Der}\, \mathfrak{g}_{S,D}.$$

induce actions of N on each weight graded quotient of $\mathcal{O}(\mathcal{G}^G_{S,D})$ and $\mathfrak{g}_{S,D}$. By Corollary 6.2, each weight graded quotient of $\mathcal{O}(\mathcal{G}^G_{S,D})$ is a direct sum of finite dimensional $\mathfrak{sp}(H)$-modules and each weight graded quotient of $\mathfrak{g}_{S,D}$ is a direct product of finite dimensional $\mathfrak{sp}(H)$-modules. Consequently, each weight graded quotient of $\mathcal{O}(\mathcal{G}^G_{S,D})$ is a direct sum of finite dimensional nilpotent N-modules and and each weight graded quotient of $\mathfrak{g}_{S,D}$ is a direct product of finite dimensional nilpotent N-modules.

The following theorem is a special case of more general results proved in [20] using Galois theory and in [22] using Hodge theory.

Theorem 8.3. *For all curve systems γ of a stable decorated surface (S,D), all subgroups G of* $\operatorname{Aut} D$, *and all $N \in C^o(\gamma)$, there is a (necessarily unique) relative weight filtration $M_\bullet^\gamma$ (denoted $M_\bullet$) of $\mathcal{O}(\mathcal{G}_{S,D})$ of the endomorphism*

$$\operatorname{ad}(N) \in \operatorname{Der}(\mathcal{O}(\mathcal{G}_{S,D}^G), W_\bullet)$$

that is compatible with the product, antipode and coproduct of $\mathcal{O}(\mathcal{G}_{S,D})$. It induces relative weight filtrations on $\mathfrak{g}_{S,D}$ and $\mathfrak{p}(S'_D, x)$, where $x \in S'_D - |\gamma|$ is an admissible base point. The bracket of each of these Lie algebras is strictly compatible with $M_\bullet$. These relative weight filtrations depends only on γ and will be denoted by $M_\bullet^\gamma$. Each $N_j \in C(\gamma)$ lies in $M_{-2}\mathfrak{g}_{S,D}$. Moreover, for each curve system γ there is a natural (though not canonical) isomorphism

$$\mathfrak{g}_{S,D} \cong \prod_{k,m} \operatorname{Gr}_k^M \operatorname{Gr}_m^W \mathfrak{g}_{S,D}$$

of completed Lie algebras. In addition, if $\widetilde{D}$ is a refinement of D that contains the base point x and $\tilde{\gamma}$ is a curve system on $(S,\widetilde{D})$ whose image in (S,D) is γ, then the natural actions

$$d\tilde{\phi}_x : \mathfrak{g}_{S,\widetilde{D}} \to \operatorname{Der} \mathfrak{p}(S'_D, x) \text{ and } d\phi : \mathfrak{g}_{S,D} \to \operatorname{OutDer} \mathfrak{p}(S'_D).$$

are strictly compatible with $W_\bullet$ and the relative weight filtrations $M_\bullet^\gamma$ and $M_\bullet^{\tilde{\gamma}}$. The filtrations $M_\bullet$ and $W_\bullet$ can be simultaneously split. That is, they can be chosen so that the diagram

$$\begin{tikzcd}
\mathfrak{g}_{S,\widetilde{D}} \arrow[r, "\simeq"] \arrow[d] & \prod_{k,m} \operatorname{Gr}_k^M \operatorname{Gr}_m^W \mathfrak{g}_{S,\widetilde{D}} \arrow[d] \\
\operatorname{Der} \mathfrak{p}(S'_D, x) \arrow[r, "\simeq"] & \operatorname{Der} \operatorname{Gr}_\bullet^M \operatorname{Gr}_\bullet^W \mathfrak{p}(S'_D, x)
\end{tikzcd}$$

commutes. There is a similar diagram for $d\phi : \mathfrak{g}_{S,D} \to \operatorname{OutDer} \mathfrak{p}(S'_D)$.

Since the diagonal $\Delta : \mathcal{O}(\mathcal{G}_{S,D}^G) \to \mathcal{O}(\mathcal{G}_{S,D}^G) \otimes \mathcal{O}(\mathcal{G}_{S,D}^G)$ preserves $M_\bullet$, the image of $M_{-1}\mathcal{O}(\mathcal{G}_{S,D}^G)$ under the diagonal is contained in

$$M_{-1}\mathcal{O}(\mathcal{G}_{S,D}^G) \otimes \mathcal{O}(\mathcal{G}_{S,D}^G) + \mathcal{O}(\mathcal{G}_{S,D}^G) \otimes M_{-1}\mathcal{O}(\mathcal{G}_{S,D}^G).$$

This implies that $M_{-1}\mathcal{O}(\mathcal{G}_{S,D}^G)$ is a Hopf ideal of $\mathcal{O}(\mathcal{G}_{S,D}^G)$. Define

$$M_0\mathcal{G}_{S,D}^G = \operatorname{Spec}\big(\mathcal{O}(\mathcal{G}_{S,D}^G)/M_{-1}\mathcal{O}(\mathcal{G}_{S,D}^G)\big).$$

This is a subgroup of $\mathcal{G}_{S,D}^G$. Since $M_\bullet$ is preserved by the bracket, $M_k\,\mathfrak{g}_{S,D}$ is a pronilpotent Lie subalgebra of $\mathfrak{g}_{S,D}$ whenever $k \leq 0$. The Lie algebra of $M_0\mathcal{G}_{S,D}^G$ is $M_0\mathfrak{g}_{S,D}$. When $k < 0$, $M_k\mathfrak{g}_{S,D}$ is pronilpotent. Denote the corresponding prounipotent subgroup of $\mathcal{G}_{S,D}^G$ by $M_k\mathcal{G}_{S,D}$.

The uniqueness of relative weight filtrations is a strong condition which implies that $M_\bullet^\gamma$ has many nice properties. Some are established in the following paragraphs.

Proposition 8.4. *Suppose that γ is a curve system on a stable decorated surface (S, D). If $\phi \in \Gamma_{S,D}^G$, then the relative weight filtrations of $\mathcal{O}(\mathcal{G}_{S,D}^G)$ and $\mathfrak{g}_{S,D}$ satisfy $M_\bullet^{\phi(\gamma)} = \operatorname{Ad}(\phi)M_\bullet^\gamma$.*

Proof. For a curve system σ, let $\tau_\sigma = \prod_{c\in\sigma}\tau_c$ where τ_c denotes the Dehn twist about c. Denote the logarithm of τ_σ by N_σ. Since $\tau_{\phi(c)} = \phi\tau_c\phi^{-1}$, it follows that $\tau_{\phi(\gamma)} = \phi\tau_\gamma\phi^{-1}$, which implies $N_{\phi(\gamma)} = \operatorname{Ad}(\phi)N_\gamma$. Since $N_\gamma(M_k^\gamma) \subseteq M_{k-2}^\gamma$, it follows that

$$\begin{aligned} N_{\phi(\gamma)}\big(\operatorname{Ad}(\phi)M_k^\gamma\big) &= \big(\operatorname{Ad}(\phi)N_\gamma\big)\big(\operatorname{Ad}(\phi)M_k^\gamma\big) \\ &= \operatorname{Ad}(\phi)\big(N_\gamma\big(M_k^\gamma\big)\big) \subseteq \operatorname{Ad}(\phi)M_{k-2}^\gamma. \end{aligned}$$

The result now follows from the uniqueness of the relative weight filtration. Q.E.D.

The subalgebra $M_0^\gamma\,\mathfrak{g}_{S,D}$ of $\mathfrak{g}_{S,D}$ behaves like a parabolic subalgebra of a semi-simple Lie algebra:

Parabolic subalgebras of semi-simple and Kac-Moody Lie algebras are self normalizing, and correspond to boundary strata in the semi-simple case. The following result suggests that when $g \geq 2$, the subalgebras $M_0^\gamma\mathfrak{g}_{S,D}$ of $\mathfrak{g}_{S,D}$ might provide a good notion of parabolic subalgebra of $\mathfrak{g}_{S,D}$.

Proposition 8.5. *If γ is a curve system on a stable decorated surface (S, D) where $g(S) \geq 3$, then the normalizer of $M_0^\gamma\mathfrak{g}_{S,D}$ in $\mathfrak{g}_{S,D}$ is $M_0^\gamma\mathfrak{g}_{S,D}$.*

Proof. Since the functor $\operatorname{Gr}_\bullet^M \operatorname{Gr}_\bullet^W$ is exact, it suffices to prove the result for the associated bigraded Lie algebra $\operatorname{Gr}_\bullet^M \operatorname{Gr}_\bullet^W \mathfrak{g}_{S,D}$. Set $M_\bullet = M_\bullet^\gamma$ and $H = H_1(S)$. Let $\xi \in \operatorname{Gr}_0^M \mathfrak{sp}(H)$ be the element defined in Example 7.4. It lies in $\operatorname{Gr}_0^M \operatorname{Gr}_0^W \mathfrak{g}_{S,D}$. Johnson's theorem [29] implies that $\operatorname{Gr}_m^W \mathfrak{g}_{S,D}$ is an $\mathfrak{sp}(H)$-quotient of $(H^n \oplus \Lambda^3 H)^{\otimes m}$, where $n = \#D - 1$. It follows from Example 7.4 that if $k > 0$ and $X \in \operatorname{Gr}_k^M \operatorname{Gr}_m^W \mathfrak{g}_{S,D}$, then $[\xi, X] = (k - m)X$, which is non-zero whenever $k > 0$. Thus, if $X \notin M_0 \operatorname{Gr}_\bullet^M \operatorname{Gr}_\bullet^W \mathfrak{g}_{S,D}$, then X does not normalize $M_0 \operatorname{Gr}_\bullet^M \operatorname{Gr}_\bullet^W \mathfrak{g}_{S,D}$. Q.E.D.

This result also holds in genus 2, but not in genus 1.

8.1. Dependence on γ

For a curve system γ of a stable decorated surface (S, D), denote the subspace of $H_1(S)$ spanned by the homology classes of the $c \in \gamma$ by $\langle\gamma\rangle$.

Proposition 8.6. *If γ is a curve system of a stable decorated surface (S, D) and $\sigma \subseteq \gamma$, then the relative weight filtrations $M_\bullet^\gamma$ and $M_\bullet^\sigma$ of $\mathcal{O}(\mathcal{G}_{S,D}^G)$, $\mathfrak{g}_{S,D}$ and $\mathfrak{p}(S'_D)$ are equal if and only if $\langle\sigma\rangle = \langle\gamma\rangle$.*

Proof. The condition that $\langle\gamma\rangle = \langle\sigma\rangle$ implies that the monodromy weight filtrations on $H_1(S)$ are equal. Since each weight graded quotient of $\mathcal{O}(\mathcal{G}_{S,D}^G)$, $\mathfrak{g}_{S,D}$ and $\mathfrak{p}(S'_D)$ is a subquotient of a tensor power of $H_1(S)$, it follows that the monodromy weight filtrations associated to γ and σ on $\mathrm{Gr}_\bullet^W \mathfrak{g}_{S,D}$ are equal if and only if $\langle\gamma\rangle = \langle\sigma\rangle$. Similarly for $\mathfrak{p}(S'_D)$ and $\mathcal{O}(\mathcal{G}_{S,D}^G)$.

To complete the proof, we now show that if the monodromy weight filtrations on $\mathrm{Gr}_\bullet^W V$ agree, where $V = \mathcal{O}(\mathcal{G}_{S,D}^G)$, $\mathfrak{g}_{S,D}$ or $\mathfrak{p}(S'_D)$, then the relative weight filtrations on V agree. Since $\sigma \subseteq \gamma$, for each $c \in \sigma$, $N_c(M_k^\gamma) \subseteq M_{k-2}^\gamma$. That is $M_\bullet^\gamma$ is a relative weight filtration on V for each $N \in C^o(\sigma)$. Uniqueness implies that $M_\bullet^\sigma = M_\bullet^\gamma$. Q.E.D.

When $g(S) = 0$, $H_1(S) = 0$ and the hypotheses of Proposition 8.6 are satisfied when σ is empty. This implies that the relative weight filtration is the existing weight filtration:

Corollary 8.7. *If $\gamma = \{c_0, \dots, c_m\}$ is a curve system on a stable decorated surface (S, D) of genus 0 and G is a subgroup of $\operatorname{Aut} D$, then the relative weight filtrations of $\mathcal{O}(\mathcal{G}_{S,D}^G)$, $\mathfrak{p}(S'_D, x)$ and $\mathfrak{g}_{S,D}$ equal their natural weight filtrations $W_\bullet$. Consequently, $\mathcal{G}_{S,D}^G = M_0^\gamma \mathcal{G}_{S,D}^G$.* □

Suppose that (S, D) is a stable decorated surface, where $D = P \cup V$. For the purposes of this definition, we will consider V as a set of boundary components. Following Hatcher-Thurston [27] we say that two curve systems γ' and γ'' of (S, D) differ by an *A-move* if

$$\{c_1, c_2, c_3, c_4\} \subseteq (\gamma' \cap \gamma'') \cup V$$

that bound a genus 0 subsurface T of S and there are $c_0' \in \gamma'$ and $c_0'' \in \gamma''$ that lie in T and

$$\gamma' = \sigma \cup \{c_0'\} \text{ and } \gamma' = \sigma \cup \{c_0'\}.$$

Figure 1 illustrates an A-move.

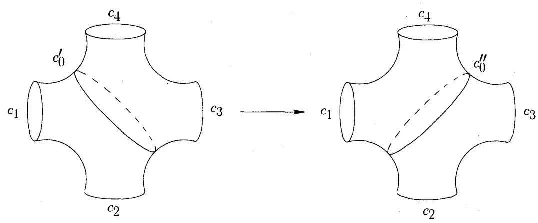

Fig. 1. An A-move

Suppose that σ and γ are curve systems on the stable decorated surface (S, D). We say that γ *is obtained from* σ *by homology neutral insertions* (or that σ is obtained from γ by homology neutral deletions) if $\sigma \subseteq \gamma$ and the subspaces $\langle\sigma\rangle$ and $\langle\gamma\rangle$ of $H_1(S)$ are equal.

The invariance of the $M_\bullet^\gamma$ under homology neutral insertions and deletions follows directly from Proposition 8.6.

Proposition 8.8. *Suppose that (S, D) is a stable decorated surface. If γ_1 and γ_2 are two curve systems on (S, D) that differ by a sequence of A-moves and by homology neutral insertions and deletions, then the relative weight filtrations $M_\bullet^{\gamma_1}$ and $M_\bullet^{\gamma_2}$ of $\mathfrak{g}_{S,D}$ are equal.* □

8.2. Glueing Lemma

Suppose that $\gamma = \{c_0, \dots, c_m\}$ is a curve system on the stable decorated surface (S, D). For each connected component T' of $S_D' - |\gamma|$ there is a compact oriented surface T and decorations $D_T = P_T \cup V_T$ such that

$$P_T = T' \cap P$$

and V_T is the union of $T' \cap V$ with the new tangent vectors obtained by collapsing the boundary components created by removing the c_j.[9]

There is a natural homomorphism (the *glueing map*)

$$\prod_T \Gamma_{T,D_T} \to \Gamma_{S,D} \tag{3}$$

whose image is the subgroup of $\Gamma_{S,D}$ that are represented by diffeomorphisms that restrict to the identity on each c_j. The kernel is isomorphic to $\mathbb{Z}^\gamma$.

[9]It is convenient and natural to choose a point on each c_j, as it will provide a marking on each boundary component of $S_D' - |\gamma|$.

Proposition 8.9 (Glueing Lemma). *The glueing map induces a homomorphism on relative completions such that the diagram*

$$\begin{tikzcd} \prod_T \Gamma_{T,D_T} \arrow[r] \arrow[d] & \Gamma_{S,D} \arrow[d] \\ \prod_T \mathcal{G}_{T,D_T} \arrow[r] & \mathcal{G}_{S,D} \end{tikzcd}$$

commutes. The induced Lie algebra homomorphism

$$\bigoplus_T \mathfrak{g}_{T,D_T} \to \mathfrak{g}_{S,D}$$

is strict with respect to the weight filtration $W_\bullet$ on the $\mathfrak{g}_{T,D_T}$ and the relative weight filtration $M_\bullet^\gamma$ on $\mathfrak{g}_{S,D}$.

Proof. Here we prove the existence of the induced homomorphism $\prod_T \mathcal{G}_{T,D_T} \to \mathcal{G}_{S,D}$. Its compatibility with the filtrations follows from the existence of limit mixed Hodge structures [22] or, alternatively, the Galois equivariance of the corresponding map of profinite mapping class groups.

Set $H = H_1(S)$ and $H_T = H_1(T)$. The relative weight filtration $M_\bullet := M_\bullet^\gamma$ of H satisfies

$$\mathrm{Gr}^M_{-1} H = \oplus_T H_T \text{ and } M_{-2}H = \langle \gamma \rangle.$$

Set

$$M_m\mathrm{Sp}(H) := \{\phi \in \mathrm{Sp}(H) : (\phi - \mathrm{id})(M_k H) \subseteq M_{k+m}H\}.$$

This is the subgroup of $\mathrm{Sp}(H)$ with Lie algebra $M_m\mathfrak{sp}(H)$. The Zariski closure of the image of $\prod_T \Gamma_{T,D_T}$ in $\mathrm{Sp}(H)$ is contained in $M_0\mathrm{Sp}(H)$ and is the extension of the subgroup $\prod_T \mathrm{Sp}(H_T)$ of $\mathrm{Gr}^M_0 \mathrm{Sp}(H)$ by a unipotent subgroup of $M_{-2}\mathrm{Sp}(H)$. Denote by $\mathcal{H}$ the inverse image of $\prod_T \Gamma_{T,D_T}$ under the surjection $M_0\mathcal{G}_{S,D} \to \mathrm{Gr}^M_0 \mathrm{Sp}(H)$. It is an extension of $\prod_T \Gamma_{T,D_T}$ by the prounipotent group $M_{-2}\mathcal{G}_{S,D}$.

The completion of $\prod_T \Gamma_{T,D_T}$ with respect to the natural homomorphism $\prod_T \Gamma_{T,D_T} \to \prod_T \mathrm{Sp}(H_T)$ is $\prod_T \mathcal{G}_{T,D_T}$. The universal mapping property of relative completion implies that the homomorphism $\prod_T \Gamma_{T,D_T} \to \mathcal{H}$ induces a homomorphism $\prod_T \mathcal{G}_{T,D_T} \to \mathcal{H}$ such that the diagram

$$\begin{tikzcd} \prod_T \Gamma_{T,D_T} \arrow[rr] \arrow[d] & & \Gamma_{S,D} \arrow[d] \\ \prod_T \mathcal{G}_{T,D_T} \arrow[r] & \mathcal{H} \arrow[r] & \mathcal{G}_{S,D} \end{tikzcd}$$

commutes. Q.E.D.

There is a more elaborate version of the Glueing Lemma that applies to groups that contain

$$\prod_T \Gamma^{G_T}_{T,D_T}$$

as a finite index subgroup, where $G_T \subseteq \operatorname{Aut} D_T$, and which map to $\Gamma^G_{S,D}$ for certain $G \subseteq \operatorname{Aut} D$. Rather than formulate such a result in general, we now state and prove a very special case that we shall need when investigating handlebody subgroups of $\Gamma_{S,D}$.

Consider the decorated surface (S, D) of genus $h-1$ with one marked boundary component that is constructed as the double covering of the 2-sphere branched at the $2h$th roots of unity, $\boldsymbol{\mu}_{2h}$, and with the disk of radius $1/2$ removed from one of the branches. This surface can be described as the Riemann surface of the algebraic function

$$y^2 = x^{2h} - 1$$

with the disk $|x| < 1/2$ removed from one of the two branches. The marked boundary point is chosen to be $x = 1/2$.

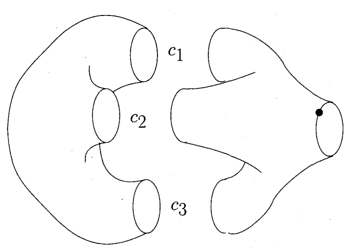

Fig. 2. The surface $S - |\gamma|$ when $h = 3$

For $j = 1, \ldots, h$, let c_j be the circle in S that is the inverse image in S of the interval $[\zeta^{2j-1}, \zeta^{2j}]$ in $\mathbb{C}$, where $\zeta = \exp 2\pi i/2h$. Then $\gamma := \{c_1, \ldots, c_h\}$ is a curve system that separates S into two genus 0 subsurfaces (T_0, D_0) and (T_1, D_1), where

$$D_0 = \{c_1, \ldots, c_h\} \text{ and } D_1 = \{\partial S\} \cup D_0.$$

The case $h = 3$ is illustrated in Figure 2.

The group $\boldsymbol{\mu}_h$ of hth roots of unity acts on S: $\zeta^{2j} : (x, y) \mapsto (\zeta^{2j} x, y)$. It acts on D_0 and D_1 by taking c_j to c_{j+1}, where the indices are considered mod h. Denote the natural homomorphism $\Gamma^G_{T_j,D_j} \to \boldsymbol{\mu}_h$ by p_j.

Set

$$[\Gamma_{T_0,D_0} \times \Gamma_{T_1,D_1}]^{\boldsymbol{\mu}_h} = \{(\phi, \phi_2) \in \Gamma_{T_0,D_0}^{\boldsymbol{\mu}_h} \times \Gamma_{T_1,D_1}^{\boldsymbol{\mu}_h} : p_1(\phi_1) = p_2(\phi_2)\}.$$

There is an obvious glueing homomorphism

$$(4) \qquad [\Gamma_{T_0,D_0} \times \Gamma_{T_1,D_1}]^{\boldsymbol{\mu}_h} \to \Gamma_{S,D}.$$

The completion of $[\Gamma_{T_0,D_0} \times \Gamma_{T_1,D_1}]^{\boldsymbol{\mu}_h}$ with respect to the natural homomorphism to $\boldsymbol{\mu}_h$ is easily seen to be the restriction of $\mathcal{G}_{T_0,D_0}^{\boldsymbol{\mu}_h} \times \mathcal{G}_{T_1,D_1}^{\boldsymbol{\mu}_h}$ to the diagonal of $\boldsymbol{\mu}_h \times \boldsymbol{\mu}_h$. Denote it by

$$[\mathcal{G}_{T_0,D_0}^{\boldsymbol{\mu}_h} \times \mathcal{G}_{T_1,D_1}^{\boldsymbol{\mu}_h}]^{\boldsymbol{\mu}_h}.$$

Proposition 8.10. *The homomorphism* (4) *induces a homomorphism*

$$[\mathcal{G}_{T_0,D_0}^{\boldsymbol{\mu}_h} \times \mathcal{G}_{T_1,D_1}^{\boldsymbol{\mu}_h}]^{\boldsymbol{\mu}_h} \to \mathcal{G}_{S,D}$$

that is strictly compatible with the induced mapping

$$(\mathcal{O}(\mathcal{G}_{S,D}), M_\bullet^\gamma) \to (\mathcal{O}([\mathcal{G}_{T_0,D_0}^{\boldsymbol{\mu}_h} \times \mathcal{G}_{T_1,D_1}^{\boldsymbol{\mu}_h}]^{\boldsymbol{\mu}_h}), W_\bullet).$$

The existence of the induced homomorphism is proved by an argument similar to the proof of Proposition 8.9. The strictness with respect to the filtrations follows from either Hodge theory or Galois theory.

Since T_0 and T_1 are spheres, Corollary 8.7 implies that $\mathcal{G}_{T_j,D_j}^{\boldsymbol{\mu}_h} = W_0\mathcal{G}_{T_j,D_j}^{\boldsymbol{\mu}_h}$. This gives the following important consequence of strictness.

Corollary 8.11. *The image of* $[\mathcal{G}_{T_0,D_0}^{\boldsymbol{\mu}_h} \times \mathcal{G}_{T_1,D_1}^{\boldsymbol{\mu}_h}]^{\boldsymbol{\mu}_h} \to \mathcal{G}_{S,D}$ *lies in* $M_0\mathcal{G}_{S,D}$. $\square$

Define $\phi_h \in \Gamma_{S,D}$ to be the mapping class of the diffeomorphism

$$(x, y) \mapsto (e^{2\pi i/h}x, y)$$

of S composed with $1/h$th of the inverse of the Dehn twist about ∂T. This fixes the boundary point $1/2$. Observe that ϕ_h^h is the inverse of the Dehn twist about ∂T. Since ϕ_h lies in the image of (4), we have:

Proposition 8.12. *The image of* ϕ_h *in* $\mathcal{G}_{S,D}$ *lies in* $M_0\mathcal{G}_{S,D}$.

We shall also need a certain diffeomorphism ψ of a surface S of genus 2 with one boundary component. It is convenient to take S to the Riemann surface

$$y^2 = (x^2 - 1)(x^2 - 4)(x^2 - 9)$$

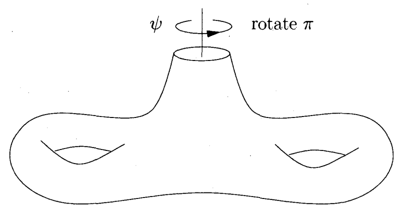

Fig. 3. The diffeomorphism ψ

with one of the two preimages of the disk $|x| < 1/2$ removed. It is illustrated in Figure 3. The element ψ of $\Gamma_{S,\partial S}$ is the composition of the diffeomorphism $(x, y) \mapsto (-x, y)$ of S with the square root of the inverse of the Dehn twist about the boundary of S. In terms of the illustration in Figure 3, it is obtained by rotating S by π about the vertical axis of the boundary circle and composing with the square root of the inverse of the Dehn twist about the boundary of S. An argument similar to the one used to prove Proposition 8.12 can be used to prove:

Proposition 8.13. *The image of ψ in $\mathcal{G}_{S,\partial S}$ lies in $M_0\mathcal{G}_{S,\partial S}$.*

§9. Handlebodies and Relative Weight Filtrations

A curve system $\gamma = \{c_0, \dots, c_m\}$ on a stable decorated surface (S, D) is said to be *rational* if each connected component of $S - |\gamma|$ is a genus 0 surface.[10]

Lemma 9.1. *If γ is a rational curve system on a stable decorated surface (S, D), then there is a unique handlebody U_γ such that $S = \partial U_\gamma$ and each curve $c \in \gamma$ bounds a disk in U.*[11]

Proof. This follows directly from the elementary fact that an oriented 2-sphere bounds a handlebody (necessarily a 3-ball) in a unique way, as the handle body is the cone over S. Q.E.D.

A maximal curve system on a stable decorated surface (S, D) is called a *pants decomposition* of S'_D. If γ is a pants decomposition of S'_D,

[10]This is equivalent to the condition that the nodal surface S/γ obtained from S by collapsing each $c \in \gamma$ to point has the topological type of a stable rational curve. It is also equivalent to the condition that $\mathrm{Gr}^M_{-1} H_1(S) = 0$.

[11]That is, if U_1 and U_2 are handlebodies with $\partial U_1 = \partial U_2 = S$ where each $c \in \gamma$ bounds in each U_j, then there is a homeomorphism $f : U_1 \to U_2$ that is the identity on S.

then each component of $S'_D - |\gamma|$ is a sphere with r boundary components and n punctures, where $r + n = 3$. Thus pants decompositions are rational and determine a handlebody U_γ. It is proved in [26]that any two pants decompositions of S can be joined by A-moves and S-moves. It is clear that A-moves leave the handlebody U_γ unchanged while S-moves change the handlebody.

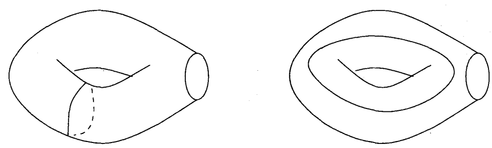

Fig. 4. An S-move

Theorem 9.2. *Two pants decompositions of a decorated stable surface determine the same handlebody if and only if they can be joined by A-moves.*

Sketch of Proof. It is clear that A-moves do not change the handlebody. We use ideas from Morse theory to prove the converse. They are an elaboration of ideas used by Hatcher and Thurston [27] (see also [26]). We will use boundary components instead of tangent vectors and will assume, without loss of generality, that the decorations D consist entirely of boundary components.

If γ is a pants decomposition of (S, D), then the boundary of the associated handlebody U equals

$$\partial U = S \cup \bigcup_{v \in \pi_0(\partial S)} \mathbb{D}_v,$$

where $\mathbb{D}_v$ is a disk that corresponds to the boundary component v of S. We regard this as a manifold with corners at the submanifold ∂S of ∂U.

To a pants decomposition γ of S, we associate a graph G_γ. This has one white vertex for each $c \in \gamma$ and one black vertex for each pair of pants. Edges connect a black and a white vertex; each black vertex is joined to the white vertex corresponding to its boundary components. All black vertices have valence 3; white vertices have valence 1 if they are boundary components of S, otherwise they have valence 2. The handlebody U is a regular neighbourhood of G_γ. The circles retract onto the white vertices.

A height function $f : G_\gamma \to [0, 1]$ on such a graph is a continuous function whose restriction to each edge has no critical points and whose local extrema occur only at white vertices. We also require that f vanish on each 1-valent white vertex. Every G_γ has a height function.

We will say that a smooth function $F : U \to [0,1]$ is *Morse* if it vanishes identically on each $\mathbb{D}_v$, it has no critical points in U, and if its restriction $F|_S$ to S is a Morse function. A Morse function $F : U \to [0,1]$ is *convex* if $F^{-1}(a)$ is a disjoint union of contractible sets for each $a \in [0,1]$.

A height function f extends to a convex Morse function $F : U \to [0,1]$ where $F|_S$ has one critical point for each black vertex and two for each 2-valent white vertex that is a local extremum of f. The critical values of $F|_S$ equal the values of f on the black vertices and the values of $f \pm \epsilon$ on the 2-valent white vertices that are local extrema. The stable or unstable manifold in S of the critical point of $F|_S$ corresponding to a 2-valent white vertex is the isotopy class of the corresponding $c \in \gamma$. The white vertices that are not local extrema of f correspond to components of the level sets of F. The isotopy class of the vertex corresponding to $c \in \gamma$ is c.

Suppose now that γ_0 and γ_1 are two pants decompositions of (S, D) that determine the same handlebody U. Choose height functions $f_j : \gamma_j \to [0,1]$. Extend these to convex Morse functions $F_j : U \to [0,1]$ using the construction in [27]. It follows from [27] that the result will follow if we can show that there is a smooth function $F : U \times [0,1] \to [0,1]$ that vanishes identically on each $\mathbb{D}_v \times [0,1]$, whose restriction to $S \times [0,1] \to [0,1]$ is a generic 1-parameter family of Morse functions, and where the restriction $F_t : U \to [0,1]$ to each $U \times \{t\}$ has no critical points and is a convex Morse functions for all but finitely many $t \in [0,1]$.

To construct such an F, first extend F_0 and F_1 to Morse functions $H_j : (M, \partial M) \to ([0,1], 0)$, where

$$M = S^3 - \cup_{v \in \pi_0(\partial S)} B^3$$

and where $\mathbb{D}_v$ is a hemisphere of the boundary of the corresponding 3-ball. Join these by a generic 1-parameter family of functions $H_t : (M, \partial M) \to ([0,1], 0)$ that have no critical points in a neighbourhood of ∂M and where $H : (M, \partial M) \times [0,1] \to ([0,1], 0) \times [0,1]$ that takes (x, t) to $(H_t(x), t)$ is smooth. The critical set $\Sigma \subset M \times [0,1]$ of H is 1-dimensional and has relative dimension 0 over $[0,1]$. On the other hand, we can choose U so that it is a regular neighbourhood of a graph Γ in M. Then $U \times [0,1]$ is a regular neighbourhood of $\Gamma \times [0,1]$ in $M \times [0,1]$. Since $\Gamma \times [0,1]$ has relative dimension 1 over $[0,1]$ and is disjoint from $\Sigma_t := \Sigma \cap M \times \{t\}$ when $t = 0, 1$, there is a vector field on $M \times [0,1]$ that is tangent to the t-slices and whose flow moves $\Gamma \times [0,1]$ in $M \times [0,1]$ to a subset J that is disjoint from Σ and intersects each fiber in a graph J_t homeomorphic to Γ. Then we may identify a regular neighbourhood W of J in M with $U \times [0,1]$ where $W_t := W \cap (M \times \{t\})$ is homeomorphic with U and does not intersect Σ. We may then deform the restriction of H to W so that it is a generic 1-parameter family of functions $\partial S \times [0,1] \subset \partial W$ and has no critical points in the interior of W.

The issue now is that the restriction $H_t : U \to [0,1]$ of H_t to $U \to [0,1]$ may not be convex as there may be $a, t \in [0,1]$ where $H_t^{-1}(a)$ is not a disjoint union of contractible sets. There are parameter values $0 < t_1 < t_2 < \cdots < t_m < 1$ where $H_t|_U$ is Morse when $t \notin \{t_1, \ldots, t_m\}$. On each interval (t_{j-1}, t_j) of $[0,1] - \{t_1, \ldots, t_m\}$, the Morse function $H_t|_S$ is represented by a graph G_j and a height function $h_j : G_j \to [0,1]$. As t moves through t_j, two vertices of G_j reverse height and G_j changes into G_{j+1} by one of the elementary moves described in [27]. The extra data of the function $H_t : U \to [0,1]$, when $t \in (t_{j-1}, t_j)$, is determined by an equivalence relation on imbedded intervals in G_j that correspond to non-critical values of H_t: two "strands" are equivalent if the corresponding circles are boundary components of the same component of $H_j^{-1}(a)$ for some non-critical value $a \in [0,1]$.

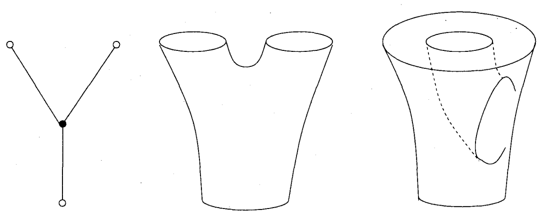

Fig. 5. Interior versus Exterior

Even if the Morse functions H_t are not convex, we can use the sequence of graphs $(G_0, h_0), \ldots, (G_m, h_m)$ to construct a sequence of A-moves that join γ_0 to γ_1. Note that at a black vertex, the level set can change, for example, from a disk to two disks or to an annulus, as illustrated in Figure 5. However, we can apply the Hatcher-Thurston construction to each graph to obtain a sequence of convex Morse functions $K_j : U \to [0,1]$ where $0 \leq j \leq m$ where K_0 and K_1 correspond to the pants decompositions γ_0 and γ_1. Denote by μ_j the pants decomposition of S that corresponds to the restriction of K_j to S. Then it follows by examinining the list of elementary moves in [27] that the pants decompositions determined by two convex Morse functions that differ by an elementary move, themselves differ by an A-move. In particular, μ_{j-1} and μ_j differ by an A-move. This shows that $\gamma_0 = \mu_0$ and $\gamma_1 = \mu_m$ are connected by a sequence of A-moves. Q.E.D.

This result has the important consequence that each way of writing S as the boundary of a handlebody U defines a relative weight filtration on invariants of (S, D), such as $\mathfrak{g}_{S,D}$.

Corollary 9.3. *Suppose that (S, D) is a stable decorated surface and that x is an admissible base point of S'_D. If U is a handlebody and $S = \partial U$, then U determines relative weight filtrations $M^U_\bullet$ on $\mathfrak{g}_{S,D}$, $\mathcal{O}(\mathcal{G}_{S,D})$, $\mathfrak{g}_{S,D\cup\{x\}}$ and $\mathfrak{p}(S'_D, x)$. The actions*

$$\mathfrak{g}_{S,D} \to \operatorname{OutDer} \mathfrak{p}(S'_D) \text{ and } \mathfrak{g}_{S,D\cup\{x\}} \to \operatorname{Der} \mathfrak{p}_{S'_D,x}$$

are strict with respect to the weight filtrations $W_\bullet$ and the relative weight filtrations $M^U_\bullet$.

Proof. Choose a pants decomposition γ of $(S, D \cup \{x\})$ such that each $c \in \gamma$ bounds a disk in U. Define $M^U_\bullet = M^\gamma_\bullet$. Theorem 9.2 and Proposition 8.8 implies that this is independent of the choice of γ. The strictness properties follow from Theorem 8.3. Q.E.D.

For a handlebody U and $x \in U$, denote $\pi_1(U, x)^{\mathrm{un}}$ by $\mathcal{F}(U, x)$ and its Lie algebra by $\mathfrak{f}(U, x)$.

Proposition 9.4. *If S bounds the handlebody U and $x \in S$, then*

$$M^U_0 \mathfrak{p}(S, x) = \mathfrak{p}(S, x) \text{ and } M^U_{-2}\mathfrak{p}(S, x) = \ker\{\mathfrak{p}(S, x) \to \mathfrak{f}(U, x)\}.$$

Consequently, $\mathfrak{f}(U, x) \cong \operatorname{Gr}^{M^U}_0 \mathfrak{p}(S, x)$.

Proof. This proof can be made in either the Galois category or the Hodge category, according to taste. Choose a pants decomposition γ of (S, x) as in the proof of Corollary 9.3 such that $M^\gamma_\bullet = M^U_\bullet$. The relative weight filtration on $\mathfrak{p}(S, x)$ arises from a degeneration of (S, x) to the stable rational curve C_0 whose underlying surface is $(S/\gamma, x)$. The degeneration[12] can be chosen to be defined over $\mathbb{Q}$ as each of its components is a 3-pointed $\mathbb{P}^1$. The inclusion of a nearby fiber C_t (the fiber over a first order smoothing of C_0) into the total space of the local deformation $X \to \mathbb{D}$ induces the homomorphism $\pi_1(S, x) \cong \pi_1(C_t, x) \to \pi_1(C_0, x) \cong \pi_1(U, x)$. This implies that the homomorphism $\mathfrak{p}(S, x) \to \mathfrak{f}(U, x)$ is Galois equivariant and a morphism of mixed Hodge structures.

[12]We shall denote the degeneration by $X \to \mathbb{D}$, where $\mathbb{D}$ is an analytic disk in the Hodge case and a formal disk $\operatorname{Spec} \mathbb{Q}[[t]]$ in the Galois case.

Since the Galois action on the algebraic fundamental group of (C_0, x) is trivial (resp., the MHS on $\mathfrak{f}(U,x)$ is pure of weight 0 and type $(0,0)$), it follows that $\mathfrak{f}(U,x)$ is a trivial Galois module (resp., is also pure of weight 0 and type $(0,0)$). Since $\mathfrak{p}(S,x) \to \mathfrak{f}(U,x)$ is a morphism (Galois, Hodge), it is strict with respect to the weight filtration $M_\bullet^U$. This and the strictness of the bracket of $\mathfrak{p}(S,x)$ with respect to both $W_\bullet$ and $M_\bullet^U$ imply that

$$M_{-2}^U H_1(\mathfrak{p}(S,x)) = \ker\{H_1(\mathfrak{p}(S,x)) \to H_1(\mathfrak{f}(U,x))\}.$$

Q.E.D.

§10. Handlebody Groups

Suppose that (S,D) is a stable decorated surface and that S is the boundary of a handlebody U. Define the *handlebody group* of (U,D) by

$$\Lambda_{U,D} = \pi_0 \operatorname{Diff}^+(U,D),$$

where each diffeomorphism acts trivially on D.[13] Griffiths [14], Suzuki [49], Luft [35], and Pitsch [43] have proved fundamental results about handlebody groups and found generating sets of Λ_U. For example, Griffiths [14] proved that homomorphisms

$$\Lambda_{U,x} \to \operatorname{Aut} \pi_1(U,x) \text{ and } \Lambda_U \to \operatorname{Out} \pi_1(U)$$

are surjective. Luft proved that the kernels of each of these is generated by twists on imbedded disks $(\mathbb{D}, \partial\mathbb{D}) \hookrightarrow (U,S)$.[14]

Restriction to the boundary defines a homomorphism

$$r_{U,D} : \Lambda_{U,D} \to \Gamma_{S,D}$$

It is straightforward to show that if $\widetilde{D}$ is a refinement of D, then $r_{U,\widetilde{D}}$ and $r_{U,D}$ induce an isomorphism

$$r : \ker\{\Lambda_{U,\widetilde{D}} \to \Lambda_{U,D}\} \xrightarrow{\simeq} \ker\{\Gamma_{U,\widetilde{D}} \to \Gamma_{U,D}\}. \tag{5}$$

[13] For a subgroup G of $\operatorname{Aut} D$, one can also define $\Lambda_{U,D}^G$ as we did for mapping class groups. However, we shall not need such groups.

[14] The twist on an imbedded disk is the isotopy class of a smoothing of the homeomorphism $(re^{i\theta}, t) \mapsto \left(re^{i\left(\theta+2\pi t\right)}, t\right)$ of a tubular neighbourhood $\mathbb{D} \times [0,1]$ of $\mathbb{D}$ in U. Its restriction to S is the Dehn twist about the loop $\partial\mathbb{D}$ in S.

For this reason, we will mainly restrict our attention to the cases where $\#D \leq 1$.

The following appears to be well-known to the experts. I am grateful to Alan Hatcher for communicating a proof.

Proposition 10.1. *If (S, D) is a stable surface, then the homomorphism $r_{U,D}$ is injective.*

Sketch of Proof. By the isomorphism (5), it suffices to prove the result when D is empty and $g \geq 2$ and when (g, n) is $(0, 3)$ and $(1, 1)$.

In genus 0 (with any number of points), the result follows directly from a result of Cerf [4]. The general case is proved by induction on g. Suppose $g \geq 1$. Suppose that $\phi \in \mathrm{Diff}^+ U$ is a diffeomorphism whose restriction to the boundary is isotopic to the identity. Then ϕ is isotopic to a diffeomorphism whose restriction to S is the identity. We will assume that this is the case. Now choose an imbedded disk $(\mathbb{D}, \partial\mathbb{D}) \subset (U, S)$. Let $(\mathbb{D}', \partial\mathbb{D}')$ be a disk imbedded in (U, S) parallel to and disjoint from $\mathbb{D}$. After altering ϕ by an isotopy fixing S if necessary, we may assume that the restriction of ϕ to $\mathbb{D}$ is transverse to $\mathbb{D}'$. Since U is an irreducible 3-manifold [28], $\mathbb{D}' \cup \phi(\mathbb{D}) \cup A$, where $A \subset S$ is the annulus between $\partial\mathbb{D}$ and $\partial\mathbb{D}'$, bounds a 3-ball. We can then deform ϕ by an isotopy fixing S so that the number of connected components of the complement in U of $\mathbb{D}' \cup \phi(\mathbb{D}) \cup A$ is reduced by one. We may therefore assume that $\mathbb{D}' \cup \phi(\mathbb{D}) \cup A$ bounds a ball. By further modifying ϕ by an isotopy fixing S, we may assume that ϕ fixes $\mathbb{D}$ pointwise. Now cut U apart along $\mathbb{D}$ to obtain a diffeomorphism ϕ' of a handlebody U' of genus one less whose restriction to $\partial U'$ is the identity. The result now follows by induction. Q.E.D.

The handlebody group Λ_U is bounded by the 0th term M_0^U of the relative weight filtration.

Lemma 10.2. *If (S, D) is a stable decorated surface and S bounds the handlebody U, then the image of $\Lambda_{U,D} \to \mathcal{G}_{S,D}$ lies in $M_0^U \mathcal{G}_{S,D}$.*

Proof. The proof would be straightforward if $\mathfrak{u}_{S,x} \to \mathrm{Der}\, \mathfrak{p}(S, x)$ were injective. Since this is not known, we need a direct proof. As noted above, Luft [35] proved that $\ker\{\Lambda_{U,x} \to \mathrm{Aut}\, \pi_1(U, x)\}$ is generated by twists on imbedded disks. If c is a simple closed curve in S that bounds an imbedded disk in U, then there is a pants decomposition γ of (S, D) that contains c where each $c' \in \gamma$ bounds in U. It follows from Theorem 8.3 that that the Dehn twist on c lies in $M_{-2}^U \mathcal{G}_{S,D} := M_{-2}^\gamma \mathcal{G}_{S,D}$.

Thus, to prove the result, it suffices to show that there are elements of $M_0 \Lambda_{U,x} := \Lambda_{U,x} \cap M_0^U \mathcal{G}_{S,x}$ whose images in $\mathrm{Aut}\, \pi_1(U, x)$ generate $\mathrm{Aut}\, \pi_1(U, x)$. To do this, we use elements of $\Lambda_{U,x}$ closely related to those used by Luft in [35]. That these generators lie in $M_0^U \mathcal{G}_{S,x}$ follows from Propositions 8.12 and 8.13.

Represent $\pi_1(U,x)$ as a free group $\langle a_1, \dots, a_g \rangle$, where each a_j is a simple closed curve on the boundary of S. Note that the automorphisms conjugate to $\phi_2 \in M_0 \Lambda_{U,x}$ constructed in Section 8.2 can be used to invert any generators a_j of $\pi_1(U,x)$ while leaving the remaining generators fixed. The automorphism $\psi \in M_0 \Lambda_{U,x}$ defined there can be used to define an automorphism that fixes all but two of the generators a_j and a_k and acts on them via

$$a_j \mapsto a_k^{-1} \text{ and } a_k \mapsto a_j^{-1}.$$

Composing this with the first kind of automorphism, we see that there are elements of $M_0 \Lambda_{U,x}$ that transpose any two of the generators a_j. We can therefore realize all permutations of the generators a_j by elements of $M_0 \Lambda_{U,x}$. Finally, the elements $\phi_3 \in M_0 \Lambda_{U,x}$ realize the automorphism that fixes a_j when $j > 2$ and satisfies

$$a_1 \mapsto a_2 \text{ and } a_2 \mapsto (a_1 a_2)^{-1}.$$

By a Theorem of Nielsen [41] (cf. [35]) these automorphisms of $\pi_1(U,x)$ generate $\operatorname{Aut} \pi_1(U,x)$. This completes the proof. Q.E.D.

When $\#D \geq 1$, the homomorphism $\Gamma_{S,D} \to \mathcal{G}_{S,D}$ is injective.

Theorem 10.3. *If (S,D) is a stable decorated surface that bounds the handlebody U, where $\#D = 1$, then*

(i) $\Lambda_{U,D} = \Gamma_{S,D} \cap M_0^U \mathcal{G}_{S,D}$;
(ii) $\ker\{\Lambda_{U,D} \to \operatorname{Aut} \pi_1(U,x)\} = \Lambda_{U,D} \cap M_{-2}^U \mathcal{G}_{S,D}$;
(iii) $\Lambda_{U,D} \cap M_{-2}^U \mathcal{U}_{S,D}$ *is generated by opposite twists on disjoint bounding pairs of imbedded disks* $(\mathbb{D}_j, \partial \mathbb{D}_j) \hookrightarrow (U,S)$ $(j = 1,2)$ *whose images avoid D.*

The second assertion is a consequence of a result of Griffith [14] and the third a restatement of a result of Pitsch [43, Prop. 6].

Proof. We prove the result when x is a point of S. The case where D is a non-zero tangent vector $v \in T_x S$ follows as $\mathfrak{g}_{S,v} \to \mathfrak{g}_{S,x}$ is strict with respect to $M_\bullet^U$ and the kernel is central and generated by a Dehn twist on a curve that bounds a disk in U, and therefore lies in M_{-2}^U.

Proposition 9.4 combined with the fact that the homomorphisms $\pi_1(S,x) \to \mathcal{P}(S,x)$ and $\pi_1(U,x) \to \mathcal{F}(U,x)$ are injective implies that the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(S,x) \cap M_{-2}^U \mathcal{P}(S,x) \arrow[r] \arrow[d] & \pi_1(S,x) \arrow[r] \arrow[d] & \pi_1(U,x) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & M_{-2}^U \mathcal{P}(S,x) \arrow[r] & \mathcal{P}(S,x) \arrow[r] & \mathcal{F}(U,x) \arrow[r] & 1 \end{tikzcd}$$

has exact rows. Since $\Gamma_{S,x}$ is a subgroup of $\operatorname{Aut}\pi_1(S,x)$, it follows that $\Gamma_{S,x}$ is a subgroup of $\operatorname{Aut}\mathcal{P}(S,x)$. Consequently $\Gamma_{S,x} \cap M_0^U \operatorname{Aut}\mathcal{P}(S,x)$ consists of those automorphisms of $\pi_1(S,x)$ that preserve $\ker\{\pi_1(S,x) \to \pi_1(U,x)\}$. By a result of Griffiths [14] this is $\Lambda_{U,x}$, so that

$$\Gamma_{S,x} \cap M_0^U \operatorname{Aut}\mathcal{P}(S,x) = \Lambda_{U,x}. \tag{6}$$

Since $\mathcal{P}(S,x) = M_0^U \mathcal{P}(S,x)$, and since $\pi_1(U,x) \to \mathcal{F}(U,x)$ is injective, the commutativity of the diagram above implies that

$$\Gamma_{S,x} \cap M_{-2}^U \operatorname{Aut}\mathcal{P}(S,x) \subseteq \ker\{\Lambda_{U,x} \to \operatorname{Aut}\pi_1(U,x)\}. \tag{7}$$

Since the homomorphism $\mathfrak{g}_{S,x} \to \operatorname{Der}\mathfrak{p}(S,x)$ preserves the filtration $M_\bullet^U$, it follows that for all k we have

$$\Gamma_{S,x} \cap M_k^U \mathcal{G}_{S,x} \subseteq \Gamma_{S,x} \cap M_k^U \operatorname{Aut}\mathcal{P}(S,x). \tag{8}$$

Lemma 10.2 implies that $\Lambda_{U,x} \subseteq \Gamma_{S,x} \cap M_0^U \mathcal{G}_{S,x}$. The first assertion follows by combining this with the inclusions (6) and (8) with $k = 0$.

By a result of Luft [35], the kernel of $\Lambda_{U,x} \to \operatorname{Aut}\pi_1(U,x)$ is generated by Dehn twists on simple closed curves in S that bound a disk imbedded in U. But these lie in $M_{-2}^U \mathcal{G}_{S,x}$ by Theorem 8.3. Therefore

$$\ker\{\Lambda_{U,x} \to \operatorname{Aut}\pi_1(U,x)\} \subseteq \Gamma_{S,x} \cap M_{-2}^U \mathcal{G}_{S,x}.$$

The second assertion follows by combining this with the inclusions (7) and (8) with $k = -2$.

The final assertion follows from this and a result of Pitsch [43, Prop. 6]. Q.E.D.

Corollary 10.4. *There is a natural injective homomorphism*

$$\operatorname{Aut}\pi_1(U,x) \hookrightarrow \operatorname{Gr}_0^{M^U} \mathcal{G}_{S,D}.$$

This homomorphism induces homomorphisms on the relative completion of $\operatorname{Aut}^+\pi_1(U,x)$ and $\operatorname{Out}^+\pi_1(U,x)$. Surprisingly, these are not surjective. Equivalently, the injection in the previous corollary is not Zariski dense.

Proposition 10.5. *If $g \geq 3$, then the induced homomorphisms*

$$\mathfrak{a}_n \to \mathrm{Gr}_0^{M^U} \mathfrak{g}_{S,x} \text{ and } \mathfrak{ia}_n \to \mathrm{Gr}_0^{M^U} \mathfrak{g}_S$$

are not surjective.

Sketch of Proof. Denote $H_1(S)$ by H and $H_1(U)$ by A. Denote the relative weight filtration $M_\bullet^U$ by $M_\bullet$. Denote the kernel of $H \to A$ by B. Then $A = \mathrm{Gr}_0^M H$ and $B = \mathrm{Gr}_{-2}^M H$. Since $\mathrm{Gr}_0^W \mathfrak{g}_{S,x} = \mathfrak{sp}(H) \cong S^2 H$, it follows that

$$\mathrm{Gr}_0^M \mathrm{Gr}_0^W \mathfrak{g}_{S,x} \cong A \otimes B \cong \mathrm{End}(A) \cong \mathfrak{gl}(A).$$

There are natural $\mathfrak{sp}(H)$-equivariant isomorphisms

$$H_1(\mathfrak{u}_{S,x}) \cong \mathrm{Gr}_{-1}^W \mathfrak{g}_{S,x} \cong H_1(T_{S,x}) \cong \Lambda^3 H$$

given by the Johnson homomorphism and general results in [18]. The exactness properties of $\mathrm{Gr}_\bullet^M$ and $\mathrm{Gr}_\bullet^W$ imply that there are $\mathfrak{gl}(A)$-equivariant isomorphisms

$$\mathrm{Gr}_0^M \mathrm{Gr}_{-1}^W \mathfrak{g}_{S,x} \cong B \otimes \Lambda^2 A \cong \mathrm{Hom}(A, \mathbb{L}_2(A)),$$

where $\mathbb{L}_m(A)$ denotes the mth graded quotient of the free Lie algebra generated by A. Moreover, the mapping $IA_n \to \mathrm{Gr}_0^M \mathcal{U}_{S,x}$ induces Magnus' isomorphism

$$H_1(IA_n) \to \mathrm{Hom}(A, \mathbb{L}_2(A)).$$

By [18, (10.1),§11], the second weight graded quotient of $\mathfrak{g}_{S,x}$ is the sum of the $\mathrm{Sp}(H)$-modules that corresponds to the partitions $[2,2]$ and $[1,1]$. A straightforward linear algebra computation shows that, as $\mathfrak{gl}(A)$-modules,

$$\mathrm{Gr}_0^M \mathrm{Gr}_{-2}^W \mathfrak{g}_{S,x} \cong B \otimes \mathbb{L}_3(A) \cong \mathrm{Hom}(A, \mathbb{L}_3(A)).$$

Alternatively, it is isomorphic to the kernel of the natural surjection $S^2\Lambda^2 H \to \Lambda^4 H$ minus a copy of the trivial representation. The image of $[\mathfrak{ia}_n, \mathfrak{ia}_n]$ in this group is a quotient of

$$\Lambda^2 H_1(IA_n) = \Lambda^2 \mathrm{Hom}(A, \mathbb{L}_2(A)).$$

Since $S^2 A$ is a summand of $\mathrm{Hom}(A, \mathbb{L}_3(A))$ but not of this group, the homomorphism $\mathfrak{ia}_n \to \mathrm{Gr}_0^M \mathfrak{u}_{S,x}/W_{-3}$ is not surjective. The result follows. Q.E.D.

This result shows that the relative weight filtration of $\mathcal{G}_{S,x}$ is not simply obtained by taking the Zariski closure of a filtration of $\Gamma_{S,x}$.

At first glance, this result appears to contradict Theorem 10.3. and the fact that the image of $T_{S,x} \to \mathcal{U}_{S,x}$ is Zariski dense. However, these results simply say that given $n \geq 1$ and a $\mathbb{Q}$-rational element ϕ of $M_0\mathcal{U}_{S,x}/W_{-n}$, there exists $\psi \in T_{S,x}$ and a positive integer m such that

$$\phi^m \equiv \psi \bmod W_{-n}\mathcal{U}_{S,x}.$$

The previous two results imply that when $n > 2$, it is not always possible to choose ψ to lie in $\Lambda_{U,x} = \Gamma_{S,x} \cap M_0\mathcal{G}_{S,x}$.

Theorem 10.3 and Proposition 8.5 yield the following strengthening of Theorem 9.2. It says that the different ways of writing (S,x) as the boundary of a handlebody is faithfully represented in the set of relative weight filtrations of $\mathfrak{g}_{S,x}$.

Corollary 10.6. *For two pants decompositions γ_1 and γ_2 of a stable decorated surface (S,D), where $\#D = 1$, the following are equivalent:*

(i) *γ_1 and γ_2 are connected by A-moves;*
(ii) *$U^{\gamma_1} = U^{\gamma_2}$;*
(iii) *the associated relative weight filtrations $M_\bullet^{\gamma_1}$ and $M_\bullet^{\gamma_2}$ of $\mathfrak{g}_{S,D}$ are equal.*

Proof. Theorem 9.2 gives the equivalence of (i) and (ii). Proposition 8.8 established that (i) implies (iii). It remains to prove that (iii) implies (ii). We will show that not (ii) implies not (iii).

Set $U_j = U^{\gamma_j}$ and $M_\bullet^{\gamma_j} = M_\bullet^{U_j}$. There exists $\phi \in \Gamma_{S,D}$ that extends to a diffeomorphism $\tilde{\phi} : U_1 \to U_2$. Then

$$\Lambda_{U_2} = \phi \Lambda_{U_1} \phi^{-1} \text{ and } M_\bullet^{U_2} = \mathrm{Ad}(\phi) M_\bullet^{U_1}.$$

If $U_1 \neq U_2$, then $\phi \notin \Lambda_{U_1}$. Theorem 10.3 implies that $\phi \notin M_0^{U_1}\mathcal{G}_{S,D}$. We will prove the result by showing that ϕ does not normalize $M_0^{U_1}\mathcal{G}_{S,D}$.

Since $\mathcal{G}_{S,D}$ is connected, it suffices to prove the Lie algebra version: if $X \in \mathfrak{g}_{S,D}$ and $X \notin M_0^{U_1}\mathfrak{g}_{S,D}$, then X does not normalize $M_0^{U_1}\mathfrak{g}_{S,D}$. But this follows directly from Proposition 8.5. Q.E.D.

§11. Extending Diffeomorphisms to Handlebodies

In this section we give an application to the problem of bounding the subset of elements of $\Gamma_{S,D}$ consisting of mapping classes that extend to some handlebody. Similar results have been obtained independently by Jamie Jorgensen [30].

View $\mathcal{G}_{S,D}$ as a proalgebraic variety over $\mathbb{Q}$. It is filtered by its weight filtration

$$\mathcal{G}_{S,D} = W_0\mathcal{G}_{S,D} \supseteq W_{-1}\mathcal{G}_{S,D} \supseteq W_{-2}\mathcal{G}_{S,D} \supseteq \cdots$$

where $W_{-1}\mathcal{G}_{S,D} = \mathcal{U}_{S,D}$ and $W_{-m}\mathcal{G}_{S,D}$ is the mth term of the lower central series of $\mathcal{U}_{S,D}$. Recall from [18] that when $g \geq 3$ and $m \neq 2$, $\mathrm{Gr}^W_{-m}\,\mathcal{U}_{S,D}$ is isomorphic to the mth graded quotient of the lower central series of the Torelli group $T_{S,D}$ tensored with $\mathbb{Q}$.

Write S as the boundary of a handlebody U. Then the set of elements of $\Gamma_{S,D}$ that extend across some handlebody with boundary S is

$$C := \bigcup_{\phi\in\Gamma_{S,D}} \phi\Lambda_{U,D}\phi^{-1}.$$

For all $m \geq 1$, set $C_m = C \cap W_{-m}\mathcal{G}_{S,D}$. Denote the Zariski closure of C_m in $\mathcal{G}_{S,D}$ by X_m.

Theorem 11.1. *If (S, D) is a stable decorated surface, then X_m is a proper subvariety of $W_{-m}\mathcal{G}_{S,D}$ for all*

$$m \geq \begin{cases} 4 & \textit{when } g = 3; \\ 2 & \textit{when } g = 4, 5, 6; \\ 1 & \textit{when } g \geq 7. \end{cases}$$

In some sense, this theorem says that most elements of $W_{-m}\Gamma_{S,D}$ do not extend to any handle body.

Denote the Zariski closure of $\Lambda_{U,D}$ in $\mathcal{G}_{S,D}$ by $\mathcal{L}_{U,D}$ and its intersection with $W_{-m}\mathcal{U}_{S,D}$ by $W_{-m}\mathcal{L}_{U,D}$. Since $\Gamma_{S,D}$ is Zariski dense in $\mathcal{G}_{S,D}$, C_m is contained in the Zariski closure of the image of the map

$$F : \mathcal{G}_{S,D} \times W_{-m}\mathcal{L}_{U,D} \to \mathcal{G}_{S,D}$$

defined by $F(g, \lambda) = g\lambda g^{-1}$.

To prove the result we show that the image of C_m in $\mathrm{Gr}^W_{-m}\,\mathcal{G}_{S,D}$ is contained in a proper subvariety. Since $\Lambda_{U,D}$ is contained in $M_0^U\mathcal{G}_{S,D}$, $W_{-m}\mathcal{L}_{U,D}$ is a subgroup of $M_0^U W_{-m}\mathcal{G}_{S,D}$. Consequently, the image of C_m in $\mathrm{Gr}^W_{-m}\,\mathcal{G}_{S,D}$ is contained in the Zariski closure of the image of the map

$$\mathrm{Gr}^W_0\,\mathcal{G}_{S,D} \times M_0^U\,\mathrm{Gr}^W_{-m}\,\mathcal{G}_{S,D} \to \mathrm{Gr}^W_{-m}\,\mathcal{G}_{S,D}.$$

induced by conjugation.

The following is an immediate consequence of a theorem of Chevalley [5], which can be found in exercises 3.18 and 3.19 of [25, Chap. II, sect. 3].

Lemma 11.2. *Suppose that X is a quasi-projective variety over a field and that Y is a closed subvariety. If $G \times X \to X$ is the action of an algebraic group on X, then the image $G \cdot Y$ of the restricted action $G \times Y \to X$ is a constructable subset of X whose Zariski closure in X has dimension*

$$\dim \overline{G \cdot Y} = \dim Y + \dim G - \dim G_Y$$

where $G_Y = \{g \in G : g(Y) \subseteq Y\}$. $\square$

Recall that $\operatorname{Gr}_0^W \mathcal{G}_{S,D} \cong \operatorname{Sp}(H)$ where $H = H_1(S)$. We apply the Lemma to the adjoint action of $G = \operatorname{Sp}(H)$ on

$$X = \operatorname{Gr}_{-m}^W \mathcal{G}_{S,D} \cong \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D} \text{ where } Y = M_0^U \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}.$$

Proposition 8.5 implies that $G_Y = M_0^U \operatorname{Sp}(H)$. Applying the Lemma, and using the fact that $\mathfrak{sp}(H) = M_2^U \mathfrak{sp}(H)$, we see that the codimension of the closure of the $\operatorname{Sp}(H)$ orbit of $M_0^U \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}$ in $\operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}$ satisfies

$$\begin{aligned}
&\operatorname{codim} \overline{\operatorname{Sp}(H) \cdot M_0^U \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}} \\
&\qquad = \dim \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}/M_0^U - \dim \operatorname{Sp}(H)/M_0 \operatorname{Sp}(H) \\
&\qquad\qquad \geq \dim \operatorname{Gr}_2^M \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D} - \dim \operatorname{Gr}_2^M \mathfrak{sp}(H).
\end{aligned}$$

It remains to show this is positive for all m in the statement of the theorem. First, since $\operatorname{Gr}_2^M \mathfrak{sp}(H)$ is the symmetric square of a maximal isotropic subspace of H, it has dimension $g(g+1)/2$.

We use representation theory to find a lower bound for the other term. Each $\operatorname{Gr}_k^M \operatorname{Gr}_m^W \mathfrak{g}_{S,D}$ is a $\operatorname{Gr}_0^M \operatorname{Gr}_0^W \mathfrak{g}_{S,D}$-module. Recall from the proof of Corollary 10.6 that $\operatorname{Gr}_0^M \operatorname{Gr}_0^W \mathfrak{g}_{S,D}$ is isomorphic to $\mathfrak{gl}_g$, so that its irreducible representations are given by Young diagrams with $\leq g$ rows. These are the same Young diagrams that parametrize the irreducible $\mathfrak{sp}(H)$-modules, where $H = H_1(S)$.

Proposition 11.3. *If $g \geq 3$ and $m > 1$, then $\operatorname{Gr}_2^M \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}$ contains the $\mathfrak{gl}_g$-module corresponding to the partition $[k,k]$ when $m = 2k$ and $[k,k,1]$ when $m = 2k-1$.*

Proof. Results of Oda [42] and Asada-Nakamura [1] imply that if $m > 0$, then the $\operatorname{Sp}(H)$-module $\operatorname{Gr}_{-m}^W \mathfrak{u}_{S,D}$ contains the representation $[k,k]$ when $m = 2k$ and $[k,k,1]$ when $m = 2k-1$. If we take $\operatorname{Gr}_2^M \mathfrak{sp}(H)$ to be positive roots of $\mathfrak{sp}(H)$, then the highest weight vectors of each of these representations lies in $\operatorname{Gr}_2^M \operatorname{Gr}_{-m}^W \mathfrak{g}_{S,D}$. Since $\mathfrak{gl}_g$ is a subalgebra of $\mathfrak{sp}(H)$ with the same Cartan subalgebra, the $\mathfrak{gl}_g$-submodule of $\mathrm{Gr}_2^M \mathrm{Gr}_{-m}^W \mathfrak{g}_{S,D}$ generated by v will correspond to the same partition. Q.E.D.

Using the formula [13, (6.4)], when $k \geq 2$ we have:

$$\dim V_{[k,k]} = \frac{(g-1)(g+k-1)\prod_{j=0}^{k-2}(g+j)^2}{k!(k+1)!}$$

$$\dim V_{[k,k,1]} = \frac{(g-1)(g-2)(g+k-1)\prod_{j=0}^{k-2}(g+j)^2}{(k-1)!(k+2)!}$$

where V_λ denotes the $\mathfrak{gl}_g$-module corresponding to the partition λ. These dimensions increase monotonically with k. The proof is completed by an elementary computation, which is left to the reader.

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