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On the ABP maximum principle for L^p-viscosity solutions of fully nonlinear PDE

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Abstract.

Fully nonlinear second-order uniformly elliptic partial differential equations (PDE for short) with unbounded ingredietns are considered. The Aleksandrov–Bakelman–Pucci (ABP for short) maximum principle for L^p-viscosity solutions of fully nonlinear, second-order uniformly elliptic PDE are shown.

The results here are joint works with A. Święch in [12], [13], [14], [15].

§1. Introduction

The aim of this manuscript is to exhibit some recent results on the ABP maximum principle for L^p-viscosity solutions of (1) below under certain hypotheses.

We are concerned with fully nonlinear second-order uniformly elliptic PDE:

$$F(x, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $F : \Omega \times \mathbb{R}^n \times S^n \to \mathbb{R}$. Here, S^n denotes the set of $n \times n$ symmetric matrices with the standard order.

It is possible to discuss the case when F may depend on the unknown function u. However, since we focus our topics on the maximum principle, we shall deal with F independent of u for the sake of simplicity.

We shall also suppose

$$\Omega \subset B_1,$$

where $B_r := \{x \in \mathbb{R}^n \mid \|x\| < r\}$. We may derive a dependence on the diameter of Ω by a scaling argument.

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In what follows, we suppose

$$p > \frac{n}{2}.$$

In 1981, Crandall and Lions introduced the notion of viscosity solutions for first-order PDE of non-divergence type since we cannot use weak solutions in the distribution sense. It was extended to second-order (possibly degenerate) elliptic/parabolic PDE. Up to now, there have been many results on the viscosity solution theory and its applications when PDE possess enough continuity. See [4] for instance.

On the other hand, in order to study weak solutions of fully nonlinear PDE with discontinuous/unbounded ingredients, the notion of L^p-viscosity solutions was introduced by Caffarelli–Crandall–Kocan–Święch [3] in 1996 motivated by a celebrated work by Caffarelli [1]. See also [2].

Definition 1.1. *We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (1) if for $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$,*

$$\text{(2)} \qquad \mathit{ess}\liminf_{y \to x} \left\{ F(y, D\varphi(y), D^2\varphi(y)) - f(y) \right\} \le 0$$

$$\text{(3)} \qquad \left(\mathit{resp.,} \quad \mathit{ess}\limsup_{y \to x} \left\{ F(y, D\varphi(y), D^2\varphi(y)) - f(y) \right\} \ge 0 \right)$$

provided $u - \varphi$ attains its local maximum (resp., minimum) at $x \in \Omega$.

Remark 1.1. (i) When F and f are continuous, if we replace $W^{2,p}_{\mathrm{loc}}(\Omega)$ by $C^2(\Omega)$, the above definition is the same as the standard one by Crandall–Lions since (2) (resp., (3)) yields

$$F(x, D\varphi(x), D^2\varphi(x)) \le f(x) \quad (\text{resp.,} \ \ge f(x)).$$

In fact, under appropriate hypotheses, when F and f are continuous, the notion of viscosity solutions by Crandall–Lions coincides with that of L^p-viscosity solutions. We notice that L^p-viscosity solutions are more restricted than the standard one because of $C^2(\Omega) \subset W^{2,p}_{\mathrm{loc}}(\Omega)$.
(ii) We notice that if $u \in C(\Omega)$ is an L^p-viscosity subsolution (resp., supersolution) of (1), and $\frac{n}{2} < p < p'$, then it is an $L^{p'}$-viscosity subsolution (resp., supersolution) of (1).

We recall the definition of L^p-strong solutions:

Definition 1.2. *We call $u \in C(\Omega)$ an L^p-strong subsolution (resp., supersolution) of (1) if $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, and*

$$F(x, Du(x), D^2u(x)) \le f(x) \quad (\mathit{resp.,} \ge f(x)) \quad \mathit{a.e.} \quad \mathit{in}\ \Omega.$$

We will write $\|\cdot\|_p$ for $\|\cdot\|_{L^p(\Omega)}$ etc. if there is no confusion. Also, $L^p_+(\Omega)$ denotes the set of nonnegative functions in $L^p(\Omega)$.

We use the following Pucci operators. We hope the readers not to be confused because the oposite sign in the max and min below is often used, e.g. in [2]: for $X \in S^n$,

$$\mathcal{P}^+(X) = \max\{-\mathrm{trace}(AX) \mid A \in S^n,\ \lambda I \le A \le \Lambda I\},$$

and

$$\mathcal{P}^-(X) = \min\{-\mathrm{trace}(AX) \mid A \in S^n,\ \lambda I \le A \le \Lambda I\}.$$

Now, we give a list of hypotheses for F:

$$(4)\qquad \begin{cases} (i) & \mathcal{P}^-(X-Y) \le F(x,\xi,X) - F(x,\xi,Y) \le \mathcal{P}^+(X-Y) \\ & \quad \text{for } x \in \Omega, \xi \in \mathbb{R}^n, X, Y \in S^n, \\ (ii) & \text{there is } \mu \in L^q_+(\Omega) \text{ such that } |F(x,\xi,O)| \le \mu(x)|\xi| \\ & \quad \text{for } x \in \Omega, \xi \in \mathbb{R}^n, \\ (iii) & F(x,0,O) = 0 \text{ for } x \in \Omega. \end{cases}$$

We will refer to $\mu \in L^q_+(\Omega)$ from the above definition (ii) of (4).

Remark 1.2. We notice that if $u \in C(\Omega)$ is an L^p-viscosity subsolution (resp., supersolution) of (1), then it is an L^p-viscosity subsolution (resp., supersolution) of (5) (resp., (6)) below.

For $v : \Omega \to \mathbb{R}$, we denote the upper contact set of v in Ω by

$$\Gamma[v;\Omega] := \{x \in \Omega \mid \exists \xi \in \mathbb{R}^n \ s.t.\ v(y) \le v(x) + \langle \xi, y-x \rangle \text{ for } \forall y \in \Omega\}.$$

The well-known classical ABP maximum principle is as follows:

Theorem 1. (*e.g.* [8]) *There exist $C_k = C_k(n, \lambda/\Lambda) > 0$ $(k = 1, 2)$ such that for $f \in L^n_+(\Omega)$ and $\mu \in L^n_+(\Omega)$, if $u \in C(\overline{\Omega})$ is an L^n-strong subsolution (resp., supersolution) of*

$$(5)\qquad \mathcal{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad in\ \Omega$$

$$(6)\qquad (resp.,\ \mathcal{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad in\ \Omega),$$

then it follows that

$$(7)\qquad \sup_\Omega u \le \sup_{\partial\Omega} u^+ + C_1 e^{C_2\|\mu\|_n^n} \|f\|_{L^n(\Gamma[u^+;\Omega])}$$

$$\left(resp.,\ \inf_\Omega u \ge \inf_{\partial\Omega}(-u^-) - C_1 e^{C_2\|\mu\|_n^n} \|f\|_{L^n(\Gamma[u^-;\Omega])} \right).$$

Remark 1.3. In [6], [7], for $\mu \in L^q(\Omega)$ with $q > n$, Fok obtained the ABP maximum principle for L^p-strong solutions when $p > n - \varepsilon$, where $\varepsilon > 0$ depends on $q - n > 0$. We notice that the corresponding $\varepsilon > 0$ in our results does not depend on $q - n > 0$.

In what follows, we will only present the ABP maximum principle for subsolutions since the one for supersolutions can be derived by considering $-u$.

§2. Known results

We recall known results on the ABP maximum principle for L^p-viscosity solutions.

Proposition 1. ([1], [2]) *Assume that $f \in L^n_+(\Omega) \cap C(\Omega)$. There exists $C_1 = C_1(n, \lambda/\Lambda) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of*

$$\mathcal{P}^-(D^2u) = f(x) \quad \text{in } \Omega,$$

then it follows that

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C_1 \|f\|_{L^n(\Gamma[u^+;\Omega])}.$$

Notice that we have to suppose f to be continuous in Proposition 1. Later, this hypothesis is removed in [3]. Furthermore, we may treat the case when PDE admit the first derivative terms with bounded coefficients. Moreover, we may obtain the result even when $f \in L^p(\Omega)$ for $p > \hat{p}$, where $\hat{p} \in (\frac{n}{2}, n)$ is the constant from [5].

Proposition 2. ([3]) *Assume that $\mu \in L^\infty_+(\Omega)$ and $f \in L^p_+(\Omega)$ for $p > \hat{p}$. There exists $C_1 = C_1(n, \lambda/\Lambda, p, \|\mu\|_\infty) > 0$ such that if $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of*

(8)
$$\mathcal{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega_+[u] := \{x \in \Omega \mid u(x) > \sup_{\partial\Omega} u^+\},$$

then it follows that

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C_1 \|f\|_{L^p(\Omega_+[u])}. \tag{9}$$

In Proposition 2, if $p \ge n$, then the region of the L^p-norm can be replaced by $\Gamma[u^+; \Omega_+[u]]$.

Here, we give an existence result for L^p-strong solutions. In what follows, we suppose enough regularity on $\partial\Omega$ so that the $W^{2,p}$-estimates hold up to the boundary. We refer to [20] by Winter for the regularity near $\partial\Omega$.

Proposition 3. ([3], [5]) *Assume that* $f \in L^p(\Omega)$ *for* $p > \hat{p}$, *and* $\mu_0 \geq 0$. *There exist* $C_k = C_k(n, \lambda/\Lambda, p, \mu_0) > 0$ $(k = 3,4)$ *and an* L^p*-strong subsolution (resp., supersolution) of*

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu_0|Du| = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\left(\text{resp.,} \begin{cases} \mathcal{P}^-(D^2u) - \mu_0|Du| = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \right)$$

such that

$$\|u\|_\infty \leq C_3\|f\|_p, \quad \text{and} \quad \|u\|_{W^{2,p}(\Omega)} \leq C_4\|f\|_p. \tag{10}$$

Remark 2.1. It is possible to show L^p-strong subsolutions (resp., supersolutions) in the above are indeed L^p-strong solutions via a bit more precise observation while we only need the existence of L^p-strong subsolution (resp., supersolution) for our later use. See [3] for the details.

Now, we present an existence result for L^p-strong subsolutions when the PDE has unbounded coefficients.

Proposition 4. ([12]) *Assume that* $\mu \in L^q(\Omega)$ *and* $f \in L^p(\Omega)$, *where* (p, q) *satisfies*

$$q \geq p \geq n \quad \text{and} \quad q > n. \tag{11}$$

There exist $C_k = C_k(n, \lambda/\Lambda, p, q, \|\mu\|_q) > 0$ $(k = 3,4)$ *and an* L^p*-strong subsolution of*

$$\begin{cases} \mathcal{P}^+(D^2u) + \mu(x)|Du| = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{12}$$

such that (10) *holds.*

Remark 2.2. (*i*) We can modify the argument of the proof of Proposition 3 to obtain Proposition 4. Moreover, it is possible to verify that the above constructed L^p-strong subsolutions are L^p-strong solutions as before. See [13] for the details.
(*ii*) In [7], Fok obtained the existence of L^p-strong subsolutions of (5) when $q = p > n$, and $\mu \in L^q(\Omega) \cap L^{2n}(\Omega^\varepsilon)$ for some $\varepsilon > 0$, where $\Omega^\varepsilon := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \varepsilon\}$.
(*iii*) The hypothesis (11) is equivalent to the case when $q \geq p > n$ or $q > p = n$.

§3. Main results

We shall show the ABP maximum principle for L^p-viscosity subsolutions of (5) and

$$\mathcal{P}^-(D^2u) - \mu(x)|Du|^m = f(x) \quad \text{in } \Omega, \tag{13}$$

where $m > 1$, $\mu \in L^q(\Omega)$ and $f \in L^p(\Omega)$.

3.1. Linear growth

First, we consider (5) in case when (11).

Theorem 2. ([12]) *Assume that $\mu \in L^q_+(\Omega)$ and $f \in L^p_+(\Omega)$, where (p, q) satisfies* (11). *There exist $C_k = C_k(n, \lambda/\Lambda) > 0$ $(k = 1, 2)$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of* (5), *then it follows that*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C_1 e^{C_2\|\mu\|_n^n} \|f\|_{L^n(\Omega)}. \tag{14}$$

Remark 3.1. (i) Although the classical ABP maximum principle has a slightly better estimate with the upper contact set $\Gamma[u^+; \Omega]$, this estimate is enough to use in a proof of the weak Harnack inequality.
(ii) In [7], Fok obtained the ABP maximum principle for L^p-viscosity subsolutions of (5) when $q = p > n$, and $\mu \in L^q(\Omega) \cap L^{2n}(\Omega^\varepsilon)$ for some $\varepsilon > 0$. The reason why $\mu \in L^{2n}$ was needed is that we used the Hopf–Cole transformation in [7] (and also [8]) to cancel the quadratic terms $|Du|^2$.

We next consider the case when

$$\hat{p} < p < n < q. \tag{15}$$

Theorem 3. ([12]) *Assume that $\mu \in L^q_+(\Omega)$ and $f \in L^p_+(\Omega)$, where (p, q) satisfies* (15). *There exist $C_1 = C_1(n, \lambda/\Lambda) > 0$, $C_2 = C_2(n, \lambda/\Lambda, p, q) > 0$ and $N = N(n, p, q) \in \mathbb{N}$ such that if $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of* (5), *then it follows that*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + C_1 \left\{ e^{C_2\|\mu\|_n^n} \|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k \right\} \|f\|_{L^p(\Omega)}.$$

To prove Theorem 3, we established an "iterated comparison function" method. Thanks to this maximum principle, we may extend Proposition 4 to the case of (15).

Proposition 5. ([12]) *Assume that $\mu \in L^q(\Omega)$ and $f \in L^p(\Omega)$, where (p,q) satisfies* (15). *There exist $C_k = C_k(n, \lambda/\Lambda, p, q, \|\mu\|_q) > 0$ $(k = 3, 4)$ and an L^p-strong subsolution of* (12) *such that* (10) *holds.*

In case of $q = n$, we need to suppose that $\|\mu\|_n$ is small to obtain the ABP maximum principle.

Theorem 4. ([15]) *Assume that $\mu \in L^q(\Omega)$ and $f \in L^p(\Omega)$, where (p,q) satisfies*

$$q = n > p > \hat{p}. \tag{16}$$

There exist $\delta_0 = \delta_0(n, \lambda/\Lambda, p) > 0$ and $C_1 = C_1(n, \lambda/\Lambda, p) > 0$ such that if

$$\|\mu\|_n \leq \delta_0, \tag{17}$$

and $u \in C(\overline{\Omega})$ is an L^n-viscosity subsolution of (5), *then it follows that*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + C_1 \|f\|_p. \tag{18}$$

To prove Theorem 4, under (17) for some $\delta_0 > 0$, we have first to construct L^p-strong subsolutions of (12). See [15] for this result.

3.2. Superlinear growth

We shall consider (13) with $m > 1$ instead of (5).

It is impossible to establish the ABP maximum principle in general provided the PDE may have superlinear growth in Du. In fact, if it were true with no restrition, we may construct strong/classical solutions of

$$- \triangle u + |Du|^2 = f(x)$$

under the Dirichlet condition, where $f \in C^\infty$. Indeed, once we obtain L^∞-estmates, we could show the existence of solutions, which contradicts to the fact that we cannot expect the existnece of solutions with quadratic nonlinear terms in Du because we know an example of non-existence by Nagumo [17].

In general, there are counter examples so that the maximum principle fails when the PDE have superlinear growth terms in Du. We refer to [11] and [12] for such examples.

When $p > n$, we do not need any restriction for $m > 1$.

Theorem 5. ([12]) *Assume that $\mu \in L^q_+(\Omega)$ and $f \in L^p_+(\Omega)$, where (p,q) satisfies*

$$q \geq p > n, \ q > n \quad \text{and} \quad m > 1. \tag{19}$$

There exist $\delta_1 = \delta_1(n,\lambda,\Lambda,p,m) > 0$ *and* $C_1 = C_1(n,\lambda,\Lambda,p,m) > 0$ *such that if*

$$\|f\|_p^{m-1}\|\mu\|_q \leq \delta_1, \tag{20}$$

and $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity subsolution of* (12)*, then* (18) *holds.*

When $p \in (\hat{p}, n]$, we need some restriction for $m > 1$.

Theorem 6. ([12]) *Assume that* $\mu \in L^q_+(\Omega)$ *and* $f \in L^p_+(\Omega)$*, where* (p,q,m) *satisfies*

$$q > n \geq p > \hat{p}, \quad \textit{and} \quad 1 < m < 2 - \frac{n}{q}. \tag{21}$$

There exist $\delta_1 = \delta_1(n,\lambda,\Lambda,p,q,m) > 0$ *and* $C_1 = C_1(n,\lambda,\Lambda,p,q,m) > 0$ *such that if* (20) *holds, and* $u \in C(\overline{\Omega})$ *is an* L^p*-viscosity subsolution of* (12)*, then* (18) *holds.*

Remark 3.2. As in the linear growth case, it is possible to use the existence of L^p-strong subsolutions of the associated PDE:

$$\mathcal{P}^+(D^2u) + 2^{m-1}\mu(x)|Du|^m = f(x) \quad \text{in } \Omega,$$

where 2^{m-1} comes from the inequality $(a+b)^m \leq 2^{m-1}(a^m + b^m)$ for $a, b \geq 0$. See [14] for the details.

§4. Applications

We shall give some applications of the ABP maximum principle. In order to prove the assertions below, we have to use the argument in [1], [2], [3] with our ABP maximum principle in the preceeding section.

4.1. Relation between L^p-viscosity and L^p-strong solutions

When $q = \infty$, in [3], the following equivalence holds. If $u \in C(\Omega)$ is an L^p-strong subsolution of (1) if and only if it is an L^p-viscosity subsolution of (1) such that $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. This relation holds true for PDE with unbounded ingredients.

If we allow F to have superlinear terms in Du as in (12), then the following hypotheses are reasonable for F in place of *(ii)* of (4): Fix $m \geq 1$.

$$\left\{ \begin{array}{l} \text{There is } \mu \in L^q_+(\Omega) \text{ such that, for } x \in \Omega, \xi, \eta \in \mathbb{R}^n, X \in S^n, \\ |F(x,\xi,X) - F(x,\eta,X)| \leq \mu(x)(|\xi|^{m-1} + |\eta|^{m-1})|\xi - \eta|. \end{array} \right. \tag{22}$$

We will consider the following cases:

$$
(23) \qquad \begin{cases} (i) & q \geq p \geq n,\ q > n,\ m \geq 1, \\ (ii) & q > n > p > \hat{p},\ 1 < m < 1 + \dfrac{p(q-n)}{q(n-p)}, \\ (iii) & p = q = n,\ m = 1, \\ (iv) & q = n > p > \hat{p},\ m = 1. \end{cases}
$$

We notice that if p is enough close to n in (ii) of (23), then we may treat the case of $m = 2$, which is important from a view point of applications.

Theorem 7. ([13]) *Assume* $(i), (iii)$ *of* (4) *and* (22).
(I) Assume that one of (i), (ii), (iii) *in* (23) *holds. If* $u \in C(\Omega)$ *is an* L^p*-strong subsolution of* (1), *then it is an* L^p*-viscosity subsolution of* (1).
(II) Assume that one of (i), (ii), (iv) *in* (23) *holds. If an* L^p*-viscosity subsolution* $u \in C(\Omega)$ *belongs to* $W^{2,p}_{\mathrm{loc}}(\Omega)$ *of* (1), *then it is an* L^p*-strong subsolution of* (1).

Remark 4.1. To prove the cases of $m > 1$, we need the ABP maximum principle for

$$
\mathcal{P}^-(D^2u) - \mu_1(x)|Du| - \mu_m(x)|Du|^m = f(x)
$$

with precise estimates. See Nakagawa [18] for the details.

4.2. Weak Harnack inequality

In view of the ABP maxmimum principle, we can prove the weak Harnack inequality, which implies the Hölder continuity of L^p-viscosity solutions of (1). We refer to Sirakov [19] by a different approach for the Hölder continuity of L^p-viscosity solutions of (1) with unbounded ingredients.

We can apply the weak Harnack inequality to show the strong maximum principle. See Section 5 in [13] for this application.

First, we consider the case when PDE have linear growth in Du.

Theorem 8. *Assume that* $\mu \in L^q_+(B_2)$ *and* $f \in L^p_+(B_2)$, *where* (p, q) *satisfies one of*

$$
(24) \qquad \begin{cases} (i) & q \geq p > \hat{p},\ q > n, \\ (ii) & q = n > p > \hat{p}. \end{cases}
$$

There exist $C_5 = C_5(n, \lambda/\Lambda, p, q, \mu) > 0$ *and* $r = r(n, \lambda/\Lambda) > 0$ *such that if* $u \in C(B_2)$ *is a nonnegative* L^p*-viscosity supersolution of*

$$
\mathcal{P}^+(D^2u) + \mu(x)|Du| = -f(x) \quad \text{in } B_2,
$$

then it follows that

$$\left(\int_{B_1} u^r dx\right)^{\frac{1}{r}} \le C_5 \left(\inf_{B_1} u + \|f\|_{L^p(B_2)}\right). \tag{25}$$

Remark 4.2. (*i*) We refer to [13] for a precise dedendence on $\|\mu\|_q$ in C_5 particularly in case of (15).
(*ii*) Under (*i*) in (24), C_5 depends on $\|\mu\|_n$ while it depends on μ itself under (*ii*) of (24). Because in both cases, we need to assume $\|\mu\|_n$ is small at the first step.
(*iii*) In [7], Fok obtained the weak Harnack inequality for L^p-viscosity supersolutions assuming $\mu \in L^{2n}$.

We discuss the weak Harnack ineqaulity for PDE containg superlinear terms in Du.

Theorem 9. ([14]) *Fix $M > 0$ and $m > 1$. Assume that $\mu \in L^q_+(B_2)$ and $f \in L^p_+(B_2)$, where (p,q) satisfies (i) of (24) and*

$$1 < m < 2 - \frac{n}{q}. \tag{26}$$

There exist $\delta_2 = \delta_2(n,\lambda,\Lambda,p,m,M) > 0$, $C_5 = C_5(n,\lambda,\Lambda,p,q,R) > 0$ and $r = r(n,\lambda,\Lambda,p,q,m) > 0$ such that if

$$\|\mu\|_q(1 + \|f\|_p^{m-1}) \le \delta_2,$$

and $u \in C(B_2)$ is a nonnegative L^p-viscosity supersolution of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du|^m = -f(x) \quad \text{in } B_2$$

such that $0 \le u \le M$ in B_2, then it follows that (25) holds.

We refer to [16] for the Hölder continuity of viscosity solutions when PDE have superlinear growth terms in Du.

It is easy to establish the weak Harnack inequality near the boundary, which could be used to show some maximum principle in unbounded domains. See Section 8 in [13] for this. See also Koike–Nakagawa [10] and the references theirin for an application to the Phragmén–Lindelöf theorem.

4.3. Local maximum principle

Although the weak Harnack inequality shows that L^p-viscosity solutions of (1) satisfy Hölder continuity, it is natural to ask if the local maximum principle for L^p-viscosity subsolutions holds or not. In fact, when we have unbounded coefficients to Du, we cannot apply the standard method as in [8]. However, we may modify the argument in [2]. See a recent work [9] by Imbert.

Theorem 10. *Assume that $\mu \in L^q_+(B_2)$ and $f \in L^p_+(B_2)$, where (p,q) satisfies one of* (24). *For $s>0$, there is $C_6 = C_6(n,\lambda/\Lambda,p,q,\mu,s) > 0$ such that if $u \in C(B_2)$ is an L^p-viscosity subsolution of*

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } B_2,$$

then it follows that

$$\sup_{B_1} u \le C_6 \left\{ \left(\int_{B_1} u_+^s dx \right)^{\frac{1}{s}} + \|f\|_{L^{p\wedge n}(B_2)} \right\}. \tag{27}$$

Remark 4.3. When (i) in (24) holds, C_6 depends on $\|\mu\|_q$ while it depends on μ itself under (ii) of (24).

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