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On the bifurcation structure of radially symmetric positive stationary solutions for a competition-diffusion system

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Abstract.

In this paper, we consider radially symmetric positive stationary solutions for the competition-diffusion system which describes the dynamics of population for a two-competing-species community, and discuss the bifurcation structure of solution by employing the comparison principle and the bifurcation theory.

§1. Introduction

In this paper, we consider the bifurcation structure of positive stationary solution for the competition-diffusion system

$$\text{(1.1)} \qquad \begin{cases} \mathbf{u}_t = \varepsilon\, D\, \Delta \mathbf{u} + \mathbf{f}(\mathbf{u}), & x \in \Omega, \quad t > 0, \\ \frac{\partial}{\partial \nu}\mathbf{u} = \mathbf{0}, & x \in \partial\Omega, \quad t > 0 \end{cases}$$

which describes the dynamics of population for a two-competing-species community, where $\varepsilon > 0$, $d_u > 0$, $d_v > 0$, $D = \mathrm{diag}(d_u, d_v)$, $\mathbf{u} = (u, v)$,

$$\mathbf{f}(\mathbf{u}) = (f, g)(\mathbf{u}), \qquad f(\mathbf{u}) = u\, f_0(\mathbf{u}), \qquad g(\mathbf{u}) = v\, g_0(\mathbf{u}),$$

$\mathbf{f}_0(\mathbf{u}) = (f_0, g_0)(\mathbf{u})$ is a smooth function in $\mathbf{u}$, and we call $\mathbf{u}(x)$ *positive* if $\mathbf{u}(x)$ is in the first quadrant for any $x \in \mathrm{Cl}\,\Omega$. For the sake of simplicity, we take Ω as a ball with center origin and radius π, and we restrict our discussion to the radially symmetric positive solution for the stationary

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problem of the system (1.1). It turns out that the solution $\mathbf{u}(r)$ satisfies

$$(1.2) \qquad \begin{cases} \varepsilon\, D\,\mathcal{K}(\mathbf{u};\ell) = \mathbf{f}(\mathbf{u}), & r \in (0,\pi), \\ \mathbf{u}'(0) = \mathbf{0}, \quad \mathbf{u}'(\pi) = \mathbf{0}, \end{cases}$$

where $r = |\,x\,|$, and $\mathcal{K}(u;\ell) = -r^{1-\ell}\left[r^{\ell-1}\,u'\right]'$ is a linear operator from $X = \{\, u \in C^2([0,\pi]) \mid u'(0) = 0 = u'(\pi)\,\}$ to $C^0([0,\pi])$. Moreover, although ℓ is a positive integer, we treat ℓ as a real-valued parameter with $\ell \geq 1$.

§2. Assumption

To mention assumptions and results, we define the order relations $\preceq_s$ and $\preceq_o$ on $\mathbb{R}^2$ in the following manner:

$$\begin{aligned} (u,v) \preceq_s (\tilde{u},\tilde{v}) \quad &\Longleftrightarrow \quad u \leq \tilde{u},\ v \leq \tilde{v}, \\ (u,v) \preceq_o (\tilde{u},\tilde{v}) \quad &\Longleftrightarrow \quad u \leq \tilde{u},\ v \geq \tilde{v}. \end{aligned}$$

We denote by $\prec_s$ and $\prec_o$ the relations obtained from the above definition by replacing $\leq$ with $<$, and we set $\mathbb{R}_+ = (0,+\infty)$. From the competitive interaction, we assume that

(A.1) $f_0(\mathbf{0}) > 0$ and $g_0(\mathbf{0}) > 0$ hold, and there exists $\delta > 0$ such that

$$\max\left(f_{0u}(\mathbf{u}), f_{0v}(\mathbf{u}), g_{0u}(\mathbf{u}), g_{0v}(\mathbf{u})\right) < -\delta$$

is satisfied for any $\mathbf{u} \in \mathbb{R}^2_+$,

(A.2) there exist the zeros $\mathbf{e}_-$, $\hat{\mathbf{e}}$ and $\mathbf{e}_+$ of $\mathbf{f}(\mathbf{u})$ on $\mathrm{Cl}\,\mathbb{R}^2_+ \setminus \{\mathbf{0}\}$ such that

$$\mathbf{e}_- \prec_o \hat{\mathbf{e}} \prec_o \mathbf{e}_+, \qquad \det \mathbf{f}_{\mathbf{u}}(\mathbf{e}_\pm) > 0, \qquad \det \mathbf{f}_{\mathbf{u}}(\hat{\mathbf{e}}) < 0$$

hold and the equation $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ with the condition

$$\mathbf{u} \in \mathcal{D} \equiv \{\, \mathbf{u} \in \mathbb{R}^2 \mid \mathbf{e}_- \prec_o \mathbf{u} \prec_o \mathbf{e}_+ \,\}$$

has no solution other than $\hat{\mathbf{e}}$, and

(A.3) there exists a solution $\phi(r)$ of

$$(2.1) \qquad \begin{cases} D\,\mathcal{K}(\mathbf{u};\ell) = \mathbf{f}(\mathbf{u}), & r \in \mathbb{R}_+, \\ \mathbf{u}(r) \in \mathcal{D}, \quad \mathbf{u}'(r) \prec_o \mathbf{0}, & r \in \mathbb{R}_+, \\ \mathbf{u}'(0) = \mathbf{0}, \quad \mathbf{u}(+\infty) = \mathbf{e}_- \end{cases}$$

for the case where $\ell = 1$.

The assumption (A.1) and the comparison principle say that the problem (1.2) falls into a class of weakly coupled elliptic systems with respect to the order relation $\preceq_o$, and the assumption (A.2) implies that $\mathbf{e}_-$ and $\mathbf{e}_+$ are stable equilibrium points of the ODE $\mathbf{u}_t = \mathbf{f}(\mathbf{u})$ and $\hat{\mathbf{e}}$ is an unstable one.

Let us consider the nonlinear term $\mathbf{f}(\mathbf{u})$ with

$$f_0(\mathbf{u}) = 1 - u - c\,v, \qquad g_0(\mathbf{u}) = a - b\,u - v \tag{2.2}$$

for the case where $0 < 1/c < a < b$. We set

$$\mathbf{e}_- = (0, a)\,, \qquad \hat{\mathbf{e}} = \left(\frac{1 - a\,c}{1 - b\,c}, \frac{a - b}{1 - b\,c}\right), \qquad \mathbf{e}_+ = (1, 0)\,.$$

Proposition 2.1 ([1]). *Under the condition $\ell = 1$ and the nonlinear term (2.2), there exist a constant $a_0 \in (1/c, b)$ and a continuous function $\phi(\cdot, a)$ defined on $(1/c, a_0)$ such that $\phi(r, a)$ is a positive solution of the problem (2.1) for each a. Furthermore, if $\mathbf{u}(r)$ is an arbitrary nonconstant positive solution of the problem (2.1) for $\ell = 1$ and $a \in (1/c, a_0)$, then there exists $\tau \in \mathbb{R}$ such that $\mathbf{u}(r) = \phi(r + \tau, a)$ holds for any $r \in \mathbb{R}$.*

The above proposition means that for each $a \in (1/c, a_0)$, the nonlinear term (2.2) is most simple example satisfying the assumptions (A.1), (A.2) and (A.3). Moreover we remark here that for our nonlinear term $\mathbf{f}(\mathbf{u})$, we can prove the uniqueness result as shown in the latter part of Proposition 2.1, by employing the argument in [1].

§3. Local Structure

We denote by $\mathbb{N}_0$ the set of nonnegative integers. Let $\{\,\lambda_k(\ell)\,\}_{k\in\mathbb{N}_0}$ be the set of eigenvalues of $\mathcal{K}(\cdot;\ell)$ satisfying

$$0 = \lambda_0(\ell) < \lambda_k(\ell) \le \lambda_{k+1}(\ell) \qquad \text{for each } k \in \mathbb{N},$$

and let $\phi_k(r, \ell)$ $(k \in \mathbb{N}_0)$ be an eigenfunction of $\mathcal{K}(\cdot;\ell)$ corresponding to the eigenvalue $\lambda_k(\ell)$. Here we may assume $\phi_k(0, \ell) = 1$ for each $k \in \mathbb{N}_0$ without loss of generality. It is well-known that the following property holds for any $\ell \ge 1$:

(i) $\lim_{k\to\infty} \lambda_k(\ell) = +\infty$ is satisfied,
(ii) $\phi_1'(r, \ell) < 0$ holds for any $r \in (0, \pi)$, and
(iii) $\phi_k(r, \ell)$ $(k \in \mathbb{N})$ is represented as

$$\phi_k(r, \ell) = \begin{cases} \cos k\,r & (\ell = 1)\,, \\ C\,r^{\frac{2-\ell}{2}}\,J_{\frac{\ell-2}{2}}\left(\sqrt{\lambda_k(\ell)}\,r\right) & (\ell > 1)\,, \end{cases}$$

where $J_\nu(z)$ is the Bessel function of the first kind, and C is a suitable positive constant.

Setting

$$\Phi(\ell, k, n) = \int_0^\pi \phi_k(r, \ell)^n \, r^{\ell-1} \, dr,$$

we can easily obtain $\Phi(\ell, k, 2) > 0$ for any $\ell \geq 1$ and $k \in \mathbb{N}_0$, and $\Phi(1, k, 3) = 0$ for any $k \in \mathbb{N}$.

Lemma 3.1 ([2]). *$\Phi(\ell, k, 3) > 0$ holds for any $\ell > 1$ and $k \in \mathbb{N}$.*

Let $\ell \geq 1$ be arbitrarily fixed. From the assumption (A.2), it follows that the equation $\det(\varepsilon D - \mathbf{f_u}(\hat{\mathbf{e}})) = 0$ has a unique positive solution $\varepsilon = \bar{\varepsilon}$. Let $\mathbf{v}$ and $\mathbf{v}^*$ be nontrivial solutions of

$$(\bar{\varepsilon} D - \mathbf{f_u}(\hat{\mathbf{e}})) \, \mathbf{v} = \mathbf{0} \qquad \text{and} \qquad \left(\bar{\varepsilon} D - \mathbf{f_u}(\hat{\mathbf{e}})^T\right) \mathbf{v}^* = \mathbf{0},$$

respectively, where A^T is the transposed matrix of the matrix A. After simple calculations, we can check that $\mathbf{v} \succ_o \mathbf{0}$, $\mathbf{v}^* \succ_o \mathbf{0}$ and $(D\,\mathbf{v}, \mathbf{v}^*) > 0$ are satisfied, and that for each $k \in \mathbb{N}$,

(i) the linearized operator $\mathcal{L}_k$ of the problem (1.2) around $\mathbf{u} = \hat{\mathbf{e}}$ for

$$\varepsilon = \hat{\varepsilon}_k(\ell) \equiv \frac{\bar{\varepsilon}}{\lambda_k(\ell)}$$

has the simple eigenvalue 0 with the corresponding eigenfunction $\phi_k(r, \ell)\,\mathbf{v}$, and

(ii) $\phi_k(r, \ell)\,\mathbf{v}^*$ is an eigenfunction for the adjoint operator of $\mathcal{L}_k$ corresponding to the eigenvalue 0.

Setting

$$\begin{aligned} \varepsilon = \quad & \tilde{\varepsilon}_k(\ell, \mu) && \equiv \hat{\varepsilon}_k(\ell) + \mu\, \tilde{\varepsilon}_{k,1}(\ell) + \mu^2\, \tilde{\varepsilon}_{k,2}(\ell, \mu), \\ \mathbf{u} = \quad & \tilde{\mathbf{u}}_k(r, \ell, \mu) && \equiv \hat{\mathbf{u}} + \mu\, \phi_k(r, \ell)\,\mathbf{v} + \mu^2\, \tilde{\mathbf{u}}_{k,2}(r, \ell, \mu) \end{aligned}$$

and employing usual bifurcation theory, we have

$$\tilde{\varepsilon}_{k,1}(\ell) = \frac{(\mathbf{f}_2(\mathbf{v}, \mathbf{v}), \mathbf{v}^*)\ \Phi(\ell, k, 3)}{\lambda_k(\ell)\ (D\,\mathbf{v}, \mathbf{v}^*)\ \Phi(\ell, k, 2)} \tag{3.1}$$

for each $k \in \mathbb{N}$, where $\mathbf{f}_2(\mathbf{u}_1, \mathbf{u}_2)$ is a bilinear map obtained from the second derivative of $\mathbf{f}(\mathbf{u})$. Moreover the above expansion says that either $\tilde{\mathbf{u}}_k'(r, \ell, \mu) \prec_o \mathbf{0}$ on $(0, \pi)$ or $\tilde{\mathbf{u}}_k'(r, \ell, \mu) \succ_o \mathbf{0}$ on $(0, \pi)$ holds for small $|\mu| \neq 0$, because $\mathbf{v} \succ_o \mathbf{0}$ and $\phi'(r) < 0$ on $(0, \pi)$ are satisfied. We should remark that when $\ell = 1$ and/or $(\mathbf{f}_2(\mathbf{v}, \mathbf{v}), \mathbf{v}^*) = 0$ holds, we need to study the property of $\tilde{\varepsilon}_{k,2}(\ell, \mu)$ to determine the local structure of solution for the problem (1.2) in a neighborhood of $(\varepsilon, \mathbf{u}) = (\hat{\varepsilon}_k(\ell), \hat{\mathbf{e}})$.

Let $\ell \geq 1$ be arbitrary, and let $\mathbf{u}(r)$ be an arbitrary monotone positive solution of the problem (1.2) for $\varepsilon > 0$ satisfying $\mathbf{u}(r) \in \mathcal{D}$ on $[0, \pi]$, where we call $\mathbf{u}(r) = (u, v)(r)$ *monotone* if $u'(r)\, v'(r) < 0$ is satisfied for each $r \in (0, \pi)$. From

$$\begin{aligned} \mathbf{0} =& \mathcal{K}(\mathbf{u}'; \ell)(r) + \frac{\ell - 1}{r^2}\, \mathbf{u}'(r) - (\varepsilon\, D)^{-1}\, \mathbf{f}_{\mathbf{u}}(\mathbf{u}(r))\, \mathbf{u}'(r), \\ 0 =& \mathcal{K}(\phi_1'; \ell)(r) + \frac{\ell - 1}{r^2}\, \phi_1'(r) - \lambda_1(\ell)\, \phi_1'(r), \qquad r \in (0, \pi), \end{aligned}$$

we have

$$0 = \int_0^\pi h(r)\, \phi_1'(r)\, r^{\ell - 1}\, dr,$$

where

$$\begin{aligned} h(r) = & \left(\frac{f_u(\mathbf{u}(r))}{\varepsilon\, d_u} - \frac{g_u(\mathbf{u}(r))}{\varepsilon\, d_v} - \lambda_1(\ell) \right) u'(x) \\ & + \left(\lambda_1(\ell) + \frac{f_v(\mathbf{u}(r))}{\varepsilon\, d_u} - \frac{g_v(\mathbf{u}(r))}{\varepsilon\, d_v} \right) v'(x). \end{aligned}$$

We set

$$M = \frac{2 \max_{\mathbf{u} \in \operatorname{Cl} \mathcal{D}} (|\, f_u(\mathbf{u})\, |, |\, f_v(\mathbf{u})\, |, |\, g_u(\mathbf{u})\, |, |\, g_v(\mathbf{u})\, |)}{\min(d_u, d_v)}.$$

Since

$$u'(r)\, h(r) \leq \left(\frac{M}{\varepsilon} - \lambda_1(\ell) \right) \left(u'(r)^2 - u'(r)\, v'(r) \right) < 0$$

holds for any $r \in [0, \pi]$ when $\varepsilon > M/\lambda_1(\ell)$ is satisfied, it follows that $\varepsilon \leq M/\lambda_1(\ell)$ must be satisfied. The comparison principle and the assumptions (A.1) and (A.2) give us the following for any positive solution $\mathbf{u}(r) = (u, v)(r)$ of (1.2):

(i) If $u'(\tau)\, v'(\tau) = 0$ for some $\tau \in (0, \pi)$ and either $\mathbf{u}'(r) \succeq_o \mathbf{0}$ on $[0, \pi]$ or $\mathbf{u}'(r) \preceq_o \mathbf{0}$ on $[0, \pi]$ hold, then $\mathbf{u}(r)$ must be a constant function on $[0, \pi]$;

(ii) If $\mathbf{u}(\tau) \in \partial\mathcal{D}$ for some $\tau \in [0, \pi]$ and $\mathbf{u}(r) \in \operatorname{Cl} \mathcal{D}$ for any $r \in [0, \pi]$ hold, then either $\mathbf{u}(\cdot) = \mathbf{e}_-$ or $\mathbf{u}(\cdot) = \mathbf{e}_+$ is satisfied.

Combining the above facts and Theorem 1.3 in Rabinowitz [3], we have the following:

Lemma 3.2. *Let $\ell \geq 1$ be arbitrary. Then there exists a maximal connected continuum $\mathcal{C}(\ell) \subset \mathbb{R}_+ \times X^2$ such that (i) $\mathcal{C}(\ell)$ contains $(\hat{\varepsilon}_1(\ell), \hat{\mathbf{e}})$ and meets $\{ 0 \} \times X^2$, and (ii) for each $(\varepsilon, \mathbf{u}(\cdot)) \in \mathcal{C}(\ell) \backslash \{ (\hat{\varepsilon}, \hat{\mathbf{e}}) \}$, $\mathbf{u}(r)$ is a monotone positive solution of the problem (1.2) for ε and satisfies $\mathbf{u}(r) \in \mathcal{D}$ for any $r \in [0, \pi]$.*

§4. Global Structure

Let $\ell \geq 1$ be arbitrarily fixed, and let $\mathbf{u}_j(r)$ $(j = 1, 2)$ be an arbitrary monotone positive solution of the problem (1.2) for $\varepsilon = \varepsilon_j > 0$. We denote by $[\mathbf{u}]_j$ the jth element of the vector $\mathbf{u}$. Here we consider the case where $[\mathbf{u}_1(0)]_1 = [\mathbf{u}_2(0)]_1$. With $j \in \{1, 2\}$, setting

$$\gamma_j = \frac{\pi}{\sqrt{\varepsilon_j}}, \qquad \mathbf{w}_j(r)\,(= (w_j, z_j)(r)) = \mathbf{u}_j\left(\sqrt{\varepsilon_j} r\right),$$

we see that $\mathbf{w}_j(r)$ is a monotone positive solution of

$$D\,\mathcal{K}(\mathbf{w}; \ell) = \mathbf{f}(\mathbf{w}) \tag{4.1}$$

in $(0, \gamma_j)$ with the conditions $\mathbf{w}'(0) = \mathbf{0}$ and $\mathbf{w}'(\gamma_j) = \mathbf{0}$.

We assume $z_1(0) > z_2(0)$, and set $\gamma_0 = \min(\gamma_1, \gamma_2)$. Since

$$\ell\, d_u\, [w_1 - w_2]''(0) = f(\mathbf{w}_2(0)) - f(\mathbf{w}_1(0)) > 0$$

holds due to the assumption (A.1), it follows that there exists $\tau \in (0, \gamma_0]$ such that $\mathbf{w}_1(r) \succ_s \mathbf{w}_2(r)$ is satisfied for any $r \in [0, \tau)$. Since $\mathbf{f}_0(\mathbf{w}_1(r)) \prec_s \mathbf{f}_0(\mathbf{w}_2(r))$ holds for any $r \in [0, \tau)$ because of the assumption (A.1), the problem (4.1) gives us the estimates

$$\begin{aligned} w_2(r)^2 \left(\frac{w_1}{w_2}\right)'(r) &= w_1'(r)\, w_2(r) - w_1(r)\, w_2'(r) \\ &= \frac{r^{1-\ell}}{d_u} \int_0^r (f_0(\mathbf{w}_2(s)) - f_0(\mathbf{w}_1(s)))\, w_1(s)\, w_2(s)\, s^{\ell-1}\, ds > 0, \end{aligned} \tag{4.2}$$

$$\begin{aligned} z_2(r)^2 \left(\frac{z_1}{z_2}\right)'(r) &= z_1'(r)\, z_2(r) - z_1(r)\, z_2'(r) \\ &= \frac{r^{1-\ell}}{d_v} \int_0^r (g_0(\mathbf{w}_2(s)) - g_0(\mathbf{w}_1(s)))\, z_1(s)\, z_2(s)\, s^{\ell-1}\, ds > 0 \end{aligned} \tag{4.3}$$

for any $r \in [0, \tau]$, which imply that $w_1(r)/w_2(r)$ and $z_1(r)/z_2(r)$ are both increasing on $[0, \tau]$. Since

$$1 = \frac{w_1(0)}{w_2(0)} < \frac{w_1(\tau)}{w_2(\tau)} = 1 \qquad \text{or} \qquad 1 < \frac{z_1(0)}{z_2(0)} < \frac{z_1(\tau)}{z_2(\tau)} = 1$$

holds for the case where $\tau < \gamma_0$, it turns out that $\tau = \gamma_0$ must be satisfied. From the estimates (4.2) and (4.3), we have $\mathbf{w}_2'(\gamma_1) \prec_s \mathbf{0}$ for the case where $\gamma_1 \leq \gamma_2$, and $\mathbf{w}_1'(\gamma_2) \succ_s \mathbf{0}$ for the case where $\gamma_1 \geq \gamma_2$. These contradict that $\mathbf{w}_1(r)$ and $\mathbf{w}_2(r)$ are both monotone on $(0, \gamma_0)$. Hence we obtain $z_1(0) \leq z_2(0)$. Since we can derive a contradiction when we assume $z_1(0) < z_2(0)$, we have $\mathbf{w}_1(0) = \mathbf{w}_2(0)$. By the uniqueness of solutions for the problem (4.1), we obtain $\mathbf{w}_1(r) = \mathbf{w}_2(r)$ for any $r \in \mathbb{R}_+$.

Lemma 4.1. *Let $\ell \geq 1$ be arbitrarily fixed, and let $\mathbf{u}_j(r)$ $(j = 1, 2)$ be an arbitrary monotone positive solution of the problem* (1.2) *for $\varepsilon = \varepsilon_j > 0$. If $[\mathbf{u}_1(0)]_1 = [\mathbf{u}_2(0)]_1$ is satisfied, then $\varepsilon_1 = \varepsilon_2$ and $\mathbf{u}_1(\cdot) = \mathbf{u}_2(\cdot)$ hold.*

Let $\ell \geq 1$ be arbitrarily fixed. Setting

$$\mathcal{P}(\ell) = \{ \, [\mathbf{u}(0)]_1 \mid (\varepsilon, \mathbf{u}(\cdot)) \in \mathcal{C}(\ell) \, \},$$
$$p_-(\ell) = \inf \mathcal{P}(\ell), \qquad p_+(\ell) = \sup \mathcal{P}(\ell),$$

we have

$$[\mathbf{e}_-]_1 \leq p_-(\ell) < [\hat{\mathbf{e}}]_1 < p_+(\ell) \leq [\mathbf{e}_+]_1 \qquad \text{for any } \ell \geq 1.$$

It follows from Lemma 4.1 that there exist continuous functions $\hat{\varepsilon}(p, \ell)$ and $\hat{\mathbf{u}}(\cdot, p, \ell)$ defined on $\mathcal{P}(\ell)$ such that (i) $[\hat{\mathbf{u}}(0, p, \ell)]_1 = p$ holds for each $p \in \mathcal{P}(\ell)$ and (ii) $\mathcal{C}(\ell)$ is represented as

$$\mathcal{C}(\ell) = \{ \, (\hat{\varepsilon}(p, \ell), \hat{\mathbf{u}}(\cdot, p, \ell)) \mid p \in \mathcal{P}(\ell) \, \},$$

which implies that the secondary bifurcation of monotone positive solution for the problem (1.2) is of saddle-node type even if it occurs. By Lemma 3.2, we have

$$\lim_{p \to p_\pm(\ell)} \hat{\varepsilon}(p, \ell) = 0 \qquad \text{for any } \ell \geq 1.$$

From the assumption (A.2) and the comparison principle, we obtain

$$\hat{\mathbf{u}}(0, p, \ell) \prec_o \hat{\mathbf{e}} \prec_o \hat{\mathbf{u}}(\pi, p, \ell) \qquad \text{for } p < [\hat{\mathbf{e}}]_1,$$
$$\hat{\mathbf{u}}(0, p, \ell) \succ_o \hat{\mathbf{e}} \succ_o \hat{\mathbf{u}}(\pi, p, \ell) \qquad \text{for } p > [\hat{\mathbf{e}}]_1.$$

From the above estimate, we can take $r_-(p, \ell) \in (0, \pi]$ as satisfying

$$[\hat{\mathbf{u}}(r_-(p, \ell), p, \ell)]_1 = \frac{[\hat{\mathbf{e}}]_1 + p_-(\ell)}{2} \; (\equiv \hat{u}_-)$$

for any p in a neighborhood of $p = p_-(\ell)$. Setting

$$\xi(p, \ell) = \frac{r_-(p, \ell)}{\sqrt{\hat{\varepsilon}(p, \ell)}}, \qquad \mathbf{w}(y, p) = (w, z)(y, p) = \hat{\mathbf{u}}\left(\sqrt{\hat{\varepsilon}(p, \ell)}\, y, p, \ell \right),$$

we see that $\mathbf{w}(y, p)$ is a solution of (4.1) in $\mathbb{R}_+$ satisfying $\mathbf{w}'(0, p) = \mathbf{0}$ and $w(\xi(p, \ell), p) = \hat{u}_-$. From the Ascoli-Arzela theorem, it follows that for any compact subset of $\mathbb{R}_+$, there exists a decreasing sequence $\{\, p_n \,\}_{n\in\mathbb{N}}$ such that the limits

$$\lim_{n\to\infty} p_n = p_-(\ell), \qquad \hat{\mathbf{w}}(\cdot) = (\hat{w}, \hat{z})(\cdot) = \lim_{n\to\infty} \mathbf{w}(\cdot, p_n)$$

exist and $\hat{\mathbf{w}}(y)$ is a positive solution of (4.1) in $\mathbb{R}_+$ satisfying

$$\mathbf{e}_- \preceq_o \hat{\mathbf{w}}(y) \preceq_o \mathbf{e}_+, \qquad \hat{w}(y) \le \hat{u}_-, \qquad \hat{\mathbf{w}}'(y) \succeq_o \mathbf{0}, \qquad y \in \mathbb{R}_+.$$

Since the limit $\hat{\mathbf{w}}_+ = \lim_{y\to+\infty} \hat{\mathbf{w}}(y)$ exists, we have

$$\frac{\ell\,\hat{\mathbf{w}}'(y)}{y} = -\frac{\ell}{y^\ell} \int_0^y D^{-1}\,\mathbf{f}(\hat{\mathbf{w}}(s))\, s^{\ell-1}\, ds \to -D^{-1}\,\mathbf{f}(\hat{\mathbf{w}}_+)$$

as $y \to +\infty$. By the boundedness of $\hat{\mathbf{w}}(y)$, we obtain $\mathbf{f}(\hat{\mathbf{w}}_+) = \mathbf{0}$. From $[\hat{\mathbf{w}}_+]_1 \le \hat{u}_- < [\hat{\mathbf{e}}]_1$ and the assumption (A.2), we have $\hat{\mathbf{w}}_+ = \mathbf{e}_-$, and then we obtain $p_-(\ell) = [\mathbf{e}_-]_1$. In a similar manner with the above argument, we can show that for each $\ell \ge 1$, if $p_+(\ell) < [\mathbf{e}_+]_1$ holds, then there exists a monotone solution of (4.1) in $\mathbb{R}_+$ such that $u(0) = p_+(\ell)$, $\mathbf{u}'(0) = \mathbf{0}$ and $\mathbf{u}(+\infty) = \mathbf{e}_-$ are satisfied. Moreover we employ the comparison principle and Lemma 4.1, we can prove that $p_+(\ell)$ is a lower semi-continuous function in $\ell \ge 1$.

Theorem 4.2. *$p_-(\ell) = [\mathbf{e}_-]_1$ holds for any $\ell \ge 1$, and $p_+(\ell)$ is a lower semi-continuous function in $\ell \ge 1$.*

From the above theorem, it follows that when $p_+(\ell)$ has a jump discontinuity at $\ell = \ell_0 \ge 1$, there exists a monotone positive solution, which satisfies $\mathbf{u}(r) \in \mathcal{D}$ for any $r \in [0, \pi]$ and does not belong to $\mathcal{C}(\ell_0)$, of the problem (1.2) for $\ell = \ell_0$. Since the study of $p_+(\ell)$ is important for determining the bifurcation structure of monotone positive solution, we shall discuss the property of $p_+(\ell)$ in the near future.

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