Advanced Studies in Pure Mathematics 64, 2015
Nonlinear Dynamics in Partial Differential Equations
pp. 253–261

A numerical scheme for the Hele-Shaw flow with a time-dependent gap by a curvature adjusted method

Shigetoshi Yazaki

Abstract.

We will sketch a curvature adjusted method for a moving plane curves and apply it to the Hele-Shaw flow in a time-dependent gap together with the boundary element method.

§1. Introduction

In this paper we study evolution of a family of closed smooth plane curves $\Gamma(t)$: $\boldsymbol{x}(u,t) \in \mathbb{R}^2$ for $u \in [0,1] \subset \mathbb{R}/\mathbb{Z}$ and $t \in [0,T)$ starting from a given initial curve $\Gamma(0) = \Gamma_0$, and driven by the evolution law:

$$\partial_t \boldsymbol{x} = \alpha \boldsymbol{t} + \beta \boldsymbol{n},$$

where $\boldsymbol{t} = \partial_u \boldsymbol{x}/|\partial_u \boldsymbol{x}|$ is the unit tangent vector, and $\boldsymbol{n}$ is the unit outward normal vector which satisfies $\det(\boldsymbol{n}, \boldsymbol{t}) = 1$. Here and hereafter, we denote $\partial_\xi \mathsf{F} = \partial \mathsf{F}/\partial \xi$ and $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ where $\boldsymbol{a} \cdot \boldsymbol{b}$ is Euclidean inner product between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The solution curves are immersed or embedded such that $|\partial_u \boldsymbol{x}| > 0$ holds. The normal velocity β may depend on many factors. The classical curvature flow $\beta = -k$ is the typical example. Here k is the curvature in the direction $-\boldsymbol{n}$, which is defined from $\partial_s \boldsymbol{t} = -k\boldsymbol{n}$, $\partial_s \boldsymbol{n} = k\boldsymbol{t}$, and described as $k = \det(\partial_s \boldsymbol{x}, \partial_{ss} \boldsymbol{x})$, and $\partial_s \boldsymbol{x} = \boldsymbol{t}$ is the unit tangent vector, where we denoted $\partial_{\xi\xi} \mathsf{F} = \partial(\partial_\xi \mathsf{F})/\partial \xi$. Note that ∂_s is not partial differentiation. It means the operator $\partial_s \mathsf{F}(u,t) = g(u,t)^{-1} \partial_u \mathsf{F}(u,t)$, where $g(u,t) = |\partial_u \boldsymbol{x}(u,t)| > 0$ is called the local length and s is the arc length parameter determined from $ds = g(u,t)\, du$.

Received February 1, 2012.
Revised March 4, 2013.
2010 *Mathematics Subject Classification.* 76D27, 35R37, 65M38, 76M15.
Key words and phrases. Moving tracking method, non-trivial tangential velocity, Hele-Shaw flow, curvature adjusted method, boundary element method.

We will mention a brief sketch of the utilization of a non-trivial tangential velocity α, and will apply one of the utilization to the Hele-Shaw flow in a time-dependent gap.

§2. Curvature adjusted tangential velocity

The tangential velocity α has no effect of the shape of evolving curves, and the shape is determined by the value of the normal velocity β only. Therefore a tangential velocity α will be chosen in a non-trivial way depending on purpose. For example, the simplest setting $\alpha \equiv 0$ can be chosen. However, in general, such a choice of α may lead to various numerical instabilities caused by either undesirable concentration and/or extreme dispersion of numerical grid points. Therefore, to obtain stable numerical computation, several choices of a non-trivial tangential velocity have been emphasized and developed by many authors. We will present a brief review of development of non-trivial tangential velocities as follows. Kimura [4] proposed a uniform redistribution scheme in the case $\beta = -k$ by using a special choice of α which satisfies discretization of an average condition and the uniform distribution condition (U): $\mathsf{r}(u,t) = g(u,t)/L(t) \equiv 1$ for any u. Here $L(t)$ is the total length of $\Gamma(t)$. Hou, Lowengrub and Shelley [3] utilized condition (U) directly (especially for $\beta = -k$) starting from $\mathsf{r}(u,0) \equiv 1$, and derived

$$\partial_s \alpha = \langle k\beta \rangle - k\beta, \tag{1}$$

which comes from $\partial_t \mathsf{r} = \big(\partial_s \alpha + k\beta - \langle k\beta \rangle\big) g/L \equiv 0$. Here and hereafter $\langle \mathsf{F} \rangle = \frac{1}{\mathsf{F}} \int_\Gamma \mathsf{F}\, ds$ is the average of F along Γ. It was proposed independently by Mikula and Ševčovič [5]. In [3, Appendix 2], Hou et al. also pointed out generalization of (1) as follows:

$$\frac{\partial_s(\varphi(k)\alpha)}{\varphi(k)} = \frac{\langle f \rangle}{\langle \varphi(k) \rangle} - \frac{f}{\varphi(k)}, \quad f = \varphi(k)k\beta - \varphi'(k)\left(\partial_{ss}\beta + k^2\beta\right) \tag{2}$$

for a given function φ. If $\varphi \equiv 1$, then this is nothing but (1). (2) is derived from the following calculation. Let a generalized relative local length be $\mathsf{r}_\varphi(u,t) = \mathsf{r}(u,t)\varphi(k(u,t))/\langle \varphi(k(\cdot,t)) \rangle$. Then preserving condition $\partial_t \mathsf{r}_\varphi(u,t) \equiv 0$ leads (2).

As mentioned above, in the paper [5] the authors arrived (1) in general frame work of the so-called intrinsic heat equation for $\beta = \beta(\theta, k)$, where θ is the angle of $\boldsymbol{n}$, i.e., $\boldsymbol{n} = (\cos\theta, \sin\theta)^{\mathrm{T}}$ and $\boldsymbol{t} = (-\sin\theta, \cos\theta)^{\mathrm{T}}$. After these results, for example, in the paper [6], they proposed method of asymptotically uniform redistribution, i.e., derived

$$\partial_s \alpha = \langle k\beta \rangle - k\beta + \big(\mathsf{r}^{-1} - 1\big)\omega(t) \tag{3}$$

for quite general normal velocity $\beta = \beta(\boldsymbol{x}, \theta, k)$, where $\omega \in L^1_{loc}[0,T)$ is a relaxation function satisfying $\int_0^T \omega(t)\,dt = +\infty$.

Besides these uniform distribution methods, the so-called crystalline curvature flow is known, which is not uniform distribution method but its numerical computation is quite stable. Under the method, grid points are distributed dense (resp. sparse) on the subarc where the absolute value of curvature is large (resp. small). In the paper [10], the author shows that the tangential velocity $\alpha = \partial_s\beta/k$ is utilized in the crystalline curvature flow equation implicitly and the above non-uniform distribution is realized by the tangential velocity. Note that under the crystalline curvature flow, polygonal curves are restricted in an admissible class, which is generalized in a prescribed class, and several polygonal curvature flows were proposed in the class [2].

The asymptotically uniform redistribution is quite effective and valid for wide range of application. However, from approximation point of view, unless solution curve is a circle, there is no reason to take uniform redistribution. Hence the redistribution will be desired in a way of taking into account the shape of evolution curves, i.e., depending on size of curvatures. In the paper [7], it is proposed that a method of redistribution which takes into account the shape of limiting curve:

$$\frac{\partial_s(\varphi(k)\alpha)}{\varphi(k)} = \frac{\langle f\rangle}{\langle\varphi(k)\rangle} - \frac{f}{\varphi(k)} + \left(\mathsf{r}_\varphi^{-1} - 1\right)\omega(t). \tag{4}$$

If $\varphi \equiv 1$, then this is nothing but (3), and if $\varphi = k$ and Γ is convex, we have $\alpha = \partial_s\beta/k$ in the case $\omega = 0$. Therefore, this is a combination of method of asymptotic uniform redistribution and the crystalline tangential velocity as mentioned above. This method—curvature adjusted method—was applied to various curvature-dependent flows and an image segmentation, and nice results were confirmed [1], [7], [8].

§3. Hele-Shaw flow in a time-dependent gap

The so-called Hele-Shaw flow is flow of viscous fluid which is contained in the narrow gap between two parallel plates, that is, in the Hele-Shaw cell. Fig. 1 indicates the Hele-Shaw cell settled in the xyz-

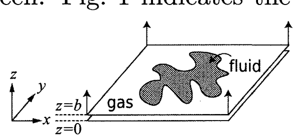


Fig. 1. Hele-Shaw cell

coordinate. Let b be the gap between two parallel plates in the z-direction. In classical Hele-Shaw experiments, b is fixed.

Taking average of fluid region in z-direction, we deduce the problem to two dimensional problem. Let Ω be the two dimensional viscous fluid region enclosed by Γ. In the Hele-Shaw approximation, the two dimensional velocity vector is given by $\boldsymbol{u} = -\frac{b^2}{12\mu}\nabla p$, where μ is viscosity, ∇ means the two dimensional gradient: $\nabla = (\partial_x, \partial_y)^{\mathrm{T}}$, $\boldsymbol{x} = (x, y)^{\mathrm{T}}$, and the pressure $p = p(\boldsymbol{x}, t)$ is harmonic: $\Delta p = 0$ in Ω. Since the boundary $\Gamma = \partial\Omega$ moves with the fluid, deformation velocity $\beta = \dot{\boldsymbol{x}} \cdot \boldsymbol{n}$ in the normal direction $\boldsymbol{n}$ for $\boldsymbol{x} \in \Gamma$ is given as $\beta = \boldsymbol{u} \cdot \boldsymbol{n} = -\frac{b^2}{12\mu}\frac{\partial p}{\partial \boldsymbol{n}}$, $\frac{\partial p}{\partial \boldsymbol{n}} = \nabla p \cdot \boldsymbol{n}$, where we denoted $\dot{\mathsf{F}} = \partial_t \mathsf{F}$. On the boundary Γ, we use the Laplace's relation $p - p_* = \tau k$, $\tau > 0$. Here p_* is the atmospheric pressure and τ is a surface tension coefficient. Since p_* is a constant, we can assume $p_* = 0$ without loss of generality by replacing $p - p_*$ with p. Consequently, we have the following classical Hele-Shaw problem:

$$
\text{(CHS)} \qquad \begin{cases} \Delta p = 0, & \boldsymbol{x} \in \Omega(t), \quad t > 0, \\ p = \tau k, & \boldsymbol{x} \in \Gamma(t), \quad t > 0, \\ \beta = -\frac{b^2}{12\mu}\frac{\partial p}{\partial \boldsymbol{n}}, & \boldsymbol{x} \in \Gamma(t), \quad t > 0. \end{cases}
$$

Shelley, Tian and Wlodarski [9] proposed a problem in the case where b depends on the time t, i.e., the upper plate is lifted uniformly at a specific rate. The following is the Hele-Shaw problem in a time-dependent gap $b(t)$:

$$
\text{(THS)} \qquad \begin{cases} \Delta p = 12\mu\frac{\dot{b}(t)}{b(t)^3}, & \boldsymbol{x} \in \Omega(t), \quad t > 0, \\ p = \tau k, & \boldsymbol{x} \in \Gamma(t), \quad t > 0, \\ \beta = -\frac{b(t)^2}{12\mu}\frac{\partial p}{\partial \boldsymbol{n}}, & \boldsymbol{x} \in \Gamma(t), \quad t > 0. \end{cases}
$$

They established the existence, uniqueness and regularity of solutions in the case where the surface tension is zero. They also studied numerical computation by means of the small-scale decomposition investigated in [3]. In the present paper, we will propose another scheme by means of boundary element method with a curvature adjusted tangential velocity.

In the case where the plates are fixed, then $\dot{b}(t) = 0$ and (THS) is nothing but the classical Hele-Shaw problem (CHS). The problem (THS) can be dimensionalized, and retaining the same variable names, we have the non-dimensional (THS), which is (THS) without 2μ apparently. Note that r.h.s. of the Poisson equation depends only on time. Then it can be erased by means of a special solution $p_\star$ satisfying $\Delta p_\star = \dot{b}(t)/b(t)^3$. For instance, if we put $\hat{p} = p - p_\star$, $p_\star = \dot{b}(t)|\boldsymbol{x}|^2/(4b(t)^3)$, then the non-dimensional (THS) becomes

$$\text{(HS)} \qquad \begin{cases} \Delta p = 0, & \boldsymbol{x} \in \Omega(t), \quad t > 0, \\ p = \tau k - \frac{\dot{b}(t)}{4b(t)^3}|\boldsymbol{x}|^2, & \boldsymbol{x} \in \Gamma(t), \quad t > 0, \\ \beta = -b(t)^2 \frac{\partial p}{\partial \boldsymbol{n}} - \frac{\dot{b}(t)}{2b(t)}\boldsymbol{x} \cdot \boldsymbol{n}, & \boldsymbol{x} \in \Gamma(t), \quad t > 0. \end{cases}$$

Here we have denoted $\hat{p}$ again by p.

§4. Numerical experiments

We show a numerical simulation of (HS). Algorithm is as follows.

Step 1. For a given data, say γ on Γ, solve the following Dirichlet problem (D): $\Delta p = 0$ in Ω with $p = \gamma$ on Γ, and obtain the data $\partial p/\partial \boldsymbol{n}$ on Γ, by means of boundary element method (BEM).

Step 2. Move Γ with the normal velocity β in (HS) by using a technique of curvature adjusted tangential velocity [7].

We explain a brief sketch of BEM. Let p be a solution of $\Delta p = \varepsilon$ in Ω with the boundary condition $p = \gamma$ on Γ. Here $\varepsilon = \varepsilon(\boldsymbol{x})$ $(\boldsymbol{x} \in \Omega)$ is an error. Then p is an approximated solution of (D). One of the typical strategy of BEM is to find the data $\partial p/\partial \boldsymbol{n}$ on Γ under the constraint $\iint_\Omega w\varepsilon \, d\Omega = 0$ for a given weighted function w. Then from $\varepsilon = \Delta p$, we have $B = -\iint_\Omega p\Delta w \, d\Omega = \int_\Gamma w \frac{\partial p}{\partial \boldsymbol{n}} \, ds - \int_\Gamma \gamma \frac{\partial w}{\partial \boldsymbol{n}} \, ds$. For the choice of w, we use the fundamental solution $w(\boldsymbol{x}, \boldsymbol{\xi}) = \frac{1}{2\pi} \log \frac{1}{|\boldsymbol{x} - \boldsymbol{\xi}|}$. Hence we obtain $B = \frac{\theta(\boldsymbol{\xi})}{2\pi} p(\boldsymbol{\xi})$ $(\boldsymbol{\xi} \in \Gamma)$, where $\theta(\boldsymbol{\xi})$ is the inner angle at $\boldsymbol{\xi} \in \Gamma$ if $\boldsymbol{\xi}$ is a corner, or $\theta(\boldsymbol{\xi}) = \pi$ if $\boldsymbol{\xi}$ is a smooth point.

To construct the numerical scheme, Γ would be a closed N-sided polygonal curve such as $\Gamma = \bigcup_{i=1}^N \Gamma_i$, where $\Gamma_i = [\boldsymbol{x}_{i-1}, \boldsymbol{x}_i]$ is the i-th edge with the vertices $\boldsymbol{x}_{i-1}$ and $\boldsymbol{x}_i$. We assume that $p(\boldsymbol{x}) \equiv \gamma(\boldsymbol{x}_i^*) = \gamma_i$, $\frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{n}} \equiv \frac{\partial p(\boldsymbol{x}_i^*)}{\partial \boldsymbol{n}} = q_i$ for $\boldsymbol{x} \in \Gamma_i$ and $\boldsymbol{x}_i^* = (\boldsymbol{x}_{i-1} + \boldsymbol{x}_i)/2$. Then we have

$$\frac{\theta(\boldsymbol{\xi})}{2\pi} p(\boldsymbol{\xi}) = \int_\Gamma w(\boldsymbol{x}, \boldsymbol{\xi}) \frac{\partial p(\boldsymbol{x})}{\partial \boldsymbol{n}} \, ds - \int_\Gamma \frac{\partial w(\boldsymbol{x}, \boldsymbol{\xi})}{\partial \boldsymbol{n}} \gamma(\boldsymbol{x}) \, ds,$$

and for the choice of $\boldsymbol{\xi} = \boldsymbol{x}_i^*$ on Γ_i $(i = 1, 2, \cdots, N)$ we obtain

$$\frac{\theta(\boldsymbol{x}_i^*)}{2\pi} p(\boldsymbol{x}_i^*) = \frac{1}{2}\gamma_i = \sum_{j=1}^N \int_{\Gamma_j} w(\boldsymbol{x}, \boldsymbol{x}_i^*) q_j \, ds - \sum_{j=1}^N \int_{\Gamma_j} \frac{\partial w(\boldsymbol{x}, \boldsymbol{x}_i^*)}{\partial \boldsymbol{n}} \gamma_j \, ds.$$

Hence we solve the linear equation $G\boldsymbol{q} = H\boldsymbol{\gamma}$ and obtain the solution $\boldsymbol{q} = (q_1, q_2, \cdots, q_N)^{\mathrm{T}} = G^{-1}H\boldsymbol{\gamma}$, where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_N)^{\mathrm{T}}$, and $G = (G_{ij})$ and $H = (H_{ij})$ are $G_{ij} = \int_{\Gamma_j} w(\boldsymbol{x}, \boldsymbol{x}_i^*) \, ds$, $H_{ij} = \frac{1}{2}\delta_{ij} + \int_{\Gamma_j} \frac{\partial w(\boldsymbol{x}, \boldsymbol{x}_i^*)}{\partial \boldsymbol{n}} \, ds$. Note that G and H can be calculated analytically.

As a consequence, for $i = 1, 2, \cdots, N$ we solve $\Delta p = 0$ in $\Omega(t)$ with

$$p_i = \tau k_i - \frac{\dot{b}(t)}{4b(t)^3}|\boldsymbol{x}_i^*|^2 \quad \text{on } \Gamma_i(t) = [\boldsymbol{x}_{i-1}(t), \boldsymbol{x}_i(t)]$$

by BEM, and obtain q_i and

$$\beta_i = -b(t)^2 q_i - \frac{\dot{b}(t)}{2b(t)} \frac{\partial |\boldsymbol{x}_i^*|^2}{\partial \boldsymbol{n}} \quad \text{on } \Gamma_i(t).$$

Here Ω is enclosed region by Γ and k_i is a discretized curvature defined on Γ_i as a constant value:

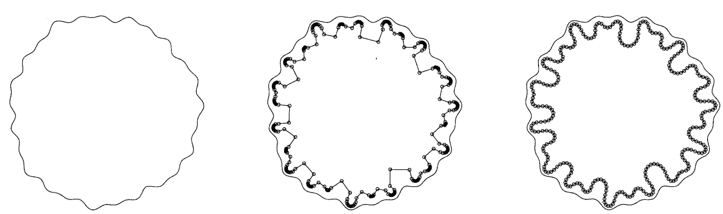
$$k_i = \frac{1}{|\Gamma_i|}\left(\tan\frac{\nu_{i+1} - \nu_i}{2} + \tan\frac{\nu_i - \nu_{i-1}}{2}\right), \quad |\Gamma_i| = |\boldsymbol{x}_i - \boldsymbol{x}_{i-1}|,$$

where ν_i's are computed from $\boldsymbol{t}_i = (\boldsymbol{x}_i - \boldsymbol{x}_{i-1})/|\Gamma_i| = (\cos\nu_i, \sin\nu_i)^{\mathrm{T}}$. Then we track the polygonal curve $\Gamma(t) = \bigcup_{i=1} \Gamma_i(t)$ by the evolution equation $\dot{\boldsymbol{x}}_i = \alpha_i \boldsymbol{t}_i^* + \beta_i \boldsymbol{n}_i^*$ $(i = 1, 2, \cdots, N)$, where $\boldsymbol{t}_i^*$ and $\boldsymbol{n}_i^*$ are the unit tangential vector and the unit normal vector at $\boldsymbol{x}_i$:

$$\boldsymbol{t}_i^* = (\cos\nu_i^*, \sin\nu_i^*)^{\mathrm{T}}, \quad \det(\boldsymbol{n}_i^*, \boldsymbol{t}_i^*) = 1, \quad \nu_i^* = \frac{\nu_i + \nu_{i+1}}{2},$$

and α_i is a given non-trivial curvature adjusted tangential velocity computed from discretization of (4) with $\varphi(k) = 1 - \varepsilon^2\sqrt{1 - \varepsilon + \varepsilon k^2}$, $\omega = \kappa_1 - \kappa_2 \frac{1}{L}\sum_{i=1}^{N} k_i \beta_i |\Gamma_i|$, $L = \sum_{i=1}^{N} |\Gamma_i|$, $\kappa_1 = 1000$, $\kappa_2 = 100$, $\varepsilon = 0.1$. Finally, we discretize the above evolution equation in time semi-implicitly and construct a numerical algorithm. See [7] in detail.

As an initial curve, we use $\boldsymbol{x}(u, 0) = R(u)(\cos(2\pi u), \sin(2\pi u))^{\mathrm{T}}$, $R(u) = 1 + 0.02(\cos(6\pi u) + \sin(14\pi u) + \cos(30\pi u) + \sin(50\pi u))$ $(u \in [0, 1])$, which is shown in Fig. 2 (a). Fig. 2 (b) and (c) show effect of usage of the



(a) Initial curve (b) Without α (c) With α

Fig. 2. Initial curve and effect of tangential velocity α

tangential velocity. Without the tangential velocity α, the computation will collapse immediately as in Fig. 2 (b), otherwise the appropriate α makes a stable computation as in Fig. 2 (c).

In what follows, we will show three examples in the case where the surface tension coefficients are $\tau = 2, 1, \frac{1}{2} \times 10^{-4}$ with the number of grid points being $N = 300$. For comparison, the initial curve and the three values of surface tension coefficients τ are the same as in [9]. In each Figs. 3–5, top and left figure indicates selected curves within the time $t = 0 \sim 2.6$. Even with a small number of grid points $N = 300$, fingering phenomena can be realized stably and the simulation patterns are the same as in numerical computation [9, Figure 2–4]. This is advantage of our scheme, i.e., one can catch small or delicate wave patterns along curve with small and necessary number of grid points. See [1], [7], [8] for several features of the curvature adjusted method and many examples.

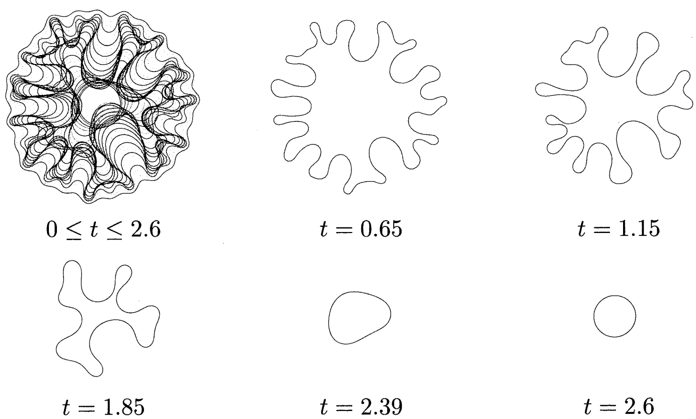


Fig. 3. Case $\tau = 2 \times 10^{-4}$

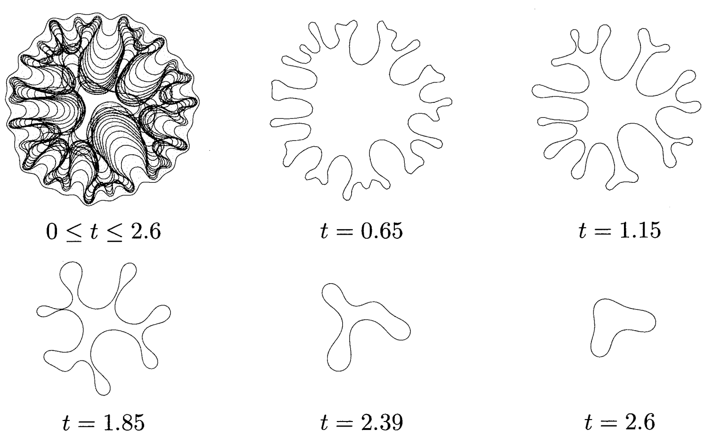


Fig. 4: Case $\tau = 10^{-4}$

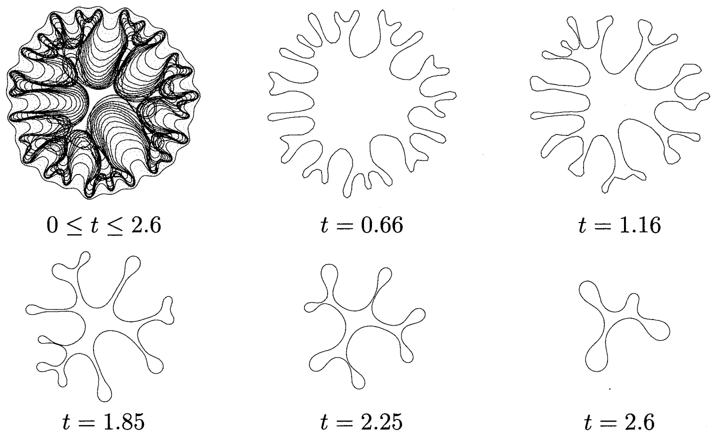


Fig. 5. Case $\tau = \frac{1}{2} \times 10^{-4}$

Remark. We have two properties. One is that the volume is preserved in the sense $b(t)|\Omega(t)| \equiv b(0)|\Omega(0)|$. The other is that the center of mass is preserved in time $\frac{d}{dt}\frac{1}{|\Omega(t)|}\iint_{\Omega(t)} \boldsymbol{x}\, d\Omega = \mathbf{0}$. It is to be desired that numerical scheme should satisfy the above two properties in some sense, e.g. in discrete sense. However, it is still open problem.

Acknowledgements. The author would like to thank the referee very much for his or her valuable comments and suggestions. This research is supported by Grant-in-Aid for Scientific Research (C) 23540150.

References

[1] M. Beneš, M. Kimura, P. Pauš, D. Ševčovič, T. Tsujikawa and S.Yazaki, Application of a curvature adjusted method in image segmentation, Bull. Inst. Math. Acad. Sin. (N.S.), **3** (2008), 509–523.

[2] M. Beneš, M. Kimura and S. Yazaki, Second order numerical scheme for motion of polygonal curves with constant area speed, Interfaces Free Bound., **11** (2009), 515–536.

[3] T. Y. Hou, J. S. Lowengrub and M. J. Shelley, Removing the stiffness from interfacial flows with surface tension, J. Comput. Phys., **114** (1994), 312–338.

[4] M. Kimura, Numerical analysis for moving boundary problems using the boundary tracking method, Japan J. Indust. Appl. Math., **14** (1997), 373–398.

[5] K. Mikula and D. Ševčovič, Evolution of plane curves driven by a nonlinear function of curvature and anisotropy, SIAM J. Appl. Math., **61** (2001), 1473–1501.

[6] K. Mikula and D. Ševčovič, Evolution of curves on a surface driven by the geodesic curvature and external force, Appl. Anal., **85** (2006), 345–362.

[7] D. Ševčovič and S. Yazaki, Evolution of plane curves with a curvature adjusted tangential velocity, Jpn J. Ind. Appl. Math., **28** (2011), 413–442.

[8] D. Ševčovič and S. Yazaki, Computational and qualitative aspects of motion of plane curves with a curvature adjusted tangential velocity, Math. Methods Appl. Sci., **35** (2012), 1784–1798.

[9] M. J. Shelley, F.-R. Tian and K. Wlodarski, Hele-Shaw flow and pattern formation in a time-dependent gap, Nonlinearity, **10** (1997), 1471–1495.

[10] S. Yazaki, On the tangential velocity arising in a crystalline approximation of evolving plane curves, Kybernetika (Prague), **43** (2007), 913–918.

Meiji University
1-1-1 Higashi-Mita, Tama-ku
Kawasaki-shi 214-8571
Japan
E-mail address: syazaki@meiji.ac.jp