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# Another canonical compactification of the moduli space of abelian varieties

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**Abstract.**

We construct a canonical compactification $SQ_{g,K}^{\mathrm{toric}}$ of the moduli space $A_{g,K}$ of abelian varieties over $\mathbf{Z}[\zeta_N, 1/N]$ by adding certain reduced singular varieties along the boundary of $A_{g,K}$, where $K$ is a symplectic finite abelian group, $N$ is the maximal order of elements of $K$, and $\zeta_N$ is a primitive $N$-th root of unity. In [18] a canonical compactification $SQ_{g,K}$ of $A_{g,K}$ was constructed by adding possibly non-reduced GIT-stable (Kempf-stable) degenerate abelian schemes. We prove that there is a canonical bijective finite birational morphism $\mathrm{sq} : SQ_{g,K}^{\mathrm{toric}} \to SQ_{g,K}$. In particular, the normalizations of $SQ_{g,K}^{\mathrm{toric}}$ and $SQ_{g,K}$ are isomorphic.

Contents



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## §1. Introduction

In [18] a canonical compactification $SQ_{g,K}$ of the moduli space $A_{g,K}$ of abelian varieties with level structure was constructed by applying geometric invariant theory [17]. It is a compactification of $A_{g,K}$ by all Kempf-stable degenerate abelian schemes, that is, those degenerate abelian schemes whose Hilbert points have closed SL-orbits in the semi-stable loci. However some of the Kempf-stable degenerate abelian schemes are *non-reduced* in contrast with Deligne–Mumford stable curves. See [20] for a non-reduced Kempf-stable degenerate abelian scheme.

The purpose of this article is to construct another canonical compactification $SQ_{g,K}^{\mathrm{toric}}$ of $A_{g,K}$ by adding to $A_{g,K}$ certain *reduced* singular degenerate abelian schemes instead of *non-reduced* Kempf-stable ones. The new compactification $SQ_{g,K}^{\mathrm{toric}}$ is very similar to $SQ_{g,K}$. In fact, their normalizations are canonically isomorphic (see Section 12). The compactifications are, as functors, the same if $g \leq 4$, and different if $g \geq 8$ (or maybe if $g \geq 5$ because it is believed that there are non-reduced Kempf-stable degenerate abelian schemes of dimension $g$ for any $g \geq 5$). An advantage of $SQ_{g,K}^{\mathrm{toric}}$ is that the *reduced* degenerate abelian schemes on the boundary $SQ_{g,K}^{\mathrm{toric}} \setminus A_{g,K}$ are much simpler than those Kempf-stable ones lying on the boundary $SQ_{g,K} \setminus A_{g,K}$. See also Alexeev [1] for related topics.

Let $R$ be a complete discrete valuation ring and $k(\eta)$ the fraction field of $R$. Given an abelian variety $(G_\eta, \mathcal{L}_\eta)$ over $k(\eta)$ with an ample line bundle $\mathcal{L}_\eta$, we have Faltings–Chai degeneration data for it by a finite base change if necessary. In [18] for the Faltings–Chai degeneration data, we constructed two natural $R$-flat projective degenerating families $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ of abelian varieties with generic fiber isomorphic to $(G_\eta, \mathcal{L}_\eta)$. The family $(Q, \mathcal{L})$ is the most naive choice with $\mathcal{L}$ an ample line bundle, while the family $(P, \mathcal{L})$ with $\mathcal{L}$ $(= \mathcal{L}_P)$ the pull back of $\mathcal{L}$ $(= \mathcal{L}_Q)$ on $Q$ is the normalization of $(Q, \mathcal{L})$ after a certain finite minimal base change so that the closed fiber $P_0$ of $P$ may be reduced.

We call the closed fiber $(P_0, \mathcal{L}_0)$ of $(P, \mathcal{L})$ a torically stable quasi-abelian scheme (abbr. TSQAS), while we call the closed fiber $(Q_0, \mathcal{L}_0)$ of $(Q, \mathcal{L})$ a projectively stable quasi-abelian scheme (abbr. PSQAS) [18].

Let $(K, e_K)$ be a finite symplectic abelian group. Since we have $K \simeq \oplus_{i=1}^{g}((\mathbf{Z}/e_i\mathbf{Z}) \oplus \mu_{e_i})$ for some positive integers $e_i$ such that $e_i | e_{i+1}$, we define $e_{\min}(K) = e_1$ and $e_{\max}(K) = e_g$. Let $N = e_{\max}(K)$. The Heisenberg group $G(K)$ is, by definition, a central extension of $K$ by the group $\mu_N$ of all $N$-th roots of unity. The classical level-$K$ structures on abelian varieties are generalized as level-$G(K)$ structures on

TSQASes. The group scheme $G(K)$ has an essentially unique irreducible representation of weight one over $\mathbf{Z}[\zeta_N, 1/N]$. In [18] this fact played a substantial role in constructing a canonical compactification $SQ_{g,K}$ of the moduli space $A_{g,K}$ of abelian varieties with (non-classical and non-commutative) level-$K$ structure. We note that, for any closed field $k$ over $\mathbf{Z}[\zeta_N, 1/N]$, $A_{g,K}(k)$ is the same as the set of all isomorphism classes of abelian varieties with level-$K$ structure in the classical sense.

The following is the main theorem of the present article.

**Theorem.** *If $e_{\min}(K) \geq 3$, the functor of $g$-dimensional torically stable quasi-abelian schemes with level-$G(K)$ structure over reduced base algebraic spaces has a complete separated reduced-coarse (hence reduced) moduli algebraic space $SQ_{g,K}^{\text{toric}}$ over $\mathbf{Z}[\zeta_N, 1/N]$. Moreover, there is a canonical bijective finite birational morphism $\mathrm{sq} : SQ_{g,K}^{\text{toric}} \to SQ_{g,K}$. In particular, the normalization of $SQ_{g,K}^{\text{toric}}$ is isomorphic to that of $SQ_{g,K}$.*

Here is an outline of our article. In Section 2, we recall from [18] a couple of basic facts about degenerating families of abelian varieties. In Section 3, we show how to recover $P_0$ from $Q_0$, and $Q$ from $P$. In Section 4, first we define Heisenberg group schemes $G(K)$ and $\mathcal{G}(K)$, finite or infinite, then we discuss in detail the relation between level-$G(K)$ structures and $G(K)$-linearizations. Moreover we recall irreducible $G(K)$-modules of weight one, which will play a substantial role in compactifying the moduli. We notice that the finite Heisenberg group scheme $G(K)$ acts on $\Gamma(P_0, \mathcal{L}_0^m)$ with weight one if $m \equiv 1 \mod N$.

In Section 5, we define level-$G(K)$ structures on TSQASes $(P_0, \mathcal{L}_0)$ or their family, and then define the functor $\mathcal{SQ}_{g,K}^{\text{toric}}$ of TSQASes. In Section 6, we also give a precise definition of the functor $\mathcal{SQ}_{g,K}$ of PSQASes, using [20]. In Section 7, we discuss rigid $\rho$-structures for any irreducible representation $\rho$. In Section 8, we recall from [18] the stable reduction theorem for TSQASes with rigid level-$G(K)$ structure. In Sections 9, 10 and 11, we prove existence of the reduced-coarse moduli $SQ_{g,K}^{\text{toric}}$. In the course of the proof, we characterize TSQASes by the conditions (i)-(x) in Sections 9.3, 9.5 and 9.6. In Section 12, we prove that there is a canonical bijective finite birational morphism from $SQ_{g,K}^{\text{toric}}$ to $SQ_{g,K}$ extending the identity of $A_{g,K}$.

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## §2. Degenerating families of abelian varieties

The purpose of this section is to recall basic facts about degenerating families of abelian varieties. To minimize the article we try to keep the same notation as in [18].

### 2.1. Grothendieck's stable reduction

Let $R$ be a complete discrete valuation ring, $I$ the maximal ideal of $R$ and $S = \mathrm{Spec}\, R$. Let $\eta$ be the generic point of $S$, $k(\eta)$ the fraction field of $R$ and $k(0) = R/I$ the residue field.

Suppose we are given a polarized abelian variety $(G_\eta, \mathcal{L}_\eta)$ of dimension $g$ over $k(\eta)$ such that $\mathcal{L}_\eta$ is symmetric, ample and rigidified (that is trivial) along the unit section. Then by Grothendieck's stable reduction theorem [4], $(G_\eta, \mathcal{L}_\eta)$ can be extended to a polarized semiabelian $S$-scheme $(G, \mathcal{L})$ with $\mathcal{L}$ a rigidified relatively ample invertible sheaf on $G$ as the connected Néron model of $G_\eta$ by taking a finite extension of $k(\eta)$ if necessary. The closed fiber $G_0$ is a semiabelian scheme over $k(0)$, namely an extension of an abelian variety $A_0$ by a split torus $T_0$.

*From now on, we restrict ourselves to the totally degenerate case, that is, the case when $A_0$ is trivial*, because by [18] there is no essentially new difficulty when we consider the case when $A_0$ is nontrivial. Hence we assume that $G_0$ is a split $k(0)$-torus. Let $\lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta$ be the polarization (epi)morphism. By the universal property of the (connected) Néron model $G^t$ of $G^t_\eta$, we have an epimorphism $\lambda : G \to G^t$ extending $\lambda(\mathcal{L}_\eta)$. Hence the closed fiber of $G^t$ is also a split $k(0)$-torus.

Let $S_n = \mathrm{Spec}\, R/I^{n+1}$ and $G_n = G \times_S S_n$. Associated to $G$ and $\mathcal{L}$ are the formal scheme $G_{\mathrm{for}} = \varprojlim G_n$ and an invertible sheaf $\mathcal{L}_{\mathrm{for}} = \varprojlim (\mathcal{L} \otimes R/I^{n+1})$. By our assumption that $G_0$ is a $k(0)$-split torus, $G_n$ turns out to be a multiplicative group scheme for every $n$ by [5, p. 7]. Thus the scheme $G_{\mathrm{for}}$ is a formal split $S$-torus. Similarly $G^t_{\mathrm{for}}$ is a formal split $S$-torus. Let $X := \mathrm{Hom}_{\mathbf{Z}}(G_{\mathrm{for}}, (\mathbf{G}_{m,S})_{\mathrm{for}})$, $Y := \mathrm{Hom}_{\mathbf{Z}}(G^t_{\mathrm{for}}, (\mathbf{G}_{m,S})_{\mathrm{for}})$ and $\widetilde{G} := \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{G}_{m,S})$, $\widetilde{G}^t = \mathrm{Hom}(Y, \mathbf{G}_{m,S})$. Then $\widetilde{G}$ (resp. $\widetilde{G}^t$) algebrizes $G_{\mathrm{for}}$ (resp. $G^t_{\mathrm{for}}$). The morphism $\lambda : G \to G^t$ induces an injective homomorphism $\phi : Y \to X$ and an algebraic epimorphism $\widetilde{\lambda} : \widetilde{G} \to \widetilde{G}^t$. For simplicity we identify the injection $\phi : Y \to X$ with the inclusion $Y \subset X$.

## 2.2. Fourier expansions

In the totally degenerate case, $G_{\text{for}}$ (resp. $\widetilde{G}$) is a formal split $S$-torus (resp. a split $S$-torus). We choose the coordinate $w^x$ of $\widetilde{G}$ satisfying $w^x w^y = w^{x+y}$ $(\forall x, y \in X)$. Since $\mathcal{L}_{\text{for}}$ is trivial on $G_{\text{for}}$, we have

$$\Gamma(G_\eta, \mathcal{L}_\eta) = \Gamma(G, \mathcal{L}) \otimes k(\eta) \hookrightarrow \Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \otimes k(\eta) \hookrightarrow \prod_{x \in X} k(\eta) \cdot w^x.$$

Therefore, any element $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$ can be written as a formal Fourier series $\theta = \sum_{x \in X} \sigma_x(\theta) w^x$ with $\sigma_x(\theta) \in k(\eta)$, which converges $I$-adically.

**Theorem 2.3.** [Faltings–Chai90] *Let $k(\eta)^\times = k(\eta) \setminus \{0\}$. There exists a function $a : Y \to k(\eta)^\times$ and a bimultiplicative function $b : Y \times X \to k(\eta)^\times$ with the following properties:*

(1) $b(y, x) = a(x+y) a(x)^{-1} a(y)^{-1}$, $a(0) = 1$ $(\forall x, y \in Y)$,
(2) $b(y, z) = b(z, y) = a(y+z) a(y)^{-1} a(z)^{-1}$ $(\forall y, z \in Y)$,
(3) $b(y, y) \in I$ $(\forall y \neq 0)$, *and for every* $n \geq 0$, $a(y) \in I^n$ *for almost all* $y \in Y$,
(4) $\Gamma(G_\eta, \mathcal{L}_\eta)$ *is identified with the* $k(\eta)$ *vector subspace of formal Fourier series* $\theta = \sum_{x \in X} \sigma_x(\theta) w^x$ *which satisfy the relations* $\sigma_{x+y}(\theta) = a(y) b(y, x) \sigma_x(\theta)$ *and* $\sigma_x(\theta) \in k(\eta)$ $(\forall x \in X, \forall y \in Y)$.

## 2.4. The bilinear form $B(x, y)$ on $X \times X$

By taking a finite base change of $S$ if necessary, the functions $b$ and $a$ can be extended respectively to $X \times X$ and $X$ so that the previous relations between $b$ and $a$ are still true on $X \times X$. Let $R^\times = R \setminus \{0\}$ and $k(0)^\times = k(0) \setminus \{0\}$. Then we define integer-valued functions $A : X \to \mathbf{Z}$, $B : X \times X \to \mathbf{Z}$ and $\bar{b}(y, x) \in R^\times$, $\bar{a}(y) \in R^\times$ by

$$\begin{aligned} B(y, x) &= val_s(b(y, x)), \quad dA(\alpha)(x) = B(\alpha, x) + r(x)/2, \\ A(x) &= val_s(a(x)) = B(x, x)/2 + r(x)/2, \\ b(y, x) &= \bar{b}(y, x) s^{B(y,x)}, \quad a(x) = \bar{a}(x) s^{(B(x,x)+r(x))/2} \end{aligned}$$

for some $r \in \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$, where $B$ is positive definite by Theorem 2.3. Let $a_0 = \bar{a} \mod I$ and $b_0 = \bar{b} \mod I$, where $a_0(x), b_0(y, x) \in k(0)^\times$.

## 2.5. Delaunay cells and Delaunay decompositions

Let $X$ be a lattice of rank $g$, $X_{\mathbf{R}} = X \otimes \mathbf{R}$, and let $B : X \times X \to \mathbf{Z}$ be a positive definite symmetric integral bilinear form, which determines a distance $\| \ \|_B$ on $X_{\mathbf{R}}$ by $\|x\|_B := \sqrt{B(x, x)}$ $(x \in X_{\mathbf{R}})$. For any $\alpha \in X_{\mathbf{R}}$ we say that $a \in X$ is $\alpha$-nearest if $\|a - \alpha\|_B = \min\{\|b - \alpha\|_B ; b \in X\}$.

For an $\alpha \in X_{\mathbf{R}}$, we define *a Delaunay cell* $\sigma$ to be the closed convex hull of all lattice elements which are $\alpha$-nearest. Two different $\alpha$ and $\alpha'$ could give the same $\sigma$. If $\alpha \in \sigma$ satisfies the condition $\|a-\alpha\|_B = \min\{\|b-\alpha\|_B; b \in X\}$ for any $a \in \sigma \cap X$, then we call $\alpha$ *the center of* $\sigma$, which we denote by $\alpha(\sigma)$. The center of $\sigma$ is uniquely determined by $\sigma$.

All the Delaunay cells constitute a locally finite decomposition of $X_{\mathbf{R}}$, which we call *the Delaunay decomposition* $\mathrm{Del}_B$. Let $\mathrm{Del} := \mathrm{Del}_B$, and $\mathrm{Del}(c)$ the set of all the Delaunay cells containing $c \in X$. For $\sigma \in \mathrm{Del}(c)$, we define $C(c,\sigma) := c + C(0, -c+\sigma)$, and define $C(0,-c+\sigma)$ to be the cone spanned over $\mathbf{R}^+$ by all $a-c$, $(a \in \sigma \cap X)$. See [18, p. 662].

### 2.6. The semi-universal covering $\widetilde{Q}$

Let $k(\eta)$ be the fraction field of $R$ as before, and $k(\eta)[X]$ the group algebra of the additive group $X$ over $k(\eta)$. Let

$$k(\eta)[X][\vartheta]$$

be the graded algebra over $k(\eta)[X]$ with $\vartheta$ indeterminate of degree one, where by definition $\deg(z) = 0$ for any $z \in k(\eta)[X]$. We denote by $w^x$ the generator of $k(\eta)[X]$ corresponding to $x \in X$, where $w^x \cdot w^y = w^{x+y}$ for $x, y \in X$. Then we define a graded subalgebra $\widetilde{R}$ of $k(\eta)[X][\vartheta]$ by

$$\widetilde{R} := R[a(x)w^x\vartheta; x \in X] = R[\xi_x\vartheta; x \in X],$$

where $\xi_x := s^{B(x,x)/2+r(x)/2}w^x$, and $a(x)$ the $a$-part of the degeneration data in Theorem 2.3.

Let $\widetilde{Q} := \mathrm{Proj}(\widetilde{R})$ and $\widetilde{P}$ the normalization of $\widetilde{Q}$. For $y \in Y$, we define an action $S_y$ on $\widetilde{Q}$ by

$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta,$$

which induces a natural action on $\widetilde{P}$, denoted by the same $S_y$. By $\widetilde{\mathcal{L}}$ we denote $O_{\mathrm{Proj}}(1)$ on $\widetilde{Q}$ as well as its pullback to $\widetilde{P}$.

**Theorem 2.7.** *Let $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})$ (resp. $(\widetilde{Q}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})$) be the formal completion of $(\widetilde{P}, \widetilde{\mathcal{L}})$ (resp. $(\widetilde{Q}, \widetilde{\mathcal{L}})$). Then*

(1) *The quotient formal schemes $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ and $(\widetilde{Q}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ are flat projective formal $S$-schemes.*

(2) *There exist flat projective $S$-schemes $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ such that their formal completions $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ and $(Q_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ along the closed fibers are respectively isomorphic to the quotient formal schemes $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ and $(\widetilde{Q}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$.*

(3) *$P$ is the normalization of $Q$.*

*Proof.* This follows from [3, III, 5.4.5]. See also [2] and [18]. Q.E.D.

### 2.8. A torically stable quasi-abelian scheme $(P_0, \mathcal{L}_0)$

Let $\mathrm{Del}(P_0)$ be the Delaunay decomposition corresponding to $P_0$. By taking a finite base change of $S$ if necessary, we may assume that $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any Delaunay cell $\sigma \in \mathrm{Del}(P_0)$. By [2] this implies that $P_0$ *is reduced.* We call the closed fiber $(P_0, \mathcal{L}_0)$ of $(P, \mathcal{L})$ a torically stable quasi-abelian scheme (abbr. a TSQAS) over $k(0) := R/I$.

In what follows, we always assume that $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any $\sigma \in \mathrm{Del}(\widetilde{P}_0)$. Hence $P_0$ is reduced.

We quote two theorems from [2] and [18].

**Theorem 2.9.** *Let $P_0$ (resp. $\widetilde{P}_0$) be the closed fiber of $P$ (resp. $\widetilde{P}$). Let $\sigma$ and $\tau$ be Delaunay cells in $\mathrm{Del}(\widetilde{P}_0)$.*

(1) *For each $\sigma \in \mathrm{Del}(\widetilde{P}_0)$, there exists a subscheme $O(\sigma)$ of $\widetilde{P}_0$, which is a torus of dimension $\dim_{\mathbf{R}} \sigma$ over $k(0)$,*
(2) *$\tau \subset \sigma$ iff $O(\tau)$ is contained in $\overline{O(\sigma)}$, the closure of $O(\sigma)$ in $P_0$, and $\overline{O(\sigma)}$ is the union of all $O(\tau)$ with $\tau \subset \sigma$, $\tau \in \mathrm{Del}(\widetilde{P}_0)$,*
(3) *$P_0 = \bigcup_{\sigma \in \mathrm{Del}(\widetilde{P}_0) \bmod Y} O(\sigma)$.*

**Theorem 2.10.** *Let $P_0$ be the closed fiber of $P$, and $n > 0$. Then*

(1) *$h^0(P_0, \mathcal{L}_0^n) = [X:Y]n^g$, $h^i(P_0, \mathcal{L}_0^n) = 0$ $(i > 0)$, and*

$$\Gamma(P_0, \mathcal{L}_0) = \left\{ \sum_{x \in X} c(x)\xi_x ; \begin{matrix} c(x+y) = b_0(y,x)a_0(y)c(x) \\ c(x) \in k, \forall x \in X, \forall y \in Y \end{matrix} \right\},$$

(2) *$\Gamma(P_0, \mathcal{L}_0^n) = \Gamma(P, \mathcal{L}^n) \otimes k(0)$,*
(3) *$\mathcal{L}_0^n$ is very ample for $n \geq 2g+1$.*

### 2.11. The group schemes $G$ and $G^\sharp$

We review [18, 4.12] to recall the notation. By choosing a suitable base change of $S$, we assume $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any $\sigma \in \mathrm{Del}_B$. Then $P_0$ is reduced. Then $G$ is realized as an open subscheme of $P$. In fact, for any Delaunay $g$-cell $\sigma \in \mathrm{Del}(0)$, there is an open smooth subscheme $G(\sigma) \subset P$ such that

(i) $G(\sigma) \simeq G$, $G(\sigma)_\eta = P_\eta$, $G(\sigma)_0 = O(\sigma)$,
(ii) $G(\sigma)_{\mathrm{for}}$ is a formal $S$-torus of dimension $g$.

We define $G^\sharp = G^\sharp(\sigma) := \cup_{x \in (X/Y)} S_x(G(\sigma)) \subset P$. Then $G^\sharp$ is a group scheme over $S$ such that $G^\sharp_\eta = P_\eta$. It is an $S$-group scheme uniquely determined by $P$, independent of the choice of $\sigma$, though $G(\sigma)$ are in general distinct as $S$-subschemes of $P$. We note that each stratum $O(\tau)$ is $G_0$-invariant for any $\tau \in \mathrm{Del}_B$. See [18, 4.12] for the detail.

### 2.12. The Heisenberg group scheme $\mathcal{G}(\mathcal{L}_\eta)$ of $\mathcal{L}_\eta$

Let $K(\mathcal{L}_\eta)$ be the kernel of $\lambda(\mathcal{L}_\eta) : G_\eta \to G_\eta^t$. It is the subgroup scheme of $G_\eta$ representing the functor defined by

$$U \mapsto K(\mathcal{L}_\eta)(U) = \left\{ x \in G_\eta(U);\ \begin{matrix} \mathcal{L}_{\eta,U} \otimes p_2^*(N) \simeq T_x^*(\mathcal{L}_{\eta,U}) \\ \text{for some } N \in \mathrm{Pic}(U) \end{matrix} \right\}$$

for a $k(\eta)$-scheme $U$, where $\mathcal{L}_{\eta,U}$ is the pullback of $\mathcal{L}_\eta$ to $G_{\eta,U} := (G_\eta) \times_{k(\eta)} U$. We note that $N$ is given by the restriction of $\mathcal{L}_{\eta,U}$ to the subscheme $x(U)$ ($\simeq U$) of $G_{\eta,U}$. In other words,

$$x \in K(\mathcal{L}_\eta)(U) \Longleftrightarrow \mathcal{L}_{\eta,U} \otimes p_2^*(\mathcal{L}_{\eta,U|x(U)}) \simeq T_x^*(\mathcal{L}_{\eta,U})$$

See [15, § 13] for the details.

Let $\mathcal{L}_\eta^\times := \mathcal{L}_\eta \setminus \{\text{the zero section}\}$ be the $\mathbf{G}_m$-torsor on $G_\eta$ associated with the line bundle $\mathcal{L}_\eta$. Let $\mathcal{G}(\mathcal{L}_\eta) := (\mathcal{L}_\eta^\times)_{|K(\mathcal{L}_\eta)}$ be the restriction of $\mathcal{L}_\eta^\times$ to $K(\mathcal{L}_\eta)$. We call $\mathcal{G}(\mathcal{L}_\eta)$ *the Heisenberg group scheme of* $\mathcal{L}_\eta$. See [15, § 23, Theorem 1]. Then we define a functor $\mathrm{Aut}(\mathcal{L}_\eta/P_\eta)$ similar to $\underline{\mathrm{Aut}}(L/X)$ in [15, § 23, Theorem 1]:

$$\begin{aligned} U \mapsto\ & \mathrm{Aut}(\mathcal{L}_\eta/P_\eta)(U) \\ := & \left\{ (x,\phi);\ \begin{matrix} x \in K(\mathcal{L}_\eta)(U) \text{ and} \\ \phi : \mathcal{L}_{\eta,U} \to T_x^*(\mathcal{L}_{\eta,U}) \quad U\text{-isom. on } G_{\eta,U} \end{matrix} \right\} \end{aligned}$$

for any $k(\eta)$-scheme $U$.

An obvious difference from the definition of $K(\mathcal{L}_\eta)$ is that the definition of $\mathrm{Aut}(\mathcal{L}_\eta/P_\eta)$ lacks $N \in \mathrm{Pic}(U)$. This difference enables us to define the action of $\mathrm{Aut}(\mathcal{L}_\eta/P_\eta)$ on $\Gamma(G_\eta, \mathcal{L}_\eta)$.

In the same manner as in [15, § 23, Theorem 1], we see the functor $\mathrm{Aut}(\mathcal{L}_\eta/P_\eta)$ is represented by the $k(\eta)$-scheme $\mathcal{G}(\mathcal{L}_\eta)$, which admits therefore naturally a structure of a group $k(\eta)$-scheme over $K(\mathcal{L}_\eta)$.

The group scheme structure of $\mathcal{G}(\mathcal{L}_\eta)$ is given by [13, p. 289] as follows. Let $(x,\phi)$ and $(y,\psi)$ be any $T$-valued points of $\mathcal{G}(\mathcal{L}_\eta)$, $T$ a $k(\eta)$-scheme. Equivalently, $\phi : \mathcal{L}_\eta \to T_x^*(\mathcal{L}_\eta)$ and $\psi : \mathcal{L}_\eta \to T_y^*(\mathcal{L}_\eta)$ are $T$-isomorphisms of line bundles on $G_{\eta,T}$. The group law of $\mathcal{G}(\mathcal{L}_\eta)$ is

$$(y,\psi)\cdot(x,\phi) = (x+y, T_x^*\psi \circ \phi)$$

where we note the composition $T_x^*\psi \circ \phi$ is an isomorphism of $\mathcal{L}_\eta$ onto $T_{x+y}^*(\mathcal{L}_\eta)$. There is a natural epimorphism of $\mathcal{G}(\mathcal{L}_\eta)$ onto $K(\mathcal{L}_\eta)$ with fiber $\mathbf{G}_{m,k(\eta)}$, where $\mathbf{G}_{m,k(\eta)}$ is the center of the group scheme $\mathcal{G}(\mathcal{L}_\eta)$.

Thus $\mathcal{G}(\mathcal{L}_\eta)$ is a central extension of $K(\mathcal{L}_\eta)$ by the $k(\eta)$-split torus $\mathbf{G}_{m,k(\eta)}$. We define the commutator form $e^{\mathcal{L}_\eta}$ of $\mathcal{G}(\mathcal{L}_\eta)$ by

$$e^{\mathcal{L}_\eta}(\bar{g},\bar{h}) = [g,h] := ghg^{-1}h^{-1}, \quad \text{for } \forall g,h \in \mathcal{G}(\mathcal{L}_\eta)$$

where $\bar{g}$, $\bar{h}$ are the images of $g$ and $h$ in $K(\mathcal{L}_\eta)$. It is a nondegenerate and alternating bimultiplicative form on $K(\mathcal{L}_\eta)$.

Applying [18, Lemma 7.4], we see that the isomorphism class of $\mathcal{G}(\mathcal{L}_\eta)$ as a central extension is uniquely determined by the commutator form $e^{\mathcal{L}_\eta}$ by taking a finite extension of $k(\eta)$ if necessary. In other words, suppose that we are given two central extensions $\mathcal{G}$ and $\mathcal{G}'$ of $K(\mathcal{L}_\eta)$ by $\mathbf{G}_{m,k(\eta)}$. If they have the same commutator form, then by taking a finite extension $K'$ of $k(\eta)$ if necessary, the pullbacks of $\mathcal{G}$ and $\mathcal{G}'$ to $K'$ are isomorphic as central extensions of $K(\mathcal{L}_\eta) \times_{k(\eta)} K'$ by $\mathbf{G}_{m,K'}$.

### 2.13. The action of $\mathcal{G}(\mathcal{L}_\eta)$ on $\Gamma(G_\eta, \mathcal{L}_\eta)$

The group scheme $\mathcal{G}(\mathcal{L}_\eta)$ acts on $\Gamma(G_\eta, \mathcal{L}_\eta)$ as follows: for $z = (x, \phi)$ any $T$-valued point of $\mathcal{G}(\mathcal{L}_\eta)$, $T$ any $k(\eta)$-scheme,

$$\rho_{\mathcal{L}_\eta}(z)(\theta) := T^*_{-x}(\phi(\theta))$$

where $\theta \in \Gamma(G_{\eta,T}, \mathcal{L}_{\eta,T})$. For any $w = (y, \psi) \in \mathcal{G}(\mathcal{L}_\eta)(T)$, one checks

$$\rho_{\mathcal{L}_\eta}(w)\rho_{\mathcal{L}_\eta}(z)(\theta) = \rho_{\mathcal{L}_\eta}(x + y, T^*_x(\psi) \cdot \phi)(\theta) = \rho_{\mathcal{L}_\eta}(w \cdot z)(\theta).$$

See [13, p. 295]. Thus $\Gamma(G_\eta, \mathcal{L}_\eta)$ is a $\mathcal{G}(\mathcal{L}_\eta)$-module.

By [12, V, 2.5.5] (See also [15, § 23]), $\Gamma(G_\eta, \mathcal{L}_\eta)$ is an irreducible $\mathcal{G}(\mathcal{L}_\eta)$-module of weight one, unique up to isomorphism by taking a finite extension of $k(\eta)$ if necessary.

If the characteristic of $k(\eta)$ and the order of $K(\mathcal{L}_\eta)$ are coprime, then $\mathcal{G}(\mathcal{L}_\eta) \simeq \mathcal{G}(K) \otimes k(\eta)$ by taking a finite extension of $k(\eta)$ if necessary. Moreover if $\mathcal{O}_N \subset k(\eta)$, then $\Gamma(G_\eta, \mathcal{L}_\eta) \simeq V(K) \otimes k(\eta)$ as $\mathcal{G}(K) \otimes k(\eta)$-modules, which is therefore irreducible. See § 4 for the precise definitions of $\mathcal{G}(K)$, $\mathcal{O}_N$ and $V(K)$.

**Lemma 2.14.** *The flat closure $K^\sharp_S(\mathcal{L})$ of $K(\mathcal{L}_\eta)$ in $G^\sharp$ is finite and flat over $S$.*

*Proof.* See [18, Lemma 4.14]. Caution : $K^\sharp_S(\mathcal{L})$ is the same as $K^\sharp_S(\mathcal{L}_\eta)$ in [18, Lemma 4.14].

### 2.15. The Heisenberg group scheme $\mathcal{G}^\sharp_S(\mathcal{L})$ of $\mathcal{L}$

Now we shall extend $\mathcal{G}(\mathcal{L}_\eta)$ *relatively completely* over $S$. Let $\mathcal{L}^\times := \mathcal{L} \setminus \{\text{the zero section}\}$ be the $\mathbf{G}_m$-torsor on $P$ associated with the invertible sheaf $\mathcal{L}$, and $\mathcal{G}^\sharp_S(\mathcal{L}) := \mathcal{L}^\times_{|K^\sharp_S(\mathcal{L})}$ the restriction of $\mathcal{L}^\times$ to $K^\sharp_S(\mathcal{L})$. We note that $\mathcal{G}^\sharp_S(\mathcal{L})$ is the same as $\mathcal{G}^\sharp_S(\mathcal{L}_\eta)$ in [18, Definition 4.15].

Let $e^\sharp_S$ be an extension of $e^{\mathcal{L}_\eta}$ to $K^\sharp_S(\mathcal{L})$. By [12, IV, 7.1 (ii)] $\mathcal{G}^\sharp_S(\mathcal{L})$ is a group scheme over $S$ extending $\mathcal{G}(\mathcal{L}_\eta)$, which is a central extension of $K^\sharp_S(\mathcal{L})$ by $\mathbf{G}_{m,S}$ with $e^\sharp_S$ the commutator form. The bimultiplicative form $e^\sharp_S$ on $K^\sharp_S(\mathcal{L})$ is nondegenerate alternating by [12, IV, 2.4] and by Lemma 2.14.

We note that in view of [18, Lemma 7.4], the isomorphism class of $\mathcal{G}^\sharp_S(\mathcal{L})$ as a central extension is uniquely determined by the commutator form $e^\sharp_S$ by taking a finite cover of $S$ if necessary.

**Lemma 2.16.** *We define a functor* $\mathrm{Aut}(\mathcal{L}/P)$ *as follows:*

$$U \mapsto \mathrm{Aut}(\mathcal{L}/P)(U) := \left\{ (x,\phi);\ \begin{matrix} x \in K^\sharp_S(\mathcal{L})(U) \text{ and} \\ \phi : \mathcal{L}_{P_U} \to T^*_x(\mathcal{L}_{P_U})\ U\text{-isom. on } P_U \end{matrix} \right\}$$

*for any $S$-scheme $U$. The functor* $\mathrm{Aut}(\mathcal{L}/P)$ *is represented by* $\mathcal{G}^\sharp_S(\mathcal{L})$.

*Proof.* Similar to that of [15, § 23, Theorem 1]. Q.E.D.

**Definition 2.17.** We define

$$K(P,\mathcal{L}) := K^\sharp_S(\mathcal{L}), \quad \mathcal{G}(P,\mathcal{L}) := \mathcal{G}^\sharp_S(\mathcal{L}),$$
$$\mathcal{G}(G_\eta,\mathcal{L}_\eta) := \mathcal{G}(\mathcal{L}_\eta) = \mathcal{G}(P,\mathcal{L}) \otimes k(\eta),$$
$$K(P_0,\mathcal{L}_0) := K(P,\mathcal{L}) \otimes k(0), \quad \mathcal{G}(P_0,\mathcal{L}_0) := \mathcal{G}(P,\mathcal{L}) \otimes k(0).$$

The natural projection from $\mathcal{L}^\times$ to $G^\sharp$ makes $\mathcal{G}(P,\mathcal{L})$ a central extension of $K(P,\mathcal{L})$ by $\mathbf{G}_{m,S}$ with its commutator form $e^\sharp_S$

$$1 \to \mathbf{G}_{m,S} \to \mathcal{G}(P,\mathcal{L}) \to K(P,\mathcal{L}) \to 0.$$

We call $\mathcal{G}(P,\mathcal{L})$ (resp. $\mathcal{G}(P_0,\mathcal{L}_0)$) *the Heisenberg group scheme of* $(P,\mathcal{L})$ (resp. $(P_0,\mathcal{L}_0)$). See also Section 4.6.

**Lemma 2.18.** *Let $G^\sharp \subset P$ be the group $S$-scheme in 2.11. Then*

(1) $\Gamma(Q,\mathcal{L}) = \Gamma(P,\mathcal{L}) = \Gamma(G^\sharp,\mathcal{L})$,

(2) $\Gamma(P_0,\mathcal{L}_0) = \Gamma(P,\mathcal{L}) \otimes k(0)$ *and*

(3) *it is an irreducible $\mathcal{G}(P,\mathcal{L})$-module of weight one, in other words (by definition), any $\mathcal{G}(P,\mathcal{L})$-submodule of $\Gamma(P,\mathcal{L})$ of weight one is of the form $J\Gamma(P,\mathcal{L})$ for some ideal $J$ of $R$.*

*Proof.* See [12, V, 2.4.2; VI, 1.4.7], [18, Theorem 3.9, Lemma 5.12]. See also Theorem 2.10. Q.E.D.

**Lemma 2.19.** *We define a morphism* $\lambda(\mathcal{L}_0) : G^\sharp_0 \to \mathrm{Pic}^0(P_0)$ *by*

$$\lambda(\mathcal{L}_0)(a) = T^*_a(\mathcal{L}_0) \otimes \mathcal{L}_0^{-1}$$

*for any $U$-valued point $a$ of $G^\sharp_0$, and $U$ any $k(0)$-scheme. Then*

(1) $K(P_0, \mathcal{L}_0) = \ker \lambda(\mathcal{L}_0)$,
(2) *$\mathcal{G}(P_0, \mathcal{L}_0)$ is determined uniquely by $(P_0, \mathcal{L}_0)$.*

*Proof.* First we note that $G_0^\sharp \simeq G_0 \times (X/Y)$ in general, and that in the totally degenerate case $G_0 \simeq \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{G}_m)$, while in the general case $G_0$ is a $\mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{G}_m)$-torsor over an abelian variety $A_0$, whose extension class is determined uniquely by $(P_0, \mathcal{L}_0)$. The proof of the first assertion is proved in the same manner as [18, Lemma 5.14].

Next we prove the second assertion. We see as in the case of abelian varieties that $K(P_0, \mathcal{L}_0)$ is the maximal subscheme of $G_0^\sharp$ such that the sheaf $m^*(\mathcal{L}) \otimes p_2^*(\mathcal{L})^{-1}$ is trivial on $K(P_0, \mathcal{L}_0) \times P_0$, where $m : G_0^\sharp \times P_0 \to P_0$ is the action of $G_0^\sharp$, and $p_2 : G_0^\sharp \times P_0 \to P_0$ is the second projection. This is proved in the same manner as in [15, § 13].

Now we define a functor $\mathrm{Aut}(\mathcal{L}_0/P_0)$:

$$\begin{aligned} U \mapsto\ & \mathrm{Aut}(\mathcal{L}_0/P_0)(U) \\ & := \left\{ (x, \phi);\ \begin{array}{l} x \in K(P_0, \mathcal{L}_0)(U) \text{ and} \\ \phi : \mathcal{L}_{0,U} \to T_x^*(\mathcal{L}_{0,U})\ U\text{-isom. on } P_{0,U} \end{array} \right\} \end{aligned}$$

for any $k(0)$-scheme $U$. Then in the same manner as in [15, § 23, Theorem 1], we see the functor $\mathrm{Aut}(\mathcal{L}_0/P_0)$ is represented by $\mathcal{G}(P_0, \mathcal{L}_0)$.

By the first assertion and Section 2.12, $K(P_0, \mathcal{L}_0)$ and $\mathcal{G}(P_0, \mathcal{L}_0)$ are independent of the choice of a Delaunay $g$-cell $\sigma$. Q.E.D.

**Definition 2.20.** Let $k$ be an algebraically closed field, and let $(P_0, \mathcal{L}_0)$ be a TSQAS over $k = k(0)$. Then we define

$$\begin{aligned} e_{\min}(K(P_0, \mathcal{L}_0)) &= \max\{n > 0; \ker(n \cdot \mathrm{id}_{G_0^\sharp}) \subset K(P_0, \mathcal{L}_0)\}, \\ e_{\max}(K(P_0, \mathcal{L}_0)) &= \min\{n > 0; \ker(n \cdot \mathrm{id}_{G_0^\sharp}) \supset K(P_0, \mathcal{L}_0)\}, \end{aligned}$$

where $G_0^\sharp$ is the closed fiber of $G^\sharp$ in 2.11. If the order of $K(P_0, \mathcal{L}_0)$ and the characteristic of $k(0)$ are coprime, then $K(P_0, \mathcal{L}_0) \simeq \oplus_{i=1}^g (\mathbf{Z}/e_i\mathbf{Z})^{\oplus 2}$ for some positive integers $e_i$ with $e_i | e_{i+1}$. Hence $e_{\min}(K(P_0, \mathcal{L}_0)) = e_1$ and $e_{\max}(K(P_0, \mathcal{L}_0)) = e_g$.

**Theorem 2.21.** *Let $(P_0, \mathcal{L}_0)$ be a (not necessarily totally degenerate) torically stable quasi-abelian scheme over $k(0)$. Then*

(1) $\Gamma(P_0, \mathcal{L}_0^n) = \Gamma(P, \mathcal{L}^n) \otimes k(0)$ *for any* $n \geq 1$,
(2) $h^0(P_0, \mathcal{L}_0^n) = n^g \sqrt{|K(P_0, \mathcal{L}_0)|}$,
(3) $H^q(P_0, \mathcal{L}_0^n) = H^q(P, \mathcal{L}^n) = 0$ *for any* $q, n \geq 1$,
(4) *if* $n \geq 2g + 1$, $\mathcal{L}_0^n$ *is very ample on* $P_0$.

See [2] and [18].

**Theorem 2.22.** *Let $(Q_0, \mathcal{L}_0)$ be a projectively stable quasi-abelian scheme over $k(0)$, by definition, a closed fiber of $(Q, \mathcal{L})$ in Theorem 2.7. We define $K(Q_0, \mathcal{L}_0) = K(P_0, \mathcal{L}_0)$ (see [18, Definition 5.11]). Then*

(1) $\Gamma(Q_0, \mathcal{L}_0^n) = \Gamma(Q, \mathcal{L}^n) \otimes k(0)$ *for any* $n \geq 1$,
(2) $h^0(Q_0, \mathcal{L}_0^n) = n^g \sqrt{|K(Q_0, \mathcal{L}_0)|}$,
(3) $H^q(Q_0, \mathcal{L}_0^n) = H^q(Q, \mathcal{L}^n) = 0$ *for any* $q, n \geq 1$, *and*
(4) *if* $e_{\min}(K(Q_0, \mathcal{L}_0)) \geq 3$, $\mathcal{L}_0$ *is very ample on* $Q_0$.

*Proof.* The first and the second assertions are corollaries to [20, Theorem 5.17]. We prove the third assertion. If $Q_0$ is an abelian variety $A$ over $k(0)$ and if $n := e_{\min}(K(P_0, \mathcal{L}_0))$, then $P_0 \simeq Q_0 = A$ and $A[n] = \mathrm{Ker}\,(n\,\mathrm{id}_A)$ is a closed subscheme of $K(P_0, \mathcal{L}_0)$. This implies that $\mathcal{L}_0 = M^n$ for some ample line bundle $M$ on $A$ in view of [15, p. 231, Theorem 3]. It follows from Lefschetz's theorem that $\mathcal{L}_0$ is very ample. The general case follows from [18, Theorem 6.3], using (1). Q.E.D.

**Theorem 2.23.** *Suppose* $e_{\min}(K(P_0, \mathcal{L}_0)) \geq 3$. *Then*

(1) $\Gamma(P, \mathcal{L})$ *is base-point free and defines a finite morphism* $\phi$ *of* $P$ *into the projective space* $\mathbf{P}(\Gamma(P, \mathcal{L}))$. *The image of* $P$ *by* $\phi$ *with reduced structure is isomorphic to* $Q$, *and*
(2) $\phi$ *coincides with the normalization morphism* $\nu : P \to Q$,
(3) *letting* $\mathrm{Sym}(\phi)$ *be the graded subalgebra of* $\oplus_{n=0}^{\infty}\Gamma(P, \mathcal{L}^n)$ *generated by* $\Gamma(P, \mathcal{L}) = \nu^*\Gamma(Q, \mathcal{L}_Q)$, *and* $\mathcal{L}_{\mathrm{Sym}(\phi_P)}$ *the tautological line bundle, then* $(Q, \mathcal{L}_Q) \simeq (\mathrm{Proj}(\mathrm{Sym}(\phi_P)), \mathcal{L}_{\mathrm{Sym}(\phi_P)})$.

*Proof.* Let $\nu : P \to Q$ be the normalization. We note that both $P$ and $Q$ are reduced by the construction in Section 2.6.

By definition $\mathcal{L} := \nu^*(\mathcal{L}_Q)$. By Lemma 2.18 we have $\Gamma(P, \mathcal{L}) = \nu^*\Gamma(Q, \mathcal{L}_Q)$. Hence $\Gamma(P, \mathcal{L})$ is base-point free by Theorem 2.22 so that it defines a finite $S$-morphism $\phi : P \to \mathbf{P}(\Gamma(P, \mathcal{L}))$. Since $\Gamma(Q, \mathcal{L}_Q) \otimes k(0)$ is very ample on $Q_0$ by Theorem 2.22, so is $\Gamma(Q, \mathcal{L}_Q)$ on $Q$. Let $\phi_Q : Q \to \mathbf{P}(\Gamma(Q, \mathcal{L}_Q))$ be the natural morphism defined by $\Gamma(Q, \mathcal{L}_Q)$. Then since $\Gamma(P, \mathcal{L}) = \nu^*\Gamma(Q, \mathcal{L}_Q)$, $\phi$ factors through $\phi_Q(Q) \simeq Q \subset \mathbf{P}(\Gamma(Q, \mathcal{L}_Q)) \simeq \mathbf{P}(\Gamma(P, \mathcal{L}))$. Thus $\phi : P \to \phi_Q(Q) \simeq Q$ coincides with $\nu$. This proves (2). Since $Q$ is reduced, we have $(\phi(P))_{\mathrm{red}} = Q_{\mathrm{red}} = Q$. This proves (1). In particular, $\phi^* : \Gamma(Q, \mathcal{L}_Q^n) \to \Gamma(P, \mathcal{L}^n)$ is injective.

Since $\Gamma(Q, \mathcal{L}_Q)$ is very ample by Theorem 2.22, $S^n\Gamma(Q, \mathcal{L}_Q) \to \Gamma(Q, \mathcal{L}_Q^n)$ is surjective for any $n > 0$. It follows from $\phi^*(\Gamma(Q, \mathcal{L}_Q)) = \Gamma(P, \mathcal{L})$ that the degree $n$ part of $\mathrm{Sym}(\phi)$ coincides with $\phi^*\Gamma(Q, \mathcal{L}_Q^n)$, hence $Q \simeq \mathrm{Proj}(\mathrm{Sym}(\phi))$. This proves (3). Q.E.D.

**Remark 2.24.** We note that if $Q_0$ is non-reduced, then $Q_0 = \mathrm{Proj}(\mathrm{Sym}(\phi)) \otimes k(0))$ can be different from $\mathrm{Proj}(\mathrm{Sym}(\phi_{|P_0}))$, where

$\mathrm{Sym}(\phi_{|P_0})$ is the subalgebra of $\oplus_{n=0}^{\infty}\Gamma(P_0, \mathcal{L}_0^n)$ generated by $\Gamma(P_0, \mathcal{L}_0)$. In fact, if $Q_0$ is non-reduced, there is an $n$ such that $\nu_P^* : \Gamma(Q_0, L_0^n) \to \Gamma(P_0, L_0^n)$ has a nontrivial kernel, and $\mathrm{Ker}\,(S^n\Gamma(Q_0, L_0) \to \Gamma(P_0, L_0^n))$ can be strictly smaller than $\mathrm{Ker}\,(S^n\Gamma(Q_0, L_0) \to \Gamma(Q_0, L_0^n))$.

## §3. The schemes $P_0$ and $Q_0$

### 3.1. An amalgamation of an admissible scheme

Let $k$ be a field, and we consider $k$-schemes locally of finite type. Let $\Lambda$ be a partially ordered set with $\leq$ a partial order, where we understand that $\lambda \leq \nu$ if and only if either $\lambda = \nu$ or $\lambda < \nu$ (that is, $\lambda$ is strictly smaller than $\nu$).

We assume that $\Lambda$ satisfies the following

(a) $\Lambda$ has a unique minimal element $\phi$ with $\phi \leq \lambda$ for any $\lambda \in \Lambda$, and if $\lambda < \nu$ for infinitely many mutually distinct $\nu$, then $\lambda = \phi$,
(b) any totally ordered sequence in $\Lambda$ has a maximum,
(c) for any pair of maximal elements $\lambda$, $\nu$ ($\lambda \neq \nu$), there is an element $\lambda \cap \nu$, called the intersection of $\lambda$ and $\nu$, which is the maximal element among $\sigma \in \Lambda$ such that $\sigma < \lambda$, $\sigma < \nu$,
(d) for any pair of maximal elements $\lambda$, $\nu$, we have incidence numbers $[\lambda : \lambda \cap \nu]$ and $[\nu : \lambda \cap \nu]$, both being $\pm 1$ with distinct signs.

For the ordered set $\Lambda$, we suppose that we are given a set of irreducible reduced $k$-schemes $Z_\lambda$ of finite type ($\lambda \in \Lambda$), and that there exists a closed immersion $i_{\nu,\lambda} : Z_\lambda \to Z_\nu$ for any ordered pair $\lambda \leq \nu$.

Let $Z_\Lambda$ be the disjoint union of all $Z_\lambda$ ($\lambda \in \Lambda$), and $I_\Lambda$ the set of $i_{\lambda,\mu}$ for all ordered pairs $\lambda \leq \mu$. The pair $(Z_\Lambda, I_\Lambda)$ is called *an admissible system* if the conditions (i)–(iv) are satisfied:

(i) $Z_\phi$ is empty,
(ii) $Z_{\lambda\cap\nu} = Z_\lambda \cap Z_\nu$ for any pair of maximal element $\lambda$, $\nu$,
(iii) for any ordered pair $\lambda \leq \nu$, the closed immersion $i_{\nu,\lambda} : Z_\lambda \to Z_\nu$ is not an isomorphism if $\lambda \neq \nu$, and $i_{\lambda,\lambda} = \mathrm{id}_{Z_\lambda}$,
(iv) $i_{\mu,\lambda} = i_{\mu,\nu} \circ i_{\nu,\lambda}$ for any ordered triple $\lambda \leq \nu \leq \mu$.

A reduced scheme $Z$ (locally of finite type) is called *an amalgamation of the admissible system* $(Z_\Lambda, I_\Lambda)$ if the following conditions are satisfied:

(v) there is a closed immersion $i_\lambda : Z_\lambda \to Z$,
(vi) $i_\lambda = i_\nu \circ i_{\lambda,\nu}$ for any ordered pair $\lambda \leq \nu$,
(vii) there is a finite surjective morphism $i : Z_\Lambda \to Z$ such that $i_\lambda = i_{|Z_\lambda}$, the restriction of $i$ to $Z_\lambda$,

(viii) if there is a reduced scheme $Y$ (locally of finite type) with closed immersions $j_\lambda : Z_\lambda \to Y$, and a finite surjective morphism $j : Z_\Lambda \to Y$ such that $j_\lambda = j_{|Z_\lambda}$, $j_\lambda = j_\nu \circ i_{\nu,\lambda}$ for $\lambda \leq \nu$, then there is a morphism $h : Z \to Y$ such that $h \circ i = j$.

### 3.2. An example of an amalgamation

Let $\Lambda = \{\phi, a, b, c\}$ be an ordered set with $\phi < a < b$, $\phi < a < c$. We note that $b$ and $c$ are maximal in $\Lambda$. Let $Z_a = \text{Spec } k$, $Z_b = \text{Spec } k[x]$, $Z_c = \text{Spec } k[y, z]$. We define

$$i_{a,b} : Z_a \simeq \text{Spec } k[x]/(x) \subset Z_b,$$
$$i_{a,c} : Z_a \simeq \text{Spec } k[y,z]/(y,z) \subset Z_c.$$

Let $Z = \text{Spec } k[x,y,z]/(xy, xz)$. Then $Z$ is an amalgamation of $(Z_\Lambda, I_\Lambda)$. In fact, there is an exact sequence

$$\begin{array}{rcl} 0 \to O_Z \to O_{Z_b} \oplus O_{Z_c} & \to & O_{Z_a} \\ (f\ ,\ g) & \mapsto & [b:a]f + [c:a]g. \end{array}$$

We infer from this exact sequence that $Z$ is an amalgamation of $(Z_\Lambda, I_\Lambda)$, as we will see soon in the proof of Theorem 3.3.

**Theorem 3.3.** *There exists an amalgamation $Z$ of $(Z_\Lambda, I_\Lambda)$. Moreover if $Z_\lambda$ is normal for any $\lambda$, then $Z$ is seminormal, that is, any finite bijective morphism $f : W \to Z$ with $W$ reduced is an isomorphism.*

*Proof.* Let $Z_{\max}$ be the disjoint union of all $Z_\mu$ for $\mu$ maximal. Let $Z_{\max^2}$ be the disjoint union of $Z_{\lambda\cap\nu}$ for all pairs $\lambda \neq \nu$ both maximal.

Now we define an equivalence relation $\equiv$ on $Z_{\max}$ as follows. For $p \in Z_\mu$, $p' \in Z_\nu$, we define $p \equiv p'$ if one of the following equivalent conditions is satisfied:

(s) there exists $q \in Z_\lambda$ such that $p = i_{\mu\cap\nu,\nu}(q)$ and $p' = i_{\mu\cap\nu,\mu}(q)$,
(t) there exists $q \in Z_\lambda$ for some $\lambda \leq \nu \cap \mu$ such that $p = i_{\lambda,\nu}(q)$ and $p' = i_{\lambda,\mu}(q)$.

Let $Z^{\text{top}}$ be the quotient space of $Z_{\max}$ by the equivalence relation $\equiv$. Thus there is a finite-to-one continuous map $t_{\max} : Z_{\max} \to Z^{\text{top}}$. And there is a finite morphism $i_{\max^2} : Z_{\max^2} \to Z_{\max}$ such that for any pair $\lambda \neq \nu$ both maximal

$$(i_{\max^2})_{|Z_{\lambda\cap\nu}} : Z_{\lambda\cap\nu}\ (\subset Z_{\max^2}) \to Z_\lambda \cup Z_\nu\ (\subset Z_{\max})$$

is the disjoint union of $i_{\lambda,\lambda\cap\nu}$ and $i_{\nu,\lambda\cap\nu}$. It is obvious that $t_{\max}(p) = t_{\max}(p') \in Z^{\text{top}}$ iff either $p = p' \in Z_{\max}$ or $\exists\ q \in Z_{\max^2}$ such that

$t_{\max}(p) = t_{\max}(p') = t_{\max}(i_{\max^2}(q)) \in Z^{\mathrm{top}}$. Thus $Z_\lambda$ is set-theoretically a subset of $Z^{\mathrm{top}}$.

It remains to define a scheme structure of $Z^{\mathrm{top}}$. For this purpose, we define a sheaf homomorphism $i^*_{\max^2} : O_{Z_{\max}} \to O_{Z_{\max^2}}$ by

$$\bigoplus_{\lambda:\max} (a_\lambda) \mapsto \bigoplus_{\substack{\lambda \neq \nu \\ \text{both max}}} ([\lambda : \lambda \cap \nu] i^*_{\lambda, \lambda\cap\nu} a_\lambda + [\nu : \lambda \cap \nu] i^*_{\nu, \lambda\cap\nu} a_\nu).$$

We define $O_Z$ to be the kernel of $i^*_{\max^2} : O_{Z_{\max}} \to O_{Z_{\max^2}}$. Then $O_Z$ inherits a natural algebra structure from $O_{Z_{\max}}$, which defines a scheme $Z$ of locally of finite type by $Z = \mathrm{Spec}\,(O_Z)$ with its underlying topological space $Z^{\mathrm{top}}$.

Next we show that there is a natural closed immersion $i_\lambda$ of $Z_\lambda$ into $Z$ such that the underlying continuous map $i_\lambda^{\mathrm{top}}$ of $i_\lambda$ coincides with $(t_{\max})_{|Z_\lambda}$ for any maximal $\lambda$. Let $\Lambda_\lambda$ be the subset of $\Lambda$ consisting of all maximal $\nu \in \Lambda$ with $\lambda \leq \nu$. There is a natural epimorphism $i^*_{\lambda,\nu} : O_{Z_\nu} \to O_{Z_\lambda}$ for any $\nu \in \Lambda_\lambda$. Suppose $\bigoplus_{\nu\in\Lambda_\lambda} (a_\nu) \in O_Z$. Then $i^*_{\lambda,\nu}(a_\nu) = i^*_{\lambda,\mu}(a_\mu)$ for any maximal $\nu$ and $\mu$ because $\lambda \leq \mu \cap \nu$. Hence we define $i^*_\lambda : O_Z \to O_{Z_\lambda}$ by $i^*_\lambda(\bigoplus_{\nu\in\Lambda_\lambda} (a_\nu)) = i^*_{\lambda,\nu}(a_\nu)$, independent of $\nu$. Thus $i^*_\lambda$ is a well-defined epimorphism, which induces a closed immersion of $Z_\lambda$ into $Z$.

Suppose that there is a reduced scheme $Y$ (locally of finite type) with closed immersions $j_\lambda : Z_\lambda \to Y$, and a finite surjective morphism $j : Z_\Lambda \to Y$ such that such that $j_\lambda = j_{|Z_\lambda}$, $j_\lambda = j_\nu \circ i_{\nu,\lambda}$ for $\lambda \leq \nu$.

Let $j_{\max} = j_{|Z_{\max}}$. Then we have a sequence of $k$-modules

$$O_Y \overset{j^*_{\max}}{\to} O_{Z_{\max}} \overset{i^*_{\max^2}}{\to} O_{Z_{\max^2}}$$

such that $i^*_{\max^2} j^*_{\max} = 0$. Hence $j^*_{\max}$ induces a homomorphism of $O_Y$ into $O_Z$, which defines a morphism $h : Z \to Y$ as desired. We note that an amalgamation is unique locally, hence local amalgamations are patched together globally to define a global amalgamation.

Finally we prove that $Z$ is seminormal if $Z_\lambda$ is normal for any $\lambda \in \Lambda$. Suppose that there is a finite bijective morphism $f : W \to Z$ with $W$ reduced. Then we define $W_\lambda := f^{-1}(i_\lambda(Z_\lambda))$. Since $f_{|W_\lambda} : W_\lambda \to i_\lambda(Z_\lambda) \simeq Z_\lambda$ is finite bijective and $Z_\lambda$ is normal, we see that $f_{|W_\lambda}$ is an isomorphism, $W_\lambda \simeq Z_\lambda$. It follows that there is a finite morphism $g : Z_\Lambda \to W$ such that $g_{|Z_\lambda} = (f_{|W_\lambda})^{-1} \circ i_\lambda$. Then $g$ is surjective because $f$ is bijective, and $g$ satisfies the condition (viii). Since $Z$ is an amalgamation of $Z_\Lambda$, there is a surjective morphism $h : Z \to W$ such that $h \circ i = g$. It is obvious that $f \circ h = \mathrm{id}_Z$. Hence $W \simeq Z$. This proves that $Z$ is seminormal. Q.E.D.

### 3.4. The coordinates of $P_0$ and $Q_0$

Let $k(\eta)$ be the fraction field of $R$ as before, and $\widetilde{R}$ the graded subalgebra of $k(\eta)[X][\vartheta]$ defined in Section 2.6

$$\widetilde{R} := R[a(x)w^x\vartheta; x \in X] = R[\xi_x\vartheta; x \in X],$$

where $\xi_x := s^{B(x,x)/2+r(x)/2}w^x$. Let $\widetilde{Q} := \mathrm{Proj}(\widetilde{R})$, $\widetilde{P}$ the normalization of $\widetilde{Q}$ and $S_y$ the action of $Y$ on both $\widetilde{Q}$ and $\widetilde{P}$ defined in Section 2.6.

We always assume that $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any $\sigma \in \mathrm{Del}(P_0)$, where $dA(\alpha)(x) = B(\alpha, x) + r(x)/2$. Hence $P_0$ is reduced.

We set

$$\begin{aligned} \xi_{x,c} :&= \xi_{x+c}/\xi_c = s^{B(x,x)/2+B(x,c)+r(x)/2}w^x \quad (\forall x),\\ \zeta_{x,c} :&= s^{B(\alpha(\sigma),x)+r(x)/2}w^x \quad (x \in C(0, -c+\sigma)),\end{aligned}$$

where $\sigma \in \mathrm{Del}(c)$ stands for a Delaunay $g$-cell with $c \in \sigma$.

**Lemma 3.5.** ([18, Theorem 5.7]) *Let $(\widetilde{Q}_0, \widetilde{\mathcal{L}}_0)$ be the closed fiber of $(\widetilde{Q}, \widetilde{\mathcal{L}})$ and $\bar{\xi}_{x,c} := \xi_{x,c} \otimes k(0)$ the restriction to $Q_0$. Then*

(1) *$S_0(c) := k(0)[\bar{\xi}_{x,c}; x \in X]$ is a $k(0)$-algebra of finite type,*

(2) *$\widetilde{Q}_0$ is covered with affine $k(0)$-schemes of finite type*

$$W_0(c) := \mathrm{Spec}\ k(0)[\bar{\xi}_{x,c}; x \in X] \quad (c \in X).$$

**Lemma 3.6.** ([18, Theorem 4.9]) *Let $(\widetilde{P}_0, \widetilde{\mathcal{L}}_0)$ be the closed fiber of $(\widetilde{P}, \widetilde{\mathcal{L}})$ and $\bar{\zeta}_{x,c} := \zeta_{x,c} \otimes k(0)$ the restriction to $P_0$. Then*

(1) *$R_0(c) := k(0)[\bar{\zeta}_{x,c}; x \in X]$ is a $k(0)$-algebra of finite type,*

(2) *Let $x_i \in X$. If $x_i \in C(0, -c+\sigma)$ for one and the same Delaunay cell $\sigma \in \mathrm{Del}(c)$ (resp. otherwise), then*

$$\bar{\zeta}_{x_1,c} \cdots \bar{\zeta}_{x_m,c} = \bar{\zeta}_{x_1+\cdots+x_m,c} \quad (\textit{resp. } 0),$$

(3) *$\widetilde{P}_0$ is covered with affine $k(0)$-schemes of finite type*

$$U_0(c) := \mathrm{Spec}\ k(0)[\bar{\zeta}_{x,c}; x \in X] \quad (c \in X),$$

(4) *let $O(\sigma)$ be a torus stratum of $\widetilde{P}_0$ in Theorem 2.9, and $\overline{O(\sigma)}$ its closure in $\widetilde{P}_0$. If $\sigma \in \mathrm{Del}(c)$, then*

$$\Gamma(\overline{O(\sigma)} \cap U_0(c), O_{\overline{O(\sigma)} \cap U_0(c)}) \simeq k(0)[\bar{\zeta}_{x,c}; x \in C(0, -c+\sigma) \cap X],$$

*which is the semigroup ring of $C(0, -c+\sigma) \cap X$.*

**Lemma 3.7.** *For $\sigma \in \mathrm{Del}(c)$, let $\mathrm{Semi}(-c+\sigma)$ be the subsemigroup of $X$ generated by $(-c+\sigma) \cap X$. Then*

(1) *defining $O(\sigma, (Q_0)_{\mathrm{red}}) := (\widetilde{\nu}_0)_{\mathrm{red}}(O(\sigma))$ for any Delaunay cell $\sigma$, $\Gamma(O_{\overline{O(\sigma,(Q_0)_{\mathrm{red}})} \cap V_0(c)})$ is the semigroup ring of $\mathrm{Semi}(-c+\sigma)$, where $V_0(c) = (W_0(c))_{\mathrm{red}}$,*

(2) *$\Gamma(O_{\overline{O(\sigma)} \cap U_0(c)})$ is the semigroup ring of $C(0, -c+\sigma) \cap X$, where the semigroup $C(0, -c+\sigma) \cap X$ is the saturation of $\mathrm{Semi}(-c+\sigma)$ in $X$, that is, the subset of $X$ consisting of all $a \in X$ such that $na \in \mathrm{Semi}(-c+\sigma)$ for some positive integer $n$,*

(3) *In particular, the subscheme $\overline{O(\sigma)}$ of $P_0$ is uniquely determined by the subscheme $\overline{O(\sigma, (Q_0)_{\mathrm{red}})}$ of $(Q_0)_{\mathrm{red}}$,*

(4) *if $\tau \subset \sigma$, $\tau, \sigma \in \mathrm{Del}(c)$, the natural immersion $i_{\sigma,\tau} : \overline{O(\tau, Q_0)} \subset \overline{O(\sigma, Q_0)}$ induces a unique immersion $\iota_{\sigma,\tau} : \overline{O(\tau)} \subset \overline{O(\sigma)}$ through the saturation of semigroups.*

*Proof.* It suffices to prove the assertions for $c = 0$. Let $T_0(c)$ be the residue ring of $S_0(c)$ by the nilradical of $S_0(c)$. In view of [18, Lemma 5.5] $T_0(0)$ is generated by $\xi_x$, and

$$A := \Gamma(O_{\overline{O(\sigma,(Q_0)_{\mathrm{red}})} \cap V_0(0)}) = k(0)[\xi_x; x \in \sigma],$$

where $\sigma \in \mathrm{Del}(0)$. It is not normal in general because the semi-group $C(0, \sigma) \cap X$ is not generated by $\sigma \cap X$ as a semi-group.

We recall that $Q_0$ determines a unique Delaunay decomposition $\mathrm{Del} := \mathrm{Del}(\widetilde{Q}_0)$, hence the set $\mathrm{Del}(0)$ of Delaunay cells containing 0, though it does not determine the bilinear form $B$ uniquely. Let $\sigma \in \mathrm{Del}(0)$, and let $\mathrm{Semi}(C(0, \sigma))$ be the semigroup of $C(0, \sigma) \cap X$, $\mathrm{Semi}(\sigma)$ the subsemigroup of $C(0, \sigma) \cap X$ generated by $\sigma \cap X$. Let $n_\sigma = [C(0, \sigma)) \cap X : \mathrm{Semi}(\sigma)]$.

By Lemmas 3.5, $A$ is generated over $k(0)$ by $\xi_{b,0}$, $(b \in \sigma \cap X)$, while $B := \Gamma(O_{\overline{O(\sigma)} \cap U_0(0)})$ is generated over $k(0)$ by $\zeta_{b,0}$, $(b \in C(0, \sigma) \cap X)$. Since $\xi_{b,0} = \zeta_{b,0}$ and $\zeta_{nb,0} = \zeta_{b,0}^m$ for $b \in \sigma \cap X$, $A$ is generated over $k(0)$ by $\zeta_{b,0}$, $(b \in \mathrm{Semi}(\sigma))$. Hence $A$ is the semigroup ring of $\mathrm{Semi}(\sigma)$, while $B$ is the semigroup ring of $C(0, \sigma) \cap X$. The intersection $C(0, \sigma) \cap X$ is the subset of $X$ consisting of all $a \in X$ such that $n_\sigma a \in \mathrm{Semi}(\sigma)$. It follows that $\Gamma(\overline{O(\sigma)} \cap U_0(0))$ is isomorphic to the semigroup ring of $C(0, \sigma) \cap X$, where $C(0, \sigma) \cap X$ is the saturation of $\mathrm{Semi}(\sigma)$ in $X$.

This proves (2). The assertions (3) and (4) are obvious. Q.E.D.

**Theorem 3.8.** *Assume $e_{\min}(K) \geq 2$. Then $P_0$ is an amalgamation of closed orbits $\overline{O(\sigma)}$ ($\sigma \bmod Y$), each $\overline{O(\sigma)}$ being a torus embedding associated with the $X$-saturation of $\sigma \cap X$, where the image of $\overline{O(\sigma)}$ in $Q_0$*

*with reduced structure is the toric variety associated with the semigroup ring of* $\sigma \cap X$.

*Proof.* In view of [18, Theorem 3.9] we have an exact sequence

$$0 \to O_{P_0} \to \oplus O_{V(\sigma_g)} \xrightarrow{\partial_g} \cdots \xrightarrow{\partial_2} \oplus O_{V(\sigma_1)} \xrightarrow{\partial_1} \oplus O_{V(\sigma_0)} \to 0$$

where $\sigma_i \in \mathrm{Del}^{(i)} \bmod Y$. Since $e_{\min}(K) \geq 2$, $\iota_\sigma$ is a closed immersion of $\overline{O(\sigma)}$ into $P_0$. We note that $\iota_\sigma$ is always a closed immersion into $\widetilde{P}_0$. Hence $P_0$ is an amalgamation of $\overline{O(\sigma)}$ for all $\sigma \in \mathrm{Del}(\widetilde{Q}_0) \bmod Y$ by the proof of Theorem 3.3. Q.E.D.

**Corollary 3.9.** *The scheme* $P_0$ *is uniquely determined by the reduced scheme* $(Q_0)_{\mathrm{red}}$.

*Proof.* The stratification of $P_0$ by $\overline{O(\sigma)}$ for all $\sigma \in \mathrm{Del} \bmod Y$ is uniquely determined by $(Q_0)_{\mathrm{red}}$ by Lemma 3.7. Since $P_0$ is an amalgamation of $O(\sigma) \bmod Y$ ($\sigma \in \mathrm{Del}$), it is uniquely determined by $(Q_0)_{\mathrm{red}}$. Q.E.D.

## §4. Level-$G(K)$ structures

Let $\zeta_N$ be a primitive $N$-th root of unity and $\mathcal{O} := \mathbf{Z}[\zeta_N, 1/N]$.

### 4.1. The group schemes $G(K)$ and $\mathcal{G}(K)$

Let $H$ be a finite abelian group such that $e_{\max}(H) = N$, the maximal order of elements in $H$, is equal to $N$. Now we regard $H$ as a constant finite abelian group $\mathcal{O}$-scheme. Let $H^\vee := \mathrm{Hom}_{\mathcal{O}}(H, \mathbf{G}_{m,\mathcal{O}})$ be the Cartier dual of $H$. We set $K := K(H) = H \oplus H^\vee$ and define a bimultiplicative (or simply a *bilinear*) form $e_K : K \times K \to \mathbf{G}_{m,\mathcal{O}}$ by

$$e_K(z \oplus \alpha, w \oplus \beta) = \beta(z)\alpha(w)^{-1},$$

where $z, w \in H$, $\alpha, \beta \in H^\vee$. We note that $H$ is a maximally isotropic subgroup of $K$, unique up to isomorphism.

Let $\mu_N := \mathrm{Spec}\, \mathcal{O}[x]/(x^N - 1)$ be the group scheme of all $N$-th roots of unity. We define group $\mathcal{O}$-schemes $\mathcal{G}(K)$ and $G(K)$ by

$$\mathcal{G}(K) := \{(a, z, \alpha); a \in \mathbf{G}_{m,\mathcal{O}}, z \in H, \alpha \in H^\vee\},$$
$$G(K) := \{(a, z, \alpha); a \in \mu_N, z \in H, \alpha \in H^\vee\}$$

endowed with group scheme structure

$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta).$$

We denote the natural projections of $\mathcal{G}(K)$ to $K$, and of $G(K)$ to $K$ by the same letter $p_K$. Let $V(K)$ be the group algebra $\mathcal{O}[H^\vee]$ of $H^\vee$ over $\mathcal{O}$ (equivalently, a free $\mathcal{O}$-module generated by $H^\vee$), and an $\mathcal{O}$-basis $v(\chi)$ $(\chi \in H^\vee)$ of $V(K)$. Hence they are subject to the relation

$$v(\chi + \chi') = v(\chi)v(\chi')$$

for any $\chi, \chi' \in H^\vee$.

We define an action $U(K)$ of $G(K)$ and $\mathcal{G}(K)$ on $V(K)$ by

$$U(K)(a, z, \alpha)(v(\chi)) = a\chi(z)v(\chi + \alpha)$$

where $a \in \mu_M$ or $a \in \mathbf{G}_{m,\mathcal{O}}$, $z \in H$ and $\alpha \in H^\vee$.

**Lemma 4.2.** *Let $k$ be an algebraically closed field, let $K$ be a symplectic group with $e_K$ symplectic form. Suppose that the characteristic of $k$ is prime to the order of $K$. Then there exists a polarized abelian variety $(A, L)$ over $k$ such that $G(A, L) \otimes k \simeq G(K) \otimes k$.*

*Proof.* First we take a prinicipally polarized abelian variety $(A, L)$ over $k$. Let $N = e_{\max}(K)$ as before. Then $K(A, L^N) \simeq (\mathbf{Z}/N\mathbf{Z})^{2g}$. Let $e_N$ be the symplectic form (the Weil pairing) of $K(A, L^N)$. Let $I$ be a maximally isotropic subgroup of $K(A, L^N)$, namely, a maximal subgroup of $K(A, L^N)$ which is totally isotropic with respect to $e_N$. We note that $K(A, L^N) \simeq I \oplus I^\vee$, and that $e_N$ is just the alternating bilinear form $e_{I\oplus I^\vee}$ in Section 4.1. We also choose a maximally isotropic subgroup $I(K)$ of $K$ with respect to $e_K$ by the definition in Section 4.1 so that $K \simeq I(K) \oplus I(K)^\vee$. It follows that there is an epimorphism $s : I \to I(K)$ such that $e_K(s(a), b) = e_N(a, s^\vee(b))$ for $\forall a \in I$ and $\forall b \in I(K)^\vee$, where $s^\vee$ is the adjoint of $s$. Let $J = \mathrm{Ker}\,(s)$. Let $B = A/J$ and $f : A \to B$ the natural morphism. Since $J$ is also a totally isotropic subgroup of $K(A, L^N)$, we have a descent $M$ of $L^N$ to $B$ by [13, p. 291, Proposition 1], in other words, a line bundle $M$ on $B$ such that $L^N = f^*(M)$.

Then we shall show that $G(B, M) \simeq G(K) \otimes k$ and that $K(B, M) \simeq K \otimes k$. In fact, we choose a subgroup $\tilde{J}$ of $G(A, L^N)$ isomorphic to $J$. Let $G(A, L^N)^*$ be the centralizer of $\tilde{J}$ in $G(A, L^N)$. Then we see that there is an exact sequence

$$1 \to \mu_N \to G(A, L^N)^* \to I \oplus J^\perp \to 0,$$

where $J^\perp$ is the maximal subgroup of $I^\vee$ orthogonal to $J$. Since $0 \to I(K)^\vee \to I^\vee \to J^\vee \to 0$ is exact, we have $J^\perp \simeq I(K)^\vee$. Hence by applying [13, p. 291, Proposition 2] to our situation, $G(B, M) \simeq G(A, L^N)^*/\tilde{J}$. Thus $G(B, M)$ is a unique central extension of $(I \oplus I(K)^\vee)/(J \oplus 0) \simeq I(K) \oplus I(K)^\vee \simeq K$ by $\mu_N$. Hence $G(B, M) \simeq G(K) \otimes k$ and $K(B, M) \simeq K \otimes k$ . Q.E.D.

### 4.3. $G(K)$-modules of weight one

Let $R$ be any commutative $\mathcal{O}$-algebra. Then any $G(K)\otimes R$-module $V$ is called *of weight one* if every $a \in \mu_N \subset G(K)\otimes R$ acts on $V$ as scalar multiplication $a\cdot \mathrm{id}_V$. In this case, we also say that the action of $G(K)$ on $V$ is of weight one.

**Lemma 4.4.** *Let $\mathcal{O} := \mathbf{Z}[\zeta_N, 1/N]$ and $R$ any commutative $\mathcal{O}$-algebra. Let $H$ be a finite abelian group with $e_{\max}(H) = N$, $H^\vee$ the (Cartier-)dual of $H$ and $K = H\oplus H^\vee$. Then $V(K)\otimes R$ is an irreducible $G(K)$-module of weight one, unique up to equivalence. If $V$ is an $R$-free $G(K)$-module of weight one of finite rank, then $V$ is $G(K)$-equivalent to $V(0)\otimes V(K)\otimes R$ where $V(0) := \{v\in V; h\cdot v = 0\ (\forall h\in H)\}$ is regarded as an $R$-module with trivial $G(K)$-action.*

*Proof.* We prove the lemma in the standard way. The point is that we can prove it over $R$ without assuming $R$ is a field. In what follows, we denote the $G(K)$-action as follows: $G(K)\times V \ni (g,v)\mapsto g\cdot v\in V$.

Since $e_{\max}(H) = N$, any character $\chi$ of $H$ has values in $\mu_N$, the group of $N$-th roots of unity, hence in the ring $R$. Now we recall

$$\sum_{\chi\in H^\vee}\chi(h) = \begin{cases} |H| & \text{if } h = 1\\ 0 & \text{if } h\neq 1\end{cases} \tag{1}$$

First we prove

$$V = \bigoplus_{\chi\in H^\vee} V(\chi)$$

where $V(\chi) := \{v\in V; h\cdot v = \chi(h)v\ (\forall h\in H)\}$ is the eigenspace of $V$ with character $\chi$. To prove it, for $v\in V$, we define

$$v_\chi = \frac{1}{|H|}\sum_{h\in H}\chi(h)^{-1}(h\cdot v),$$

where we note $v_\chi\in V$ because $1/|H|\in R$. We see $v_\chi\in V(\chi)$ because

$$\begin{aligned} h\cdot v_\chi &= \frac{1}{|H|}\sum_{h'\in H}\chi(h')^{-1}(hh'\cdot w)\\ &= \frac{1}{|H|}\chi(h)\sum_{h\in H}\chi(hh')^{-1}(hh'\cdot w) = \chi(h)v_\chi. \end{aligned}$$

Moreover we see by (1)

$$\sum_{\chi\in H^\vee} v_\chi = \frac{1}{|H|}\sum_{\chi\in H^\vee}\left(\sum_{h\in H}\chi(h^{-1})\right)(h\cdot v) = v,$$

which shows $V \subset \sum_{\chi \in H^\vee} V(\chi)$.

It remains to prove $V = \bigoplus_{\chi \in H^\vee} V(\chi)$. For this purpose, we suffice to prove that if $\sum_{\chi \in H^\vee} w_\chi = 0$ for $w_\chi \in V(\chi)$, then $w_\chi = 0$ for any $\chi \in H^\vee$. In fact, since $h \cdot w_\chi = \chi(h) w_\chi$ for any $h \in H$, we have $\sum_{\chi \in H^\vee} \chi(h) w_\chi = h \cdot \sum_{\chi \in H^\vee} w_\chi = 0$. It follows from (1) that

$$0 = \sum_{h \in H} \chi(h)^{-1} \sum_{\rho \in H^\vee} \rho(h) w_\rho = \sum_{\rho \in H^\vee} \left( \sum_{h \in H} (\chi^{-1}\rho)(h) \right) w_\rho = |H| \cdot w_\chi.$$

Hence $w_\chi = 0$, whence $V = \bigoplus_{\chi \in H^\vee} V(\chi)$.

Next we prove that if $V \neq 0$, then $V(0) \neq 0$. In fact, if $V \neq 0$, then $V(\chi) \neq 0$ for some $\chi \in H^\vee$. By definition of $G(K)$, if $z \in H$, then $(1, z, 0) \in G(K)$. Let $w \in V(\chi), w \neq 0$ and set $w_0 = (1, 0, -\chi) \cdot w$. Then we see $0 \neq w_0 \in V(0)$. In fact, we check $(1, z, 0) \cdot w_0 = w_0$ as follows:

$$\begin{aligned} (1, z, 0) \cdot w_0 &= (1, z, 0)(1, 0, -\chi) \cdot w \\ &= \chi(z)^{-1} (1, 0, -\chi)(1, z, 0) \cdot w \\ &= \chi(z)^{-1} (1, 0, -\chi) \cdot \chi(z) w = w_0. \end{aligned}$$

Thus we see $V(0) \neq 0$. Now we prove $V \simeq V(0) \otimes_R V(K) \otimes R$ as $G(K) \otimes R$-modules. First we define $v(\chi, w) = (1, 0, \chi) \cdot w$ for any $w \in V(0)$. Then we see

$$\begin{aligned} (1, z, 0) \cdot v(\chi, w) &= \chi(z) v(\chi, w), \\ (1, 0, \alpha) \cdot v(\chi, w) &= (1, 0, \chi + \alpha) \cdot w = v(\chi + \alpha, w), \\ (a, z, \alpha) \cdot v(\chi, w) &= (1, 0, \alpha)(1, z, 0)(1, 0, \chi)(a, 0, 0) \cdot w \\ &= a(1, 0, \chi + \alpha) \chi(z) (1, z, 0) \cdot w \\ &= a \chi(z) v(\chi + \alpha, w). \end{aligned}$$

Let $F(v(\chi, w)) = w \otimes v(\chi)$ for $w \in V(0)$ and $\chi \in H^\vee$. Then $F$ is a $G(K)$-isomorphism of $V$ onto $V(0) \otimes_R (V(K) \otimes R)$ with $V(0)$ regarded as an $R$-module with trivial $G(K)$-action. This proves $V \simeq V(0) \otimes_R (V(K) \otimes R)$ as $G(K)$-modules. Q.E.D.

**Lemma 4.5.** (*Schur's lemma*) *Let $R$ be a commutative $\mathcal{O}$-algebra.*

(1) *Let $V_1$ and $V_2$ be two $R$-free irreducible $G(K)$-modules of finite rank of weight one. Assume $f : V_1 \to V_2$ and $g : V_1 \to V_2$ to be $G(K)$-isomorphisms. Then there exists a constant $c$ such that $f = cg$,*

(2) *Let $V$ be an $R$-free $G(K)$-module of finite rank of weight one. Assume $f : V \to V$ to be a $G(K)$-isomorphism. Then there*

*exists a $G(K)$-trivial module $W$ of finite rank and $g \in \mathrm{GL}(W \otimes R)$ such that $V = W \otimes V(K) \otimes R$ and $f = g \otimes \mathrm{id}_{V(K)}$.*

*Proof.* We prove (1) in the standard way. The point is that we can prove it over $R$ without assuming that $R$ is a field.

In view of Lemma 4.4, $V_1 \simeq V_2 \simeq V(K) \otimes R$ by the assumption of (1). Thus we choose an $\mathcal{O}$-basis $v(\chi)$ of $V(K)$. Let $F : V(K) \otimes R \to V(K) \otimes R$ be a $G(K)$-isomorphism. We show that $F$ is a scalar multiplication. In fact, since $F$ is $G(K)$-equivariant,

$$\begin{aligned}(a, z, \alpha) \cdot F(v(\chi)) &= F((a, z, \alpha) \cdot v(\chi)) \\ &= F(a\chi(z)v(\chi + \alpha)) \\ &= a\chi(z)F(v(\chi + \alpha)),\end{aligned}$$

whence, in particular, $(1, z, 0) \cdot F(v(\chi)) = \chi(z)F(v(\chi))$. This shows that $F(v(\chi)) = c_\chi v(\chi)$ for some $c_\chi$. Since $v(\chi) = (1, 0, \chi) \cdot v(0)$, we have

$$\begin{aligned}F(v(\chi)) &= F((1, 0, \chi) \cdot v(0)) \\ &= (1, 0, \chi) \cdot F(v(0)) \\ &= (1, 0, \chi) \cdot c_0 v(0) = c_0 v(\chi),\end{aligned}$$

whence $c_\chi = c_0$ for any $\chi \in H^\vee$. It follows $F = c_0 \cdot \mathrm{id}$. This proves (1).

(2) is an immediate corollary to Lemma 4.4 and Lemma 4.5 (1). Q.E.D.

### 4.6. The finite Heisenberg group scheme $G(P, \mathcal{L})$

Let $R$ be a complete discrete valuation ring, $k(0) = R/I$ and $S = \operatorname{Spec} R$. Let $(P, \mathcal{L})$ a TSQAS over $S$.

**The first assumption.** In what follows, we always assume that the order of $K(P, \mathcal{L})$ and the characteristic of $k(0)$ are coprime.

In other words, we consider only good primes. See Section 5.1 for the second assumption.

The group scheme $K(P, \mathcal{L})$ is a reduced flat finite group $S$-scheme, étale over $S$. Hence by taking a finite cover of $S$ if necessary, we may assume by [18, Lemma 7.4] that $(K(P, \mathcal{L}), e^\sharp_S) \simeq (K_S, e_{K,S})$ and $\mathcal{G}(P, \mathcal{L}) \simeq \mathcal{G}(K)_S$ for a suitable $K$, where $G(K)_S$ is the unique subgroup scheme of $\mathcal{G}(K)$ mapped onto $K$ such that $G(K)_S \cap \mathbf{G}_{m,S} = \mu_{N,S}$.

Thus by taking a finite cover of $S$ if necessary, we may assume $\mathcal{G}(P, \mathcal{L}) \simeq \mathcal{G}(K)_S$. So we suppose $\mathcal{G}(P, \mathcal{L}) \simeq \mathcal{G}(K)_S$.

We define *the (finite) Heisenberg group scheme* $G(P, \mathcal{L})$ of $(P, \mathcal{L})$ to be the unique subgroup scheme of $\mathcal{G}(P, \mathcal{L})$ mapped isomorphically onto $G(K)_S$. Similarly we define $G(P_0, \mathcal{L}_0) = G(P, \mathcal{L}) \otimes k(0)$.

Thus $\mathcal{G}(P,\mathcal{L}) \simeq \mathcal{G}(K)_S$ and $G(P,\mathcal{L}) \simeq G(K)_S$ by taking a finite cover of $S$, hence by taking a finite extension of $k(0)$.

### 4.7. Reformulation via the action of $G(K)_S$

We reformulate Section 4.6 via the action of $G(K)_S$. Via the isomorphism $G(P,\mathcal{L}) \simeq G(K)_S$, for any $S$-scheme $T$, we have

$$G(P,\mathcal{L})(T) = \{(x(g),\phi_g); g \in G(K)(T)\}$$

satisfying the following conditions:

(i) $x(g) \in K(P,\mathcal{L})(T)$, and

(ii) $\phi_g : \mathcal{L}_{P_T} \to T^*_{x(g)}(\mathcal{L}_{P_T})$ is an isomorphism on $P_T$,

(iii) $\phi_{gh} = T^*_{x(h)}\phi_g \circ \phi_h$ for any $g, h \in G(K)(T)$.

Since $\phi_g$ is fiberwise a (linear) isomorphism, it is multiplication by some invertible element $\phi_g(z)$, whence we may write $\phi_g(z,\xi) = \phi_g(z)\xi$ with fiber coordinate $\xi \in \mathcal{L}_z$.

In general, we replace $T_{x(g)}$ with a transformation $T_g$ of $P$ labeled by $g$, which may not be translation by $x(g)$. And with $T_{x(g)}$ so understood as $T_g$, we say in general that $(P,\mathcal{L})$ is $G(K)_S$*-linearized* if the conditions (ii) and (iii) are satisfied. See [17, pp. 30–31]. There are $G(K)_S$-linearized smooth cubic curves such that $T_g$ is not a translation by any $x \in K(P,\mathcal{L})$ when $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$. See [18, pp. 711–712].

Hence $G(P,\mathcal{L}) \simeq G(K)_S$ if and only if $(P,\mathcal{L})$ is $G(K)_S$-linearized and $x(g) \in K(P,\mathcal{L}) \subset G^\sharp$. It follows that the set $\{\phi_{(1\oplus h\oplus 0)}; h \in H\}$ is a descent data for $\mathcal{L}$, where we note that $H$ is a maximally isotropic subgroup of $K = K(H)$ in Section 4.1.

We note that $x : G(K)(T) \to K(P,\mathcal{L})(T)$ is a homomorphism, so that $x(1) = 0$ and $x(g^{-1}) = -x(g)$. Since we have isomorphisms

$$\mathcal{L} \xrightarrow{\phi_h} T^*_{x(h)}(\mathcal{L}) \xrightarrow{T^*_{x(h)}\phi_g} T^*_{x(h)}(T^*_{x(g)}(\mathcal{L})) = T^*_{x(gh)}(\mathcal{L}),$$

we see

$$\begin{aligned}(x(gh),\phi_{gh}) &= (x(g)+x(h), T^*_{x(h)}\phi_g \circ \phi_h)\\ &= (x(g),\phi_g)\cdot(x(h),\phi_h).\end{aligned}$$

Thus we can also reformulate the above as follows. Suppose that $G(K)_S \simeq G(P,\mathcal{L})_S$. For any $S$-scheme $T$, we have

$$G(P,\mathcal{L})(T) = \{(x(g),\phi_g); g \in G(K)(T)\}$$

satisfying the conditions (i)(ii)(iii) if and only if

(iv) we are given a commutative diagram of $G(K)_S$-actions:

$$\begin{array}{ccc} G(K)_S \times \mathcal{L} & \xrightarrow[\Sigma_P]{} & \mathcal{L} \\ {\scriptstyle \mathrm{id}\times p_{\mathcal{L}}}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{L}}} \\ G(K)_S \times P & \xrightarrow{\sigma_P} & P, \end{array}$$

where $p_{\mathcal{L}} : \mathcal{L} \to P$ is the natural projection, and the actions $\Sigma_P$ and $\sigma_P$ of $G(K)_S$ in (iv) are explicitly given as follows:

$$\Sigma_P(h, z, \xi) = (T_{x(h)}(z), \phi_h(z) \cdot \xi), \quad \sigma_P(h, z) = T_{x(h)}(z),$$

where $h \in G(K)(T)$, $x(h) \in K(P, \mathcal{L})(T)$, $z \in P(T)$, and the fiber-coordinate $\xi \in \mathcal{L}_z(T)$.

The condition (iii) is translated into the composition rule:

$$\begin{aligned} g \cdot (h \cdot (z, \xi)) &= g \cdot (T_{x(h)}(z), \phi_h(z) \cdot \xi) \\ &= (T_{x(g)}(T_{x(h)}(z)), \phi_g(T_{x(h)}(z))\phi_h(z) \cdot \xi) \\ &= (T_{x(gh)}(z), (T^*_{x(h)}\phi_g \cdot \phi_h)(z) \cdot \xi) \\ &= (T_{x(gh)}(z), \phi_{gh}(z) \cdot \xi) = (gh) \cdot (z, \xi). \end{aligned}$$

This shows that the existence of a $G(K)$-linearization on $\mathcal{L}$ is equivalent to the existence of compatible $G(K)$-actions on both $P$ and $\mathcal{L}$. This is true in general, not only for $(P, \mathcal{L})$. We also note that any $G(K)$-linearization on $\mathcal{L}$ is restated as an isomorphism $G(K) \times \mathcal{L} \simeq \sigma_P{}^*\mathcal{L}$. See Section 4.12.

We define the action $\rho_{\mathcal{L}}$ of $G(K)(T)$ on $\Gamma(P_T, \mathcal{L}_{P_T})$ as in Remark 2.13. For any $T$-valued point $g$ of $G(K)(T)$, $T$ any $k(\eta)$-scheme,

$$\rho_{\mathcal{L}}(g)(\theta) := T^*_{-x(g)}(\phi_g(\theta)) \tag{2}$$

where $\theta \in \Gamma(P_T, \mathcal{L}_{P_T})$. We easily see $\rho_{\mathcal{L}}(gh) = \rho_{\mathcal{L}}(g)\rho_{\mathcal{L}}(h)$. In what follows, if there is no fear of confusion, we denote $\rho_{\mathcal{L}}$ by $\rho$ for simplicity.

Consider the case $T = \mathrm{Spec}\ k(0)$.

**Theorem 4.8.** *Let $k = k(0)$. Suppose $(K(P_0, \mathcal{L}_0), e^{\sharp}_{S,0}) \simeq (K, e_K)$, and the order of $K(P_0, \mathcal{L}_0)$ and the characteristic of $k$ are coprime. Then $\rho \otimes k : G(K) \otimes k \to \mathrm{GL}(\Gamma(P_0, \mathcal{L}_0))$ is an irreducible representation of $G(K) \otimes k$ of weight one, hence $\Gamma(P_0, \mathcal{L}_0)$ is equivalent to $V(K) \otimes k$ as a $G(K) \otimes k$-module.*

*Proof.* By [18, Theorem 4.10], $\dim_k \Gamma(P_0, \mathcal{L}_0) = \dim_k V(K) \otimes k$. By Lemma 4.4, $\Gamma(P_0, \mathcal{L}_0) \simeq V(K) \otimes k$. See also [20, Theorem 5.18]. Q.E.D.

We call $\rho : G(K)_S \to \mathrm{GL}(\Gamma(P, \mathcal{L}))$ *the Schrödinger representation of* $G(K)_S$. It is obvious that we have a natural counterpart for $\mathcal{G}(K)_S$.

We also call $\rho : G(K) \otimes k(0) \to \mathrm{GL}(\Gamma(P_0, \mathcal{L}_0))$ *the Schrödinger representation of* $G(K) \otimes k(0)$.

**Lemma 4.9.** *Let $\mathcal{O}$ be any commutative algebra, and $G$ a finite reduced flat group $\mathcal{O}$-scheme. Let $Z$ be a positive-dimensional $\mathcal{O}$-flat projective scheme. $L$ an ample $G$-linearized line bundle on $Z$. Then for any point $z \in Z$, there exists a $G$-invariant open affine $\mathcal{O}$-subscheme $U$ of $Z$ such that $z \in U$ and $L$ is trivial on $U$.*

*Proof.* We may assume that $G$ is a constant group $\mathcal{O}$-scheme by taking an open affine covering of Spec $(\mathcal{O})$ *fine enough* if necessary. We choose a coprime pair of positive integers $a$ and $b$ such that

(i) $L^a$ and $L^b$ are both very ample,
(ii) $h^0(Z, L^a) \geq N + 1$ and $h^0(Z, L^b) \geq N + 1$, where $N = |G|$.

Then we have projective embeddings $\phi_k : Z \to \mathbf{P}(V_k)$ where $V_k = H^0(Z, L^k)$ $(k = a, b)$ are both $\mathcal{O}$-finite $\mathcal{O}$-flat modules. We may assume that $V_k$ are $\mathcal{O}$-free by shrinking Spec $(\mathcal{O})$ if necessary. Let $x \in Z$ and $G \cdot x$ the $G$-orbit of $x$. Since $|G \cdot x| \leq N$, we can find a hyperplane $H'$ of $\mathbf{P}(V_a)$ defined over $\mathcal{O}$ such that $H' \cap (G \cdot x)$ is empty. Let $f' \in V_a$ be a defining equation of the hyperplane section $H' \cap Z$, and $U' := \{z \in Z; f' \neq 0\}$. Then $U'$ is the inverse image of the complement of $H'$ in $\mathbf{P}(V_a)$, whence $L^a$ is trivial on $U'$. Let $F' = \prod_{g \in G} g^*(f')$ and $V' := \{z \in Z; F'(z) \neq 0\}$. Then $F'$ is $G$-invariant, and $V' \subset U'$. Therefore $V'$ is a $G$-invariant affine open $\mathcal{O}$-subscheme of $Z$ on which $L^a$ is trivial.

Similarly we choose a hyperplane $H''$ of $\mathbf{P}(V_a)$ defined over $\mathcal{O}$ such that $H'' \cap (G \cdot x)$ is empty. We let $f'' \in V_b$ be a defining equation of $H'' \cap Z$, $F'' = \prod_{g \in G} g^*(f')$, $U'' := \{z \in Z; f''(z) \neq 0\}$, and $V'' := \{z \in Z; F''(z) \neq 0\}$. Then $V''$ is also a $G$-invariant affine open $\mathcal{O}$-subscheme of $Z$ on which $L^b$ is trivial. Let $V := V' \cap V''$. It is clear that $x \in V$. Then $V$ is a $G$-invariant affine open subset of $Z$ such that both $L^a$ and $L^b$ are trivial. Choosing a suitable pair of integers $s$ and $t$ such that $as + bt = 1$, we have $L = (L^a)^s (L^b)^t$, whence $L$ is trivial on $V$. This proves the lemma. Q.E.D.

**Remark 4.10.** We note that if $\mathcal{O}$ is a field, then Lemma 4.9 is true for any finite group scheme $G$. In fact, for a given point $x \in Z$, we first apply Lemma 4.9 to $G_{\mathrm{red}}$, the reduced part of $G$, to find a $G_{\mathrm{red}}$-invariant affine subscheme $U$ of $Z$ containing $x$. Let $G^0$ be the connected component of the identity of $G$. Since $G^0$ acts trivially on $Z_{\mathrm{red}}$, any Zariski open subset of $Z$ is $G^0$-invariant. Hence $U$ is also $G$-invariant.

### 4.11. The $G$-linearization in down-to-earth terms

Let $\mathcal{O}$ be any commutative algebra, and $G$ a finite reduced flat group $\mathcal{O}$-scheme. Let $Z$ be a positive-dimensional $\mathcal{O}$-flat projective scheme. Let $m : G \times_{\mathcal{O}} G \to G$ be the multiplication of $G$, and $\sigma : G \times_{\mathcal{O}} Z \to Z$ an action of $G$ on $Z$. Let $L$ be an ample $G$-linearized line bundle on $Z$. The action $\sigma$ satisfies the condition:

$$\sigma(m \times \mathrm{id}_Z) = \sigma(\mathrm{id}_G \times \sigma). \tag{3}$$

Now we shall give an alternative description of the $G$-linearization of $(Z, L)$ by using a nice open affine covering of $Z$. By Lemma 4.9, we can choose an affine open covering $U_j := \mathrm{Spec}\,(R_j)$ $(j \in J)$ of $Z$ such that each $U_j$ is $G$-invariant and the restriction of $L$ is trivial on each $U_j$.

The induced bundles $\sigma^* L$, (resp. $(\mathrm{id}_G \times \sigma)^* \sigma^*(L)$, $(m \times \mathrm{id}_Z)^* \sigma^*(L)$) are all trivial on $G \times_{\mathcal{O}} U_j$ (resp. $G \times_{\mathcal{O}} G \times_{\mathcal{O}} U_j$ or $G \times_{\mathcal{O}} G \times U_j$) with the same fiber-coordinate as $L_{U_j}$. Let $\zeta_j$ be a fiber-coordinate of $L_{U_j}$.

Now we assume that $G$ is a constant finite group $\mathcal{O}$-scheme. Since $G$ is affine, let $A_G := \Gamma(G, O_G)$ be the Hopf algebra of $G$. Then the isomorphism $\Psi : p_2^* L \to \sigma^*(L)$ over $U_j$ is multiplication by a unit $\psi_j(g, x) \in (A_G \otimes_{\mathcal{O}} R_j)^{\times}$ at $(g, x) \in G \times_{\mathcal{O}} U_j$. Let $A_{jk}(x)$ be the one-cocycle defining $L$. Then $\sigma^*(L)$ is defined by the one-cocycle $\sigma^* A_{jk}(x)$. Hence $\Psi : p_2^* L \to \sigma^*(L)$ over $U_j$ and $U_k$ are related by

$$\psi_j(g, x) = \frac{A_{jk}(gx)}{A_{jk}(x)} \psi_k(g, x).$$

Now we write down the isomorphism over $G \times_{\mathcal{O}} G \times_{\mathcal{O}} U_j$:

$$p_3^* p_2^* L (\simeq (\mathrm{id}_G \times \sigma)^* p_2^* L) \simeq (\mathrm{id}_G \times \sigma)^* \sigma^* L.$$

This is written on $G \times_{\mathcal{O}} G \times_{\mathcal{O}} Z$ in two ways via (3):

$$\begin{aligned} &(g, h, x, \zeta_j) \mapsto (g, h, x, \psi_j(gh, x)\zeta_j), \\ &(g, h, x, \zeta_j) \mapsto (g, h, x, \psi_j(g, hx)\psi_j(h, x)\zeta_j), \end{aligned}$$

from which we infer Section 4.7 (iii) (compare also Section 4.12 (5))

$$\psi_j(gh, x) = \psi_j(g, hx)\psi_j(h, x).$$

### 4.12. The $G(K)$-action on $\mathbf{P}(V(K))$

Let $\mathbf{P}(V(K))$ be the projective space parametrizing one dimensional quotients of $V(K)$, $\mathbf{L}(V(K))$ the hyperplane bundle of it, and $\mathrm{GL} = \mathrm{GL}(V(K))$. For brevity, we denote $\mathbf{P}(V(K))$ and $\mathbf{L}(V(K))$ by $\mathbf{P}$ and $\mathbf{L}$

if no confusion is possible. Let $\mathrm{Sym}(V(K))$ be the symmetric algebra of $V(K)$ over $\mathcal{O}$. Then as $\mathcal{O}$-schemes,

$$\mathbf{P} = \mathrm{Proj}(\mathrm{Sym}(V(K))), \quad \mathbf{A}^{n+1} = \mathrm{Spec}\,(\mathrm{Sym}(V(K)),$$

where $n+1 = \mathrm{rank}_{\mathcal{O}}\, V(K)$. The dual bundle $\mathbf{L}^{\vee}$ of $\mathbf{L}$ is the blowing-up of $\mathbf{A}^{n+1}$ at the origin as an $\mathcal{O}$-scheme. The action of GL on $V(K)$ thus induces an action on $\mathbf{L}^{\vee}$, hence actions $S$ and $s$ on $\mathbf{L}$ and $\mathbf{P}$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{GL} \times \mathbf{L} \arrow[r, "S"'] \arrow[d, "\mathrm{id} \times p_{\mathbf{L}}"'] & \mathbf{L} \arrow[d, "p_{\mathbf{L}}"] \\
\mathrm{GL} \times \mathbf{P} \arrow[r, "s"] & \mathbf{P},
\end{tikzcd}$$

such that $s^*\mathbf{L} \simeq p_2^*\mathbf{L} = \mathrm{GL} \times \mathbf{L}$ where $p_2$ is the second projection of $\mathrm{GL} \times \mathbf{P}$. The isomorphism $\Psi : \mathrm{GL} \times \mathbf{L} \to s^*\mathbf{L}$ can be given by

$$\Psi^* s^*(X_i) = \sum_{i=0}^{n} p_1^*(a_{ij}) \otimes p_2^*(X_j)$$

using the standard coordinates $a_{ij}$ of GL and $X_j$ of $\mathbf{P}$ as in [17, pp. 32–33]. Thus $(\mathbf{P}, \mathbf{L})$ is GL-linearized ([17, p. 30]). The GL-linearization $\{(S_g, \psi_g)\}$ of $\mathbf{L}$ is explicitly given by

$$S_g^*(X_j) = s^*(X_j)_{|g \times \mathbf{P}}, \quad \psi_g = \Psi_{|g \times \mathbf{L}},$$

where $j = 1, \ldots, n+1$. Moreover $\Psi$ and $S$ are related by

$$S^*(X_i) = \Psi^* s^*(X_i) \quad \text{for any } i.$$

We also have a commutative diagram

$$\begin{tikzcd}
\mathrm{GL} \times \mathrm{GL} \times \mathbf{L} \arrow[r, "\mathrm{id} \times S"'] \arrow[d, "m \times \mathrm{id}_{\mathbf{L}}"'] & \mathrm{GL} \times \mathbf{L} \arrow[d, "S"] \\
\mathrm{GL} \times \mathbf{L} \arrow[r, "S"] & \mathbf{L},
\end{tikzcd} \tag{4}$$

where $m$ is the multiplication of GL, and $p_2$ is the second projection. The commutativity of (4) implies that $\psi_g$ is a GL-linearization on $\mathbf{L}$ :

$$S_h^* \psi_g \circ \psi_h = \psi_{gh} \quad \text{for any } g, h \in \mathrm{GL}\,. \tag{5}$$

Suppose we are given an irreducible representation of weight one $\rho : G(K) \to \mathrm{GL}$, where we do not assume $\rho = U(K)$. Then $(\mathbf{P}, \mathbf{L})$

is $G(K)$-linearized. In fact, by fiber-product, we infer a commutative diagram of $G(K)$-actions:

$$\begin{array}{ccc} G(K)\times\mathbf{L} & \xrightarrow[\Sigma]{} & \mathbf{L} \\ {\scriptstyle \mathrm{id}\times p_{\mathbf{L}}}\downarrow & & \downarrow{\scriptstyle p_{\mathbf{L}}} \\ G(K)\times\mathbf{P} & \xrightarrow{\sigma} & \mathbf{P}, \end{array} \tag{6}$$

so that $\sigma^*\mathbf{L}\simeq G(K)\times\mathbf{L}$.

### 4.13. A $G(K)$-linearization induces a $G(K)$-morphism

We come back to the situation of Section 4.7 and Section 4.12. We assume we are given a $G(K)$-linearization on $(P,\mathcal{L})$. To be more precise, we assume an isomorphism $\tau: G(K)_S\simeq G(P,\mathcal{L})$ for *an affine $\mathcal{O}$-scheme* $S$, which was functorially given by

$$\tau(g)=(x(g),\phi_g)\in G(P,\mathcal{L})(T)$$

for $g\in G(K)(T)$ and any $S$-scheme $T$. Hence by Section 4.7 and Section 4.12, we have an isomorphism $\widetilde{T}: G(K)_S\times\mathcal{L}\to\sigma_P^*\mathcal{L}$ where $\sigma_P: G(K)_S\times P\to P$ is the action of $G(K)$ on $P$.

Let $\phi: P\to\mathbf{P}_S=\mathbf{P}(V(K))_S$ be the rational map defined by the linear system $\Gamma(P,\mathcal{L})$. Suppose $\phi$ to be an $S$-morphism. Then we have a $\Gamma(O_S)$-isomorphism

$$\phi^*:\Gamma(\mathbf{P}_S,\mathbf{L}_S)=V(K)\otimes\Gamma(O_S)\to\Gamma(P,\mathcal{L}),$$

which enables us to define, with the help of $\rho_{\mathcal{L}}$, a homomorphism $\rho(\phi): G(K)_S\to\mathrm{GL}(V(K))_S$:

$$\rho(\phi)(g)(\theta):=ad((\phi^*)^{-1})(\rho_{\mathcal{L}}(g))(\theta):=(\phi^*)^{-1}\circ\rho_{\mathcal{L}}(g)\circ\phi^*(\theta), \tag{7}$$

where $g\in G(K)(S)$, $\theta\in V(K)\otimes\Gamma(O_S)$. Recall that a $G(K)_S$-action $\rho_{\mathcal{L}}$ on $\Gamma(P,\mathcal{L})$ was given by $\rho_{\mathcal{L}}(g)(\theta)=T^*_{-x(g)}(\phi_g(\theta))$ for $\theta\in\Gamma(P,\mathcal{L})$.

Then we shall show that $\phi$ *induces a unique pair of compatible $G(K)$-morphisms $(\phi,\Phi):(P,\mathcal{L})\to(\mathbf{P}_S,\mathbf{L}_S)$ such that $\rho(\phi)$ coincides with* $\rho_{\mathbf{L}}: G(K)_S\to\mathrm{GL}(V(K))_S$. In fact, for a given $\rho=\rho(\phi): G(K)_S\to\mathrm{GL}(V(K))_S$, we have a $G(K)$-linearization of $\mathbf{L}$, that is, an isomorphism $\Psi_{|G(K)\times\mathbf{L}}: G(K)\times\mathbf{L}\to s^*\mathbf{L}$ by Section 4.12. In other words, we have a commutative diagram (6) with

$$\Psi^*s^*(X_i)=\sum_{i=0}^{n}p_1^*(a_{ij})\otimes p_2^*(X_j).$$

This also gives an action of $G(K)$ on $\mathbf{P}$ which makes $\phi$ a $G(K)$-morphism, that is, $\phi(T_{x(g)} \cdot z) = S_{\rho(g)} \cdot \phi(z)$ for any $g \in G(K)_S$. Since $\mathcal{L} = \phi^*\mathbf{L}$, we have a natural (unique) morphism $\Phi : \mathcal{L} \to \mathbf{L}$ which is compatible with $\phi : P \to \mathbf{P}$. Hence we have a $G(K)_S$-equivariant Cartesian diagram

$$\begin{CD} \mathcal{L} @>{}>{\Phi}> \mathbf{L} \\ @V{p_{\mathcal{L}}}VV @VV{p_{\mathbf{L}}}V \\ P @>{\phi}>> \mathbf{P}. \end{CD}$$

Therefore we have an isomorphism

$$(\mathrm{id}_{G(K)} \times \Phi)^* \Psi_{|G(K) \times \mathbf{L}} : G(K) \times \mathcal{L} \to T^*(\mathcal{L})$$

by $T = s \circ (\mathrm{id}_{G(K)} \times \phi)$ on $G(K) \times P$, which coincides with the given $\widetilde{T}$.

Thus we have by Section 4.12 compatible $G(K)$-linearizations of $\mathcal{L}$ and $\mathbf{L}$. Hence we have a subgroup scheme of $\mathrm{Aut}(\mathbf{L}_S/\mathbf{P}_S)$

$$\{(S_{\rho(g)}, \psi_{\rho(g)}); \psi_{\rho(g)} : \mathbf{L} \simeq S^*_{\rho(g)}(\mathbf{L}), g \in G(K)_S\}$$

with $S_{\rho(g)} = \sigma_{|\{\rho(g)\} \times \mathbf{P}_S} \in \mathrm{Aut}(\mathbf{P}_S)$ and $\psi_{\rho(g)} = \Psi_{|\{\rho(g)\} \times \mathbf{P}_S}$, which are subject to the compatibility condition

$$\phi_g = \phi^* \psi_{\rho(g)} : \mathcal{L} = \phi^*\mathbf{L} \simeq \phi^*(S^*_{\rho(g)}\mathbf{L}) = T^*_{x(g)}\mathcal{L}, \tag{8}$$

because $\phi \circ T_{x(g)} = S_{\rho(g)} \circ \phi$ by the $G(K) \otimes k$-equivariance of $\phi$.

Since $\mathcal{L} \simeq \phi^*\mathbf{L}$, we can define a $G(K)_S$-action on $\mathbf{L}$ and a morphism $\Phi : \mathcal{L} \to \mathbf{L}$ as follows:

$$\begin{aligned} &G(K)_S \times \mathbf{L} \ni (g, w, \xi) \mapsto (S_{\rho(g)}(w), \psi_{\rho(g)}(w)(\xi)), \\ &\Phi(z, \xi) = (\phi(z), \xi), \quad (z, \xi) \in \mathcal{L}, \quad (w, \xi) \in \mathbf{L}. \end{aligned}$$

Since the $G(K)$-linearizations and $G(K)$-actions of $\mathcal{L}$ and $\mathbf{L}$ are compatible, $\phi^* : \Gamma(\mathbf{P}, \mathbf{L}) \to \Gamma(P, \mathcal{L})$ is a $G(K)_S$-homomorphism. In fact, applying (2) to $(\mathbf{P}, \mathbf{L})$, we define

$$\rho_{\mathbf{L}}(g)(\theta) := S^*_{\rho(g^{-1})}(\psi_{\rho(g)}(\theta)), \quad \theta \in \Gamma(\mathbf{P}, \mathbf{L}).$$

Then *we prove* $\rho(g) := \rho(\phi)(g) = \rho_{\mathbf{L}}(g)$ *for* $g \in G(K)$. In fact, as $\phi^*(S^*_{\rho(g^{-1})}) = T^*_{-x(g)}\phi^*$ and $\phi^*\psi_{\rho(g)} = \phi_g$, we see

$$\begin{aligned} \phi^*(\rho_{\mathbf{L}}(g)(\theta)) &= \phi^*(S^*_{\rho(g^{-1})}(\psi_{\rho(g)}(\theta))) \\ &= T^*_{-x(g)}(\phi^*(\psi_{\rho(g)}(\theta))) = T^*_{-x(g)}(\phi^*(\psi_{\rho(g)})(\phi^*\theta)) \\ &= T^*_{-x(g)}(\phi_g(\phi^*(\theta))) = \rho_{\mathcal{L}}(g)(\phi^*(\theta)), \end{aligned}$$

which shows $\rho(\phi)(g) = \rho_{\mathbf{L}}(g)$ by (7).

## §5. The functor of TSQASes

Let $(K, e_K)$ be a finite symplectic abelian group, $N = e_{\max}(K)$, $\mathcal{O} := \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$ and $k$ any field over $\mathcal{O}$. We keep the first assumption in Section 4.6.

### 5.1. A TSQAS $(P_0, \mathcal{L}_0)$ of level $K$

Let $(Z, L)$ be a pair of a $g$-dimensional scheme $Z$ over $k$, and $L$ a line bundle on $Z$ over $k$, which we refer to simply as a pair in what follows. Two pairs $(Z, L)$ and $(Z', L')$ are defined to be isomorphic over $k$, if there is a $k$-isomorphism $\phi : Z \to Z'$ such that $\phi^*(L') \simeq L$. A pair $(Z, L)$ is called *a torically stable quasi-abelian scheme* over $k$ if it is isomorphic to the closed fiber $(P_0, \mathcal{L}_0)$ of some $(P, \mathcal{L})$ in Theorem 2.7 with $k = k(0)$.

A pair $(Z, L)$ is called a $g$-dimensional torically stable quasi-abelian scheme of level $K$ over $k$, or *a TSQAS of level $K$ over $k$* if

(i) $(Z, L)$ is a $g$-dimensional torically stable quasi-abelian scheme over $k = k(0)$,
(ii) $(K(Z, L), e^{\sharp}_{S,0}) \simeq (K, e_K) \otimes k$ as finite abelian group $k$-schemes with bilinear forms.

See Definition 2.17 for the notation.

We note that $(K(Z, L), e^{\sharp}_{S,0})$ is independent of the choice of $(P, \mathcal{L})$ with $(P_0, \mathcal{L}_0) \simeq (Z, L)$. In fact, in view of Lemma 2.19, $\mathcal{G}(Z, L)$ is uniquely determined by $(Z, L)$, whose commutator form $e^{\sharp}_{S,0}$ is therefore uniquely determined by $(Z, L)$.

**The second assumption.** In what follows, assume $e_{\min}(K) \geq 3$.

Summarizing Section 4.7, Section 4.12 and Section 4.13, we infer

**Theorem 5.2.** *Let $(Z, L)$ be a $g$-dimensional torically stable quasi-abelian scheme of level $K$ over $k$ with $e_{\min}(K) \geq 3$. Suppose that the order of $K$ and the characteristic of $k$ are coprime. Then*

(1) *there is $(P, \mathcal{L})$, projective flat over $S$, such that $(P_0, \mathcal{L}_0) \simeq (Z, L)$, and $(P_\eta, \mathcal{L}_\eta)$ is a polarized abelian variety, where $S =$ Spec $R$, $R$ is a complete discrete valuation ring with $\eta$ a generic point of $S$, and with residue field $k(0)$ of $R$ $k$,*
(2) *$G(P, \mathcal{L}) \simeq G(K)_S$, $G(Z, L) = G(P_0, \mathcal{L}_0) \simeq G(K) \otimes k$, whence $\mathcal{L}$ (resp. $L$) is $G(K)_S$-linearized (resp. $G(K) \otimes k$-linearized),*
(3) *Let $\phi_P : P \to \mathbf{P}(V(K))_S$ (resp. $\phi : Z \to \mathbf{P}(V(K) \otimes k)$) be the morphism associated with the linear system $\Gamma(P, \mathcal{L})$ (resp. $\Gamma(Z, L)$). We define $\rho(\phi_P) := \mathrm{ad}((\phi_P^*)^{-1})\rho_{\mathcal{L}}$ and $\rho(\phi) = \rho(\phi_P) \otimes$*

*$k$ with the help of Eq.(2). Then $\phi_P$ (resp. $\phi$) is a $G(K)_S$-morphism (resp. a $G(K) \otimes k$-morphism) with regards to the $G(K)_S$-action on $\mathbf{P}(V(K))_S$ (resp. on $\mathbf{P}(V(K) \otimes k)$) induced from $\rho(\phi_P)$,*

(4) *We have a pair of compatible $G(K)_S$-morphisms and a pair of compatible $G(K) \otimes k$-morphisms*

$$(\phi_P, \Phi_P) : (P, \mathcal{L}) \to (\mathbf{P}(V(K))_S, \mathbf{L}(V(K))_S),$$
$$(\phi, \Phi) : (Z, L) \to (\mathbf{P}(V(K) \otimes k), \mathbf{L}(V(K) \otimes k)).$$

The second assumption implies in view of Theorem 2.22 that $\mathcal{L}_0$ is very ample on $Q_0$, hence $\mathcal{L}_0$ is ample on $P_0$. Note that $\mathcal{L}_0$ is not very ample on $P_0$ in general, for instance, when $P_0$ is of type $E_8$. See [20].

### 5.3. Level-$G(K)$ structures over $k$

Let $k$ be any field over $\mathcal{O}$ and $(Z, L)$ a TSQAS of level $K$ over $k$. The finite Heisenberg group scheme $G(Z, L)$, a subgroup scheme of $\mathrm{Aut}(L/Z)$, was given by

$$G(Z, L) = \{\tau(g) = (x(g), \phi_g); g \in G(K) \otimes k\}.$$

*A level-$G(K)$ structure* $(\phi, \tau)$ on $(Z, L)$ is defined to be a pair of a finite $k$-morphism $\phi : Z \to \mathbf{P} \otimes k = \mathbf{P}(V(K)) \otimes k$ and a group scheme isomorphism $\tau : G(K) \otimes_{\mathcal{O}} k \to G(Z, L)$ such that

(i) $\phi$ is a $G(K) \otimes k$-morphism with regards to $\tau$, and $\phi^* : \Gamma(\mathbf{P}, \mathbf{L}) \otimes k = V(K) \otimes k \simeq \Gamma(Z, L)$ as $G(K)$-modules.

In this situation, in view of Theorem 5.2, $\phi$ always becomes a $G(K) \otimes k$-morphism, that is, $\phi(T_{x(g)} \cdot z) = S_{\rho(\phi)(g)} \cdot \phi(z)$ for any $g \in G(K) \otimes k$ and $z \in Z$. See Theorem 5.2 for $\rho(\phi)$. By Theorem 5.2, we always have a pair of compatible $G(K) \otimes k$-morphisms $(\phi, \Phi) : (Z, L) \to (\mathbf{P}, \mathbf{L})$. Hence in view of Lemma 5.5, $\phi^*$ is always a $G(K) \otimes k$-homomorphism, namely,

$$\rho_L(g)\phi^*(\theta) = \phi^* \rho_{\mathbf{L}}(g)(\theta) = \phi^* \rho(\phi)(g)(\theta).$$

For a given level-$G(K)$ structure $(\phi, \tau)$ we define

$$\rho_L(g)(\theta) := T^*_{-x(g)}(\phi_g(\theta)), \quad g \in G(K) \otimes k,$$
$$\rho(\phi, \tau) := \mathrm{ad}((\phi^*)^{-1})\rho_L : G(K) \otimes k \to \mathrm{GL}(V(K) \otimes k).$$

We note $\rho(\phi, \tau) = \rho(\phi)$ with the notation in Theorem 5.2. Since $\rho_L$ is injective, $\rho(\phi, \tau)$ is also an injective homomorphism. This is only conjugate to $U(K) \otimes k$ by an element of $\mathrm{GL}(V(K) \otimes k)$ by a lemma of Schur, because we do not require $\rho(\phi, \tau)$ to be the same as $U(K) \otimes k$.

Let $(\phi, \tau)$ be a level-$G(K)$ structure. Then $(\phi, \tau)$ is called *a rigid level-$G(K)$ structure* if

(ii) $\rho(\phi, \tau) = U(K) \otimes_{O} k$.

We use this *linguistically more correct* terminology following the advices of Professors A. King and G. Sankaran, changing our previous ones in [18]; level $G(K)$-structure and rigid $G(K)$-structure.

However, both to simplify the terminology and to compromise with [18], in what follows, we call a rigid level-$G(K)$ structure *a rigid $G(K)$-structure.* We do so because the level-$G(K)$ structure is a set of certain structures on a polarized scheme with compatible $G(K)$-actions. In this sense, a level-$G(K)$ structure (resp. a PSQAS $(P, \phi, \tau)$ with level-$G(K)$ structure) might be called simply a $G(K)$-structure (resp. a $G(K)$-triple).

The given TSQAS $(Z, L, \phi, \tau)$ with level-$G(K)$ structure is denoted simply $(Z, \phi, \tau)_{\mathrm{LEV}}$ because $L = \phi^*(\mathbf{L})$ by (i). If (i) and (ii) are true, we denote it by $(Z, \phi, \tau)_{\mathrm{RIG}}$.

### 5.4. Morphisms of level-$G(K)$ structures over $k$

Let $(Z_i, L_i, \phi_i, \tau_i)$ be $k$-TSQASes with level-$G(K)$ structure ($i = 1, 2$). Let $\pi_i : L_i \to Z_i$ be the natural projection. Suppose that there is a $k$-morphism $f : Z_1 \to Z_2$ such that $L_1 \simeq f^*L_2$. Then there is a $k$-isomorphism $H = H(f) : L_1 \simeq Z_1 \times_{Z_2} L_2 = f^*L_2$ as $Z_1$-schemes. Then we define a $k$-morphism $F = F(f) : L_1 \to Z_1 \times_{Z_2} L_2 \to L_2$ as the composite $F(f) = p_2 \circ H(f)$, where $p_2$ is the second projection. We note $f \circ \pi_1 = \pi_2 \circ F(f)$. In fact, with $L_1$ as $Z_1 \times_{Z_2} L_2$ understood, we have $F(f)(z, \xi) = (f(z), \xi) \in L_2$ for $(z, \xi) \in L_1$ so that $f(z) = f \circ \pi_1(z, \xi) = \pi_2 \circ F(f)(z, \xi)$.

With this preparation, for a pair of $k$-TSQASes $(Z_i, L_i, \phi_i, \tau_i)$ with level-$G(K)$ structure, $f : (Z_1, L_1, \phi_1, \tau_1) \to (Z_2, L_2, \phi_2, \tau_2)$ is defined to be *a $k$-morphism of $k$-TSQASes with level-$G(K)$ structure* if the following conditions are satisfied:

(i) $f$ is a $G(K) \otimes k$-morphism over $k$ such that $\phi_1 = \phi_2 \circ f$,

(ii) $F(f)$ is also a $G(K) \otimes k$-morphism compatible with $f$, namely,

$$F(f) \circ \tau_1(g) = \tau_2(g) \circ F(f) \quad \text{for any } g \in G(K) \otimes k.$$

Since $L_i \simeq \phi_i^*(\mathbf{L})$, the condition $\phi_1 = \phi_2 \circ f$ in (i) implies $L_1 \simeq f^*L_2$.

In terms of $G(K)$-linearizations $\phi'_g$ of $L_1$ and $\phi''_g$ of $L_2$, the conditions (i) and (ii) are given explicitly as follows:

(iii) $\phi_1 = \phi_2 \circ f$,

(iv) $f(T_{x(g)}(z)) = T_{y(g)}(f(z))$,

(v) $\phi'_g(z) = \phi''_g(f(z)), \quad g \in G(K) \otimes k$, where we understand a $G(K)$-linearization $\phi'_g$ (resp. $\phi''_g$) as multiplication by an invertible element $\phi'_g(z)$ (resp. $\phi''_g(z)$). See Section 4.7

Let us suppose the cocycle conditions for $\phi'_g$ and $\phi''_g$ as follows:

$$\phi'_{gh} = T^*_{x(h)}\phi'_g \cdot \phi'_h, \quad \phi''_{gh} = T^*_{y(h)}\phi''_g \cdot \phi''_h.$$

By (v) $\phi'_g(z) = \phi''_g(f(z))$, whence $\phi'_g(T_{x(h)}z) = \phi''_g(f(T_{x(h)}z))$. By (iv) $T^*_{x(h)}\phi'_g(z) = \phi''_g(f(T_{x(h)}z)) = \phi''_g(T_{y(h)}f(z)) = T^*_{y(h)}\phi''_g(f(z))$. Hence

$$\begin{aligned}\phi'_{gh}(z) &= T^*_{x(h)}\phi'_g(z) \cdot \phi'_h(z)\\ &= T^*_{y(h)}\phi''_g(f(z)) \cdot \phi''_h(f(z)) = \phi''_{gh}(f(z)),\end{aligned}$$

which shows the compatibility of (v).

**Lemma 5.5.** *Suppose that $f : (Z_1, L_1, \phi_1, \tau_1) \to (Z_2, L_2, \phi_2, \tau_2)$ is a $k$-morphism of $k$-TSQASes with level-$G(K)$ structure. Then we have*

$$\rho_{L_1}(g)(f^*\theta) = f^*\rho_{L_2}(g)(\theta)$$

*for any $g \in G(K) \otimes k$ and $\theta \in \Gamma(Z_2, L_2)$.*

*Proof.* By (iii) and (iv), we see

$$\begin{aligned}\rho_{L_1}(g)(f^*\theta) :&= (T^*_{-x(g)}\phi'_g)(f^*\theta) := T^*_{-x(g)}\phi'_g(z)(T^*_{-x(g)}f^*\theta)\\ &= \phi'_g(T_{-x(g)}(z))(T^*_{-x(g)}f^*\theta)\\ &= \phi''_g(f(T_{-x(g)}(z)))(T^*_{-x(g)}f^*\theta) \quad \text{(by (v))}\\ &= \phi''_g(T_{-y(g)}(f(z)))(f^*T^*_{-y(g)}\theta) \quad \text{(by (iv))}\\ &= f^*T^*_{-y(g)}\phi''_g(z)(f^*T^*_{-y(g)}\theta)\\ &= f^*(T^*_{-y(g)}\phi''_g(z)(T^*_{-y(g)}\theta))\\ &= f^*\rho_{L_2}(g)(\theta).\end{aligned}$$

This completes the proof. Q.E.D.

### 5.6. Morphisms of level-$G(K)$ structures

Let $f : (Z_1, L_1, \phi_1, \tau_1) \to (Z_2, L_2, \phi_2, \tau_2)$ be a $k$-morphism of $k$-TSQASes with level-$G(K)$ structure. Those $k$-TSQASes $(Z_i, L_i, \phi_i, \tau_i)$ are defined to be *isomorphic as $k$-TSQASes with level-$G(K)$ structure* if $f$ is a $k$-isomorphism. In this case we write

$$(Z_1, \phi_1, \tau_1)_{\mathrm{LEV}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{LEV}}.$$

A pair of rigid $G(K)$-structures $(Z_i, \phi_i, \tau_i)$ is defined to be isomorphic if $(Z_i, \phi_i, \tau_i)_{\mathrm{LEV}}$ are isomorphic.

**Lemma 5.7.** *Let $(Z, L)$ be a TSQAS with level-$G(K)$ structure over $k$. For any level-$G(K)$ structure $(\phi, \tau)$ on $(Z, L)$, there exists a unique rigid $G(K)$-structure $(\phi(\tau), \tau)$ such that*

$$(Z, \phi(\tau), \tau)_{\mathrm{LEV}} \simeq (Z, \phi, \tau)_{\mathrm{LEV}}.$$

*Proof.* By our assumption we have $\rho(\phi, \tau)$ is conjugate to $U(K) \otimes k$. Hence there is a $S \in \mathrm{GL}(V(K) \times k)$ such that $\rho(\phi, \tau) = S(U(K) \otimes k)S^{-1}$. Let $\phi^*_{\mathrm{new}} = \phi^* \circ S$. Then we have $\rho(\phi_{\mathrm{new}}, \tau) = S^{-1}\rho(\phi, \tau)S = U(K) \otimes k$. We note that $\phi^*_{\mathrm{new}}$ defines a finite $G(K) \otimes k$-morphism into $\mathbf{P}$ with regards to the $G(K) \otimes k$-action induced from $\rho(\phi_{\mathrm{new}}, \tau)$. Thus $(\phi_{\mathrm{new}}, \tau)$ is a rigid $G(K) \otimes k$-structure of $(Z, L)$.

Suppose $(\psi, \tau)$ is another rigid $G(K)$-structure of $(Z, L)$ such that $\rho(\psi, \tau) = U(K) \otimes k$. Then we have $\mathrm{ad}(\phi^*)U(K) \otimes k = \mathrm{ad}(\psi^*)U(K) \otimes k$, whence $(\phi^*)^{-1}\psi^* U(K) \otimes k = U(K) \otimes k(\phi^*)^{-1}\psi^*$. Then $(\phi^*)^{-1}\psi^*$ is a scalar matrix by Schur's lemma (see Lemma 4.5) because $U(K) \otimes k$ is irreducible. Hence $\phi = \psi$, which define the same morphism of $Z$ into $\mathbf{P} \otimes k$. Thus $\phi(\tau)$ is unique, and $(Z, \phi(\tau), \tau)_{\mathrm{RIG}} \simeq (Z, \phi, \tau)_{\mathrm{RIG}}$. Q.E.D.

**Lemma 5.8.** *Let $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ be $k$-TSQASes with rigid $G(K)$-structure $(i = 1, 2)$. Then the following are equivalent:*

(1) $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$,
(2) *there is a $G(K) \otimes k$-equivariant (in fact, $K \otimes k$-equivariant) isomorphism $f : Z_1 \simeq Z_2$ with $\phi_1 = \phi_2 \circ f$.*

*Proof.* It is clear that (1) implies (2). Next we prove (2) implies (1). Since both are rigid $G(K)$-structures, we have $\rho(\phi_1, \tau_1) = \rho(\phi_2, \tau_2) = U(K) \otimes k$, which we denote by $\rho$. Hence we have $G(K) \otimes k$-morphisms $(\phi_i, \Phi_i) : (Z_i, L_i) \to (\mathbf{P}, \mathbf{L}) \otimes k$ with regards to the same $G(K) \otimes k$-action on $(\mathbf{P}, \mathbf{L}) \otimes k$. Let $\psi_{\rho(g)}$ be a $G(K) \otimes k$-linearization of $\mathbf{L}$. In view of Theorem 5.2 we can apply Section 5.4 (v) to the cases with the target pair $(\mathbf{P}, \mathbf{L}) \otimes k$ to infer that $G(K) \otimes k$-linearizations of $L_i$ are given by $\phi'_g(z) = \psi_{\rho(g)}(\phi_1(z))$ and $\phi''_g(w) = \psi_{\rho(g)}(\phi_2(w))$. It follows

$$\phi'_g(z) = \psi_{\rho(g)}(\phi_1(z)) = \psi_{\rho(g)}(\phi_2 \circ f(z)) = \phi''_g(f(z)),$$

which proves Section 5.4 (v) for the morphism $f$. This completes the proof of (1). Q.E.D.

We note that if $L$ is not very ample, then there might be an automorphism of $(Z, L)$ which keeps $\phi$ and $\tau$ invariant. For instance, an elliptic curve $(Z, L)$ with $\deg L = 2$ is an example of it, in which case $e_{\max}(K(Z, L)) = 2$.

### 5.9. TSQASes over $T$

The arguments of this section together with those of Section 5.10 apply to $T$-PSQASes too, which will supplement the argument in [18, pp. 701–702, Definition 9.16].

Let $T$ be an $\mathcal{O}$-scheme. A quadruplet $(P, \mathcal{L}, \phi, \tau)$ is called *a torically stable quasi-abelian $T$-scheme* (abbr. *a $T$-TSQAS*) of relative dimension $g$ with level-$G(K)$ structure if the following conditions are satisfied:

(i) $P$ is a proper flat $T$-scheme with $\pi : P \to T$ the projection,
(ii) $\mathcal{L}$ is a relatively ample line bundle of $P$,
(iii) $\phi : P \to \mathbf{P}(V(K))_T$ is a finite $T$-morphism such that

$$\phi^* : V(K) \otimes_{\mathcal{O}} M \simeq \pi_* \mathcal{L}$$

for some line bundle $M$ on $T$ with trivial $G(K)_T$-action,
(iv) $\tau : G(K)_T \to \mathrm{Aut}_T(\mathcal{L}/P)$ is a closed immersion of a group $T$-scheme, which makes $\phi$ a $G(K)_T$-morphism in the sense that

$$\phi(\tau(g) \cdot z) = S_{\rho(\phi,\tau)(g)} \phi(z) \quad (z \in P)$$

and that $\phi^* : V(K) \otimes_{\mathcal{O}} M \simeq \pi_* \mathcal{L}$ in (iii) is a $G(K)$-isomorphism, (see below for $\mathrm{Aut}_T(\mathcal{L}/P)$ and $\rho(\phi, \tau)$, and see Section 4.13 for $S_{\rho(\phi,\tau)(g)}$),
(v) for any prime point $s$ of $T$, the fiber at $s$ $(P_s, \mathcal{L}_s, \phi_s, \tau_s)$ is a TSQAS of dimension $g$ over $k(s)$ with level-$G(K)$ structure.

We denote a $T$-TSQAS $(P, \mathcal{L}, \phi, \tau)$ with level-$G(K)$ structure by $(P, \mathcal{L}, \phi, \tau)_{\mathrm{LEV}}$ or $(P, \phi, \tau)_{\mathrm{LEV}}$ for brevity.

We remark that $\rho(\phi, \tau) : G(K)_T \to \mathrm{GL}(V(K) \otimes M)$ in (iv) and (vi) is defined in the same manner as before by

$$\begin{aligned} \rho(\phi, \tau)(g)(\theta) : &= \mathrm{ad}((\phi^*)^{-1} \rho_{\mathcal{L}}(g)(\theta) \\ &= (\phi^*)^{-1} \tau(g^{-1})^* (\phi_g)(\phi^* \theta), \end{aligned}$$

with $\phi_g$ a $G(K)_T$-linearization of $\mathcal{L}$, $\theta \in V(K) \otimes O_T$, and $g \in G(K)_T$.

We call $(P, \mathcal{L}, \phi, \tau)_{\mathrm{LEV}}$ a $T$-TSQAS with rigid $G(K)$-structure if

(vi) $\rho(\phi, \tau) = U(K)_T$,

which we denote by $(P, \mathcal{L}, \phi, \tau)_{\mathrm{RIG}}$ or $(P, \phi, \tau)_{\mathrm{RIG}}$.

We see as in Theorem 5.2 that there is a pair of compatible $G(K)_T$-morphisms $(\phi, \Phi) : (P, \mathcal{L}) \to (\mathbf{P}, \mathbf{L})_T$ with regards to the $G(K)_T$-action on $(\mathbf{P}, \mathbf{L})_T$ induced from $\rho(\phi, \tau)$.

### 5.10. $\mathrm{Aut}_T(\mathcal{L}/P)$

Here we insert the general facts. The symbol $\mathrm{Aut}_T(\mathcal{L}/P)$ stands for a scheme (locally of finite type) which represents the functor

$$\begin{aligned} U \mapsto\ & \mathrm{Aut}_T(\mathcal{L}/P)(U) \\ &:= \left\{(g,\phi);\ \begin{array}{l} g \in \mathrm{Aut}_T(P)(U) \text{ and} \\ \phi : \mathcal{L}_U \to g^*(\mathcal{L}_U)\ U\text{-isom. on } P_U \end{array} \right\}, \end{aligned}$$

where $U$ is a $T$-scheme, and $\mathrm{Aut}_T(P)$ is the relative automorphism group of $P$. Since $P$ is projective over $T$, $\mathrm{Aut}_T(P)$ is a $T$-scheme locally of finite type. Suppose $(g,\phi) \in \mathrm{Aut}(\mathcal{L}/P)(U)$. Then $\phi$ induces a $T$-isomorphism $\Phi := \Phi(g,\phi)$ of $\mathbf{P}(O_T \oplus \mathcal{L})$ mapping the subschemes $\mathbf{P}((O_T \oplus \mathcal{L})/\mathcal{L})$ and $\mathbf{P}((O_T \oplus \mathcal{L})/O_T)$ onto themselves. Let $\mathbf{P}_{\mathcal{L}} := \mathbf{P}(O_T \oplus \mathcal{L})$, $\mathbf{S}_0 := \mathbf{P}((O_T \oplus \mathcal{L})/O_T)$ and $\mathbf{S}_\infty := \mathbf{P}((O_T \oplus \mathcal{L})/\mathcal{L})$ temporarily.

Let $g \in \mathrm{Aut}(P)(U)$ and $\phi \in \mathrm{Aut}(\mathcal{L})(U)$. Then $(g,\phi) \in \mathrm{Aut}(\mathcal{L}/P)(U)$ if and only if $\Phi(g,\phi) \in \mathrm{Aut}_T(\mathbf{P}_{\mathcal{L}}) \times_T \mathrm{Aut}_T(\mathbf{S}_0) \times_T \mathrm{Aut}_T(\mathbf{S}_\infty)(U)$ and $p_{\mathcal{L}}\phi = g$, where $p_{\mathcal{L}} : \mathcal{L} \to P$ is the natural projection. Thus $\mathrm{Aut}(\mathcal{L}/P)$ is (represented by) a closed subscheme of $\mathrm{Aut}_T(\mathbf{P}_{\mathcal{L}})$.

Since $G(K)_T$ is finite (proper) over $T$, the image scheme $\tau(G(K)_T)$ is a closed subscheme of $\mathrm{Aut}_T(\mathcal{L}/P)$, which we denote by $G(P,\mathcal{L})$. Thus the condition (v) implies that $G(P_s,\mathcal{L}_s) = G(P,\mathcal{L})_s$ and $\tau_s : G(K) \otimes k(s) \simeq G(P_s,\mathcal{L}_s)$.

In other words, $\mathcal{L}$ has a $G(K)_T$-linearization $\phi_g$ such that $\phi_g \otimes k(s)$ is a $G(K) \otimes k(s)$-linearization of $\mathcal{L}_s$ for any prime point $s$ of $T$, where any fiber $(P_s,\mathcal{L}_s)$ is a $k(s)$-TSQAS with level-$G(K) \otimes k(s)$ structure.

### 5.11. Morphisms of $T$-TSQASes

Let $(P_i,\phi_i,\tau_i)_{\mathrm{LEV}} := (P_i,\mathcal{L}_i,\phi_i,\tau_i)$ $(i = 1,2)$ be $T$-TSQASes with level-$G(K)$ structure and $p_i : P_i \to T$ the projection (structure morphism). We call $f : P_1 \to P_2$ a morphism of $T$-TSQASes level-$G(K)$ structure if there exist a pair of compatible morphisms

$$(f, F(f)) : (P_1,\mathcal{L}_1) \to (P_2,\mathcal{L}_2)$$

and a line bundle $M$ on $T$ with trivial $G(K)_T$-action such that

(i) $\mathcal{L}_1 \simeq p_1^*(M) \otimes f^*(\mathcal{L}_2)$,

(ii) $f$ is a $G(K)_T$-morphism with $\phi_1 = \phi_2 \circ f$,

(iii) $F(f)$ is a $G(K)_T$-morphism, namely,

$$F(f) \circ \tau_1(g) = \tau_2(g) \circ F(f), \quad g \in G(K)_T.$$

Note that $\rho(\phi_1,\tau_1) = \rho(\phi_2,\tau_2)$ by Lemma 5.5 if $(P_1,\phi_1,\tau_1)_{\mathrm{LEV}} \simeq (P_2,\phi_2,\tau_2)_{\mathrm{LEV}}$.

### 5.12. TSQASes over an algebraic space $T$

We call that *an algebraic space* $T$ is by definition the isomorphism class of an étale representative $U \to T$ with étale equivalence relation $R \subset U \times U$. See [9]. Let $p_i : R \to U$ be the composite of the immersion $R \subset U \times U$ with $i$-th projection $(i = 1, 2)$.

For $T$ an algebraic space, *a $T$-TSQAS $(Z, \psi, \tau)_{\mathrm{LEV}}$ with level-$G(K)$ structure* is defined to be a $U$-TSQAS $(Z_U, \psi_U, \tau_U)_{\mathrm{LEV}}$ whose pullbacks by $p_i$ are isomorphic as $R$-TSQASes with level $G(K)$-structure.

To the following two lemmas, we can apply the same proof as in Lemmas 5.7 and 5.8 by replacing $G(K) \otimes k$ with $G(K)_T$.

**Lemma 5.13.** *For any $T$-TSQAS $(P, \phi, \tau)$ with level-$G(K)$ structure, there exists a unique rigid $G(K)$-structure $(\phi(\tau), \tau)$ such that*

$$(P, \phi(\tau), \tau)_{\mathrm{LEV}} \simeq (P, \phi, \tau)_{\mathrm{LEV}}.$$

**Lemma 5.14.** *Let $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ be $T$-TSQASes with rigid $G(K)$-structure $(i = 1, 2)$. Then the following are equivalent:*

(1) $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$,
(2) *there is a $G(K)_T$-isomorphism $f : Z_1 \simeq Z_2$ with $\phi_1 = \phi_2 \circ f$.*

### 5.15. The functor of TSQASes

Now we define the contravariant functor $\mathcal{SQ}_{g,K}^{\mathrm{toric}}$ from the category of algebraic $\mathcal{O}$-spaces to the category of sets as follows. For any $\mathcal{O}$-scheme $T$, we set

$$\begin{aligned}\mathcal{SQ}_{g,K}^{\mathrm{toric}}(T) = {} & \text{the set of torically stable quasi-abelian} \\ & T\text{-schemes } (P, \phi, \tau)_{\mathrm{LEV}} \text{ of relative dimension } g \\ & \text{with level-} G(K) \text{ structure modulo } T\text{-isom}.\end{aligned}$$

In view of Lemma 5.13 and Lemma 5.14, we see

$$\begin{aligned}\mathcal{SQ}_{g,K}^{\mathrm{toric}}(T) = {} & \text{the set of torically stable quasi-abelian} \\ & T\text{-schemes } (P, \phi, \tau)_{\mathrm{RIG}} \text{ of relative dimension } g \\ & \text{with rigid } G(K)\text{-structure modulo } T\text{-isom}.\end{aligned}$$

## §6. PSQASes

In this section we always assume $e_{\min}(K) \geq 3$.

### 6.1. A PSQAS of level $K$

In what follows, we abbreviate a projectively stable quasi-abelian scheme as a PSQAS. Any PSQAS $(Q_0, \mathcal{L}_0)$ is called *a PSQAS of level $K$ over $k(0)$* if $(Q_0, \mathcal{L}_0)$ is a $k(0)$-scheme with $(K(Q_0, \mathcal{L}_0), e^{\sharp}_{S,0}) \simeq (K, e_K) \otimes k(0)$. We have a theorem for $(Q_0, \mathcal{L}_0)$ similar to Theorem 5.2, where $\phi_P$ in the assertion (3) is replaced with a closed $T$-immersion $\phi_Q : Q \to \mathbf{P}(V(K))_S$.

### 6.2. PSQASes over $T$

Let $T$ be an $\mathcal{O}$-scheme. A quadruplet $(Q, \mathcal{L}, \phi, \tau)$ is called *a projectively stable quasi-abelian $T$-scheme* (abbr. *a $T$-PSQAS*) of relative dimension $g$ with level-$G(K)$ structure if the conditions (i)–(v) are true:

(i) $Q$ is a projective flat $T$-scheme with $\pi : Q \to T$ the projection,

(ii) $\mathcal{L}$ is a relatively very ample line bundle of $Q$,

(iii) $\phi : Q \to \mathbf{P}(V(K))_T$ is a closed $T$-immersion such that

$$\phi^* : V(K) \otimes_{\mathcal{O}} M \simeq \pi_* \mathcal{L}$$

for some line bundle $M$ on $T$ with trivial $G(K)_T$-action,

(iv) $\tau : G(K)_T \to \mathrm{Aut}_T(\mathcal{L}/Q)$ is a closed immersion of a group $T$-scheme, which makes $\phi$ a $G(K)_T$-morphism in the sense that

$$\phi(\tau(g) \cdot z) = S_{\rho(\phi,\tau)(g)} \phi(z) \quad (z \in Q),$$

and that $\phi^* : V(K) \otimes_{\mathcal{O}} M \simeq \pi_* \mathcal{L}$ in (iii) is a $G(K)$-isomorphism, where $\rho(\phi, \tau)$ and $S_{\rho(\phi,\tau)(g)}$ are defined similarly to those for $T$-TSQASes,

(v) for any prime point $s$ of $T$, the fiber at $s$ $(Q_s, \mathcal{L}_s, \phi_s, \tau_s)$ is *a PSQAS* of dimension $g$ over $k(s)$ with level-$G(K)$ structure.

We denote a $T$-PSQAS $(Q, \mathcal{L}, \phi, \tau)$ by $(Q, \mathcal{L}, \phi, \tau)_{\mathrm{LEV}}$ or $(Q, \phi, \tau)_{\mathrm{LEV}}$. We call $(Q, \mathcal{L}, \phi, \tau)_{\mathrm{LEV}}$ a $T$-PSQAS with rigid $G(K)$-structure if

(vi) $\rho(\phi, \tau) = U(K)_T$,

which we denote $(Q, \mathcal{L}, \phi, \tau)_{\mathrm{LEV}}$ by $(Q, \mathcal{L}, \phi, \tau)_{\mathrm{RIG}}$ or $(Q, \phi, \tau)_{\mathrm{RIG}}$.

We see that there is a pair of compatible $G(K)_T$-morphisms $(\phi, \Phi) : (P, \mathcal{L}) \to (\mathbf{P}, \mathbf{L})_T$ with regards to the $G(K)_T$-action on $(\mathbf{P}, \mathbf{L})_T$ induced from $\rho(\phi, \tau)$.

### 6.3. Morphisms of $T$-PSQASes

Let $(Q_i, \mathcal{L}_i, \phi_i, \tau_i)_{\mathrm{LEV}}$ $(i = 1, 2)$ be $T$-PSQASes with level-$G(K)$ structure and $p_i : Q_i \to T$ the projection (structure morphism). Then $f : Q_1 \to Q_2$ is called *a morphism of $T$-PSQASes with level-$G(K)$ structure* if there exist a pair of compatible morphisms $(f, F(f)) : (Q_1, \mathcal{L}_1) \to (Q_2, \mathcal{L}_2)$ and a line bundle $M$ on $T$ with trivial $G(K)_T$-action such that

(i) $\mathcal{L}_1 \simeq p_1^*(M) \otimes f^*(\mathcal{L}_2)$,
(ii) $f$ is a $G(K)_T$-morphism with $\phi_1 = \phi_2 \circ f$,
(iii) $F(f)$ is a $G(K)_T$-morphism, namely,

$$F(f) \circ \tau_1(g) = \tau_2(g) \circ F(f), \quad g \in G(K)_T.$$

We note $\rho(\phi_1, \tau_1) = \rho(\phi_2, \tau_2)$ if $(Q_1, \phi_1, \tau_1)_{\mathrm{LEV}} \simeq (Q_2, \phi_2, \tau_2)_{\mathrm{LEV}}$. This is proved similarly to Lemma 5.5.

With these definitions of PSQASes and morphisms between them, we will have the functor of PSQASes similar to that of TSQASes. We omit the details. See [18].

We quote from [18] two lemmas similar to Lemmas 5.14 and 5.13. See [18, Lemma 9.7, Lemma 9.8].

**Lemma 6.4.** *For a $T$-PSQAS $(Z, \phi, \tau)$ with level-$G(K)$ structure, there exists a unique rigid $G(K)$-structure $(\phi(\tau), \tau)$ such that*

$$(Z, L, \phi(\tau), \tau)_{\mathrm{LEV}} \simeq (Z, L, \phi, \tau)_{\mathrm{LEV}}.$$

*Proof.* The proof is the same as Lemma 5.13. Q.E.D.

**Lemma 6.5.** *Let $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ be $T$-PSQASes with rigid $G(K)$-structure $(i = 1, 2)$. Then the following are equivalent:*

(1) $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$,
(2) *there is a $T$-isomorphism $f : Z_1 \simeq Z_2$ with $\phi_1 = \phi_2 \circ f$.*

*Proof.* For simplicity we denote any $G(K)$-action as $z \mapsto g \cdot z$ below. Suppose (2). Then $f$ is a $G(K)_T$-morphism. In fact, since $\phi_i$ is a $G(K)_T$-morphism, we have

$$\phi_2(g \cdot f(z)) = g \cdot \phi_2(f(z)) = g \cdot \phi_1(z) = \phi_1(g \cdot z) = \phi_2(f(g \cdot z)),$$

whence $\phi_2(g \cdot f(z)) = \phi_2 \circ f(g \cdot z)$. Since $\phi_2$ is injective, $g \cdot f(z) = f(g \cdot z)$. This shows that $f$ is a $G(K)_T$-morphism. The rest of the proof is the same as Lemma 5.14. Q.E.D.

The following lemma has already been proved essentially in [18, Theorem 11.4].

**Lemma 6.6.** (*The first valuative lemma for separatedness*) *We assume $e_{\min}(K) \geq 3$. Let $R$ be a discrete valuation ring, $S :=$ Spec $R$, $\eta$ the generic point of $S$ and $k(\eta)$ the fraction field of $R$. Let $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ be $S$-PSQASes with rigid $G(K)$-structure. If $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ are isomorphic over $k(\eta)$, then they are isomorphic over $S$.*

*Proof.* We first note that $\phi_i$ in this lemma is a closed immersion. Let $H = \mathrm{Hilb}^{P(n)}_{\mathbf{P}(V(K))}$ be the Hilbert scheme parametrizing all closed subschemes of $\mathbf{P}(V(K))$, whose Hilbert polynomial are equal to $P(n) =$

$n^g\sqrt{|K|}$, and $X_{\mathrm{univ}}$ the universal subscheme of $\mathbf{P}(V(K))$ over $H$. Then by the universality of $X_{\mathrm{univ}}$, $\phi_i$ induces a unique morphism $\mathrm{Hilb}(\phi_i)$ : $S \to H$ such that $Z_i$ is the pullback by $\mathrm{Hilb}(\phi_i)$ of $X_{\mathrm{univ}}$.

By the assumption and Lemma 6.5, there is a $k(\eta)$-isomorphism (in fact, $G(K) \otimes k(\eta)$-isomorphism) $f_\eta : Z_{1,\eta} \to Z_{2,\eta}$ such that $\phi_{1,\eta} = \phi_{2,\eta} \circ f_\eta$. It follows from the very definition of $H = \mathrm{Hilb}_{\mathbf{P}(V(K))}$ that $\mathrm{Hilb}(\phi_{1,\eta}) = \mathrm{Hilb}(\phi_{2,\eta})$. Since $H$ is separated, $\mathrm{Hilb}(\phi_1) = \mathrm{Hilb}(\phi_2)$, hence $\phi_1(Z_1) = \phi_2(Z_2)$. This implies that there is an $S$-isomorphism $f : Z_1 \to Z_2$ extending $f_\eta$ such that $\phi_1 = \phi_2 \circ f$. It is clear that $f$ is a $G(K)_S$-morphism because $f$ is a $G(K) \otimes k(\eta)$-morphism. This proves $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$ by Lemma 6.5. Q.E.D.

**Lemma 6.7.** (*The second valuative lemma for separatedness*) *We assume* $e_{\min}(K) \geq 3$. *Let $R$ be a discrete valuation ring,* $S :=$ Spec $R$, *and $k(\eta)$ the fraction field of $R$. Let $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ be $S$-TSQASes with rigid $G(K)$-structure whose generic fibers are abelian varieties. Suppose that $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ are isomorphic over $k(\eta)$. Then they are isomorphic over $S$.*

*Proof.* By Theorem 2.23, we have two $S$-PSQASes $(Q_i, \phi_{Q_i}, \tau_{Q_i})_{\mathrm{RIG}}$ such that $(Q_i, \phi_{Q_i}, \tau_{Q_i}) \otimes k(\eta) \simeq (Z_i, \phi_i, \tau_i) \otimes k(\eta)$, where $Z_i$ is the normalization of $Q_i$. In view of Lemma 6.6, there is a $G(K)$-isomorphism $h : (Q_1, \phi_{Q_1}, \tau_{Q_1}) \to (Q_2, \phi_{Q_2}, \tau_{Q_2})$, which induces an isomorphism of their normalizations $h^{\mathrm{norm}} : (Z_1, \phi_{Z_1}, \tau_{Z_1}) \to (Z_2, \phi_{Z_2}, \tau_{Z_2})$. Q.E.D.

### 6.8. The functor of PSQASes

The functor of PSQASes is defined in a manner similar to that of TSQASes. We define the contravariant functor $\mathcal{SQ}_{g,K}$ from the category of $\mathcal{O}$-schemes to the category of sets as below, which is almost the same as in [18] except the point that we use [20, Theorem 5.17] (see also Theorem 2.22). In view of Lemma 6.4 and Lemma 6.5, we see for any $\mathcal{O}$-scheme $T$,

$$\begin{aligned}\mathcal{SQ}_{g,K}(T) : &= \text{the set of projectively stable quasi-abelian}\\ &\quad T\text{-schemes } (P,\phi,\tau)_{\mathrm{LEV}} \text{ of relative dimension } g\\ &\quad \text{with level-}G(K) \text{ structure modulo } T\text{-isom}\\ &= \text{the set of projectively stable quasi-abelian}\\ &\quad T\text{-schemes } (P,\phi,\tau)_{\mathrm{RIG}} \text{ of relative dimension } g\\ &\quad \text{with rigid } G(K)\text{-structure modulo } T\text{-isom}.\end{aligned}$$

## §7. Rigid $\rho$-structures

### 7.1. Examples

It is worthy of further study in the cases of other irreducible representations $\rho : G \to \mathrm{GL}(V)$ of $G$ connected or discrete, finite or infinite, where $(Z, L)$ is no longer a TSQAS nor a PSQAS. The moduli of those schemes embedded in $\mathbf{P}(V)$ with rigid $\rho$-structure is just the subset of the Hilbert scheme $\mathrm{Hilb}\,\mathbf{P}(V)$ consisting of all $G$-invariant closed subschemes of $\mathbf{P}(V)$. Any $G$-invariant closed subscheme of $\mathbf{P}(V)$ is known to have Hilbert points, each of which is Kempf-stable, in other words, each of which has a closed $\mathrm{SL}(V)$-orbit in the semi-stable locus if $\rho$ is an irreducible representation. In this sense, it is worthy of further study even in some of particular cases. Our moduli $SQ_{g,K}$ [18] gives an example of it. See also [18, Section 13]. The moduli of $(1,5)$-polarized abelian surfaces embedded in $\mathbf{P}^4$ gives another example [8].

### 7.2. Start

Let $T$ be an $\mathcal{O}$-scheme. Let $G$ be a group $\mathcal{O}$-scheme, $V$ a free $\mathcal{O}$-module of finite rank. Suppose that $V \otimes k$ is an irreducible $G \otimes k$-module for any field $k$ over $\mathcal{O}$. Let $\rho : G \to \mathrm{GL}(V)$ be a homomorphism induced from the $G$-module structure of $V$. We fix $\rho$ for all.

We assume the lemma of Schur for $\rho$. In other words, if $a \in \mathrm{GL}(V)$ commutes with any $\rho(g)$ $(g \in G)$, then $a$ is a scalar matrix.

### 7.3. Rigid $\rho$-structures

A quadruplet $(Z, L, \phi, \tau)$ is called a $T$-scheme with $\rho$-structure if the conditions (i)–(v) are true:

(i) $Z$ is a projective flat $T$-scheme with $\pi : Z \to T$ the projection,
(ii) $L$ is a relatively very ample $G_T$-linearized line bundle of $Z$,
(iii) $\phi : Z \to \mathbf{P}(V)_T$ is a closed $G_T$-immersion such that

$$\phi^* : V \otimes_{\mathcal{O}} M \simeq \pi_* L$$

for some line bundle $M$ on $T$ with trivial $G_T$-action,
(iv) $\tau : G_T \to \mathrm{Aut}_T(L/Z)$ is a closed immersion of a group $T$-scheme, which makes $\phi$ a $G_T$-morphism in the sense that

$$\phi(\tau(g) \cdot z) = S_{\rho(\phi,\tau)(g)} \phi(z) \quad (z \in Z),$$

where $\rho(\phi, \tau)$ is defined similarly to those for $T$-TSQASes,
(v) $\rho(\phi, \tau)$ is $\mathrm{GL}(V \otimes_{\mathcal{O}} M)$-equivalent to $\rho_T$.

By $(Z, L, \phi, \tau)$ or $(Z, \phi, \tau)$ we denote a $T$-scheme with $\rho$-structure $(Z, L, \phi, \tau)$. Let $(\mathbf{P}, \mathbf{L}) := (\mathbf{P}(V), \mathbf{L}(V))$. For a given $(Z, L, \phi, \tau)$, there is a pair of compatible $G_T$-morphisms $(\phi, \Phi) : (Z, L) \to (\mathbf{P}, \mathbf{L})_T$ with regards to the $G_T$-action on $(\mathbf{P}, \mathbf{L})_T$ induced from $\rho(\phi, \tau)$.

We call a $\rho$-structure $(Z, L, \phi, \tau)$ a rigid $\rho$-structure if

(vi) $\rho(\phi, \tau) = \rho_T$,

which we denote by $(Z, L, \phi, \tau)_{\mathrm{RIG}}$ or $(Z, \phi, \tau)_{\mathrm{RIG}}$.

### 7.4. Morphisms of rigid $\rho$-structures

Let $(Z_i, \phi_i, \tau_i)$ $(i = 1, 2)$ be $T$-schemes with $\rho$-structures. Then $f : Z_1 \to Z_2$ is called *a morphism of $T$-schemes with $\rho$-structure* if there exists a pair of compatible morphisms $(f, F(f)) : (Z_1, L_1) \to (Z_2, L_2)$ and a line bundle $M$ on $T$ with trivial $G(K)_T$-action such that

(i) $\mathcal{L}_1 \simeq p_1^*(M) \otimes f^*(\mathcal{L}_2)$,
(ii) $f$ is a $G_T$-morphism with $\phi_1 = \phi_2 \circ f$,
(iii) $F(f)$ is a $G_T$-morphism, namely,

$$F(f) \circ \tau_1(g) = \tau_2(g) \circ F(f), \quad g \in G_T.$$

We note $\rho(\phi_1, \tau_1) = \rho(\phi_2, \tau_2)$ if $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$. This is proved similarly to Lemma 5.5.

**Remark 7.5.** Let $\rho : G \to \mathrm{GL}(V)$ be the irreducible representation we start with. Let $Z$ be a $G$-stable subscheme of $\mathbf{P}(V)$ with regards to the $\rho$-action of $G$ on $\mathbf{P}$. Let $i_Z : Z \to \mathbf{P}$ be the natural inclusion of $Z$, and $L = \mathbf{L}_Z$ the restriction of $\mathbf{L} := \mathbf{L}(V)$. Since $\mathrm{GL}(V) = \mathrm{Aut}(\mathbf{L}/\mathbf{P})$, and $Z$ is $G$-stable, $G$ acts on the pair $(Z, L)$ in the compatible manner. In other words, $L$ has a $G$-linearization via $\rho$. This implies that $\rho$ induces a closed immersion $\tau_Z : G \to \mathrm{Aut}(L/Z)$. Then the triple $(Z, i_Z, \tau_Z)$ is a rigid $\rho$-structure.

The following are analogous to Lemmas 5.13 and Lemmas 5.14.

**Lemma 7.6.** *For any $T$-scheme $(Z, \phi, \tau)$ with $\rho$-structure, there exists a unique rigid $\rho$-structure $(\phi(\tau), \tau)$ such that $(Z, \phi(\tau), \tau)$ is isomorphic to $(Z, \phi, \tau)$ as $T$-schemes with $\rho$-structure.*

*Proof.* The proof is the same as Lemma 5.13. Q.E.D.

**Lemma 7.7.** *Let $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ be $T$-schemes with rigid $\rho$-structure $(i = 1, 2)$. Then the following are equivalent:*

(1) $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$,
(2) *there is a $T$-isomorphism $f : Z_1 \simeq Z_2$ with $\phi_1 = \phi_2 \circ f$.*

*Proof.* The proof is the same as Lemma 6.5. It suffices to replace $G(K)$ in Lemma 6.5 by $G$. Q.E.D.

The following lemma is an analogue to Lemma 6.6.

**Lemma 7.8.** (*The third valuative lemma for separatedness*) *Let $R$ be a complete discrete valuation ring, $S :=$ Spec $R$, and $\eta$ the generic point of $S$. If rigid $\rho_S$-structures $(Z_i, \phi_i, \tau_i)_{\mathrm{RIG}}$ $(i = 1, 2)$ are isomorphic over $k(\eta)$, then they are isomorphic over $S$.*

*Proof.* The proof is quite analogous to that of Lemma 6.6. Let $k(\eta)$ be the fraction field of $R$. Let $H = \mathrm{Hilb}_{\mathbf{P}(V(K))}$ be the Hilbert scheme parametrizing all closed subschemes of $\mathbf{P}(V(K))$, and $X_{\mathrm{univ}}$ the universal subscheme of $\mathbf{P}(V(K))$ over $H$. We note that $H$ is locally of finite type. By the universality of $X_{\mathrm{univ}}$, $\phi_i$ induces a unique morphism $\mathrm{Hilb}(\phi_i) : S \to H$ such that $Z_i$ is the pullback by $\mathrm{Hilb}(\phi_i)$ of $X_{\mathrm{univ}}$. By the assumption and Lemma 7.7, there is a $k(\eta)$-isomorphism (in fact, $G \otimes k(\eta)$-isomorphism) $f_\eta : Z_{1,\eta} \to Z_{2,\eta}$ such that $\phi_{1,\eta} = \phi_{2,\eta} \circ f_\eta$.

It follows from the definition of $H = \mathrm{Hilb}_{\mathbf{P}(V(K))}$ that $\mathrm{Hilb}(\phi_{1,\eta}) = \mathrm{Hilb}(\phi_{2,\eta})$. Since $H$ is separated, $\mathrm{Hilb}(\phi_1) = \mathrm{Hilb}(\phi_2)$, hence $\phi_1(Z_1) = \phi_2(Z_2)$. This implies that there is an $S$-isomorphism $f : Z_1 \to Z_2$ extending $f_\eta$ such that $\phi_1 = \phi_2 \circ f$. It is clear that $f$ is a $G_S$-morphism because $f$ is a $G \otimes k(\eta)$-morphism. Hence $(Z_1, \phi_1, \tau_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \tau_2)_{\mathrm{RIG}}$ by Lemma 7.7. Q.E.D.

## §8. The stable reduction theorem

### 8.1. The rigid $G(K)$-structure we start from

Let $R$ be a complete discrete valuation ring, $k(\eta)$ (resp. $k(0)$) the fraction field (resp. the residue field) of $R$, and $S =$ Spec $R$. Let $(G_\eta, \mathcal{L}_\eta)$ be a polarized abelian variety over $k(\eta)$ with $\mathcal{L}_\eta$ ample and $K(\mathcal{L}_\eta) := \ker \lambda(\mathcal{L}_\eta)$. Let $e^{\mathcal{L}_\eta}$ be the Weil pairing of $K(\mathcal{L}_\eta)$. Since $e^{\mathcal{L}_\eta}$ is nondegenerate, $R$ contains a primitive $N$-th root $\zeta_N$ of unity.

Suppose that the order of $K(\mathcal{L}_\eta)$ and the characteristic of $k(0)$ are coprime. Then there exists a finite symplectic constant abelian group $\mathbf{Z}$-scheme $(K, e_K)$ such that $(K, e_K) \otimes_{\mathbf{Z}} k(\eta) \simeq (K(\mathcal{L}_\eta), e^{\mathcal{L}_\eta})$. Moreover by taking a finite extension of $k(\eta)$, we may assume that the Heisenberg group scheme $\mathcal{G}(\mathcal{L}_\eta)$ is isomorphic to $\mathcal{G}(K) \otimes k(\eta)$, hence it has a subgroup scheme $G(K) \otimes k(\eta)$.

Let $N = e_{\max}(K)$. If $e_{\min}(K) \geq 3$, then by Theorem 5.2 $(G_\eta, \mathcal{L}_\eta)$ admits a level-$G(K)$ structure $(\phi_\eta, \tau_\eta)$ such that $\tau_\eta : G(K) \otimes k(\eta) \simeq G(G_\eta, \mathcal{L}_\eta)$. It follows from Lemma 5.7 that $(G_\eta, \mathcal{L}_\eta)$ has a unique rigid $G(K)$-structure $(\phi_\eta(\tau_\eta), \tau_\eta)$ such that

$$(G_\eta, \mathcal{L}_\eta, \phi_\eta(\tau_\eta), \tau_\eta)_{\mathrm{LEV}} \simeq (G_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{LEV}}.$$

In other words, we have a $G(K) \otimes k(\eta)$-linearization of $\mathcal{L}_\eta$ and a pair of compatible $G(K) \otimes k(\eta)$-morphisms

$$(9) \qquad (\phi_\eta, \Phi_\eta) : (G_\eta, \mathcal{L}_\eta) \to (\mathbf{P}(V(K)), \mathbf{L}(V(K))) \otimes k(\eta)$$

with $G(K) \otimes k(\eta)$-action on $(\mathbf{P}(V(K)), \mathbf{L}(V(K)))$ via $U(K) \otimes k(\eta)$.

By combining [18, Lemma 7.8], Section 2, Section 4 and Theorem 5.2 all together, we infer

**Theorem 8.2.** *Let $R$ be a complete discrete valuation ring and $S =$ Spec $R$. Let $(G_\eta, \mathcal{L}_\eta)$ be a polarized abelian variety over $k(\eta)$, $K(\mathcal{L}_\eta) :=$ $\ker \lambda(\mathcal{L}_\eta)$, $(G_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}$ a rigid $G(K)$-structure and $(\phi_\eta, \Phi_\eta)$ the pair (9) of compatible $G(K)$-morphisms. Assume that*

(i) *the characteristic of $k(0)$ and the order of $K(\mathcal{L}_\eta)$ are coprime,*
(ii) $e_{\min}(K(\mathcal{L}_\eta)) \geq 3$.

*Then after a suitable finite base change if necessary, there exist flat projective schemes $(P, \mathcal{L})$ and $(Q, \mathcal{L})$, semiabelian group schemes $G$ and $G^\sharp$, the flat closure $K(P, \mathcal{L})$ of $K(\mathcal{L}_\eta)$ in $G^\sharp$, a symplectic form $e^\sharp_S$ on $K(P, \mathcal{L})$ extending $e^{\mathcal{L}_\eta}$ and the Heisenberg group schemes $\mathcal{G}(P, \mathcal{L})$ and $G(P, \mathcal{L})$ of $(P, \mathcal{L})$, all of these being defined over $S$, such that*

(1) *$P$ is projective flat and reduced over $S$, and normal,*
(2) *$(G, \mathcal{L})$ and $(G^\sharp, \mathcal{L})$ are open subschemes of $(P, \mathcal{L})$,*
(3) $G^\sharp = K(P, \mathcal{L}) \cdot G$,
(4) $(G_\eta, \mathcal{L}_\eta) \simeq (G^\sharp_\eta, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta) \simeq (Q_\eta, \mathcal{L}_\eta)$,
(5) *there exists a constant finite symplectic abelian group $\mathbf{Z}$-scheme $(K, e_K)$ such that $(K(P, \mathcal{L}), e^\sharp_S) \simeq (K, e_K)_S$, $\mathcal{G}(P, \mathcal{L}) \simeq \mathcal{G}(K)_S$, hence we have an isomorphism $\tau_P : G(K)_S \to G(P, \mathcal{L})$. In particular, $\mathcal{L}$ is $G(K)_S$-linearized,*
(6) *$\phi_P : P \to \mathbf{P}(V(K))_S$ be the morphism associated with the linear system $\Gamma(P, \mathcal{L})$ such that $\phi_P \otimes k(\eta) = \phi_\eta$. Then $(P, \mathcal{L}, \phi_P, \tau_P)$ is a rigid $G(K)$-structure extending $(G_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}$,*
(7) *We have a pair of compatible $G(K)_S$-morphisms*

$$(\phi_P, \Phi_P) : (P, \mathcal{L}) \to (\mathbf{P}(V(K))_S, \mathbf{L}(V(K))_S),$$

*which extend $(\phi_\eta, \Phi_\eta)$,*

(8) $\Gamma(G^\sharp, \mathcal{L}) \simeq \Gamma(P, \mathcal{L}) \simeq \Gamma(Q, \mathcal{L}) \simeq V(K) \otimes_{O_N} R.$

Here we restate the stable reduction theorem for $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ by adjusting [18, Theorem 10.4] to the definitions of PSQASes given in Section 6.

**Theorem 8.3.** *Let $R$ be a complete discrete valuation ring and $S =$ Spec $R$. Let $(G_\eta, \mathcal{L}_\eta)$ be a polarized abelian variety over $k(\eta)$, and $(G_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}$ a rigid $G(K)$-structure on it. Assume that*

(i) *the characteristic of* $k(0)$ *and the order of* $K(\mathcal{L}_\eta)$ *are coprime,*
(ii) $e_{\min}(K(\mathcal{L}_\eta)) \geq 3$.

*Then after a suitable finite base change if necessary, there exist an S-TSQAS* $(P, \mathcal{L}, \phi_P, \tau_P)_{\mathrm{RIG}}$ *with rigid* $G(K)$*-structure and an S-PSQAS* $(Q, \mathcal{L}, \phi_Q, \tau_Q)_{\mathrm{RIG}}$ *with rigid* $G(K)$*-structure such that*

$$\begin{aligned}(P, \mathcal{L}, \phi_P, \tau_P)_{\mathrm{RIG}} \otimes k(\eta) &\simeq (Q, \mathcal{L}, \phi_Q, \tau_Q)_{\mathrm{RIG}} \otimes k(\eta)\\ &\simeq (G_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}.\end{aligned}$$

## §9. The scheme parametrizing TSQASes

Let $K$ be a symplectic finite abelian group with symplectic form $e_K$. We choose and fix a maximally $e_K$-isotropic subgroup $I(K)$ of $K$ such that $K = I(K) \oplus I(K)^\vee$. Let $N = e_{\max}(K) = e_{\max}(I(K))$ and $\mathcal{O} = \mathcal{O}_N = \mathcal{O}[\zeta_N, 1/N]$.

Assume $e_{\min}(K) = e_{\min}(I(K)) \geq 3$.

### 9.1. $\mathrm{Hilb}^P(X/T)$

Let $(X, L)$ be a polarized $\mathcal{O}$-scheme with $L$ very ample and $P(n)$ an arbitrary polynomial. Let $\mathrm{Hilb}^P(X)$ be the Hilbert scheme parametrizing all closed subschemes $Z$ of $X$ with $\chi(Z, nL_Z) = P(n)$. As is well known $\mathrm{Hilb}^P(X)$ is a projective $\mathcal{O}$-scheme.

Let $T$ be a projective scheme, $(X, L)$ a flat projective $T$-scheme with $L$ an ample line bundle of $X$, and $\pi : X \to T$ the projection. Then for an arbitrary polynomial $P(n)$, let $\mathrm{Hilb}^P(X/T)$ be the scheme parametrizing all closed subschemes $Z$ of $X$ with $\chi(Z, nL_Z) = P(n)$ such that $Z$ is contained in fibers of $\pi$. Let $M$ be a very ample line bundle of $T$. Then $\mathrm{Hilb}^P(X/T)$ is the $\mathcal{O}$-subscheme of $\mathrm{Hilb}^P(X)$ parametrizing all closed subschemes $Z$ with $(\pi^*M)_Z$ trivial. Hence $\mathrm{Hilb}^P(X/T)$ is a closed subscheme of $\mathrm{Hilb}^P(X)$.

Let $\mathrm{Hilb}^P_{\mathrm{conn}}(X/T)$ be the subscheme of $\mathrm{Hilb}^P(X/T)$ consisting of connected subschemes $Z \in \mathrm{Hilb}^P(X/T)$ of $X$. Then $\mathrm{Hilb}^P_{\mathrm{conn}}(X/T)$ is an open and closed $\mathcal{O}$-subscheme of $\mathrm{Hilb}^P(X/T)$.

### 9.2. The scheme $H_1 \times H_2$

Choose and fix a coprime pair of natural integers $d_1$ and $d_2$ such that $d_1 > d_2 \geq 2g+1$ and $d_i \equiv 1 \mod N$. This pair does exist because it is enough to choose prime numbers $d_1$ and $d_2$ large enough such that $d_i \equiv 1 \mod N$ and $d_1 > d_2$. We choose integers $q_i$ such that $q_1d_1 + q_2d_2 = 1$.

Now consider a $G(K)$-module $W_i(K) := W_i \otimes V(K) \simeq V(K)^{\oplus N_i}$ where $N_i = d_i^g$ and $W_i$ is a free $\mathcal{O}$-module of rank $N_i$ with trivial $G(K)$-action. Let $\sigma_i$ be the natural action of $G(K)$ on $W_i(K)$. In what follows we always consider $\sigma_i$.

Let $H_i$ $(i = 1, 2)$ be the Hilbert scheme parametrizing all closed polarized subschemes $(Z_i, L_i)$ of $\mathbf{P}(W_i(K))$ such that

(a) $Z_i$ is $G(K)$-stable,

(b) $\chi(Z_i, nL_i) = n^g d_i^g \sqrt{|K|}$, where $L_i = \mathbf{L}(W_i(K)) \otimes O_{Z_i}$.

Let $X_i$ be the universal subscheme of $\mathbf{P}(W_i(K))$ over $H_i$. Let $X = X_1 \times_{\mathcal{O}} X_2$ and $H = H_1 \times_{\mathcal{O}} H_2$. Let $p_i : X_1 \times_{\mathcal{O}} X_2 \to X_i$ be the $i$-th projection, $\pi : X \to H$ the natural projection. Hence $X$ is a subscheme of $\mathbf{P}(W_1(K)) \times_{\mathcal{O}} \mathbf{P}(W_2(K)) \times_{\mathcal{O}} H$, flat over $H = H_1 \times_{\mathcal{O}} H_2$.

We note that $\mathbf{L}(W_i(K))$ has a $G(K)$-linearization $\{\psi_g^{(i)}\}$, which we fix for all. Since $G(K)$ transforms any closed $G(K)$-stable subscheme $Z$ of $\mathbf{P}(W_i(K))$ onto itself, it follows that $G(K)$ acts on $H_i$ trivially, while $G(K)$ acts on $X_i$ non-trivially. Hence $G(K)$ acts on $H$ trivially, and on $X$ non-trivially.

### 9.3. The scheme $U_1$

The aim of this and the subsequent sections is to construct a new compactification of the moduli space of abelian varieties as the quotient of a certain $\mathcal{O}$-subscheme of $\mathrm{Hilb}^P_{\mathrm{conn}}(X/H)$ by $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$.

Let $B$ be the pullback to $X$ of a very ample line bundle on $H$. Let $M_i = p_i^*(\mathbf{L}(W_i(K))) \otimes O_X$ and $M = d_2M_1 + d_1M_2 + B$. Then $M$ is a very ample line bundle on $X$. Since $M_i$ is $G(K)$-linearized and $B$ is trivially $G(K)$-linearized, $M$ is $G(K)$-linearized. Since $G(K)$ acts on $H$ trivially, $G(K)$ transforms any fiber $X_u$ of $\pi : X \to H$ into $X_u$ itself.

Let $P(n) = (2nd_1d_2)^g \sqrt{|K|}$. Let $\mathrm{Hilb}^P_{\mathrm{conn}}(X/H)$ be the Hilbert scheme parametrizing all connected closed subschemes $Z$ of $X$ contained in the fibers of $\pi : X \to H$ with $\chi(Z, nM_Z) = P(n)$, and $Z^P_{\mathrm{conn}}$ be the universal subscheme of $X$ over it. We denote $\mathrm{Hilb}^P_{\mathrm{conn}}(X/H)$ by $H^P_{\mathrm{conn}}$ for brevity. Now using the double polarization trick of Viehweg, we define $U_1$ to be the subset of $H^P_{\mathrm{conn}}$ consisting of all subschemes $Z$ of $X$ with the properties

(i) $p_{i|Z}$ is an isomorphism $(i = 1, 2)$,

(ii) $d_2L_1 = d_1L_2$, where $L_i = M_i \otimes O_Z$,

(iii) $Z$ is $G(K)$-stable.

We prove that $U_1$ is a nonempty closed $\mathcal{O}$-subscheme of $H^P_{\mathrm{conn}}$.

The condition (i) that $p_{i|Z}$ is an isomorphism is open and closed, while the condition (ii) $d_2L_1 = d_1L_2$ is closed. The condition (iii), the $G(K)$-stability of $Z$, is equivalent to the condition that $Z \in H^P_{\mathrm{conn}}$ is a fixed point by the natural $G(K)$-action induced from those $G(K)$-actions on $X$ and $H$. Hence it is a closed condition. Hence $U_1$ is a closed, hence a projective $\mathcal{O}$-subscheme of $H^P_{\mathrm{conn}}$.

It remains to show $U_1 \neq \emptyset$. Let $k$ be an algebraically closed field over $\mathcal{O}$. By Lemma 4.2, there exists a polarized abelian variety $(A, L)$ over $k$ with $\tau_A : G(K) \otimes k \simeq G(A, L)$ an isomorphism. Hence $L$ has a $G(K) \otimes k$-linearization of weight one. Hence $d_iL$ has a $G(K) \otimes k$-linearization of weight $d_i$ too. Since $d_i \equiv 1 \mod N$, and since $a^N = 1$ for any $a \in \mu_N$, $d_iL$ has a $G(K) \otimes k$-linearization of weight one. Hence by Lemma 4.4, $\Gamma(A, d_iL) \simeq \Gamma(A, d_iL)(0) \otimes V(K) \otimes k \simeq W_i \otimes V(K) \otimes k$ because $\dim \Gamma(A, d_iL)(0) = d_i^g = N_i = \dim W_i$, where $\Gamma(A, d_iL)(0) = \{v \in \Gamma(A, d_iL); h \cdot v = 0 \ (\forall h \in I(K))\}$ is regarded as a trivial $G(K)$-module. Since $\Gamma(A, d_iL)$ is very ample, we can choose a $G(K) \otimes k$-equivariant closed immersion $\phi_i : A \to \mathbf{P}(W_i(K))$, whose image $\phi_i(A)$ is a $G(K) \otimes k$-stable subscheme of $\mathbf{P}(W_i(K))$, isomorphic to $A$. Thus $\phi_i(A) \in H_i(k)$. Let $Z := (\phi_1 \times \phi_2)(\Delta)$ $(\simeq A)$ be the image of the diagonal $\Delta$ $(\subset A \times A)$. Since $Z \simeq A$, we see that

$$\begin{aligned}&\chi(Z,\, n(d_2L_1 + d_1L_2 + B)_Z)\\ &= \chi(A, 2nd_1d_2L) = (2nd_1d_2)^g\sqrt{|K|} = P(n).\end{aligned}$$

It follows that $Z \in \mathrm{Hilb}^P_{\mathrm{conn}}(X/H)$. Since $\phi_i$ is $G(K)\otimes k$-equivariant, $Z$ is $G(K) \otimes k$-stable. Hence $Z \in U_1(k)$. It follows that $U_1 \neq \emptyset$.

**Lemma 9.4.** *Let $k$ be an algebraically closed field over $\mathcal{O}$. Let $Z \in U_1(k)$ and $L = q_1L_1 + q_2L_2$. Then $L_i = d_iL$.*

*Proof.* One sees $d_1L = d_1(q_1L_1 + q_2L_2) = (d_1q_1 + d_2q_2)L_1 = L_1$, while $d_2L = d_2(q_1L_1 + q_2L_2) = (d_1q_1 + d_2q_2)L_2 = L_2$. Q.E.D.

### 9.5. The scheme $U_2$

Let $X = X_1 \times_{\mathcal{O}} X_2$, $L = q_1L_1 + q_2L_2$ and $q_i$ the integers with $d_1q_1 + d_2q_2 = 1$. Let $U_2$ be the open subscheme of $U_1$ consisting of all subschemes $Z$ of $X$ such that besides (i)–(iii) the following are satisfied:

(iv) $Z$ is reduced,
(v) $L_Z$ is ample,
(vi) $\chi(Z, nL_Z) = n^g\sqrt{|K|}$,
(vii) $H^q(Z, nL_Z) = 0$ for $q > 0$ and $n > 0$,
(viii) $\Gamma(Z, L_Z)$ is base point free,
(ix) $H^0(p_i^*) : W_i(K) \otimes k(u) \to \Gamma(Z, L_i \otimes O_Z)$ is surjective for $i = 1, 2$,

where $u \in \mathrm{Hilb}^P_{\mathrm{conn}}(X/H)$ is the point defined by $(Z, L_Z)$. It is clear that (iv)–(ix) are open conditions. Note that surjectivity of $H^0(p_i^*)$ in

(ix) implies isomorphism of $H^0(p_i^*)$ in view of (vi) and (vii). In fact, by Lemma 9.4, $L_i = d_iL$. Hence $H^q(Z, L_i \otimes O_Z) = H^q(Z, d_iL_Z) = 0$ for $q > 0$, whence $h^0(Z, L_i \otimes O_Z) = d_i^g\sqrt{|K|}$ by (vi). Since $\operatorname{rank}_{\mathcal{O}} W_i(K) = d_i^g\sqrt{|K|}$, this implies that $H^0(p_i^*)$ is an isomorphism.

We note $U_2 \neq \emptyset$. In fact, letting $k$ be an algebraically closed field over $\mathcal{O}$, we choose a polarized abelian variety $(A, L)$ over $k$ with $G(A, L) \simeq G(K) \otimes k$. Since $e_{\min}(K) \geq 3$, $L$ is very ample and $(A, d_iL) \in H_i(k)$, $A$ being identified with $\phi_i(A)$. The image $Z := (\phi_1 \times \phi_2)(\Delta)$ of the diagonal $\Delta$ ($\subset A \times A$) belongs to $U_1(k)$ as we saw in Section 9.3. Since $L_i = d_iL$ by Lemma 9.4, all the conditions (iv)–(ix) are true for $Z$ as is well known. Hence $Z \in U_2(k)$. Hence $U_2 \neq \emptyset$.

### 9.6. The schemes $U_{g,K}^{\dagger}$ and $U_3$

First we note that if $(Z, L) \in U_2$, then we have a $G(K)$-action on $(Z, L)$, which is induced from the $G(K)$-action on $Z_{\mathrm{conn}}^P$ induced from those $G(K)$-actions on $\mathbf{P}(W_i(K))$. In what follows, we mean the above $G(K)$-action on $Z$ or $(Z, L)$ by the $G(K)$-action on $(Z, L)$ when $(Z, L) \in U_2$.

Next we recall that the locus $U_{g,K}$ of abelian varieties (with the zero not necessarily chosen) is an open subscheme of $U_2$. In fact, $U_{g,K}$ is the largest open $\mathcal{O}$-subscheme among all the open $\mathcal{O}$-subschemes $H'$ of $U_2$ such that

(a) the projection $\pi_{H'} : Z_{\mathrm{conn}}^P \times_{H_{\mathrm{conn}}^P} H' \to H'$ is smooth over $H'$,
(b) at least one geometric fiber of $\pi_{H'}$ is an abelian variety for each irreducible component of $H'$.

In general, the subset $H''$ of $U_2$ over which the projection $\pi_{H''} : Z_{\mathrm{conn}}^P \times_{H_{\mathrm{conn}}^P} H'' \to H''$ is smooth is an open $\mathcal{O}$-subscheme of $U_2$. By [17, Theorem 6.14], any geometric fiber of $\pi_{U_{g,K}}$ is a polarized abelian variety. This is proved as follows (see [18, p. 705]). Let $U = U_{g,K}$ and $Z' = Z_{\mathrm{conn}}^P \times_{U_2} U_{g,K}$. By the base change $U' \to U$, we may assume $Z'' := Z' \times_U U'$ has a section $e$ over $U'$. For instance, choose $U' = Z'$ and $e$ the diagonal of $Z' \times_U Z'$. Then by [17, Theorem 6.14] $Z''$ is an abelian scheme over $U'$ with $e$ unit section. It follows that any geometric fiber of $Z''$, a fortiori, any geometric fiber of $Z'$ is an abelian variety.

Next we define $U_{g,K}^{\dagger}$ to be a nonempty open reduced $\mathcal{O}$-subscheme of $U_{g,K}$, which will be proved in Lemma 9.7, parametrizing all subschemes $(A, L) \in U_{g,K}$ such that

(x) the $K$-action on $A$ induced from the $G(K)$-action on $(A, L)$ is effective and contained in $\operatorname{Aut}^0(A)$.

In general, $U_{g,K}^{\dagger}$ is strictly smaller than $U_{g,K}$. See [18, pp. 711–712].

Finally we define $U_3$ to be the closure of $U^\dagger_{g,K}$ in $U_2$. It is the smallest closed $\mathcal{O}$-subscheme of $U_2$ containing $U^\dagger_{g,K}$. In other words, it is the intersection of all closed $\mathcal{O}$-subschemes of $U_2$ which contain the $\mathcal{O}$-subscheme $U^\dagger_{g,K}$. In particular, $U_3$ is reduced because $U^\dagger_{g,K}$ is proved to be reduced (see the proof of Theorem 11.6).

It remains to prove:

**Lemma 9.7.** *Let $k$ be a closed field over $\mathcal{O}$. Then*

(1) *$U^\dagger_{g,K}$ is a nonempty open $\mathcal{O}$-subscheme of $U_{g,K}$,*

(2) *$U^\dagger_{g,K}(k)$ is the set of all abelian varieties $(A, L)$ over $k$ with $G(A, L) = G(K) \otimes k$ for the $G(K)$-action on $(A, L) \in U_2(k)$ induced from that on $\mathbf{P}(W_i(K))$.*

*Proof.* There exists a polarized abelian variety $(A, L)$ over a closed field $k$ with $G(A, L) \simeq G(K) \otimes k$ by Lemma 4.2. Then $(A, L) \in U^\dagger_{g,K}(k)$. Hence $U^\dagger_{g,K}$ is nonempty. The condition on $(A, L) \in U_{g,K}$ that the $K$-action on $A$ is effective is an open condition. In fact, for any element $h$ of $K$, the fixed point set by $h$ is a closed subscheme of $Z^P_{\text{conn}} \times_{H^P_{\text{conn}}} U_{g,K}$, which is mapped to a closed subscheme $F_h$ of $U_{g,K}$ by the proper morphism $\pi_{U_{g,K}}$. Thus the locus where the $K$-action on $A$ is effective is just the complement of the union of all $F_h$ ($h \neq \text{id}$) in $U_{g,K}$. Moreover, the condition that the $K$-action on $A$ is contained in $\operatorname{Aut}^0(A)$ is also an open condition, because the (relative) identity component $\operatorname{Aut}^0(Z^P_{\text{conn}} \times_{H^P_{\text{conn}}} H''/H'')$ is open in the relative automorphism group scheme $\operatorname{Aut}(Z^P_{\text{conn}} \times_{H^P_{\text{conn}}} H''/H'')$. Therefore $U^\dagger_{g,K}$ is a nonempty open $\mathcal{O}$-subscheme of $U_{g,K}$. This proves (1).

Next we prove (2). First we prove that if $(A, L) \in U^\dagger_{g,K}(k)$ for a closed field $k$ over $\mathcal{O}$, then $K(A, L) = K \otimes k$. In fact, by the condition (x), the $K$-action on $A$, which is induced from the $G(K)$-action on $Z^P_{\text{conn}}$, reduces to translation by $K(A, L)$. It follows from effectivity of the $K$-action that $K \subset K(A, L)$. By (vi) and (vii) we have $\dim \Gamma(A, L) = \sqrt{|K|}$. This shows $K(A, L) = K \otimes k$ because $\dim \Gamma(A, L) = \sqrt{|K(A, L)|}$ for $L$ very ample by [15, § 23, p. 234].

Next we show that $G(A, L) \simeq G(K) \otimes k$ for any $(A, L) \in U^\dagger_{g,K}(k)$. In fact, if $(A, L) \in U^\dagger_{g,K}(k)$, then $K(A, L) \simeq K \otimes k$ as we have seen above, and $L$ has a $G(K)$-linearization by (iii). In other words, $(A, L)$ has compatible $G(K)$-actions, which is effective on the scheme $L$. Hence $\operatorname{Aut}(L/A) \supset G(K) \otimes k$ (see Section 4.6 for $\operatorname{Aut}(L/A)$), whence $G(A, L) \supset G(K) \otimes k$. Since $|G(A, L)| = |G(K) \otimes k| = N \cdot |K|$ by $K(A, L) \simeq K \otimes k$, we have $G(A, L) \simeq G(K) \otimes k$. This proves (2). Q.E.D.

## §10. The fibers over $U_3$

### 10.1. The conditions $(\mathrm{S}_i)$ and $(\mathrm{R}_i)$

Here we recall the conditions $(\mathrm{S}_i)$ and $(\mathrm{R}_i)$:

$$
\begin{aligned}
&(\mathrm{S}_i) \qquad \operatorname{depth}(A_{\mathfrak{p}}) \geq \inf(i, \operatorname{ht}(\mathfrak{p})) \quad \text{for all } \mathfrak{p} \in \operatorname{Spec}(A),\\
&(\mathrm{R}_i) \qquad A_{\mathfrak{p}} \text{ is regular for all } \mathfrak{p} \in \operatorname{Spec}(A) \text{ with } \operatorname{ht}(\mathfrak{p}) \leq i\,.
\end{aligned}
$$

**Lemma 10.2.** *Let $A$ be a noetherian local ring. Then*

(1) *$A$ is normal if and only if* $(\mathrm{R}_1)$ *and* $(\mathrm{S}_2)$ *are true for $A$,*

(2) *$A$ is reduced if and only if* $(\mathrm{R}_0)$ *and* $(\mathrm{S}_1)$ *are true for $A$.*

See [11, Theorem 39] and [3, $\mathrm{IV}_2$, 5.8.5 and 5.8.6].

**Lemma 10.3.** *Let $R$ be a discrete valuation ring, $S :=$* Spec *$R$, and $\eta$ the generic point of $S$. Assume that $\pi : Z \to S$ is flat with $Z_0$ reduced and $Z_\eta$ nonsingular. Then $Z$ is normal.*

*Proof.* By Lemma 10.2, it suffices to check that $(\mathrm{R}_1)$ and $(\mathrm{S}_2)$ are true for any local ring $O_{Z,z}$. For simplicity write $O_Z$ instead of $O_{Z,z}$. Since $Z_0$ is reduced, it is smooth at a generic point of any irreducible component of it. Hence $Z$ is smooth at any codimension one point of $Z$ supported by $Z_0$. Since $Z_\eta$ is smooth, $Z$ is codimension one nonsingular everywhere. This is $(\mathrm{R}_1)$.

Next we prove $(\mathrm{S}_2)$. Since $\pi : Z \to S$ is flat, any generator $t$ of the maximal ideal of $R$ is not a zero divisor of $O_Z$. Hence it is not nilpotent. Let $\mathfrak{p}$ be a prime ideal of $O_Z$. If $\mathfrak{p} \cap R \neq 0$, then $t \in \mathfrak{p}$. (In fact, $\mathfrak{p} \cap R = tR$.) Moreover $\mathfrak{p}' := \mathfrak{p}/tO_Z$ is a prime ideal of $O_{Z_0}$ with $\operatorname{ht}(\mathfrak{p}') = \operatorname{ht}(\mathfrak{p}) - 1$. Otherwise, we would have $\operatorname{ht}(\mathfrak{p}') = \operatorname{ht}(\mathfrak{p})$. This implies that there is a prime ideal $\mathfrak{q}$ of $O_Z$ such that $t \in \mathfrak{q} \subset \mathfrak{p}$ and $\operatorname{ht}(\mathfrak{q}) = 0$. Hence $\mathfrak{q}(O_Z)_{\mathfrak{q}}$ is the unique prime ideal of $(O_Z)_{\mathfrak{q}}$, which is the nilradical of $(O_Z)_{\mathfrak{q}}$. Since $t \in \mathfrak{q}(O_Z)_{\mathfrak{q}}$, it follows that $t$ is nilpotent. This contradicts that $t$ is not nilpotent. This shows $\operatorname{ht}(\mathfrak{p}') = \operatorname{ht}(\mathfrak{p}) - 1$.

Since $Z_0$ is reduced, hence $(\mathrm{S}_1)$ for $Z_0$ is true by Lemma 10.2. Therefore $\operatorname{depth}(O_Z)_{\mathfrak{p}} = \operatorname{depth}(O_{Z_0})_{\mathfrak{p}'} + 1 \geq \inf(1, \operatorname{ht}(\mathfrak{p}')) + 1 = \inf(2, \operatorname{ht}(\mathfrak{p}))$. If $\mathfrak{p} \cap R = 0$, then $k(\eta) \subset (O_Z)_{\mathfrak{p}}$ and $(O_Z)_{\mathfrak{p}} = (O_{Z_\eta})_{\mathfrak{p}O_{Z_\eta}}$. Hence $\operatorname{depth}(O_Z)_{\mathfrak{p}} = \operatorname{depth}(O_{Z_\eta})_{\mathfrak{p}O_{Z_\eta}} = \dim(O_{Z_\eta})_{\mathfrak{p}O_{Z_\eta}} = \operatorname{ht}(\mathfrak{p}) \geq \inf(2, \operatorname{ht}(\mathfrak{p}))$ because $Z_\eta$ is nonsingular. This proves $(\mathrm{S}_2)$. Q.E.D.

**Theorem 10.4.** *Let $R$ be a discrete valuation ring, $S :=$* Spec *$R$, and $\eta$ the generic point of $S$. Let $h$ be a morphism from $S$ into $U_3$. Let $(Z, \mathcal{L})$ be the pullback by $h$ of the universal subscheme $Z_{\mathrm{univ}}$, universal for* $\mathrm{Hilb}^P_{\mathrm{conn}}(X/H)$*, such that $(Z_\eta, \mathcal{L}_\eta)$ is a polarized abelian variety. Then $(Z, \mathcal{L})$ is isomorphic to a (modified) Mumford's family $(P, \mathcal{L}_P)$ in Theorem 2.7 after a finite base change.*

*Proof.* By the assumption, $(Z_\eta, \mathcal{L}_\eta)$ is a polarized abelian variety over $k(\eta)$ such that the $K$-action on $Z_\eta$ induced from the $G(K)$-action on $(Z_\eta, \mathcal{L}_\eta)$ is effective and contained in $\mathrm{Aut}^0(Z_\eta)$. By (iii) $\mathcal{L}_\eta$ is $G(K) \otimes k(\eta)$-linearized, and by Lemma 9.7, we have an abelian variety with rigid $G(K)$-structure $(G_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}$ by a suitable finite base change if necessary. Then by Theorem 8.2, after a suitable finite base change if necessary, there exists an $S$-TSQAS $(P, \mathcal{L}_P, \phi_P, \tau_P)_{\mathrm{RIG}}$ with rigid $G(K)$-structure, extending $(Z_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}$. The scheme $P$ is normal by Lemma 10.3, because $P_0$ is reduced and $P$ is $S$-flat.

We note that there also exists an $S$-PSQAS $(Q, \mathcal{L}_Q, \phi_Q, \tau_Q)_{\mathrm{RIG}}$ with rigid $G(K)$-structure extending $(Z_\eta, \mathcal{L}_\eta, \phi_\eta, \tau_\eta)_{\mathrm{RIG}}$, which is unique up to $S$-isomorphism by Lemma 6.6. The scheme $Q$ was defined in Section 2. It is reduced though it may not be normal in general. Let $\phi_P : P \to \mathbf{P}(V(K))_S$ be the morphism defined by $\Gamma(P, \mathcal{L}_P)$. By Theorem 2.23, $Q$ is the image of $P$, and $\phi_P : P \to Q$ is the normalization of $Q$. Moreover $\mathcal{L}_Q$ is the restriction (the pullback) of $\mathbf{L}(V(K))_S$ to $Q$.

Let $\pi : Z \to S$ be the flat family given at the start. Hence any fiber of $Z$ satisfies the conditions (i)–(iii) in Section 9.3 and the conditions (iv)–(ix) in Section 9.5. In Section 9.3 we fix $G(K)$-actions on $W_i(K)$ once and for all. Thus we have induced $G(K)$-linearizations of $\mathbf{L}(W_i(K))$ on $\mathbf{P}(W_i(K))$ $(i = 1, 2)$, and hence those of $p_i^*\mathbf{L}(W_i(K))$ on $\mathbf{P}(W_1(K)) \times \mathbf{P}(W_2(K))$ where $p_i$ is the $i$-th projection. Hence we have induced $G(K)$-linearizations on $(Z, L_i)$ because $Z$ is $G(K)$-stable by (iii), namely the closed immersions of $Z$ into $\mathbf{P}(W_i(K))$ are $G(K)$-morphisms.

The $R$-module $\Gamma(Z, \mathcal{L})$ is free of rank $\sqrt{|K|}$ by (vi) and (vii). It is a $G(K) \otimes R$-module of weight one, hence $G(K)_S$-isomorphic to $V(K) \otimes R$ in view of Lemma 4.4. By (viii) $\Gamma(Z, \mathcal{L})$ is base point free, which defines a finite $G(K)_S$-morphism $\phi_Z : Z \to \mathbf{P}(V(K))_S$. Since $e_{\min}(K) \geq 3$, $(\phi_Z)_\eta$ is a closed immersion of $Z_\eta$.

Let $W$ be the flat closure of $(\phi_Z)(Z_\eta)$ in $\mathbf{P}(V(K))_S$, and $\mathbf{L}_W$ the restriction to $W$ of $\mathbf{L}(V(K))_S$. Since $(\phi_Z)(Z_\eta)$ is reduced, so is the flat closure of $(\phi_Z)(Z_\eta)$. Hence $W$ is reduced. Since $Z_0$ is reduced, so is $Z$, hence $\phi_Z$ factors through $W$ with $(\phi_Z)_\eta$ an isomorphism. Since $Z_\eta$ is irreducible, so is $W$. Hence $\phi_Z : Z \to W$ is a finite surjective birational morphism.

Let $\mathbf{P} = \mathbf{P}(V(K))$ and $\mathbf{L} = \mathbf{L}(V(K))$. Since both $W$ and $Q$ are $G(K)$-stable, both $\mathbf{L}_W$ and $\mathcal{L}_Q = \mathbf{L}_Q$ have $G(K)$-linearizations induced from that of $(\mathbf{P}, \mathbf{L})$. Hence $G(K)_S$ is a subgroup scheme of both $\mathrm{Aut}(\mathbf{L}_W/W)$ and $\mathrm{Aut}(\mathbf{L}_Q/Q)$. Let $i_W : W \to \mathbf{P}$ and $i_Q : Q \to \mathbf{P}$ be natural inclusions (closed immersions) of $W$ and $Q$ into $\mathbf{P}$, $\tau_W$ and $\tau_Q$ are closed immersions of the subgroup scheme $G(K)_S$ into $\mathrm{Aut}(\mathbf{L}_W/W)$ and $\mathrm{Aut}(\mathbf{L}_Q/Q)$. Then it follows from Lemma 7.6 and Lemma 7.7

that $(W, \mathbf{L}_W)$ has a unique rigid $U(K)_S$-structure $(W, i_W, \tau_W)_{\mathrm{RIG}}$ (see Remark 7.5). Meanwhile $(Q, \mathcal{L}_Q)$ has a unique rigid $G(K)_S$-structure $(Q, i_Q, \tau_Q)_{\mathrm{RIG}}$ by Theorem 8.2. We note that $(Q, i_Q, \tau_Q)_{\mathrm{RIG}}$ is also a unique rigid $U(K)$-structure by Lemma 7.7.

Since we have

$$(W_\eta, i_{W_\eta}, \tau_{W_\eta})_{\mathrm{RIG}} \simeq (Q_\eta, i_{Q_\eta}, \tau_{Q_\eta})_{\mathrm{RIG}} \simeq (Z_\eta, i_{Z_\eta}, \tau_{Z_\eta})_{\mathrm{RIG}},$$

the rigid $U(K)$-structures $(W, i_W, \tau_W)_{\mathrm{RIG}}$ and $(Q, i_Q, \tau_Q)_{\mathrm{RIG}}$ are $S$-isomorphic by Lemma 7.8. In particular, this shows that $W \simeq Q$, and that the $G(K)_S$-action on $(W, \mathbf{L}_W)$ is the same as that of the finite Heisenberg group $G(W, \mathbf{L}_W)$ (see Section 4.6). In view of Lemma 10.3, $Z$ is normal. We have a finite morphism $\phi_Z : Z \to W \simeq Q$, with $(\phi_Z)_\eta$ an isomorphism. Hence $Z$ is the normalization of $Q$, whence $Z \simeq P$. Since $\mathcal{L}_P = \phi_P^*(\mathbf{L})$ and $\mathcal{L} = \phi_Z^*(\mathbf{L}_W)$, we have $(Z, \mathcal{L}) \simeq (P, \mathcal{L}_P)$. Q.E.D.

**Corollary 10.5.** *Let $(Z_0, \mathcal{L}_0)$ be the closed fiber of $(Z, \mathcal{L})$ in Theorem 10.4. Then $(Z_0, \mathcal{L}_0)$ is a TSQAS with level-$G(K)$ structure such that the action of $G(K)$ on $(Z_0, \mathcal{L}_0)$ is that of $G(Z_0, \mathcal{L}_0)$.*

*Proof.* By the proof of Theorem 10.4, we see that $Z$ is the normalization of $W$ and $(W, i_W, \tau_W)_{\mathrm{RIG}} \simeq (Q, i_Q, \tau_Q)_{\mathrm{RIG}}$. The normalization morphism $\phi_Z : Z \to W$ is $G(K)$-equivariant, and the action of $G(K)$ on $(W, \mathcal{L}_W)$ is $G(W, \mathcal{L}_W)$ by the proof of Theorem 10.4. Hence the action of $G(K)$ on $(Z, \mathcal{L})$ is $G(Z, \mathcal{L})$. This proves the corollary. Q.E.D.

**Corollary 10.6.** *Let $k$ be a closed field over $\mathcal{O}$ and $Z \in U_3(k)$. Let $L = M \otimes O_Z$ under the notation of Section 9.3. Then $(Z, L)$ is a TSQAS with level-$G(K)$ structure such that the $G(K)$-action on $(Z, L)$ induced from that on $W_i(K)$ is that of $G(Z, L)$.*

*Proof.* By Theorem 10.4, $(Z, L) \simeq (P_0, \mathcal{L}_0) \otimes k$, where $(P_0, \mathcal{L}_0)$ is a TSQAS, a closed fiber of an $S$-TSQAS $(P, \mathcal{L})$ of level $K$. By Corollary 10.5, the action of $G(K)$ on $(P_0, \mathcal{L}_0)$ is that of $G(P_0, \mathcal{L}_0)$. Q.E.D.

## §11. The reduced-coarse moduli space $SQ_{g,K}^{\mathrm{toric}}$

Let $N = e_{\max}(K)$ and $\mathcal{O} = \mathcal{O}_N = \mathbf{Z}[\zeta_N, 1/N]$. In this section, we use the same notation as in Section 9.

**Lemma 11.1.** *Let $k$ be an algebraically closed field over $\mathcal{O}$.*

(1) *$U_3$ is $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-invariant,*
(2) *Let $(Z, L) \in U_3(k)$ and $(Z', L') \in U_3(k)$ where $L = M \otimes O_Z$ and $L' = M \otimes O_{Z'}$. If $(Z, L) \simeq (Z', L')$ as polarized schemes with $G(K)$-linearization, then $(Z', L')$ belongs to the $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-orbit of $(Z, L)$.*

*Proof.* First we prove (2). Let $f : (Z,L) \to (Z',L')$ be an isomorphism with $G(K)$-linearization. Hence we have an isomorphism $f_i : (Z, d_iL) \to (Z', d_iL')$ as polarized schemes with $G(K)$-linearization. By the assumptions on $(Z,L)$ and $(Z',L')$, we see first that $d_iL$ and $d_iL'$ are very ample. Hence $(Z, d_iL)$ and $(Z', d_iL') \in H_i(k)$ $(i = 1, 2)$. We note that $d_i \equiv 1 \mod N$. Hence as $G(K)$-modules

$$H^0(Z, d_iL) \simeq H^0(Z', d_iL') \simeq W_i \otimes V(K) \otimes k =: W_i(K)$$

for some trivial $G(K)$-module $W_i$, which is the same as $W_i$ in the statement (2) of Lemma 11.1.

By Section 4.13, we have closed $G(K)$-immersions

$$\begin{aligned} \iota_i &: (Z, d_iL) \to (\mathbf{P}(W_i(K)), \mathbf{L}(W_i(K)), \\ \iota'_i &: (Z', d_iL') \to (\mathbf{P}(W_i(K)), \mathbf{L}(W_i(K)). \end{aligned}$$

We can define $\rho_{d_iL}$ and $\rho_{d_iL'}$, and $\rho(\iota_i)$ and $\rho(\iota'_i)$ in the same manner as in Section 4.7 (2) and Section 4.12 (4). Then we may assume that $\rho(\iota_i)(g) = \rho(\iota'_i)(g) = (\mathrm{id}_{W_i} \otimes U(K))(g)$ for any $g \in G(K)$. Since $H^0(f_i^*) : H^0(Z', d_iL') \to H^0(Z, d_iL)$ is a $G(K)$-isomorphism of vector $k$-spaces, we see that there are

(i) commutative diagrams of $G(K)$-isomorphisms

$$\begin{tikzcd} (Z, d_iL) \arrow[r, "f_i"] \arrow[d, "\iota_i"] & (Z', d_iL') \arrow[d, "\iota'_i"] \\ (\mathbf{P}(W_i(K)), \mathbf{L}(W_i(K)) \arrow[r, "F_i"] & (\mathbf{P}(W_i(K)), \mathbf{L}(W_i(K))), \end{tikzcd}$$

(ii) commutative diagrams of $G(K)$-isomorphisms of vector $k$-spaces

$$\begin{tikzcd} H^0(Z, d_iL) & H^0(Z', d_iL') \arrow[l, "H^0(f_i^*)"'] \\ H^0(\mathbf{L}(W_i(K))) \otimes k \arrow[u, "H^0(\iota_i^*)"] & H^0(\mathbf{L}(W_i(K))) \otimes k \arrow[l, "H^0(F_i^*)"'] \arrow[u, "H^0((\iota'_i)^*)"'] \end{tikzcd}$$

where $H^0(\mathbf{L}(W_i(K))) = W_i(K) := W_i \otimes V(K) \otimes k$, and $H^0(F_i^*)$, hence $F_i$ is defined uniquely by the condition $H^0(\iota_i^*)H^0(F_i^*) = H^0(f_i^*)H^0((\iota'_i)^*)$.

By $G(K)$-equivariance of $(f_i, F_i)$, we have

$$\rho(\iota_i)(g) \circ H^0(F_i^*) = H^0(F_i^*) \circ \rho(\iota'_i)(g),$$

whence

$$(\mathrm{id}_{W_i} \otimes U(K)(g)) \circ H^0(F_i^*) = H^0(F_i^*) \circ (\mathrm{id}_{W_i} \otimes U(K)(g)).$$

From Lemma 4.5 (2) it follows that $H^0(F_i^*) = h_i^* \otimes \mathrm{id}_{V(K)}$ for some $h_i^* \in \mathrm{GL}(W_i)$. Let $S_{h_i}$ be the transformation of $\mathbf{P}(W_i(K))$ induced from $h_i^* \otimes \mathrm{id}_{V(K)}$. It follows from $H^0(\iota_i^*)H^0(F_i^*) = H^0(f_i^*)H^0((\iota_i')^*)$ that $\iota_i' \circ f_i = S_{h_i} \circ \iota_i$. Thus (2) is proved because

$$(Z', d_1L') \times (Z', d_2L') = (S_{h_1} \times S_{h_2}) \cdot \{(Z, d_1L) \times (Z, d_2L)\}$$

as polarized closed subschemes of $\mathbf{P}(W_1(K)) \times \mathbf{P}(W_2(K))$.

Next we prove (1) using the same notation as above, though (1) is almost clear. Let $k$ be an algebraically closed field. Let $S_{h_1} \times S_{h_2} \in (\mathrm{GL}(W_1) \times \mathrm{GL}(W_2))(k)$, $(Z, L) := (Z, d_1L) \times (Z, d_2L) \in U_3(k)$ and $(Z', L') = (S_{h_1} \times S_{h_2}) \cdot (Z, L)$. Since $f_i$ is a $G(K)$-isomorphism, if $(Z, L) \in U_1(k)$, then $(Z', L') \in U_1(k)$. The conditions (iv)–(ix) are kept under $G(K)$-isomorphism, hence if $(Z, L) \in U_2(k)$, then $(Z', L') \in U_2(k)$.

If $(Z, L) \in U_3(k)$, then by Theorem 10.4, $(Z, L)$ is a closed fiber $(\mathcal{Z}_0, \mathcal{L}_0)$ of a modified Mumford family $(\mathcal{Z}, \mathcal{L})$ of TSQASes of level $K$ with generic fiber a polarized abelian variety. Then $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-action gives a new one-parameter family $(\mathcal{Z}', \mathcal{L}') := (S_{h_1} \times S_{h_2}) \cdot (\mathcal{Z}, \mathcal{L})$ of TSQASes of level $K$ with generic fiber a polarized abelian variety such that $(\mathcal{Z}_0', \mathcal{L}_0') \simeq (Z', L')$. Hence $(Z', L') \in U_3(k)$ by the definition of $U_3$ in Section 9.6. Q.E.D.

### 11.2. The uniform geometric and categorical quotient

Let $G$ be a flat group scheme, $X$ a scheme and $\sigma : G \times X \to X$ the action. We say that *the action $\sigma$ on $X$ is proper* if the morphism $\Psi := (\sigma, p_2) : G \times X \to X \times X$ is proper. Let $Y$ be an algebraic space, $\phi : X \to Y$ a morphism and $\phi' := \phi \times_Y Y'$ for any $Y'$ over $Y$. For the pair $(Y, \phi)$ with $\phi \circ \sigma = \phi \circ p_2$, we consider the following conditions:

(i) $X(k)/G(k) \to Y(k)$ is bijective for any geometric point $\mathrm{Spec}\, k$,

(ii) for a morphism $\psi : X \to Z$ to an algebraic space $Z$ with $\psi \circ \sigma = \psi \circ p_2$, there is a unique morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \phi$,

(ii-u) $(Y', \phi')$ satisfies (ii) for any $Y$-flat $Y'$,

(iii) $\phi$ is submersive, that is, $U$ is open in $Y$ if and only if $\phi^{-1}(U)$ is open in $X$,

(iii-u) $\phi$ is universally submersive, that is, $(Y', \phi')$ satisfies (iii) for any $Y'$ over $Y$,

(iv) $O_Y \simeq (\phi_*(O_X))^{G-\mathrm{inv}}$.

The pair $(Y, \phi)$ is called *a categorical quotient* (resp. *a geometric quotient*) of $X$ if it satisfies (ii) (resp. (i), (iii-u) and (iv)). As was remarked in [10, p. 195], (ii-u) implies (iv).

The pair $(Y, \phi)$ is called *a uniform geometric quotient* (resp. *a uniform categorical quotient*) if $(Y', \phi')$ is a geometric quotient (resp. a categorical quotient) of $X \times_S Y'$ by $G$ for any $Y$-flat $Y'$.

**Theorem 11.3.** *Let* $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. *Then*

(1) *The action of* $G$ *on* $U^{\dagger}_{g,K}$ *is proper and free.*
(2) *The action of* $G$ *on* $U_3$ *is proper with finite stabilizer.*
(3) *The uniform geometric and uniform categorical quotient of* $U_3$ *resp.* $U^{\dagger}_{g,K}$ *by* $G$ *exists as a separated algebraic space, which we denote by* $SQ^{\mathrm{toric}}_{g,K}$ *resp.* $A^{\mathrm{toric}}_{g,K}$.

*Proof.* Note that (3) of the theorem follows from [10] once we prove (1) and (2). So we shall prove (1) and (2) of the theorem.

Let $k$ be a closed field, and $\widetilde{G} := \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$.

Let $(Z, L) \in U_3(k)$ and $h \in \widetilde{G}$. Suppose $h \cdot (Z, L) = (Z, L)$. Then there exist $h_i \in \mathrm{GL}(W_i)$ $(i = 1, 2)$ keeping $L_i := d_i L$ invariant such that $h = (h_1, h_2)$. Hence $h$ keeps $L = q_1(d_1 L) + q_2(d_2 L)$ invariant. This implies that $h$ is an automorphism of $(Z, L)$ with $G(K)$-linearization. In particular, $h$ induces a linear transformation $H^0(h^*)$ of $\Gamma(Z, L)$, which commutes with $U(K)(g)$ for any $g \in G(K)$. Thus $H^0(h^*)$ on $\Gamma(Z, L)$ is a scalar matrix by Lemma 4.5.

If $(Z, L)$ is a polarized abelian variety, $L$ is very ample by the assumption $e_{\min}(K) \geq 3$, so that $h$ is the identity of $Z = P_0 = Q_0$. This implies that $h_i$ is the identity of $Z$, hence the identity of $\mathrm{PGL}(W_i)$. It follows that the stabilizer of a polarized abelian variety $(Z, L)$ is trivial. Hence the $G$-action on $U^{\dagger}_{g,K}$ is free, which proves (1).

Next we consider the totally degenerate case, that is, $(Z, L)$ is a union of normal torus embeddings.

In view of Theorem 10.4, by taking a suitable finite base change if necessary, we may assume that there exists a modified Mumford family $(P, \mathcal{L})$ over $S :=$ Spec $R$, $R$ a complete discrete valuation ring, such that $(Z, L) = (P_0, \mathcal{L}_0)$. By Theorem 2.23, we have an $S$-PSQAS $(Q, \mathcal{L}_Q)$ and a finite birational morphism $\phi : (P, \mathcal{L}) \to (Q, \mathcal{L}_Q)$ associated with $\Gamma(P, \mathcal{L})$. Since $H^0(h^*)$ is a scalar matrix on $\Gamma(Z, L)$, $h$ induces the identity of $(Q_0)_{\mathrm{red}}$. With the notation in Section 2, $(Q_0)_{\mathrm{red}}$ is covered with open affines

$$V_0(c) = \mathrm{Spec}\ k(0)[\xi_{a,c}, a \in \mathrm{Del}(0)] \quad (c \in X),$$

where $\xi_{a,c} = \xi_{a+c}/\xi_c$. Hence $H^0(h^*)$ keeps $\xi_a := \xi_{a,0}$ invariant. The 0-dimensional stratum $O(c)$ of $P_0$ is also fixed by $h$ because of the bijective correspondence of strata of $P_0$ and $Q_0$. Now we look at the algebra $R_0(c)$ of $P_0$ at $O(c)$. The algebra $R_0(c)$ is given by

$$R_0(c) = k(0)[\zeta_{b,c}, b \in C(0,\sigma) \cap X, \sigma \in \mathrm{Del}^g(0)] \quad (c \in X),$$

where some power of $\zeta_{b,c}$ is a product of $\xi_{a,c}$ $(= \zeta_{a,c})$ for some $a \in \mathrm{Del}(0)$. Hence $H^0(h^*)(\zeta_{b,c}) = \alpha(b)\zeta_{b,c}$ for some $\alpha(b)$, a root of unity. Since $R_0(c)$ is finitely generated over $k(0)$, this implies that $h$ is of finite order as an automorphism of $Z = P_0$.

When both the torus part and the abelian part of $(Z, L)$ are nontrivial, then the stabilizer group of $(Z, L)$ is finite with possibly nontrivial automorphism on the torus part, and trivial on the abelian part. Hence the $G$-action on $U_3$ has finite stabilizer.

It remains to prove that the action of $G$ is proper. This is reduced to proving Claim 1:

**Claim 1.** Let $R$ be a discrete valuation ring $R$ with fraction field $k(\eta)$, $S = \mathrm{Spec}\ R$. Let $\sigma : G \times U_3 \to U_3$ be the action and $\Psi = (\sigma, p_2) : G \times U_3 \to U_3 \times U_3$. Then for any pair $(\phi, \psi_\eta)$ consisting of a morphism $\phi : S \to U_3 \times U_3$ and a morphism $\psi_\eta : \mathrm{Spec}\ k(\eta) \to G \times U_3$ such that $\psi_\eta \circ \Psi = \phi \otimes_R k(\eta)$, there is a morphism $\psi : S \to G \times U_3$ such that $\psi \circ \Psi = \phi$ and $\psi \otimes_R k(\eta) = \psi_\eta$.

Since $U_3$ is the closure of $U^\dagger_{g,K}$ in $U_2$, Claim 1 follows from Claim 2:

**Claim 2.** Let $(Z_i, \phi_{Z_i}, \tau_{Z_i})_{\mathrm{RIG}}$ $(i = 1, 2)$ be an $S$-TSQAS with rigid $G(K)$-structure, whose generic fiber is an abelian variety. If they are isomorphic over $k(\eta)$, then they are isomorphic over $S$.

Claim 2 follows from Lemma 6.7. This completes the proof of properness of the action $\Psi$, which completes the proof of (2). Q.E.D.

**Definition 11.4.** Let $W$ be an algebraic $\mathcal{O}$-space, and $h_W$ the functor defined by $h_W(T) = \mathrm{Hom}(T, W)$.

Let $F$ be a contravariant functor from the category of algebraic spaces over $\mathcal{O}$ to the category of sets.

A reduced algebraic $\mathcal{O}$-space $W$ with a morphism of functors $f : F \to h_W$ is called *a reduced-coarse moduli (algebraic $\mathcal{O}$-)space* of $F$ if the following conditions are satisfied:

(a) $f(\mathrm{Spec}\ k) : F(\mathrm{Spec}\ k) \to h_W(\mathrm{Spec}\ k)$ is bijective for any algebraically closed field $k$ over $\mathcal{O}$,

(b) For any reduced algebraic $\mathcal{O}$-space $V$, and any morphism $g : F \to h_V$, there is a unique morphism $\chi : h_W \to h_V$ such that $g = \chi \circ f$.

**Lemma 11.5.** *Assume $e_{\min}(K) \geq 3$. Let $A^{\text{toric}}_{g,K}$ be the uniform geometric quotient of $U^{\dagger}_{g,K}$ by $G := \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. Then $A^{\text{toric}}_{g,K}$ is isomorphic to the fine moduli $\mathcal{O}$-scheme $A_{g,K}$ of abelian varieties with level-$G(K)$ structure in* [18].

*Proof.* We choose and fix an pair $d_i$ of coprime positive integers such that $d_i \equiv 1 \mod N$ and $d_i \geq 2g+1$. We do so because $dL$ is very ample for $d \geq 2g+1$ by Theorem 2.21. Let $Y = Z^P_{\text{conn}} \times_{U_2} U^{\dagger}_{g,K}$ under the notation of Section 9.6. Then $Y$ is $U^{\dagger}_{g,K}$-flat with any fiber $(Z, L)$ an abelian variety with level-$G(K)$ structure, hence $L$ is very ample by the assumption $e_{\min}(K) \geq 3$. Since any fiber of $Y$ in the same $G$-orbit determines a unique abelian variety with rigid $G(K)$-structure by Lemma 5.7, we have a $G$-invariant morphism $\eta : U^{\dagger}_{g,K} \to A_{g,K}$, which induces a morphism $\bar{\eta} : A^{\text{toric}}_{g,K} \to A_{g,K}$.

Since $A_{g,K}$ is the fine moduli, there is a universal family $(Z_A, \mathcal{L}_A)$ of abelian varieties with rigid $G(K)$-structure over $A_{g,K}$. Let $\pi_A : Z_A \to A_{g,K}$ be the projection. Then $(\pi_A)_*(d_i\mathcal{L}_A)$ is a locally free $O_{A_{g,K}}$-module. It is a $G(K)$-module of weight one because $d_i \equiv 1 \mod N$. By Lemma 4.4 there is a finite locally free $O_{A_{g,K}}$-module $W_i$ such that $(\pi_A)_*(d_i\mathcal{L}_A) = W_i \otimes_{\mathcal{O}} V(K)$ as $G(K)$-modules. We choose a suitable covering of $A_{g,K}$ by affine open sets fine enough so that we have local trivializations of $W_i$. Then we have a collection of local morphisms $\eta_i : A_{g,K} \to U^{\dagger}_{g,K}$. Let $a^{\text{toric}}_{g,K} : U^{\dagger}_{g,K} \to A^{\text{toric}}_{g,K}$ be the natural projection defined by the quotient by $\mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. Then the composite $a^{\text{toric}}_{g,K} \circ \eta_i$ defines a morphism from $A_{g,K}$ to $A^{\text{toric}}_{g,K}$, which is evidently the inverse of $\bar{\eta}$. This proves that $\bar{\eta}$ is an isomorphism. Q.E.D.

**Theorem 11.6.** *Let $K$ be a finite symplectic abelian group with $e_{\min}(K) \geq 3$ and $N = e_{\max}(K)$. The functor $\mathcal{SQ}^{\text{toric}}_{g,K}$ has a reduced-coarse moduli (algebraic $\mathcal{O}_N$-)space, which we denote by $SQ^{\text{toric}}_{g,K}$. It is a complete reduced separated algebraic space.*

*Proof.* Let $G = \mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$. We choose and fix any pair of primes $d_1$ and $d_2$ with $d_i \geq 2g+1$ and $d_i \equiv 1 \mod N$. Let $SQ^{\text{toric}}_{g,K}$ be the uniform geometric and uniform categorical quotient of $U_3$ by $G$. Since local moduli of polarized deformations of any polarized abelian variety is nonsingular of dimension $g(g+1)/2$ by Grothendieck and Mumford [22, p. 244, Theorem 2.4.1] (see also [*ibid.*, p. 242, Theorem 2.3.3]), and $G$ is smooth and acts freely on $U^{\dagger}_{g,K}$ by Theorem 11.3, $U^{\dagger}_{g,K}$ is a smooth $\mathcal{O}$-scheme. In particular, $U^{\dagger}_{g,K}$ is reduced. Hence its closure $U_3$ in $U_2$ is also a reduced $\mathcal{O}$-subscheme of $U_2$. Since the action of $G$ has finite stabilizer on $U_3$, the quotient $SQ^{\text{toric}}_{g,K}$ is reduced. Since

$U^\dagger_{g,K}$ is $G$-invariant open, the uniform geometric quotient $A^{\text{toric}}_{g,K}$ of $U^\dagger_{g,K}$ by $G$ is an open algebraic $\mathcal{O}$-subspace of $SQ^{\text{toric}}_{g,K}$.

Let $W = SQ^{\text{toric}}_{g,K}$. It remains to prove that $W = SQ^{\text{toric}}_{g,K}$ is a reduced-coarse moduli for the functor $\mathcal{SQ}^{\text{toric}}_{g,K}$. To prove it, we define a morphism of functors

$$f : \mathcal{SQ}^{\text{toric}}_{g,K} \to h_W$$

as follows. Now we use the notation of Section 9. As in Section 9.2 let $H_i$ $(i = 1, 2)$ be the Hilbert scheme parametrizing all closed polarized subschemes $(Z_i, L_i)$ of $\mathbf{P}(W_i(K))$ such that

(a) $Z_i$ is $G(K)$-stable,

(b) $\chi(Z_i, nL_i) = n^g d_i^g \sqrt{|K|}$, where $L_i = \mathbf{L}(W_i(K)) \otimes O_{Z_i}$.

Let $X_i$ be the universal subscheme of $\mathbf{P}(W_i(K))$ over $H_i$. Let $X = X_1 \times_{\mathcal{O}} X_2$ and $H = H_1 \times_{\mathcal{O}} H_2$.

Let $T$ be a reduced scheme and let $\sigma := (P, L, \phi, \tau)_{\text{RIG}}$ be a $T$-TSQAS with rigid $G(K)$-structure. Then $d_iL$ has a $G(K)$-linearization, hence $\pi_*(d_iL)$ is a locally $O_T$-free $G(K)$-module of rank $d_i^g\sqrt{|K|}$. Since $d_i \equiv 1 \mod N$, it is locally isomorphic to $W_i \otimes V(K) \otimes O_T$ as a $G(K)$-module. Since $d_i \geq 2g+1$, $d_iL$ is very ample by Theorem 2.21 so that we have a closed $G(K)$-immersion $\phi_i : (P, d_iL) \to (\mathbf{P}(W_i(K))_T, \mathbf{L}(W_i(K))_T$ over $T$. Thus the image $\phi_i(P) \simeq P$ is a $T$-flat $G(K)$-stable subscheme of $\mathbf{P}(W_i(K))_T$. Hence $(\phi_1 \times \phi_2)(P)$ is a $T$-flat subscheme of the relative scheme $(X/H)_T$, any of whose fibers satisfies (a) and (b). Hence we have a morphism

$$\widetilde{f}(T)(\sigma) : T \to \text{Hilb}^P_{\text{conn}}(X/H) = H^P_{\text{conn}}.$$

First we prove that $\widetilde{f}(T)(\sigma)$ factors through $U_2$. Any of the fibers of $(\phi_1 \times \phi_2)(P)$ satisfies (i)–(ix) in Section 9. In fact, (i)–(iii) is clear from our construction, while (iv)–(ix) follow from Theorem 2.10, Lemma 2.18 and Theorem 2.23. The condition (ix) is a consequence of very-ampleness and $G(K)$-linearization of $d_iL$. Hence $\widetilde{f}(T)(\sigma)$ factors through $U_2$.

Next we prove that $\widetilde{f}(T)(\sigma)(t) \in U_3(k)$ for any geometric point $t \in T(k)$, $k$ any algebraically closed field over $\mathcal{O}$. In fact, by Theorem 5.2, there exists a complete discrete valuation ring $R$ with residue field $k$, and an $R$-TSQAS $\rho := (P', \mathcal{L}', \phi_{P'}, \tau_{P'})$ such that its generic fiber $(P'_\eta, \mathcal{L}'_\eta)$ is an abelian variety, and its closed fiber $\rho_0$ is isomorphic to the geometric fiber $\sigma_t$ of $\sigma$. Let $S = \text{Spec } R$. Then we have a morphism $\widetilde{f}(S)(\rho) : S \to U_2$ in the same manner as above. By Theorem 5.2, $G(P', \mathcal{L}') \simeq G(K)_S$, whence $G(P'_\eta, \mathcal{L}'_\eta) \simeq G(K) \otimes k(\eta)$. It follows that the $K$-action on the generic fiber $P'_\eta$ induced from the $G(K) \otimes k(\eta)$-action is effective and contained in $\mathrm{Aut}^0(P'_\eta)$. Hence $\widetilde{f}(S)(\rho)\otimes k(\eta)$ factors through $U_3$. Since $U_3$ is a closed reduced subscheme of $U_2$, $\widetilde{f}(S)(\rho)$ factors through $U_3$, hence $\widetilde{f}(T)(\sigma)(t)\in U_3(k)$.

Since $T$ is reduced, this implies that $\widetilde{f}(T)(\sigma)$ factors through $U_3$. Namely, $\widetilde{f}(T)(\sigma)$ is a morphism from $T$ into $U_3$ such that $(P,L) = \widetilde{f}(T)(\sigma)^*(Z^P_{\mathrm{conn}}, p_1^*M_X)$ with the notation of Section 9.3, where $p_1 : Z^P_{\mathrm{conn}}\ (\subset X\times H^P_{\mathrm{conn}}) \to X$ is the first projection. Hence we have a morphism

$$f(T)(\sigma) : T \to SQ^{\mathrm{toric}}_{g,K}\ (= W).$$

Next we prove that $f$ is a morphism of functors. For any morphism of reduced schemes $q : U \to T$, and a $T$-TSQAS $\sigma := (P,L,\phi,\tau)_{\mathrm{RIG}}$, we have a $U$-TSQAS $q^*(\sigma) := q^*(P,L,\phi,\tau)_{\mathrm{RIG}}$. The above construction of $\widetilde{f}(T)(\sigma)$ and $\widetilde{f}(U)(q^*(\sigma))$ in parallel leads to $\widetilde{f}(T)(\sigma)\circ q = \widetilde{f}(U)(q^*(\sigma))$, whence

$$f(T)(\sigma)\circ q = f(U)(q^*(\sigma)).$$

This proves that $f$ is a morphism of functors.

It remains to prove that the following is bijective :

$$f(\mathrm{Spec}\ k) : \mathcal{SQ}^{\mathrm{toric}}_{g,K}(\mathrm{Spec}\ k) \to h_W(\mathrm{Spec}\ k) = W(k)$$

for any algebraically closed field $k$ over $\mathcal{O}$. In fact, any $k$-TSQAS with rigid $G(K)$-structure $\sigma := (Z,L,\phi,\tau) \in \mathcal{SQ}^{\mathrm{toric}}_{g,K}(\mathrm{Spec}\ k)$ *belongs to* $U_3(k)$, to be more precise, $\sigma$ determines *non-canonically* a $k$-rational point $\widetilde{f}(\mathrm{Spec}\ k)(\sigma)$ of $U_3(k)$, and vice versa. In other words, $\mathcal{SQ}^{\mathrm{toric}}_{g,K}(\mathrm{Spec}\ k)$ is the quotient of $U_3(k)$ by the equivalence relation of $k$-isomorphism of level-$G(K)$ structures. Since $\sigma$ determines a $k$-rational point $\widetilde{f}(\mathrm{Spec}\ k)(\sigma)$ of $U_3$, so does it a $k$-rational point $f(\mathrm{Spec}\ k)(\sigma)$ of $W$. For any $\sigma = (Z,L,\phi,\tau)$ and $\sigma' := (Z',L',\phi',\tau') \in U_3(k)$, $(Z,L)\simeq(Z',L')$ if and only if $\sigma$ and $\sigma'$ belong the same $G$-orbit by Lemma 11.1. Hence $f(\mathrm{Spec}\ k)$ is injective. The surjectivity of $f(\mathrm{Spec}\ k)$ is clear. This proves that $W = SQ^{\mathrm{toric}}_{g,K}$ is a reduced-coarse moduli of the functor $\mathcal{SQ}^{\mathrm{toric}}_{g,K}$.

By Theorem 8.3, $SQ^{\mathrm{toric}}_{g,K}$ is complete. By Lemma 6.7, $SQ^{\mathrm{toric}}_{g,K}$ is separated. This completes the proof. Q.E.D.

**Corollary 11.7.** *The uniform geometric and uniform categorical quotient of* $U_3$ *by* $\mathrm{PGL}(W_1)\times\mathrm{PGL}(W_2)$ *is uniquely determined by the pair* $(g,K)$*, which is independent of the choice of the coprime pair* $(d_1,d_2)$ *and a very ample line bundle* $B$ *on* $X$*.*

*Proof.* By Theorem 11.6, the uniform geometric and uniform categorical quotient of $U_3$ by $\mathrm{PGL}(W_1)\times\mathrm{PGL}(W_2)$ is the reduced-coarse moduli for the functor $\mathcal{SQ}_{g,K}$, which is uniquely determined by $(g,K)$. Hence it is independent of the choice of $(d_1,d_2)$, and a very ample line bundle $B$ on $X$. Q.E.D.

## §12. The canonical morphism from $SQ_{g,K}^{\text{toric}}$ onto $SQ_{g,K}$

The purpose of this section is to prove that there is a canonical finite birational morphism between the moduli spaces $SQ_{g,K}^{\text{toric}}$ and $SQ_{g,K}$ (Corollary 12.4). The following is a key to the proof of it.

**Theorem 12.1.** *Assume* $e_{\min}(K) \geq 3$. *Let* $\sigma := (P,\mathcal{L},\phi,\tau)_{\mathrm{RIG}}$ *be a $T$-TSQAS with rigid $G(K)$-structure, and* $\pi : P \to T$ *the projection, $T$ a reduced scheme. Suppose that any generic fiber* $(P_\eta,\mathcal{L}_\eta)$ *of $\pi$ is an abelian variety. Let*

(a) $\mathrm{Sym}(\phi)$ *be the graded subalgebra of* $\pi_*\mathrm{Sym}(\mathcal{L})$ *generated by* $\pi_*(\mathcal{L})$, $Q = \mathrm{Proj}(\mathrm{Sym}(\phi))$, $\mathcal{L}_Q$ = *the tautological line bundle of $Q$,*

(b) $\phi_Q$ *a closed immersion of $Q$ into* $\mathbf{P}(V(K))_T \simeq \mathbf{P}(\pi_*(\mathcal{L}))$ *induced from the surjection* $\mathrm{Sym}(\pi_*(\mathcal{L})) \to \mathrm{Sym}(\phi)$, *and*

(c) $\tau_Q$ *a closed immersion of* $G(K)_T$ *into* $\mathrm{Aut}(\mathcal{L}_Q/Q)$ *which is naturally induced from $\tau$.*

*Then* $\phi(\sigma) := (Q,\mathcal{L}_Q,\phi_Q,\tau_Q)$ *is a $T$-PSQAS with rigid $G(K)$-structure. Moreover, if any fiber* $\pi^{-1}(s)$ $(s \in T)$ *is an abelian variety, then*

$$(P,\mathcal{L},\phi,\tau) \simeq (Q,\mathcal{L}_Q,\phi_Q,\tau_Q).$$

*Proof.* Let $s$ be any prime point of $T$ and $A$ the local ring of $T$ at $s$. Everything in the theorem is defined globally, hence it suffices to prove that $\phi(\sigma)$ is a $T$-PSQAS with rigid $G(K)$-structure when $T = \mathrm{Spec}\,A$.

Let $N_n := \Gamma(P,\mathcal{L}^n)$ and $M_n$ the natural image of $S^n\Gamma(P,\mathcal{L})$ in $N_n$. Let $N := \oplus_{n=0}^{\infty} N_n$ and $M := \oplus_{n=0}^{\infty} M_n = \mathrm{Sym}(\phi)$. Since $P$ is reduced, the algebra $N$ has no nilpotent elements. Since $M$ is an $R$-subalgebra of $N$, $M$ has no nilpotent elements, whence $Q$ is reduced. Since $\Gamma(P,\mathcal{L})$ is base point free, $Q$ is just the image $\phi(P)$ with reduced structure.

Let $C$ be an irreducible curve of $T$ passing through $s$ such that the pull back of $P$ to $C$ is a TSQAS with generic fiber an abelian variety. Let $R$ be the completion of the local ring at $s$ of a nonsingular model of $C$, and $S = \mathrm{Spec}\,R$. Then we have a morphism $\lambda : S \to T$ such that the unique closed point 0 of $S$ is mapped to $s$. The pullback $P_S := \lambda^*P$ is an $S$-TSQAS with its generic fiber $P_{\eta_S}$ an abelian variety. Since the closed fiber $(\lambda^*P)_0 \simeq P_s$ is reduced, $P_S$ is reduced, proper and flat over $S$. Hence $P_S$ is the flat closure in $\mathbf{P}(\Gamma(P_S,\mathcal{L}_{P_S}))$ of the open subscheme $P_{\eta_S}$. Hence $P_S$ is irreducible because $P_{\eta_S}$ is irreducible.

Let $Q_S$ be the pullback of $Q$ to $S$ by $\lambda$. Then there is a surjective morphism $\lambda^*\phi : P_S \to Q_S$, whence $Q_S$ is irreducible. Now we apply [7, III, Proposition 9.7], which says that $Q_S$ is $S$-flat if and only if every associated point (= every associated prime of the zero ideal of any local ring) of $Q_S$ maps to the generic point of $S$. In our case, since $Q_S$ is irreducible, it is clear that the unique associated point of $Q_S$ is just the generic point of $Q_S$, which is mapped to the generic point of $S$. Thus $Q_S$ is $S$-flat. Let $\mathcal{L}_{Q_S} := \lambda^*(\mathcal{L}_Q)$, $\phi_{Q_S} := \lambda^*(\phi_Q)$ and $\tau_{Q_S} := \lambda^*(\tau_Q)$. Since $M$ is generated by $M_1$, $\mathcal{L}_Q$ is a line bundle, whence $\mathcal{L}_{Q_S}$ is a line bundle. Then it is clear that $\phi(\sigma)_S := (Q_S, \mathcal{L}_{Q_S}, \phi_{Q_S}, \tau_{Q_S})$ is an $S$-scheme with rigid $U(K)$-structure (see Section 7.3), where $U(K)$ is the Schrödinger representation of $G(K)$ in Section 4.1.

Let $P_S$ (resp. $\sigma_S$) be the pullback of $P$ (resp. $\sigma$) to $S$ by $\lambda : S \to T$. By the choice of $\lambda$, the generic fiber of $P_S$ is an abelian variety, and $\sigma_S = \lambda^*(P, \mathcal{L}, \phi, \tau)$ is an $S$-TSQAS with rigid $G(K)$-structure, which we denote by $(P_S, \mathcal{L}_S, \phi_S, \tau_S)$ for brevity. By Theorem 2.23, $\phi_S(\sigma_S)$ is an $S$-PSQAS with rigid $G(K)$-structure, hence an $S$-scheme with rigid $U(K)$-structure. Then all the generic fibers of $\sigma_S$, $\phi(\sigma)_S$ and $\phi_S(\sigma_S)$ are isomorphic, whence $\phi(\sigma)_S \simeq \phi_S(\sigma_S)$ by Lemma 7.8.

Thus $(Q_s, \mathcal{L}_{Q_s}, \phi_{Q_s}, \tau_{Q_s}) = \phi(\sigma) \otimes k(s) = \phi(\sigma)_S \otimes k(0) \simeq \phi_S(\sigma_S) \otimes k(0)$ is a $k(s)$-PSQAS with rigid $G(K)$-structure. Hence $\chi(Q_s, \mathcal{L}^n_{Q_s}) = n^g\sqrt{|K|}$ is independent of $s$. By [7, III, Theorem 9.9], $Q$ is $T$-flat.

To prove that $\phi(\sigma)$ is a $T$-PSQAS with rigid $G(K)$-structure, it remains to check Section 6.2 (iv). By Section 5.9 (iv) $G(K)_T$ acts on both $\mathcal{L}$ and $P$ in a compatible manner over $T$, which acts therefore on $\mathrm{Sym}(\phi)$, hence on $Q = \mathrm{Proj}(\mathrm{Sym}(\phi))$ and $\mathcal{L}_Q$. Hence we have a closed immersion $\tau_Q$ of $G(K)_T$ into $\mathrm{Aut}(\mathcal{L}_Q/Q)$. Hence $\phi(\sigma)$ is a $T$-PSQAS with rigid $G(K)$-structure. If any fiber of $\pi$ is a polarized abelian variety, it is clear that $\phi(\sigma) = \sigma$. This completes the proof. Q.E.D.

**Theorem 12.2.** *Assume $e_{\min}(K) \geq 3$. Then there is a canonical bijective finite birational $\mathcal{O}$-morphism* $\mathrm{sq} : SQ^{\mathrm{toric}}_{g,K} \to SQ_{g,K}$ *extending the identity of $A_{g,K}$.*

*Proof.* Since $SQ^{\mathrm{toric}}_{g,K}$ is a categorical quotient of $U_3$ by $\mathrm{PGL}(W_1) \times \mathrm{PGL}(W_2)$, in order to define a morphism from $SQ^{\mathrm{toric}}_{g,K}$ to $SQ_{g,K}$, it suffices to find a $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-invariant morphism $\widetilde{h} : U_3 \to SQ_{g,K}$.

Recall that we have a universal subscheme $Z$ of $X = X_1 \times_{\mathcal{O}} X_2$ over $U_3$ in Section 9.5 with line bundles $L$, $L_1$ and $L_2$ over $U_3$ such that

(a) $L_i$ is relatively very ample, $L_i = d_i L$, $L = q_1 L_1 + q_2 L_2$,
(b) $L$ and $L_i$ are $G(K)$-linearized with weight one,
(c) $\pi_*(L) \simeq V(K) \otimes M_0$ for some line bundle $M_0$ on $T$,

(d) $\pi_*(L)$ is base point free.

Let $T$ be any subscheme of $U_3$ whose generic point is in $U^\dagger_{g,K}$, and $(P,\mathcal{L}) := (Z,L)\times_{U_3}T$. By (b) we have a closed immersion $\tau : G(K)_T \to \mathrm{Aut}(\mathcal{L}/P)$. By (b)–(d) we have a $G(K)$-morphism $\phi : P \to \mathbf{P}(V(K))_T$ with regards to $\tau$. Let $\sigma = (P,\mathcal{L},\phi,\tau)$. We may assume that $\sigma$ is a $T$-TSQAS with rigid $G(K)$-structure by rechoosing $\phi$ if necessary. In view of Theorem 12.1, $\phi(\sigma) = (Q,\mathcal{L}_Q,\phi_Q,\tau_Q)$ is a $T$-PSQAS with rigid $G(K)$-structure. So we define $\widetilde{h}(\sigma) := \phi(\sigma) \in SQ_{g,K}(T)$.

This gives a morphism from $U_3$ to $SQ_{g,K}$. If two $T$-TSQASes $\sigma = (P,\mathcal{L},\phi,\tau)$ and $\sigma' = (P',\mathcal{L}',\phi',\tau')$ with rigid $G(K)$-structure are $T$-isomorphic, then $\phi(\sigma)$ and $(\phi')(\sigma')$ are $T$-isomorphic by their construction. This shows that $\widetilde{h}$ is $\mathrm{GL}(W_1)\times\mathrm{GL}(W_2)$-invariant. Hence $\widetilde{h}$ defines a morphism from $SQ^{\mathrm{toric}}_{g,K}$ to $SQ_{g,K}$, *which we denote by* $h$. In view of Theorem 12.1, $\phi(\sigma)\simeq\sigma$ if any fiber of $\sigma$ is a polarized abelian variety. This shows that $h$ is the identity on $A^{\mathrm{toric}}_{g,K} = A_{g,K}$.

We shall prove that $h$ is bijective. For this, we prove that any geometric fiber of $h$ is a single point.

Let $\rho_0 := (Q_0,\mathcal{L}_{Q_0},\phi_{Q_0},\tau_{Q_0})_{\mathrm{RIG}}$ be a $k$-PSQAS with rigid $G(K)$-structure, $k$ a closed field. Let $R$ be a complete discrete valuation ring with its residue field $k(0) = k$, and $S = \mathrm{Spec}\ R$. Let $\rho := (Q,\mathcal{L}_Q,\phi_Q,\tau_Q)$ be an $S$-PSQAS with rigid $G(K)$-structure such that

- $\rho_0 := \rho\otimes k(0)$, and its generic fiber is an abelian variety,
- its normalization $\sigma := (P,\mathcal{L}_P,\phi_P,\tau_P)$ is an $S$-TSQAS with rigid $G(K)$-structure.

Let $\rho^\dagger := (Q^\dagger,\mathcal{L}_{Q^\dagger},\phi_{Q^\dagger},\tau_{Q^\dagger})$ be another $S$-PSQAS with rigid $G(K)$-structure such that

- $\rho^\dagger_0 := \rho^\dagger\otimes k(0)\simeq\rho_0$, and its generic fiber is an abelian variety,
- its normalization $\sigma^\dagger := (P^\dagger,\mathcal{L}_{P^\dagger},\phi_{P^\dagger},\tau_{P^\dagger})$ is an $S$-TSQAS with rigid $G(K)$-structure.

First we consider the totally degenerate case. By the assumption $\rho_0\simeq\rho^\dagger_0$, we have the same lattice $X$ and the same sublattice $Y$ of $X$ wit $K\simeq X/Y$, hence the same formal split torus $\mathbf{G}^{\mathrm{for}}_{m,S}\otimes_{\mathbf{Z}} X$ acting on $Q$ (resp. $Q^\dagger$). Hence we have the degeneration data for $\rho := (Q,\mathcal{L}_Q,\phi_Q,\tau_Q)$ (resp. $\rho^\dagger := (Q^\dagger,\mathcal{L}_{Q^\dagger},\phi_{Q^\dagger},\tau_{Q^\dagger})$) which are labelled by $X$ and $X\times X$. Let $a(x)$ and $b(x,y)$ (resp. $a^\dagger(x)$ and $b^\dagger(x,y)$) be the degeneration data for $Q$ (resp. $Q^\dagger$). Let

$$B(x,y) = val_S(b(x,y)),\quad B^\dagger(x,y) = val_S(b^\dagger(x,y)),$$
$$\bar{b}(x,y) = s^{-B(x,y)}b(x,y),\quad \bar{b}^\dagger(x,y) = s^{-B^\dagger(x,y)}b^\dagger(x,y).$$

By Section 2.6, we have a semi-universal covering $\widetilde{Q}_0$ of $Q_0$ (resp. $\widetilde{Q}_0^\dagger$ of $Q_0^\dagger$) with covering transformations $S_x$ (resp. $S_x^\dagger$) $(x \in X)$. Similarly we have a semi-universal covering $\widetilde{P}_0$ of $P_0$ (resp. $\widetilde{P}_0^\dagger$ of $P_0^\dagger$) with covering transformations $S_x$ (resp. $S_x^\dagger$) $(x \in X)$.

Let $f : \rho_0 \to \rho_0^\dagger$ be an isomorphism of $k$-PSQASes with rigid $G(K)$-structure. We may assume $\mathrm{Del}(Q_0) = \mathrm{Del}(Q_0^\dagger)$, which we write Del for brevity. Since $f$ induces a $G(K)$-isomorphism $f^* : \Gamma(Q_0^\dagger, \mathcal{L}_{Q_0^\dagger}) \to \Gamma(Q_0, \mathcal{L}_{Q_0})$, the isomorphism $(\phi_{Q_0}^*)^{-1} \cdot f^* \cdot \phi_{Q_0^\dagger}^* : V(K) \otimes k \stackrel{\simeq}{\to} V(K) \otimes k$ is multiplication by a nonzero constant $A$. Associated to $Q^\dagger$, we have a formal torus $\mathrm{Hom}(X, \mathbf{G}_{m,S}^{\mathrm{for}})$ acting on $Q^{\mathrm{for}}$, hence its closed fiber $\mathrm{Hom}(X, \mathbf{G}_m(k)) \simeq \mathrm{Hom}(X, \mathbf{G}_{m,S}^{\mathrm{for}}) \otimes k(0)$ on $Q_0$. Therefore there is at most a unique monomial term $\xi_x^\dagger$ of weight $x$ (that is, $\xi_x$ for $P_0^\dagger$ in Section 3.4) in the Fourier expansions of elements of $\Gamma(Q_0^\dagger, \mathcal{L}_{Q_0^\dagger})$. Hence we have an equality $f^*(\xi_x^\dagger) = A\xi_x$ for each weight $x$. Hence we may assume that $f$ induces the isomorphism $f(c) : W_0(c) \to W_0^\dagger(c)$ between local charts $W_0(c)$ (in Lemma 3.5) of $\widetilde{Q}_0$ and $W_0^\dagger(c)$ of $\widetilde{Q}_0^\dagger$ (that is, $W_0(c)$ for $\widetilde{Q}_0^\dagger$ in Lemma 3.5). We recall that

$$\Gamma(O_{W_0(c)}) := \Gamma(W_0(c), O_{W_0(c)}) = k[\xi_{x,c}, x \in X],$$
$$\Gamma(O_{W_0^\dagger(c)}) := \Gamma(W_0^\dagger(c), O_{W_0^\dagger(c)}) = k[\xi_{x,c}^\dagger, x \in X].$$

Then we have $f^*(c)(\xi_{x,c}^\dagger) = f^*(c)(\xi_{x+c}^\dagger/\xi_c^\dagger) = \xi_{x+c}/\xi_c = \xi_{x,c}$ for any $x \in \mathrm{Semi}(0, -c+\sigma)$, the semigroup generated by all $a - c$ $(a \in \sigma \cap X)$. Hence we have $f^*(c)(\xi_{x,c}^\dagger) = f^*(c)(\xi_{x+c}^\dagger/\xi_c^\dagger) = \xi_{x+c}/\xi_c = \xi_{x,c}$ $(\forall x \in X)$ because $\sigma$ moves freely in $\mathrm{Del}(c)$.

By Theorem 3.8, $P_0$ (and $P_0^\dagger$) is an amalgamation of those strata $\overline{O(\sigma)}$ which are in bijective correspondence with the strata of $(Q_0)_{\mathrm{red}}$. Let $U_0(c)$ (resp. $U_0^\dagger(c)$) be a local chart of $P_0$ (resp. $P_0^\dagger$). By Lemma 3.6

$$\Gamma(O_{U_0(c)}) := \Gamma(U_0(c), O_{U_0(c)}) = k[\zeta_{x,c}, x \in X],$$
$$\Gamma(O_{U_0^\dagger(c)}) := \Gamma(U_0^\dagger(c), O_{U_0^\dagger(c)}) = k[\zeta_{x,c}^\dagger, x \in X].$$

Now we define $\widetilde{f}^*(c)(\zeta_{x,c}^\dagger) = \zeta_{x,c}$. Since the relations of $\zeta_{x,c}$ or $\zeta_{x,c}^\dagger$ are given in terms of the Delaunay decomposition Del as in Lemma 3.6, $\widetilde{f}^*(c)$ is an algebra isomorphism. Since formally $S_y^*(\xi_x) = \bar{b}(x,y)\xi_x$ and $(S_y^\dagger)^*(\xi_x^\dagger) = \bar{b}^\dagger(x,y)\xi_x^\dagger$, we have in $\Gamma(O_{W_0(c)})$,

$$S_y^*(\xi_{x,c}) = b_0(x,y)\xi_{x,c+y}, \quad (S_y^\dagger)^*(\xi_{x,c}^\dagger) = b_0^\dagger(x,y)\xi_{x,c+y}^\dagger.$$

Since $\rho_0 \simeq \rho_0^\dagger$, we have $S_y^* f(c)^* = f(c+y)^*(S_y^\dagger)^*$, whence $b_0(x,y) = b_0^\dagger(x,y)$ for any $x, y \in X$. Since

$$S_y^*(\zeta_{x,c}) = b_0(x,y)\zeta_{x,c+y}, \quad (S_y^\dagger)^*(\zeta_{x,c}^\dagger) = b_0^\dagger(x,y)\zeta_{x,c+y}^\dagger,$$

we have $S_y^* \widetilde{f}(c)^* = \widetilde{f}(c+y)^*(S_y^\dagger)^*$ on $\Gamma(O_{U_0(c)})$ for any $c, y \in X$.

For any Delaunay cell $\tau \in \mathrm{Del}$, let

$$U_0(\tau) = \cap_{d \in \tau \cap X} U_0(c), \quad U_0^\dagger(\tau) = \cap_{d \in \tau \cap X} U_0^\dagger(c).$$

Then the algebras $\Gamma(O_{U_0^\dagger(\tau)})$ and $\Gamma(O_{U_0(\tau)})$ are isomorphic because the relations between the generators are described in terms of Delaunay decomposition Del. This implies that $\widetilde{f}(c)$ induces a natural isomorphism $\widetilde{f}(\tau) : U_0(\tau) \to U_0^\dagger(\tau)$ such that $S_y^* \widetilde{f}(\tau)^* = \widetilde{f}(y+\tau)^*(S_y^\dagger)^*$ ($\forall y \in X$).

Therefore, $\widetilde{f}(c)$ ($c \in X$) glue together to give rise to an isomorphism $\widetilde{f} : \widetilde{P}_0 \to \widetilde{P}_0^\dagger$, hence a well-defined global $G(K)$-isomorphism $f_P : P_0 \to P_0^\dagger$. The triple of the remaining data $(\mathcal{L}_{P_0}, \phi_{P_0}, \tau_{P_0})$ (resp. $(\mathcal{L}_{P_0^\dagger}, \phi_{P_0^\dagger}, \tau_{P_0^\dagger})$) are induced from $(\mathcal{L}_{Q_0}, \phi_{Q_0}, \tau_{Q_0})$ (resp. $(\mathcal{L}_{Q_0^\dagger}, \phi_{Q_0^\dagger}, \tau_{Q_0^\dagger})$) by the universal property of amalgamation. This proves that $\sigma_0 \simeq \sigma_0^\dagger$. Hence $h^{-1}(\rho_0)$ is a single point.

Similarly when $\rho_0$ is partially degenerate, the abelian parts and the extension classes of $\sigma_0$ and $\rho_0$ are the same. Hence the geometric fiber $h^{-1}(\rho_0)$ is a single point by the bijectivity in the totally degenerate case. Hence $h^{-1}(\rho_0)$ is a single point for any $\rho_0$. Since $SQ_{g,K}^{\mathrm{toric}}$ is proper over $\mathcal{O}$, $h$ is finite. Since $A_{g,K}^{\mathrm{toric}} \simeq A_{g,K}$ and they are Zariski open, this proves that $h$ is a bijective finite birational morphism. Q.E.D.

**Corollary 12.3.** *If* $e_{\min}(K) \geq 3$, $SQ_{g,K}^{\mathrm{toric}}$ *is projective.*

*Proof.* Since $SQ_{g,K}^{\mathrm{toric}}$ is finite over $SQ_{g,K}$ and $SQ_{g,K}$ is projective by [18, Definition 11.2, Theorem 11.4], $SQ_{g,K}^{\mathrm{toric}}$ is projective. Q.E.D.

**Corollary 12.4.** *If* $e_{\min}(K) \geq 3$, *the normalizations of* $SQ_{g,K}^{\mathrm{toric}}$ *and* $SQ_{g,K}$ *are isomorphic.*

*Proof.* The morphism $h$ is an isomorphism on $A_{g,K}^{\mathrm{toric}}$, hence it is birational. Hence $h$ induces a finite birational morphism $h^{\mathrm{norm}}$ between the normalizations of $SQ_{g,K}^{\mathrm{toric}}$ and $SQ_{g,K}$. Since any finite birational morphism between two normal schemes is an isomorphism by [16, p. 201, Theorem 3], $h^{\mathrm{norm}}$ is an isomorphism. Q.E.D.

## Notation and Terminology

| | |
|---|---|
| $A_{g,K}, A_{g,K}^{\text{toric}}$ | fine moduli of abelian varieties, Lemma 11.5 |
| $\mathrm{Aut}(\mathcal{L}/P), \mathrm{Aut}_T(\mathcal{L}/P)$ | Sections 2.12, 2.16, 5.9, 5.10 |
| $a(x), b(x,y)$ | degeneration data, Theorem 2.3 |
| $\bar{a}(x), \bar{b}(x,y)$ | Section 2.4 |
| $a_0(x), b_0(x,y)$ | Section 2.4, Proof of Theorem 12.2 |
| $\alpha(\sigma)$ | center of $\sigma$, Section 2.5 |
| $\alpha_c$ | Proof of Theorem 12.2 |
| $C(c,\sigma), C(0,-c+\sigma)$ | Section 2.5 |
| $\mathrm{Del}, \mathrm{Del}(c)$ | Section 2.5 |
| $\mathrm{Del}(P_0), \mathrm{Del}(\widetilde{Q}_0)$ | Section 2.8, $\mathrm{Del}(\widetilde{Q}_0) := \mathrm{Del} = \mathrm{Del}_B$ |
| $\phi_g, \phi_h$ | Section 4.7 |
| $e_K$ | Section 4.1 |
| $e_S^\sharp, e^{\mathcal{L}_\eta}$ | Weil pairing, Section 2.15 |
| $(\phi, \Phi)$ | Sections 4.13, 5.9 |
| $\phi(\sigma)$ | Theorem 12.1 |
| $(\phi, \tau)$ | Section 5.3 |
| $G, G^\sharp$ | semi-abelian schemes, Sections 2.15, 2.11 |
| $G(P, \mathcal{L})$ | Section 4.6 |
| $\mathcal{G}(P, \mathcal{L})$ | Definition 2.17 |
| $G(K), \mathcal{G}(K)$ | Heisenberg group, Section 4.1 |
| $\mathcal{G}_S^\sharp(\mathcal{L})$ | Section 2.15 |
| $H, K, K(H)$ | Section 4.1 |
| $H, H_1, H_2$ | Hilbert schemes, Section 9.2 |
| $H_{\text{conn}}^P, H_{\text{conn}}^P(X/H)$ | Sections 9.1, 9.3 |
| $(K, e_K)$ | Section 4.1 |
| $K(\mathcal{L}_\eta), K_S^\sharp(\mathcal{L})$ | Section 2.12, Lemma 2.14 |
| $K(P, \mathcal{L}), K(P_0, \mathcal{L}_0)$ | Definition 2.17 |
| $K(Q_0, \mathcal{L}_0)$ | $:= K(P_0, \mathcal{L}_0)$, Theorem 2.22, Lemma 2.19 |
| $\xi_x, \xi_{x,c}$ | Section 3.4 |
| $\mathbf{L}, \mathbf{L}(K), \mathbf{L}(V(K))$ | Section 4.12 |
| $\mathcal{L}^\times := \mathcal{L} \setminus \{0\}$ | Section 2.15 |
| $\lambda, \lambda(\mathcal{L}_\eta)$ | Section 2.1 |
| level-$G(K)$ structure | Sections 5.3, 6.2, 5.4 |
| $\mu_N$ | Section 4.1 |
| $\mathcal{O}, \mathcal{O}_N$ | Section 4.1 |
| $O(\sigma), O(\sigma, (Q_0)_{\text{red}})$ | Lemmas 3.6, 3.7 |
| $\mathbf{P}, \mathbf{P}(K) := \mathbf{P}(V(K))$ | Section 4.12 |

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