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# Differential Systems Associated with Simple Graded Lie Algebras

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***Dedicated to Professor Noboru Tanaka on his sixtieth birthday***

## §0. Introduction

This is a survey paper on differential systems associated with simple graded Lie algebras. By a differential system $(M, D)$, we mean a pfaffian system $D$ (or a distribution in Chevalley's sense) on a manifold $M$, that is, $D$ is a subbundle of the tangent bundle $T(M)$ of $M$. Our primary subject will be the Lie algebra (sheaf) $\mathcal{A}(M, D)$ of all infinitesimal automorphisms of $(M, D)$.

Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{R}$ of real numbers. A gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \qquad \text{for } p,\ q \in \mathbb{Z}.$$

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. We denote by $G$ the adjoint group of $\mathfrak{g}$ and let $G'$ be the normalizer of $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ in $G$;

$$G' = \{\, \sigma \in G \mid \sigma(\mathfrak{g}') = \mathfrak{g}' \,\}.$$

We consider the homogeneous space $M_{\mathfrak{g}} = G/G'$, which is a real or complex manifold ($R$-space) depending on whether the complexification $\mathbb{C}\mathfrak{g}$ of $\mathfrak{g}$ is simple or $\mathfrak{g}$ is complex simple (see Proposition 3.3 in §3.2 and §4.1). By identifying $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on $G$, the $G'$-invariant subspace $\mathfrak{f}^{-1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}'$ induces a $G$-invariant differential system $D_{\mathfrak{g}}$ on $M_{\mathfrak{g}}$, which is a holomorphic differential system when $\mathfrak{g}$ is complex simple. $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is called the standard differential system of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (§4.1).

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The main purpose of this article is to give an overview of the basic materials both on the geometry of differential systems and on the structure of simple graded Lie algebras over $K = \mathbb{R}$ or $\mathbb{C}$, which culminates to show the following (Corollary 5.4):

*The Lie algebra $\mathcal{A}(M_\mathfrak{g}, D_\mathfrak{g})$ (or more precisely, each stalk $\mathcal{A}_x(M_\mathfrak{g}, D_\mathfrak{g})$ of the Lie algebra sheaf $\mathcal{A}(M_\mathfrak{g}, D_\mathfrak{g})$) is isomorphic with $\mathfrak{g}$, except when $(M_\mathfrak{g}, D_\mathfrak{g})$ is locally isomorphic with the canonical (or contact) system on a real or complex jet space.*

For the precise statement, see Corollary 5.4 in §5.2.

Historically E. Cartan, in the course of the classification of simple Lie algebras over $\mathbb{C}$, indicated some of simple Lie algebras of exceptional type as the Lie algebras of the invariance groups of certain pfaffian systems ([C1], [C2]), (thus exihibiting the existence of simple Lie algebras of these types). These discoveries seem to be forgotten during the course of the modern development of the structure theory of semisimple Lie algebras or of the Lie group theory (cf. Introduction of [He]).

On the other hand, after E. Cartan, the equivalence problems of differential sytems, or more generally of geometric structures subordinate to differential systems were investigated and developed by N. Tanaka in [T1], [T2], [T3] and [T4]. Utilizing his theory and the structure theory of simple Lie algebras over $\mathbb{R}$ and $\mathbb{C}$, we shall show the above result, which also reestablishes Cartan's discoveries cited above (see examples in §1.3 and §5.3).

Now let us proceed to the description of the contents of this paper. In §§1 and 2, we shall review the Tanaka theory of regular differential systems. He introduced the graded algebras $\mathfrak{m}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x)$ of a regular differential system $(M, D)$ at each $x \in M$ as the first invariant for the equivalence of differential systems, which are nilpotent graded Lie algebras satisfying $\mathfrak{g}_{-1}(x) = D(x)$ and $\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)]$ for $p < -1$ (see §1.2 for the definition). Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded algebra, that is, a nilpotent graded Lie algebra satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $(M, D)$ is called of type $\mathfrak{m}$, if $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$. Moreover, given a fundamental graded algebra $\mathfrak{m}$, we can construct a model differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ group theoretically, which is called the standard differential system of type $\mathfrak{m}$ (§1.2). Here we note that, when $\mathfrak{m}$ is the negative part of a simple graded Lie algebra, $(M(\mathfrak{m}), D_\mathfrak{m})$ is naturally identified with an open dense submanifold of $(M_\mathfrak{g}, D_\mathfrak{g})$ (see §4.1).

For a fundamental graded algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, Tanaka [T2] introduced the notion of the algebraic prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$; $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathfrak{m})$ is a graded Lie algebra satisfying the following conditions:

(1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$.

(2) For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.

(3) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded algebras satisfying conditions (1) and (2) above.

Moreover, among the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ satisfying conditions (1) and (2) above, $\mathfrak{g}(\mathfrak{m})$ is characterized by the vanishing of the first cohomology groups $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$. Here $H^q(\mathfrak{m}, \mathfrak{g}) = \bigoplus_{p\in\mathbb{Z}} H^{p,q}(\mathfrak{m}, \mathfrak{g})$ is the Lie algebra cohomology associated with the representation $\mathrm{ad}\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$, which is called the generalized Spencer cohomology of the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (§2.3). The prolongation $\mathfrak{g}(\mathfrak{m})$ plays a fundamental role in the equivalence problems of regular differential systems of type $\mathfrak{m}$. Especially $\mathfrak{g}(\mathfrak{m})$ describes the structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ of all infinitesimal automorphisms of the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$. In particular $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ is isomorphic with $\mathfrak{g}(\mathfrak{m})$ when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional. We shall review these facts in §§1 and 2 following [T2] and also discuss the Hilbert-Cartan equation as an example (§1.3).

With these preparations, we shall be concerned with the following question: When does $\mathfrak{g}(\mathfrak{m})$ become finite dimensional and simple? The answer to this question (Theorems 5.2 and 5.3) gives us the result stated above. In order to answer this question, we first classify, for a simple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$, the gradations $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, which turns out to be equivalent to the classification of parabolic subalgebras $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ of $\mathfrak{g}$. This allows us to describe the gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ in terms of the root (or restricted root) space decomposition of $\mathfrak{g}$ (cf. [K-A]) and to apply the method of Kostant [K] to compute $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which is carried out in §5.2. Namely, for a complex simple Lie algebra $\mathfrak{g}$, let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system $\Phi$ relative to $\mathfrak{h}$. Take any non-empty subset $\Delta_1$ of $\Delta$ and put

$$\Phi_k^+ = \{\, \alpha = \sum_{i=1}^{\ell} n_i(\alpha)\, \alpha_i \in \Phi^+ \mid \sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k \,\} \quad \text{for } k \geqq 0.$$

Then we obtain a gradation $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for

$p < -1$ by putting

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \qquad (k > 0),$$

where $\mathfrak{g}_\alpha$ denotes the root space corresponding to $\alpha \in \Phi$. We denote the simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ obtained from $\Delta_1$ in this manner by $(X_\ell, \Delta_1)$, when $\mathfrak{g}$ is a simple Lie algebra of type $X_\ell$. Here $X_\ell$ stands for the Dynkin diagram of $\mathfrak{g}$ representing $\Delta$ and $\Delta_1$ is a subset of vertices of $X_\ell$. Then every complex graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ satisfying $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$ is conjugate to $(X_\ell, \Delta_1)$ for some $\Delta_1 \subset \Delta$ (Theorem 3.12). In the real case, we can utilize the Satake diagram to describe the gradation of $\mathfrak{g}$ (see §3.4).

Now we can state one of the main results of this paper

**Theorem 5.2′.** *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

(2) $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is a complex contact gradation, that is,* $\dim \mathfrak{g}_{-2} = 1$.

(3) $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ *or* $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

We shall obtain also the real version of this theorem (Theorem 5.3).

In §4, we shall discuss the standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$ of type $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. First we shall consider the contact gradation of $\mathfrak{g}$ and show that every complex simple Lie algebra other than $\mathfrak{sl}(2, \mathbb{C})$ admits a unique complex contact gradation up to conjugacy. We discuss the unified description of the standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ associated with this contact gradation via the adjoint representation of $\mathrm{Int}(\mathfrak{g})$, which were originally found by Boothby [Bo] as compact simply connected homogeneous complex contact manifolds. Moreover we shall reproduce the explicit matrix description, due to Takeuchi [Tk1], of the root space decompositions of simple Lie algebras over $\mathbb{C}$ of the classical type, which gives us explicit pictures of $M_\mathfrak{g}$ in these cases. With the aid of this description, we shall discuss those standard differential systems $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ which are isomorphic with the canonical systems on Grassmann bundles (geometric jet spaces). These are obviously exceptions for the assertion of our main result stated at the beginning of this introduction. More precisely we shall show that the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of types $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ and $(C_\ell, \{\alpha_1, \alpha_\ell\})$ are isomorphic with the canonical system $(J(\mathbb{C}P^\ell, i), C)$ on the Grassmann bundle $J(\mathbb{C}P^\ell, i)$ over the complex projective space $\mathbb{C}P^\ell$, consisting of $i$-dimensional contact elements to $\mathbb{C}P^\ell$, and the canonical system $(L(\mathbb{C}P^{2\ell-1}), E)$ on the Lagrange-Grassmann bundle $L(\mathbb{C}P^{2\ell-1})$ over the odd dimensional (contact) projective space $\mathbb{C}P^{2\ell-1}$ respectively.

In §5, we shall first review the harmonic theory of Kostant [K] for the Lie algebra cohomology and apply his method to compute $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ and $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which gives us the main results (Theorems 5.2, 5.3 and Corollary 5.4) of this article. Here we include the computation of $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which is important to know the fundamental invariants of the normal Cartan connection, constructed by Tanaka [T4], for the geometric structures associated with a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$. Especially, by these computations combined with Theorem of Tanaka [T4], we can find many examples of regular differential systems $(M, D)$ of type $\mathfrak{m}$ with no local invariants, whose Lie algebra $\mathcal{A}(M, D)$ of all infinitesimal automorphisms are finite dimensional and simple. Finally in §5.4, we shall discuss the reducible primitive actions of finite dimensional Lie groups, following [Go], [K-N, I and II] and [Gu], and characterize the standard differential systems $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ whose isotropy subalgebra $\mathfrak{g}'$ are maximal parabolic, as homogeneous differential systems which have nonlinear reducible primitive actions of Lie groups (cf. [O1], [Go]).

The main results of this paper (Theorems 5.2, 5.3 and Corollary 5.4) were obtained by the author around 1985 (unpublished) by a different method based on the finite dimensionality criterion of the prolongation $\mathfrak{g}(\mathfrak{m})$ (Corollary 2 to Theorem 11.1 of [T2]) due to Tanaka. The present cohomological method with the powerful theorem of Kostant has the advantage to produce the result for the second cohomology $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ at the same time.

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## §1. Symbol algebras of $(M, D)$

### 1.1. Regular differential systems

By a differential system $(M, D)$, we mean a subbundle $D$ of the tangent bundle $T(M)$ of a manifold $M$ of dimension $n$. Locally $D$ is defined by 1-forms $\omega_1, \dots, \omega_{n-r}$ such that $\omega_1 \wedge \cdots \wedge \omega_{n-r} \neq 0$ at each point, where $r$ is the rank of $D$;

$$D = \{\, \omega_1 = \cdots = \omega_{n-r} = 0 \,\}.$$

For two differential systems $(M, D)$ and $(\hat{M}, \hat{D})$, a diffeomorphism $\phi$ of $M$ onto $\hat{M}$ is called an isomorphism of $(M, D)$ onto $(\hat{M}, \hat{D})$ if the differential map $\phi_*$ of $\phi$ sends $D$ onto $\hat{D}$. Our subject will be the Lie algebra $\mathcal{A}(M, D)$ of infinitesimal automorphisms of $(M, D)$. For a vector field $X$ on $M$, $X$ belongs to $\mathcal{A}(M, D)$ if and only if

$$L_X \omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_{n-r}} \qquad \text{for } i = 1, \dots, n-r,$$

or equivalently, if and only if

$$[X, \mathcal{D}] \subset \mathcal{D},$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of $D$.

By the Frobenius theorem, we know that $D$ is completely integrable if and only if

$$d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_{n-r}} \qquad \text{for } i = 1, \dots, n-r,$$

or equivalently, if and only if

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}.$$

When $D$ is completely integrable, it is easily seen that $\mathcal{A}(M, D)$ is infinite dimensional.

Thus, for a non-integrable differential system $D$, we are led to consider the derived system $\partial D$ of $D$, which is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

In general $\partial D$ is obtained as a subsheaf of the tangent sheaf of $M$ (for the precise argument, see [T2] or [Y1]). Moreover higher derived systems $\partial^k D$ are usually defined successively by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the $k$-th weak derived system $\partial^{(k)} D$ of $D$ inductively by

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(k)}\mathcal{D}$ denotes the space of sections of $\partial^{(k)} D$.

A differential system $(M, D)$ is called regular, if $D^{-(k+1)} = \partial^{(k)} D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system $(M, D)$, we have ([T2, Proposition 1.1])

$$\begin{cases} (1) & \textit{There exists a unique integer } \mu > 0 \textit{ such that, for all } k \geqq \mu, \\ & \qquad D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D, \\ (2) & [\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q} \qquad \text{for all } p,\ q < 0. \end{cases}$$

where $\mathcal{D}^p$ denotes the space of sections of $D^p$. (2) can be checked easily by induction on $q$.

Thus $D^{-\mu}$ is the smallest completely integrable differential system, which contains $D = D^{-1}$.

### 1.2. Graded algebras associated with $(M, D)$

Let $(M, D)$ be a regular differential system such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential systems, we now define the *graded algebra* $\mathfrak{m}(x)$ *associated with a differential system* $(M, D)$ at $x \in M$, which was introduced by N. Tanaka [T2].

We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$ and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Let $\varpi_p$ be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. From

$$[f\tilde{X}, g\tilde{Y}] = f \cdot g[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X},$$

for vector fields $\tilde{X}$, $\tilde{Y}$ and functions $f$, $g$ on $M$, it follows immediately that $[X,Y] \in \mathfrak{g}_{p+q}(x)$ is well-defined for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$ (cf. [T2, Lemma 1.1]).

Endowed with this bracket operation, by (2) above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

We call $\mathfrak{m}(x)$ the *symbol algebra of* $(M, D)$ at $x \in M$ for short.

Furthermore, let $\mathfrak{m}$ be a fundamental graded Lie algebra of $\mu$-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then $(M, D)$ is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic with $\mathfrak{m}$ at each $x \in M$.

Conversely, given a fundamental graded Lie algebra $\mathfrak{m}$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the standard differential system of type $\mathfrak{m}$.

### 1.3. The Hilbert-Cartan equation

As a good illustration of our previous discussion, we shall now calculate the symbol algebras of a differential system $(R, D)$, which is associated with the following underdetermined ordinary differential equation studied by Hilbert [H] and Cartan [C3]:

$$\text{(H.C)} \qquad \frac{dv}{dx} = \left(\frac{d^2u}{dx^2}\right)^2$$

As usual, we consider a hypersurface $R'$, defined by (H.C), in the space $J^2$ of 2-jets for 2-unknown and 1-independent variables with coordinate sytem $(x, u, v, u', v', u'', v'')$;

$$R' = \{\, v' = (u'')^2 \,\}.$$

Our differential system $(R', D')$ is obtained by restricting to $R'$ the canonical (or contact) system on $J^2$;

$$D' = \{\, \omega_1' = \omega_2' = \omega_3' = \omega_4' = 0 \,\},$$

where

$$\begin{cases} \omega_1' = dv - (u'')^2 dx, \\ \omega_2' = du - u' dx, \\ \omega_3' = du' - u'' dx, \\ \omega_4' = d(u'')^2 - v'' dx = 2u'' du'' - v'' dx. \end{cases}$$

For the regularity condition, we shall work on the domain $R = \{\, u'' \neq 0 \,\}$ in $R'$ and take $(x, u, v, p, r, t)$ as a coordinate system on $R$, where $p = u'$, $r = u''$ and $t = \frac{1}{2}(u'')^{-1} v''$. Then $(R, D)$ is given on this coordinate system by

$$D = \{\, \omega_1 = \omega_2 = \omega_3 = \omega_4 = 0 \,\},$$

where $\omega_1 = dv - r^2 dx$, $\omega_2 = du - p\, dx$, $\omega_3 = dp - r\, dx$ and $\omega_4 = dr - t\, dx$.

First we calculate

$$(1.1) \qquad \begin{cases} d\omega_1 = 2r\, dx \wedge dr = 2r\, dx \wedge \omega_4, \\ d\omega_2 = dx \wedge dp = dx \wedge \omega_3, \\ d\omega_3 = dx \wedge dr = dx \wedge \omega_4, \\ d\omega_4 = dx \wedge dt. \end{cases}$$

To locate the derived system $\partial D$, we look at the equalities (1.1) modulo the ideal spanned by 1-forms $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$:

$$\begin{cases} d\omega_1 \equiv d\omega_2 \equiv d\omega_3 \equiv 0, \\ d\omega_4 \equiv dx \wedge dt. \end{cases} \qquad (\mathrm{mod}\ \omega_1, \omega_2, \omega_3, \omega_4)$$

Then, since $d\omega_i(X, Y) = -\omega_i([X, Y])$ for $X$, $Y \in \mathcal{D}$ $(i = 1, 2, 3)$, it follows that

$$D^{-2} = \partial D = \{\, \omega_1 = \omega_2 = \omega_3 = 0 \,\}.$$

To locate $\partial^2 D$, we proceed to look at $d\omega_1$, $d\omega_2$ and $d\omega_3$ modulo 1-forms $\omega_1$, $\omega_2$ and $\omega_3$. Putting $\widetilde{\omega}_1 = \omega_1 - 2r\, \omega_3$, we have

$$d\widetilde{\omega}_1 = 2\omega_3 \wedge dr = 2\omega_3 \wedge \omega_4 + 2t\, \omega_3 \wedge dx.$$

Hence we get

$$D^{-2} = \{\,\widetilde{\omega}_1 = \omega_2 = \omega_3 = 0\,\},$$

and

$$\begin{cases} d\widetilde{\omega}_1 \equiv d\omega_2 \equiv 0, \\ d\omega_3 \equiv dx \wedge \omega_4. \end{cases} \quad (\mathrm{mod}\ \widetilde{\omega}_1, \omega_2, \omega_3)$$

This implies that $\partial^2 D = \partial(D^{-2})$ is defined by $\widetilde{\omega}_1$ and $\omega_2$. However, in this case, from $\operatorname{rank} D^{-2} = \operatorname{rank} D^{-1} + 1$, we have

$$\mathcal{D}^{-2} + [\mathcal{D}^{-1}, \mathcal{D}^{-2}] = \mathcal{D}^{-2} + [\mathcal{D}^{-2}, \mathcal{D}^{-2}].$$

Namely we have $D^{-3} = \partial^2 D$. To proceed, we put $2\bar{\omega}_1 = \widetilde{\omega}_1 + 2t\,\omega_2$. Then we have

$$D^{-3} = \{\,\bar{\omega}_1 = \omega_2 = 0\,\},$$

and

$$\text{(1.2)} \qquad \begin{cases} d\bar{\omega}_1 = \omega_3 \wedge \omega_4 + \omega_6 \wedge \omega_2, \\ d\omega_2 = \omega_5 \wedge \omega_3, \\ d\omega_3 = \omega_5 \wedge \omega_4, \\ d\omega_4 = \omega_5 \wedge \omega_6. \end{cases}$$

where $\omega_5 = dx$ and $\omega_6 = dt$. From $d\bar{\omega}_1 \equiv \omega_3 \wedge \omega_4$, $d\omega_2 \equiv \omega_5 \wedge \omega_3$ $(\mathrm{mod}\ \bar{\omega}_1, \omega_2)$, we obtain

$$T(R) = \partial^3 D.$$

On the other hand, to locate $D^{-4}$, we should ignore the contributions of elements in $[\mathcal{D}^{-3}, \mathcal{D}^{-3}]$, which are not contained in $[\mathcal{D}^{-1}, \mathcal{D}^{-3}]$. Thus we must look at $d\bar{\omega}_1$ and $d\omega_2$ modulo $\bar{\omega}_1$, $\omega_2$ and $\omega_3 \wedge \omega_4$:

$$\begin{cases} d\bar{\omega}_1 \equiv 0, \\ d\omega_2 \equiv \omega_5 \wedge \omega_3. \end{cases} \quad (\mathrm{mod}\ \bar{\omega}_1, \omega_2, \omega_3 \wedge \omega_4)$$

This implies that

$$D^{-4} = \{\,\bar{\omega}_1 = 0\,\}.$$

Furthermore, we have

$$d\bar{\omega}_1 \equiv \omega_3 \wedge \omega_4 + \omega_6 \wedge \omega_2 \quad (\mathrm{mod}\ \bar{\omega}_1, \omega_2 \wedge \omega_3, \omega_2 \wedge \omega_4)$$

Hence we get

$$T(R) = D^{-5}.$$

Thus we see that $(R, D)$ is a regular differential system of type $\mathfrak{m}_6$, where

$$\mathfrak{m}_6 = \bigoplus_{p=-1}^{-5} \mathfrak{g}_p,$$

is the fundamental graded algebra of 5-th kind, whose Maurer-Cartan equation is given by (1.2). Namely $\mathfrak{m}_6$ is a 6-dimensional nilpotent graded Lie algebra, which is described as follows: There exists a basis $\{e_1, \dots, e_6\}$ of $\mathfrak{m}_6$ such that each $\mathfrak{g}_p$ is spanned by the following vectors

$$\mathfrak{g}_{-5} = \langle e_1 \rangle, \qquad \mathfrak{g}_{-4} = \langle e_2 \rangle, \qquad \mathfrak{g}_{-3} = \langle e_3 \rangle,$$
$$\mathfrak{g}_{-2} = \langle e_4 \rangle, \qquad \mathfrak{g}_{-1} = \langle e_5, e_6 \rangle,$$

and that the bracket product is given by

$$[e_6, e_5] = e_4, \qquad [e_4, e_5] = e_3, \qquad [e_3, e_5] = e_2,$$
$$[e_2, e_6] = [e_4, e_3] = e_1, \qquad [e_i, e_j] = 0 \quad \text{otherwise}.$$

A notable fact for $(R, D)$ is that we obtain the strict equalities (1.2) instead of mod equalities for defining 1-forms $\bar{\omega}_1$, $\omega_2$, $\omega_3$, $\omega_4$ of $D$, that is, $(R, D)$ is isomorphic with the standard differential system of type $\mathfrak{m}_6$. Because of this fact, we shall see later in §5.2 that the Lie algebra $\mathcal{A}(R, D)$ of infinitesimal automorphisms of $(R, D)$ is isomorphic with the 14-dimensional simple Lie algebra $G_2$ (cf. [C3], [A-K-O]). In fact we shall encounter $\mathfrak{m}_6$ in §3.4 in connection with the root space decomposition of $G_2$.

Another example of a historical interest is the following differential system $(X, E)$ on $X = \mathbb{R}^5$, which was found by E. Cartan [C2];

$$E = \{\, \omega_1 = \omega_2 = \omega_3 = 0 \,\},$$

where

$$\begin{cases} \omega_1 = dx_1 + (x_3 + \frac{1}{2} x_4 x_5)\, dx_4, \\ \omega_2 = dx_2 + (x_3 - \frac{1}{2} x_4 x_5)\, dx_5, \\ \omega_3 = dx_3 + \frac{1}{2}(x_4\, dx_5 - x_5\, dx_4), \end{cases}$$

and $(x_1, x_2, x_3, x_4, x_5)$ is a coordinate system of $X = \mathbb{R}^5$. We have

$$\text{(1.3)} \qquad \begin{cases} d\omega_1 = \omega_3 \wedge \omega_4, \\ d\omega_2 = \omega_3 \wedge \omega_5, \\ d\omega_3 = \omega_4 \wedge \omega_5, \end{cases}$$

where $\omega_4 = dx_4$ and $\omega_5 = dx_5$. In this case we may calculate symbol algebras of $(X, E)$ as follows. We take a dual basis $\{X_1, \ldots, X_5\}$ of vector fields on $X$ to a basis of 1-forms $\{\omega_1, \ldots, \omega_5\}$ given above;

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}, \quad X_3 = \frac{\partial}{\partial x_3}, \\
X_4 &= \frac{\partial}{\partial x_4} + \frac{1}{2} x_5 \frac{\partial}{\partial x_3} - (x_3 + \frac{1}{2} x_4 x_5) \frac{\partial}{\partial x_1}, \\
X_5 &= \frac{\partial}{\partial x_5} - \frac{1}{2} x_4 \frac{\partial}{\partial x_3} - (x_3 - \frac{1}{2} x_4 x_5) \frac{\partial}{\partial x_2}.
\end{aligned}
$$

Then we calculate, or from (1.3),

$$[X_5, X_4] = X_3, \quad [X_5, X_3] = X_2, \quad [X_4, X_3] = X_1,$$

and $[X_i, X_j] = 0$ otherwise. This implies that $E^{-2} = \{\omega_1 = \omega_2 = 0\}$, $E^{-3} = T(X)$ and that $(X, E)$ is isomorphic with the standard differential system of type $\mathfrak{m}_5$, where

$$\mathfrak{m}_5 = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is the fundamental graded algebra of third kind, whose Maurer-Cartan equation is given by (1.3). Here we note that the Lie algebra structure of $\mathfrak{m}_5$ is uniquely determined by the requirement that $\mathfrak{m}$ is fundamental, $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$ and $\dim \mathfrak{g}_{-2} = 1$ (cf. [C2], [T2]). In fact $\mathfrak{m}_5$ is the universal fundamental graded algebra of third kind with $\dim \mathfrak{g}_{-1} = 2$ (see [T2, §3]). We shall encounter $\mathfrak{m}_5$ in §3.4 in connection with the root space decomposition of $G_2$.

## §2. Algebraic prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$

### 2.1. Review of the prolongation of $G$-structure

We first review the notion of the algebraic prolongation of the usual $G$-structure theory (cf. [St], [K2]). Let $G$ be a Lie subgroup of $GL(V)$, where $V$ is a real vector space of dimension $n$. A $G$-structure on a manifold $M$ of dimension $n$ is, by definition, a $G$-reduction $P_G$ of the frame bundle $F(M)$ of $M$. Let $\mathfrak{g}$ be the Lie algebra of $G$. As is well-known (cf. [St], [K2]), the notion of the (algebraic) prolongation of $\mathfrak{g}$ originates from the calculation of infinitesimal automorphisms of the flat $G$-structure. A basis $\{e_1, \ldots, e_n\}$ of $V$ gives a global trivialization of the frame bundle $F(V)$ of $V$. Then the flat $G$-structure on $V$ is given as the $G$-subbundle $P_G^o = V \times G$ of $F(V) = V \times GL(V)$.

To seek infinitesimal automorphisms of $P_G^o$, we may proceed as follows: Take a linear coordinate system $(x_1, \dots, x_n)$ given by the above basis of $V$. Owing to the global trivialization of $F(V)$, every vector field $X$ on $V$ is identified with a $V$-valued function $f_X$ on $V$ by putting

$$f_X = \xi(X) = (\xi_1, \dots, \xi_n),$$

where $\xi = (dx_1, \dots, dx_n)$ is a $V$-valued 1-form on $V$ and

$$X = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}.$$

By utilizing the $V$-valued 1-form $\xi$, the derivatives of $f_X$ can be expressed as the coefficient matrix $f_X^0$ of $d\,f_X$ with respect to $\xi$, that is, the $\mathfrak{gl}(V) = V \otimes V^*$-valued function $f_X^0$ on $V$ is defined by

$$v(f_X) = d\,f_X(v) = f_X^0(\xi(v)) = f_X^0(v) \qquad \text{for } v \in V \cong T_x(V),$$

Here we regard $v$ as a tangent vector at $x \in V$ on the left side of the equalities and as a vector in $V$ on the right side. We shall write this equality, in short, as

$$d\,f_X^{-1} = [f_X^0, \xi],$$

where $f_X^{-1} = f_X$. The second derivatives of $f_X$ can be obtained as the coefficient matrix $f_X^1$ of $d\,f_X^0$ with respect to $\xi$, that is, the $\mathfrak{gl}(V) \otimes V^*$-valued function $f_X^1$ on $V$ is defined by

$$d\,f_X^0 = [f_X^1, \xi].$$

Here, by the compatibility condition for second derivatives (or by the chain rule), $f_X^1$ actually takes values in $V \otimes S^2(V^*) \subset \mathfrak{gl}(V) \otimes V^* = V \otimes V^* \otimes V^*$. Inductively the $(k+1)$-th derivatives of $f_X$ can be expressed as the coefficient matrix $f_X^k$ of $d\,f_X^{k-1}$ with respect to $\xi$, that is, the $V \otimes S^{k+1}(V^*)$-valued function $f_X^k$ on $V$ is defined by

$$d\,f_X^{k-1} = [f_X^k, \xi],$$

where $S^{k+1}(V^*)$ denotes the $(k+1)$-th symmetric power of $V^*$.

Now, for a vector field $X$ on $V$, let $\tilde{X}$ be the lift of $X$ to $F(V)$, that is, $\tilde{X}$ is a vector field on $F(V)$ generated by the differential flow $(\phi_t)_*$ of the (local) flow $\phi_t$ of $X$. Then $X$ is an infinitesimal automorphism of the flat $G$-structure $P_G^o$ if and only if $\tilde{X}$ is tangent to $P_G^o$. This is equivalent to the condition that $f_X^0$ is a $\mathfrak{g}$-valued function on $V$. Thus, for higher order derivatives, we see that $f_X^k$ takes values in

$$\mathfrak{g}^{(k)} = \mathfrak{g} \otimes \otimes^k V^* \cap V \otimes S^{k+1}(V^*).$$

Here $\mathfrak{g}^{(k)}$ is called the $k$-th prolongation of $\mathfrak{g}$. Especially the $(k+1)$-th coefficient of the Taylor expansion of $f_X$ takes values in $\mathfrak{g}^{(k)}$ at the origin of $V$. Conversely, for an element $a \in \mathfrak{g}^{(k)}$, there exists a unique polynomial (of homogeneous degree $k+1$) vector field $X$ such that $X$ is an infinitesimal automorphism of $P_G^o$ and that the coefficient of the Taylor expansion of $f_X$ at the origin coincides with $a \in \mathfrak{g}^{(k)}$.

In this way the structure of the Lie algebra of infinitesimal automorphisms of $P_G^o$ can be expressed by the graded Lie algebra;

$$\bigoplus_{p=-1}^{\infty} \mathfrak{g}^{(p)},$$

where $\mathfrak{g}^{(-1)} = V$, $\mathfrak{g}^{(0)} = \mathfrak{g}$, and the bracket operation is defined accordingly. Here we note that $\mathfrak{g}^{(-1)} = V$ corresponds to constant coefficient vector fields. For the details, we refer the reader to [K2] or [St].

### 2.2. Infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$

Let $\mathfrak{m}$ be a fundamental graded Lie algebra of $\mu$-th kind. In the same spirit as in the previous section, we are going to seek infinitesimal automorphisms of our model (flat) differential system of type $\mathfrak{m}$, that is, the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$.

Let $\xi$ be the Maurer-Cartan form on $M(\mathfrak{m})$, that is, $\xi$ is a $\mathfrak{m}$-valued 1-form on $M(\mathfrak{m})$ such that

$$\xi(X_x) = X \qquad \text{for } X \in \mathfrak{m} \text{ and } x \in M(\mathfrak{m}),$$

where $\mathfrak{m}$ is identified with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$. Then, for $p < 0$, $D_{\mathfrak{m}}^p = \partial^{(-p-1)} D_{\mathfrak{m}}$ is given by

$$D_{\mathfrak{m}}^p = \{\, \xi^{-\mu} = \cdots = \xi^{p-1} = 0 \,\} = \{\, \xi^s = 0 \quad (s < p) \,\},$$

where $\xi^p$ is the $\mathfrak{g}_p$-component of $\xi$. Namely we have a global trivialization of $F(M(\mathfrak{m}))$ by a basis of $\mathfrak{m}$. Thus every vector field $X$ on $M(\mathfrak{m})$ is identified with a $\mathfrak{m}$-valued function $f_X$ by putting

$$f_X(x) = \xi(X_x) \qquad \text{at } x \in M(\mathfrak{m}).$$

In particular $f_X$ is a constant function if and only if $X$ is left invariant. For two vector fields $X$, $Y$ on $M(\mathfrak{m})$, we have

$$f_{[X,Y]}(x) = [f_X(x), f_Y(x)] + X_x(f_Y) - Y_x(f_X), \tag{2.1}$$

at $x \in M(\mathfrak{m})$. Here the bracket product on the right side is that of $\mathfrak{m}$. Moreover, according to the decomposition of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, $f_X$ is written as a sum

$$f_X = \sum_{p<0} f_X^p,$$

where $f_X^p$ is a $\mathfrak{g}_p$-valued function on $M(\mathfrak{m})$.

Now recall that a vector field $X$ is an infinitesimal automorphism of $(M(\mathfrak{m}), D_\mathfrak{m})$ if and only if

$$[X, \mathcal{D}_\mathfrak{m}] \subset \mathcal{D}_\mathfrak{m},$$

where $\mathcal{D}_\mathfrak{m}$ is the space of sections of $D_\mathfrak{m}$. Thus $X \in \mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ if and only if

$$f^p_{[X,Y]} = 0 \qquad \text{for } p < -1 \text{ and } Y \in \mathcal{D}_\mathfrak{m}.$$

By (2.1), this condition is equivalent to the following equalities;

$$Y(f_X^p) = [f_X^{p+1}, f_Y^{-1}] \qquad \text{for } p < -1 \text{ and } Y \in \mathcal{D}_\mathfrak{m},$$

or equivalently

$$d\,f_X^p \equiv [f_X^{p+1}, \xi^{-1}] \pmod{\xi^s \quad (s < -1)} \quad \text{for } p < -1. \tag{2.2}$$

The equalities (2.2) express the condition for a vector field $X$ to be an infinitesimal automorphism of $(M(\mathfrak{m}), D_\mathfrak{m})$ in terms of $f_X$. However, from the generating condition of $\mathfrak{m}$: $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, $X \in \mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ satisfies additional equalities as follows: First we calculate

$$\begin{aligned} Y(Z(f_X^p)) &= Y([f_X^{p+1}, f_Z^{-1}]) = [Y(f_X^{p+1}), f_Z^{-1}] + [f_X^{p+1}, Y(f_Z^{-1})] \\ &= [[f_X^{p+2}, f_Y^{-1}], f_Z^{-1}] + [f_X^{p+1}, Y(f_Z^{-1})], \end{aligned}$$

for vector fields $Y$, $Z \in \mathcal{D}_\mathfrak{m}$ and $p < -2$. Then, by (2.1), we get

$$[Y, Z](f_X^p) = Y(Z(f_X^p)) - Z(Y(f_X^p)) = [f_X^{p+2}, f^{-2}_{[Y,Z]}],$$

for $p < -2$. From $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$, this implies

$$W(f_X^p) = [f_X^{p+2}, f_W^{-2}] = [f_X^{p+2}, W],$$

for $p < -2$ and $W \in \mathfrak{g}_{-2}$, since $f_W = W$ is a constant function for $W \in \mathfrak{g}_{-2}$. Proceeding by induction on $r$, we see that, for a fixed $p < 0$, the same calculation as above yields

$$Y(f_X^p) = [f_X^{p-r}, Y],$$

for $r > p$ and $Y \in \mathfrak{g}_r$. Summarizing, we obtain

$$df_X^p \equiv \sum_{r=-1}^{p+1} [f_X^{p-r}, \xi^r] \pmod{\xi^s \quad (s < p+1)} \quad \text{for } p < -1. \tag{2.3}$$

Starting from (2.3), we are going to seek all the (higher) derivatives of $f_X$. In order to do so, we first introduce a $\bigoplus_{p<0} \mathfrak{g}_p \otimes \mathfrak{g}_p^*$-valued function $f_X^0$ by

$$(f_X^0(x))(Y) = Y_x(f_X^p) \qquad \text{for } Y \in \mathfrak{g}_p \text{ and } x \in M(\mathfrak{m}).$$

Here we regard $Y$ as a vector field on $M(\mathfrak{m})$ on the right side of the equality and as a vector in $\mathfrak{g}_p$ on the left side. We write this equality in short as

$$Y(f_X^p) = [f_X^0, Y] \qquad \text{for } Y \in \mathfrak{g}_p.$$

Equivalently we can say that $f_X^0$ is defined by the following equalities;

$$df_X^p \equiv \sum_{r=-1}^{p} [f_X^{p-r}, \xi^r] \pmod{\xi^s \quad (s < p)} \quad \text{for } p < 0.$$

Namely we have strengthened the mod equalities (2.3) and add $df_X^{-1} \equiv [f_X^0, \xi^{-1}] \pmod{\xi^s \quad (s < -1)}$. From these equalities, it follows a compatibility condition for $f_X^0$: For $Y \in \mathfrak{g}_r$ and $Z \in \mathfrak{g}_s$ ($r$, $s < 0$), we calculate as above and get

$$[Y, Z](f_X^\ell) = [[f_X^k, Y], Z] - [[f_X^k, Z], Y], \tag{2.4}$$

where $k = \ell - (r + s)$. This equality is valid as far as $f_X^\ell$ and $f_X^k$ are defined. When $\ell = r + s$, by definition of $f_X^0$, we obtain

$$[f_X^0, [Y, Z]] = [[f_X^0, Y], Z] - [[f_X^0, Z], Y].$$

This implies that $f_X^0$ takes values in

$$\mathfrak{g}_0(\mathfrak{m}) = \{\, u \in \bigoplus_{p<0} \mathfrak{g}_p \otimes \mathfrak{g}_p^* \mid u([Y, Z]) = [u(Y), Z] + [Y, u(Z)] \,\}. \tag{p.0}$$

Here we note that $\mathfrak{g}_0(\mathfrak{m})$ is the Lie algebra of all (gradation preserving) derivations of the graded Lie algebra $\mathfrak{m}$.

Now we continue this procedure and introduce a $\mathfrak{g}_k(\mathfrak{m})$-valued function $f_X^k$ for positive integer $k$ inductively as follows: Assume that $\mathfrak{g}_\ell = \mathfrak{g}_\ell(\mathfrak{m})$ and $f_X^\ell$ are defined for $\ell < k$ such that

$$d\,f_X^\ell \equiv \sum_{r=-1}^{\ell-k+1} [f_X^{\ell-k}, \xi^r] \quad (\mathrm{mod}\ \xi^s \quad (s < \ell - k + 1)) \quad \text{for } \ell < k-1.$$

Here we understand that $\xi^r = 0$ for $r < -\mu$. We introduce a $\bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^*$-valued function $f_X^k$ by

$$(f_X^k(x))(Y) = Y_x(f_X^{p+k}) \qquad \text{for } Y \in \mathfrak{g}_p \text{ and } x \in M(\mathfrak{m}),$$

or equivalently by the following equalities;

$$(2.5) \qquad d\,f_X^\ell \equiv \sum_{r=-1}^{\ell-k} [f_X^{\ell-r}, \xi^r] \quad (\mathrm{mod}\ \xi^s \quad (s < \ell - k)) \quad \text{for } \ell < k.$$

Here we write $f_X^k(Y) = [f_X^k, Y]$ in short. Then, by definition of $f_X^k$ and (2.4), we have

$$[f_X^k, [Y, Z]] = [[f_X^k, Y], Z] - [[f_X^k, Z], Y],$$

for $Y \in \mathfrak{g}_r$, $Z \in \mathfrak{g}_s$ $(r,\ s < 0)$. Namely $f_X^k$ takes values in

$$(\text{p.k}) \quad \mathfrak{g}_k(\mathfrak{m}) = \{\, u \in \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^* \mid u([Y,Z]) = [u(Y), Z] - [u(Z), Y] \,\}.$$

This finishes our inductive definition of $f_X^k$ and $\mathfrak{g}_k(\mathfrak{m})$.

One should note that, for a fixed $\ell$, (2.5) becomes a strict equality when $k$ increases sufficiently large. Thus, for a family $\{f_X^k\}_{k \geqq -\mu}$ of functions on $M(\mathfrak{m})$, we obtain ([T2, Lemma 6.2])

$$(2.6) \qquad d\,f_X^k = \sum_{r=-1}^{-\mu} [f_X^{k-r}, \xi^r].$$

In this way we get the whole information of all higher derivatives of $f_X$.

### 2.3. Algebraic prolongation of $\mathfrak{m}$

Motivated by the above discussion, we now give the definition of the *algebraic prolongation* $\mathfrak{g}(\mathfrak{m})$ of the fundamental graded Lie algebra $\mathfrak{m}$, which was introduced by N. Tanaka [T2].

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of $\mu$-th kind defined over a field $K$. Here $K$ denotes the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. We put

$$\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p(\mathfrak{m}),$$

where $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$ and $\mathfrak{g}_k(\mathfrak{m})$ is defined inductively by (p.k) for $k \geqq 0$. Thus, as a vector space over $K$, $\mathfrak{g}_k(\mathfrak{m})$ is a linear subspace of $\mathrm{End}(\mathfrak{m}, \mathfrak{m}^k) = \mathfrak{m}^k \otimes \mathfrak{m}^*$, where $\mathfrak{m}^k = \mathfrak{m} \oplus \mathfrak{g}_0(\mathfrak{m}) \oplus \cdots \oplus \mathfrak{g}_{k-1}(\mathfrak{m})$. The bracket operation of $\mathfrak{g}(\mathfrak{m})$ is given as follows: First, since $\mathfrak{g}_0(\mathfrak{m})$ is the Lie algebra of all (gradation preserving) derivations of graded Lie algebra $\mathfrak{m}$, we see that $\bigoplus_{p \leqq 0} \mathfrak{g}_p(\mathfrak{m})$ becomes a graded Lie algebra by putting

$$[u, X] = -[X, u] = u(X) \qquad \text{for } u \in \mathfrak{g}_0(\mathfrak{m}) \text{ and } X \in \mathfrak{m}.$$

Similarly, for $u \in \mathfrak{g}_k(\mathfrak{m}) \subset \mathfrak{m}^k \otimes \mathfrak{m}^*$ $(k > 0)$ and $X \in \mathfrak{m}$, we put $[u, X] = -[X, u] = u(X)$ (this justifies our use of $[\,,\,]$ in the previous paragraph). Now, for $u \in \mathfrak{g}_k(\mathfrak{m})$ and $v \in \mathfrak{g}_\ell(\mathfrak{m})$ $(k,\ \ell \geqq 0)$, by induction on the integer $k + \ell \geqq 0$, we define $[u, v] \in \mathfrak{m}^{k+\ell} \otimes \mathfrak{m}^*$ by

$$[u, v](X) = [[u, X], v] + [u, [v, X]] \qquad \text{for } X \in \mathfrak{m}.$$

Here we note that, as the first case $k = \ell = 0$, this definition begins with that of the bracket product in $\mathfrak{g}_0(\mathfrak{m})$. It follows easily that $[u, v] \in \mathfrak{g}_{k+\ell}(\mathfrak{m})$. With this bracket product, $\mathfrak{g}(\mathfrak{m})$ becomes a graded Lie algebra. In fact the Jacobi identity

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0,$$

for $u \in \mathfrak{g}_p(\mathfrak{m})$, $v \in \mathfrak{g}_q(\mathfrak{m})$ and $w \in \mathfrak{g}_r(\mathfrak{m})$, follows by definition when one of $p$, $q$ or $r$ is negative, and can be shown by induction on the integer $p + q + r \geqq 0$, when all of $p$, $q$ and $r$ are non-negative.

Let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}_0(\mathfrak{m})$. We define a sbspace $\mathfrak{g}_k$ of $\mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 1$ inductively by

$$\mathfrak{g}_k = \{\, u \in \mathfrak{g}_k(\mathfrak{m}) \mid [u, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1} \,\}.$$

Then, putting

$$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \bigoplus_{k \geqq 0} \mathfrak{g}_k,$$

we see, with the generating condition of $\mathfrak{m}$, that $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

*Remark* 2.1. The notion of the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$ plays quite an important role in the equivalence problems for the geometric structures subordinate to regular differential systems of type $\mathfrak{m}$, e.g., $CR$-structures, pseudo-product structures or Lie contact structures (cf. [T3], [T5], [S-Y]). We could not touch upon the more important geometric aspect of the prolongation theory of these structures. On these subjects, we refer the reader to foundational papers [T2], [T3], [T4] of N. Tanaka, although we shall discuss some consequences of our results related to [T4] in §5.3.

Now, going back to the discussion in 2.2, we shall see how $\mathfrak{g}(\mathfrak{m})$ describes the structure of $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$, following the argument in §6 of [T2] rather closely. First let us fix a point $x \in M(\mathfrak{m})$. Then $\{f_X^\ell(x)\}_{\ell \geqq -\mu}$ has all the information of higher derivatives of $f_X$ at $x$. Conversely, given an element $a$ of $\mathfrak{g}(\mathfrak{m})$, we can construct an infinitesimal automorphism whose "Taylor expansion" at $x$ coincides with $a$. Namely we have ([T2, Lemma 6.3]):

*Let $a = \sum_{p \leqq k} a^p$ be any element of $\mathfrak{g}(\mathfrak{m})$, where $a^p \in \mathfrak{g}_p(\mathfrak{m})$. Then there is a unique $X \in \mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ such that*

$$\begin{cases} f_X^p(x) = a^p & \text{for } p \leqq k, \\ f_X^\ell \equiv 0 & \text{for } \ell > k. \end{cases}$$

By (2.6), in order to construct such $X$, we need to solve the following differential equations for $\mathfrak{g}_\ell(\mathfrak{m})$-valued functions $u^\ell = f_X^\ell$ $(-\mu \leqq \ell \leqq k)$;

$$du^\ell = \sum_{\ell < s \leqq k} [u^s, \xi^{\ell - s}] \qquad \text{for } \ell = -\mu, \ldots, k,$$

under the condition $u^\ell(x) = a^\ell \in \mathfrak{g}_\ell(\mathfrak{m})$ (here we understand that $\xi^r = 0$ for $r < -\mu$ as before). However this can be accomplished by the Frobenius theorem. In fact, on $M(\mathfrak{m}) \times \mathfrak{m}^{k+1}$, we consider a differential system $E$ defined by

$$\alpha^\ell = du^\ell - \sum_{\ell < s \leqq k} [u^s, \xi^{\ell - s}] \qquad \text{for } \ell = -\mu, \ldots, k,$$

where $u^\ell$ is the linear coordinate on $\mathfrak{g}_\ell(\mathfrak{m})$. Then it follows

$$d\alpha^\ell + \sum_{\ell < s \leqq k} [\alpha^s \wedge \xi^{\ell - s}] = 0.$$

Namely $E$ is completely integrable. Thus, since $M(\mathfrak{m})$ is simply connected, the graph of $(f_X^\ell)_{-\mu\leqq\ell\leqq k}$ is obtained as a leaf of $E$ passing through $(x,a)\in M(\mathfrak{m})\times\mathfrak{m}^{k+1}$. One should note here that, when $a\in\mathfrak{m}$, we actually obtain a right invariant vector field $X$ on $M(\mathfrak{m})$.

Thus, by fixing a point of $M(\mathfrak{m})$, we obtain a linear isomorphism of $\mathfrak{g}(\mathfrak{m})$ into $\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$. For the correspondence of bracket operation, we have ([T2, Lemma 6.4]): For $X$, $Y\in\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$,

$$f^\ell_{[X,Y]} = -\sum_{r+s=\ell}[f^r_X,f^s_Y] \qquad \text{for } \ell\geqq-\mu. \tag{2.7}$$

In fact, (2.7) follows easily from (2.1) and (2.6) when $\ell<0$. Thus, putting $g^\ell=-\sum_{r+s=\ell}[f^r_X,f^s_Y]$, we have $g^p=f^p_{[X,Y]}$ for $p<0$. Moreover, by (2.6), we calculate

$$\begin{aligned} d\,g^\ell &= -\sum_{r+s=\ell}\{[d\,f^r_X,f^s_Y]+[f^r_X,d\,f^s_Y]\} \\ &= -\sum_{s+t+u=\ell}[[f^t_X,\xi^u],f^s_Y]-\sum_{p+q+r=\ell}[f^r_X,[f^p_Y,\xi^q]] \\ &= \sum_{r<0}[g^{\ell-r},\xi^r] \qquad \text{for } \ell\geqq-\mu. \end{aligned}$$

Then, by the definition (2.5) of $f^\ell_{[X,Y]}$ for $\ell\geqq 0$, we conclude $g^\ell=f^\ell_{[X,Y]}$ for $\ell\geqq-\mu$.

In this way the structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$ can be described by $\mathfrak{g}(\mathfrak{m})$. Especially $\mathcal{A}(M(\mathfrak{m}),D_\mathfrak{m})$ is isomorphic with $\mathfrak{g}(\mathfrak{m})$, when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional. In the subsequent sections, we shall be concerned with the following question: When does $\mathfrak{g}(\mathfrak{m})$ or $\mathfrak{g}(\mathfrak{m},\mathfrak{g}_0)$ become finite dimensional and simple?

*Remark* 2.2. (1) In infinite dimensional case, the completion $\overline{\mathfrak{g}}(\mathfrak{m})$ of $\mathfrak{g}(\mathfrak{m})$ gives the formal algebra of the transitive Lie algebra sheaf $\mathcal{A}$ of infinitesimal automorphisms of $(M(\mathfrak{m}),D_\mathfrak{m})$. On this subject, we refer the reader to the further discussion in §6 of [T2].

(2) We remark here that the discussions in §§1 and 2 are valid also in the complex analytic category. Thus, for a fundamental graded Lie algebra $\mathfrak{m}=\bigoplus_{p<0}\mathfrak{g}_p$ over $\mathbb{C}$, the standard differential system $(M(\mathfrak{m}),D_\mathfrak{m})$ of type $\mathfrak{m}$ is a holomorphic differential system on a complex Lie group $M(\mathfrak{m})$. Furthermore the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ over $\mathbb{C}$ describes the stalk of the Lie algebra sheaf $\mathcal{A}$ of holomorphic infinitesimal automorphisms of $(M(\mathfrak{m}),D_\mathfrak{m})$.

### 2.4. Generarized Spencer cohomology

We now give some remarks on the algebraic prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$. First $\mathfrak{g}(\mathfrak{m})$ is characterized as the graded Lie algebra which satisfies the following conditions:

(1) $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$, where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.
(2) For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.
(3) $\mathfrak{g}(\mathfrak{m})$ is maximum among graded algebras satisfying conditions (1) and (2) above.

More precisely, let $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ be any graded algebra satisfying (1) and (2). Then $\mathfrak{h}$ is imbedded in $\mathfrak{g}(\mathfrak{m})$ as a graded subalgebra.

In fact (1) and (2) are obvious. The imbedding $\iota$ of $\mathfrak{h}$ into $\mathfrak{g}(\mathfrak{m})$ is obtained as follows: Since $\bigoplus_{p\leqq 0} \mathfrak{h}_p = \mathfrak{m} \oplus \mathfrak{h}_0$ is a graded subalgebra, we get a homomorphism $\iota_0$ of $\mathfrak{h}_0$ into $\mathfrak{g}_0(\mathfrak{m})$, which is injective by condition (2) above. Then, by definition (p.k) of $\mathfrak{g}_k(\mathfrak{m})$, we obtain a linear map $\iota_k$ of $\mathfrak{h}_k$ into $\mathfrak{g}_k(\mathfrak{m})$ by induction on $k \geqq 1$, which is also injective by (2). $\iota$ is obviously a homomorphism.

In the presence of the generating condition of $\mathfrak{m}$, the condition (2) above is equivalent to the following condition:

For $k \geqq 0$, if $X \in \mathfrak{g}_k(\mathfrak{m})$ and $[X, \mathfrak{g}_{-1}] = \{0\}$, then $X = 0$.

From this, it follows that $\mathfrak{g}_{k+1}(\mathfrak{m}) = \{0\}$ if $\mathfrak{g}_k(\mathfrak{m}) = \{0\}$, that is, $\mathfrak{g}_\ell(\mathfrak{m}) = \{0\}$ for $\ell \geqq k$ if $\mathfrak{g}_k(\mathfrak{m}) = \{0\}$. Hence $\mathfrak{g}(\mathfrak{m})$ becomes finite dimensional if and only if $\mathfrak{g}_k(\mathfrak{m}) = \{0\}$ for some $k \geqq 0$.

Now we shall turn to another characterization of $\mathfrak{g}(\mathfrak{m})$. First, recall that the prolongation $\mathfrak{g}^{(k)}$ of a linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is defined also by the following exact sequence;

$$0 \to \mathfrak{g}^{(k+1)} \to C^{k+1,1} = \mathfrak{g}^{(k)} \otimes V^* \xrightarrow{\partial} C^{k,2} = \mathfrak{g}^{(k-1)} \otimes \wedge^2 V^*,$$

where the coboundary operator $\partial\colon C^{k+1,1} \to C^{k,2}$ is given by

$$(\partial p)(X, Y) = [p(X), Y] - [p(Y), X].$$

In the same way, we can define $\mathfrak{g}_k(\mathfrak{m})$ as follows. First we decompose $\bigwedge^2 \mathfrak{m}^* = \bigoplus_{j<-1} \wedge_j^2 \mathfrak{m}^*$ according to the gradation $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, where

$$\wedge_j^2 \mathfrak{m}^* = \bigoplus_{p+q=j} \mathfrak{g}_p^* \wedge \mathfrak{g}_q^*.$$

Putting $C^{k,1} = \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^*$ and $C^{k-1,2} = \bigoplus_{j<-1} \mathfrak{g}_{j+k} \otimes \wedge_j^2 \mathfrak{m}^*$, we can define $\mathfrak{g}_k = \mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 0$ inductively by the following exact sequence;

$$0 \to \mathfrak{g}_k \to C^{k,1} \xrightarrow{\partial} C^{k-1,2},$$

where the coboundary operator $\partial\colon C^{k,1} \to C^{k-1,2}$ is given by

$$(\partial p)(X,Y) = [X, p(Y)] - [Y, p(X)] - p([X,Y]).$$

We shall utilize this characterization in the following situation. Let $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{h}_p$ is a fundamental graded algebra of $\mu$-th kind. To check whether $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{h}_0)$, we consider the Lie algebra cohomology $H^q(\mathfrak{m}, \mathfrak{h})$ associated with the representation ad: $\mathfrak{m} \to \mathfrak{gl}(\mathfrak{h})$. Namely, putting $C(\mathfrak{m}, \mathfrak{h}) = \bigoplus C^q(\mathfrak{m}, \mathfrak{h})$, $C^q(\mathfrak{m}, \mathfrak{h}) = \mathfrak{h} \otimes \bigwedge^q \mathfrak{m}^*$, we have the coboundary operator $\partial\colon C^q \to C^{q+1}$;

$$\begin{aligned}(\partial p)(X_1, \ldots, X_{q+1}) &= \sum_i (-1)^{i+1} [X_i, p(X_1, \ldots, \check{X}_i, \ldots, X_{q+1})] \\ &\quad + \sum_{i<j} (-1)^{i+j} p([X_i, X_j], X_1, \ldots, \check{X}_i, \ldots, \check{X}_j, \ldots, X_{q+1}),\end{aligned}$$

for $p \in C^q(\mathfrak{m}, \mathfrak{h})$ and $X_1, \ldots, X_{q+1} \in \mathfrak{m}$. $H^q(\mathfrak{m}, \mathfrak{h})$ is the cohomology group of this cochain complex $(C(\mathfrak{m}, \mathfrak{h}), \partial)$. According to the gradation of $\mathfrak{h}$, this complex has a bigradation given as follows ([T4, §1]): First $\bigwedge^q \mathfrak{m}^*$ has the decomposition $\bigwedge^q \mathfrak{m}^* = \bigoplus_{j \leqq -q} \wedge^q_j \mathfrak{m}^*$, where

$$\wedge^q_j \mathfrak{m}^* = \bigoplus_{i_1 + \cdots + i_q = j} \mathfrak{h}^*_{i_1} \wedge \cdots \wedge \mathfrak{h}^*_{i_q}.$$

Then the bigradation of $C(\mathfrak{m}, \mathfrak{h})$ is introduced by

$$C^{p,q}(\mathfrak{m}, \mathfrak{h}) = \bigoplus_{j \leqq -q} \mathfrak{h}_{j+p+q-1} \otimes \wedge^q_j \mathfrak{m}^*.$$

Here we note that

$$C^{p,0} = \mathfrak{h}_{p-1}, \; C^{p,1} = \bigoplus_{j<0} \mathfrak{h}_{j+p} \otimes \mathfrak{h}^*_j, \; C^{p,2} = \bigoplus_{j<-1} \mathfrak{h}_{j+p+1} \otimes \wedge^2_j \mathfrak{m}^*$$

and $\partial$ sends $C^{p,q}$ into $C^{p-1,q+1}$. With this bigradation,

$$H^q(\mathfrak{m}, \mathfrak{h}) = \bigoplus_p H^{p,q}(\mathfrak{m}, \mathfrak{h})$$

is called the *generalized Spencer cohomology group* of the graded Lie algebra $\mathfrak{h}$.

Utilizing this cohomology group, we have (cf. [T4, Lemma 1.14])

**Lemma 2.1.** *Let $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ be a graded Lie algebra such that $\mathfrak{h}_p = [\mathfrak{h}_{p+1}, \mathfrak{h}_{-1}]$ for $p < -1$. Then $\mathfrak{h}$ is the prolongation of $\mathfrak{m}$ (resp. of $(\mathfrak{m}, \mathfrak{h}_0)$) if and only if the following two conditions hold:*

(1) *For $k \geqq 0$, if $X \in \mathfrak{h}_k$ and $[X, \mathfrak{m}] = \{0\}$, then $X = 0$.*

(2) *$H^{p,1}(\mathfrak{m}, \mathfrak{h}) = \{0\}$ for $p \geqq 0$ (resp. $p \geqq 1$).*

With this criterion in mind, in order to answer the question posed at the end of 2.3, we proceed as follows: First, for a (finite dimensional) simple Lie algebra $\mathfrak{g}$, we shall classify, in §3, the gradations $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $\mathfrak{g}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Then we calculate $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ by the method of Kostant [K] in §5.2.

## §3. Simple graded Lie algebras

### 3.1. Semisimple graded Lie algebras

We begin with generalities of semisimple graded Lie algebras (cf. [Hu], [K-N], [T4]). Let $\mathfrak{g}$ be a (finite dimensional) semisimple Lie algebra over $\mathbb{R}$. A gradation of $\mathfrak{g}$ is a direct decomposition $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that

$$[\mathfrak{g}_p, \mathfrak{g}_q] \in \mathfrak{g}_{p+q} \qquad \text{for } p,\ q \in \mathbb{Z}.$$

As is well-known, there exists a unique element $E \in \mathfrak{g}_0$ such that

$$\mathfrak{g}_p = \{\, X \in \mathfrak{g} \mid [E, X] = pX \,\} \qquad \text{for } p \in \mathbb{Z}.$$

In fact, for a graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, we have a derivation $D$ of $\mathfrak{g}$ given by $D(X) = pX$ for $X \in \mathfrak{g}_p$. Then, since $\mathfrak{g}$ is semisimple, there exists a unique $E \in \mathfrak{g}$ such that $D = \operatorname{ad}(E)$. Obviously we have $E \in \mathfrak{g}_0$. In particular $\mathfrak{g}_0 \neq \{0\}$. $E$ is called the *characteristic element* of the semisimple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$.

Moreover we get easily ([K2, p. 131, Proposition 4.1], [T4, Lemma 1.2])

**Lemma 3.1.**

(1) *$B(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ if $p + q \neq 0$.*

(2) *The restriction of the Killing form $B$ to $\mathfrak{g}_p \times \mathfrak{g}_{-p}$ is non-degenerate if $\mathfrak{g}_p \neq \{0\}$.*

Namely gradations of a semisimple Lie algebra $\mathfrak{g}$ are always symmetric, that is, $\mathfrak{g}_p \neq \{0\}$ if and only if $\mathfrak{g}_{-p} \neq \{0\}$ and the Killing form $B$ gives a duality between $\mathfrak{g}_p$ and $\mathfrak{g}_{-p}$. The largest integer $\mu$ such that

$\mathfrak{g}_\mu \neq \{0\}$ is called the *depth* of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Furthermore we see that $\mathfrak{g}$ is non-compact if the gradation is not trivial, that is, if $\mathfrak{g}_p \neq \{0\}$ for some $p \neq 0$.

Now we consider the decomposition of $\mathfrak{g}$ into simple ideals;

$$\mathfrak{g} = \bigoplus_s \mathfrak{g}^s.$$

Then the characteristic element $E$ decomposes as $E = \sum_s E^s$. For $X \in \mathfrak{g}_p$, we have $X = \sum_s X^s$. Thus, from $pX = [E, X] = \sum_s [E^s, X^s]$, we get $[E^s, X^s] = pX^s$. Namely $E^s$ defines a gradation $\mathfrak{g}^s = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^s_p$ of $\mathfrak{g}^s$, where $\mathfrak{g}^s_p = \mathfrak{g}^s \cap \mathfrak{g}_p$ and

$$\mathfrak{g}_p = \bigoplus_s \mathfrak{g}^s_p.$$

Therefore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a direct sum of simple graded Lie algebras $\mathfrak{g}^s = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^s_p$.

A graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called effective if $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ contains no ideals of $\mathfrak{g}$. Then, by the above argument, we see that $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is effective if and only if none of simple ideals $\mathfrak{g}^s = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^s_p$ has a trivial gradation.

Some conditions on the gradation forces $\mathfrak{g}$ to be a simple graded Lie algebra. Among these, we quote here the following two conditions: A gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called a *contact gradation* if $\mathfrak{g}$ is effective and $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies

(1) $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ such that $\dim \mathfrak{g}_{-2} = 1$.

(2) The bracket operation $[\, ,\, ]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is nondegenerate.

In fact it follows from (1) that there exists a unique ideal $\mathfrak{g}^{s_o}$ such that $\mathfrak{g}_{-2} = \mathfrak{g}^{s_o}_{-2}$ and that $\mathfrak{g}^s = \mathfrak{g}^s_{-1} \oplus \mathfrak{g}^s_0 \oplus \mathfrak{g}^s_1$ for $s \neq s_o$. Then condition (2) forces $\mathfrak{g}^s_{-1} = \{0\}$ for $s \neq s_o$. Thus the effectiveness of $\mathfrak{g}$ implies $\mathfrak{g} = \mathfrak{g}^{s_o}$. We shall see later in §4 that each simple Lie algebra over $\mathbb{C}$ has a unique complex contact gradation up to conjugacy.

A gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, is called *primitive* if $\mathfrak{g}$ is effective and $\mathrm{ad}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is irreducible. It follows easily that $\mathfrak{g}$ is simple if it is primitive. More generally we shall discuss primitive actions of finite dimensional Lie groups in §5.4.

For simple graded Lie algebras, we prepare (cf. [T4, Lemmas 1.3, 1.6])

**Lemma 3.2.** *Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ of depth $\mu$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental.*

*Then, for every $p > -\mu$,*

(1) *If $X \in \mathfrak{g}_p$ and $[X, \mathfrak{g}_{-1}] = \{0\}$, then $X = 0$.*

(2) $\mathfrak{g}_p = [\mathfrak{g}_{p-1}, \mathfrak{g}_1]$.

*In particular the centralizer $Z_{\mathfrak{g}}(\mathfrak{m})$ of $\mathfrak{m}$ in $\mathfrak{g}$ coincides with $\mathfrak{g}_{-\mu}$.*

*Proof.* From the generating condition of $\mathfrak{m}$: $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, it follows $\mathfrak{g}_k = [\mathfrak{g}_{k-1}, \mathfrak{g}_1]$ for $k > 1$ (for this fact, see 3.3). Then we see that a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ is an ideal of $\mathfrak{g}$ if $\mathfrak{a}$ is $\mathrm{ad}(\mathfrak{g}_i)$-invariant for $i = -1$, 0, 1.

Now let us fix an integer $q$ $(-\mu \leqq q \leqq \mu)$ and put

$$\mathfrak{a}^q(q) = \{\, X \in \mathfrak{g}_q \mid [X, \mathfrak{g}_{-1}] = 0 \,\}.$$

We define a linear subspace $\mathfrak{a}^q = \bigoplus_{p=q}^{\mu} \mathfrak{a}^q(p)$ of $\mathfrak{g}$ inductively by

$$\mathfrak{a}^q(p+1) = [\mathfrak{a}^q(p), \mathfrak{g}_1] \subset \mathfrak{g}_{p+1}.$$

By the Jacobi identity, we see that $\mathfrak{a}^q(q)$ is $\mathrm{ad}(\mathfrak{g}_0)$-invariant. Moreover one can check that $\mathfrak{a}^q(p)$ is $\mathrm{ad}(\mathfrak{g}_0)$-invariant and $[\mathfrak{a}^q(p+1), \mathfrak{g}_{-1}] \subset \mathfrak{a}^q(p)$, by induction on $p \geqq q$. Thus $\mathfrak{a}^q$ is an ideal of $\mathfrak{g}$. When $q > -\mu$, $\mathfrak{a}^q$ is a proper ideal of $\mathfrak{g}$. Hence, by the simplicity of $\mathfrak{g}$, $\mathfrak{a}^q = \{0\}$, which proves (1). When $q = -\mu$, we have $\mathfrak{a}^{-\mu}(-\mu) = \mathfrak{g}_{-\mu}$. Hence $\mathfrak{a}^{-\mu} = \mathfrak{g}$, which implies (2).

This lemma shows, in particular, that condition (1) of Lemma 2.1 in §2.4 is always satisfied by a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. In other words, $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a graded subalgebra of the prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$.

### 3.2. Complexification of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$. Let $\mathbb{C}\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ be the complexification of $\mathfrak{g}$. Then $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathbb{C}\mathfrak{g}_p$ becomes a semisimple graded Lie algebra over $\mathbb{C}$. First we recall the following fact (cf. [He, p. 443, Proposition 1.5]).

*The simple Lie algebras over $\mathbb{R}$ fall into two disjoint classes:*

A. *The simple Lie algebras over $\mathbb{C}$, considered as real Lie algebras.*

B. *The real forms of simple Lie algebras over $\mathbb{C}$.*

More precisely, a real simple Lie algebra $\mathfrak{g}$ belongs to class A if $\mathbb{C}\mathfrak{g}$ is not simple and there exists a complex structure $J$ on $\mathfrak{g}$ such that $(\mathfrak{g}, J)$ is a simple Lie algebra over $\mathbb{C}$. In this case we have

$$\mathbb{C}\mathfrak{g} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0} = \{\, X - \sqrt{-1}JX \mid X \in \mathfrak{g} \,\}$ and $\mathfrak{g}^{0,1} = \{\, X + \sqrt{-1}JX \mid X \in \mathfrak{g} \,\}$ are simple ideals of $\mathbb{C}\mathfrak{g}$, which are isomorphic with $(\mathfrak{g}, J)$.

When a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ belongs to class A, we note that, since $\mathrm{ad}(E)\cdot J = J\cdot\mathrm{ad}(E)$ for the characteristic element $E$, $\mathfrak{g}_p = \{\, X \in \mathfrak{g} \mid [E, X] = pX \,\}$ is a complex subspace of $(\mathfrak{g}, J)$. Namely, for a real simple Lie algebra $\mathfrak{g}$ of class A, any gradation of $\mathfrak{g}$ as a real Lie algebra is in fact a gradation as a complex Lie algebra. Thus we obtain

**Proposition 3.3.** *The simple graded Lie algebras $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ over $\mathbb{R}$ fall into two disjoint classes:*

A. *The simple graded Lie algebras over $\mathbb{C}$, considered as real graded Lie algebras.*

B. *The real forms of simple Lie algebras over $\mathbb{C}$ so that $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ become simple graded Lie algebras over $\mathbb{C}$.*

Now we give some remarks on the generalized Spencer cohomology of $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathbb{C}\mathfrak{g}_p$. We denote by $H^q_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$ the complex cohomology group associated with the complex representation $\mathrm{ad}\colon \mathbb{C}\mathfrak{m} \to \mathfrak{gl}(\mathbb{C}\mathfrak{g})$. Namely we consider $C^q_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g}) = \mathbb{C}\mathfrak{g} \otimes_{\mathbb{C}} \wedge^q \mathbb{C}\mathfrak{m}^*$, which is naturally identified with the complexification $\mathbb{C}C^q(\mathfrak{m}, \mathfrak{g}) = \mathbb{C} \otimes_{\mathbb{R}} C^q(\mathfrak{m}, \mathfrak{g})$ of $C^q(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{R}} \wedge^q \mathfrak{m}^*$. Under this identification, the coboundary operator $\partial\colon C^q_{\mathbb{C}} \to C^{q+1}_{\mathbb{C}}$ is a real operator. Hence $H^q_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$ is naturally identified with the complexification $\mathbb{C}H^q(\mathfrak{m}, \mathfrak{g})$ of $H^q(\mathfrak{m}, \mathfrak{g})$. The bigradation is also preserved under this identification. Thus, by Lemma 3.2 and Lemma 2.1 in §2.4, we have

**Lemma 3.4.** *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental.*

*Then $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ (resp. of $(\mathfrak{m}, \mathfrak{g}_0)$) if and only if $H^{p,1}_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g}) = \{0\}$ for $p \geqq 0$ (resp. $p \geqq 1$).*

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be of class A, that is, a simple graded Lie algebra over $\mathbb{C}$. In this case we have two cohomology groups $H^q_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g})$ and $H^q_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ associated with $\mathrm{ad}\colon \mathfrak{m} \to \mathfrak{gl}_{\mathbb{C}}(\mathfrak{g}) \subset \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$. Namely $H^q_{\mathbb{C}}$ is obtained from the cochain complex $(C_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g}), \partial)$, $C_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{C}} \wedge \mathfrak{m}^*$, whereas $H^q_{\mathbb{R}}$ is obtained from the cochain complex $(C_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g}), \partial)$, $C_{\mathbb{R}} = \mathfrak{g} \otimes_{\mathbb{R}} \wedge \mathfrak{m}^*$.

From the complex structure $J$ on $\mathfrak{g}$, $C_{\mathbb{R}}$ inherits a complex structure $J \otimes_{\mathbb{R}} \mathrm{id}$ such that $\partial$ is complex linear. Hence $H^q_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ is a complex vector space. Then we have

**Lemma 3.5.** *Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ of depth $\mu$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Then*

(1) *$H^{p,1}_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g})$ and $H^{p,1}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ are isomorphic for $p > 0$.*

(2) *$H^{0,1}_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g})$ and $H^{0,1}_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ are isomorphic when $\mu > 1$.*

*Proof.* Since $\mathbb{C}H^1_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ is isomorphic with $H^1_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$, we first calculate $H^1_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g})$. Utilizing the decomposition $\mathbb{C}\mathfrak{g} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$, we have

$$C_{\mathbb{C}}(\mathbb{C}\mathfrak{m}, \mathbb{C}\mathfrak{g}) = \mathfrak{g}^{1,0} \otimes \wedge \mathbb{C}\mathfrak{m}^* \oplus \mathfrak{g}^{0,1} \otimes \wedge \mathbb{C}\mathfrak{m}^*.$$

This is the eigenspace decomposition of the complex structure $J \otimes_{\mathbb{R}} \mathrm{id}$. Obviously $\partial$ preserves this decomposition. Thus to calculate $H^1_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$, we need only to calculate the cohomology of $(\bar{C}, \partial)$, where $\bar{C} = \mathfrak{g}^{1,0} \otimes \wedge \mathbb{C}\mathfrak{m}^*$. Moreover we have the decomposition of $\wedge \mathbb{C}\mathfrak{m}^*$;

$$\wedge^q \mathbb{C}\mathfrak{m}^* = \bigoplus_{r+s=q} \wedge^{r,s} \mathbb{C}\mathfrak{m}^*,$$

which is induced from $\mathbb{C}\mathfrak{m} = \mathfrak{m}^{1,0} \oplus \mathfrak{m}^{0,1}$. Thus we have $\bar{C}^0 = \mathfrak{g}^{1,0}$, $\bar{C}^1 = \mathfrak{g}^{1,0} \otimes (\wedge^{1,0}\mathfrak{m}^* \oplus \wedge^{0,1}\mathfrak{m}^*)$ and $\bar{C}^2 = \mathfrak{g}^{1,0} \otimes (\wedge^{2,0}\mathfrak{m}^* \oplus \wedge^{1,1}\mathfrak{m}^* \oplus \wedge^{0,2}\mathfrak{m}^*)$. Then, from $[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = \{0\}$, we get

$$\begin{aligned} \partial\, \mathfrak{g}^{1,0} &\subset \mathfrak{g}^{1,0} \otimes \wedge^{1,0}\mathfrak{m}^*, \\ \partial\, (\mathfrak{g}^{1,0} \otimes \wedge^{1,0}\mathfrak{m}^*) &\subset \mathfrak{g}^{1,0} \otimes \wedge^{2,0}\mathfrak{m}^*, \\ \partial\, (\mathfrak{g}^{1,0} \otimes \wedge^{0,1}\mathfrak{m}^*) &\subset \mathfrak{g}^{1,0} \otimes (\wedge^{1,1}\mathfrak{m}^* \oplus \wedge^{0,2}\mathfrak{m}^*). \end{aligned}$$

Here we note that $\mathfrak{g}^{1,0} \otimes \wedge^{1,0}\mathfrak{m}^*$ (resp. $\mathfrak{g}^{1,0} \otimes \wedge^{0,1}\mathfrak{m}^*$) is naturally identified with the space of complex linear (resp. conjugate linear) mappings of $(\mathfrak{m}, J)$ into $(\mathfrak{g}, J)$. Hence $H^1_{\mathbb{R}}(\mathfrak{m}, \mathfrak{g})$ is isomorphic with

$$H^1_{\mathbb{C}}(\mathfrak{m}, \mathfrak{g}) \oplus \bar{Z},$$

where $\bar{Z} = \{p\colon \mathfrak{m} \to \mathfrak{g};\ \text{conjugate linear} \mid \partial p = 0\}$. For a conjugate linear map $p\colon \mathfrak{m} \to \mathfrak{g}$, we calculate

$$\begin{aligned} (\partial p)(JX, Y) &= [JX, p(Y)] - [Y, p(JX)] - p([JX, Y]) \\ &= J\{[X, p(Y)] + [Y, p(X)] + p([X, Y])\}, \end{aligned}$$

for $X$, $Y \in \mathfrak{m}$. Hence $\partial p = 0$ if and only if $[X, p(Y)] = 0$ and $p([X,Y]) = 0$ for $X$, $Y \in \mathfrak{m}$. Then, by Lemma 3.2, we get

$$\bar{Z} = \{\, p\colon \mathfrak{m} \to \mathfrak{g};\ \text{conjugate linear} \mid p(\mathfrak{g}_{-1}) \subset \mathfrak{g}_{-\mu} \text{ and } p(\mathfrak{g}_q) = \{0\} \text{ for } q < -1 \,\}.$$

Therefore we obtain

$$\bar{Z} \subset \mathfrak{g}_{-\mu} \otimes \mathfrak{g}_{-1}^* \subset C_{\mathbb{R}}^{-\mu+1,1}(\mathfrak{m}, \mathfrak{g}),$$

which completes the proof.

Thus, if $H_{\mathbb{C}}^{p,1}(\mathfrak{m}, \mathfrak{g}) = \{0\}$ for $p \geqq 0$ $(\mu > 1)$, a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ over $\mathbb{C}$, such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, is the prolongation of $\mathfrak{m}$ as a graded Lie algebra over $\mathbb{R}$ as well as over $\mathbb{C}$. In this case, the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ is a holomorphic differential system on a complex Lie group $M(\mathfrak{m})$. Then Lemma 3.5 implies that, if $H_{\mathbb{C}}^{p,1}(\mathfrak{m}, \mathfrak{g}) = \{0\}$ for $p \geqq 0$, every real infinitesimal automorphism of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is necessarily holomorphic.

In view of the discussion in this paragraph, we shall be mainly concerned with simple graded Lie algebras over $\mathbb{C}$ in the susequent discussion.

### 3.3. Gradation and the root space decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. We shall describe the gradation of $\mathfrak{g}$ in terms of the root space decomposition of $\mathfrak{g}$. Our standard reference in this section are [Hu] and [He].

Let $E$ be the characteristic element of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. Since $\mathrm{ad}(E)$ is a semisimple endomorphism of $\mathfrak{g}$, we can take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $E \in \mathfrak{h}$. Let $\Phi$ be the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then we have the root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid [H, X] = \alpha(H) X \text{ for all } H \in \mathfrak{h} \,\}$ is the root space for $\alpha \in \Phi$. It follows from $E \in \mathfrak{h}$ that

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha,$$

$$\mathfrak{g}_p = \bigoplus_{\alpha \in \Phi_p} \mathfrak{g}_\alpha \quad (p \neq 0),$$

where $\Phi_p = \{\, \alpha \in \Phi \mid \alpha(E) = p \,\}$. Moreover, since $\alpha(E) \in \mathbb{Z}$ for $\alpha \in \Phi$, $E$ belongs to the real part $\mathfrak{h}_{\mathbb{R}} = \{\, X \in \mathfrak{h} \mid \alpha(X) \in \mathbb{R}$ for $\alpha \in \Phi \,\}$ of $\mathfrak{h}$. Let $\mathfrak{h}^\sharp = \langle \Phi \rangle_{\mathbb{R}}$ be the real linear subspace of $\mathfrak{h}^*$ spanned by all roots of $\mathfrak{g}$. Identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ by the Killing form $B$ of $\mathfrak{g}$, we know that $\mathfrak{h}^\sharp$ corresponds to $\mathfrak{h}_{\mathbb{R}}$ and that the Killing form $B$ gives a positive definite inner product $(\ ,\ )$ on $\mathfrak{h}_{\mathbb{R}}$. Then, by fixing a Weyl chamber $D$ of $\mathfrak{h}_{\mathbb{R}}$ such that its closure $\bar{D}$ contains $E$, we can choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of $\Phi$ such that $\alpha(E) \geqq 0$ for all $\alpha \in \Delta$. Then $E$ determines a partition $\Phi^+ = \cup_{k \geqq 0} \Phi_k^+$ of the set $\Phi^+$ of positive roots by $\Phi_k^+ = \{\, \alpha \in \Phi^+ \mid \alpha(E) = k \,\}$ such that

$$
\begin{aligned}
&\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \\
&\mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \qquad (k > 0).
\end{aligned} \tag{3.1}
$$

This explains the symmetry of gradations of semisimple graded Lie algebras. Here we note that $\Phi_0 = \{\, \alpha \in \Phi \mid \alpha(E) = 0 \,\}$ forms a subsystem of the root system $\Phi$ with a simple root system $\Delta_0 = \{\, \alpha \in \Delta \mid \alpha(E) = 0 \,\}$.

Conversely let us fix a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ and choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system $\Phi$ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then, given a $\ell$-tuple $(a_1, \ldots, a_\ell)$ of non-negative integers, we see that an element $E \in \mathfrak{h}_{\mathbb{R}}$, which is defined by $\alpha_i(E) = a_i$, gives a gradation $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $\mathfrak{g}$ such that (3.1) holds.

With the above choice of $\mathfrak{h}$ and $\Delta$, putting $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$, we have

$$
\mathfrak{g}' = \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha = \bigoplus_{\alpha \in \Phi_0^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{B}(\Delta),
$$

where $\mathfrak{B}(\Delta) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is a standard Borel subalgebra of $\mathfrak{g}$ relative to $\mathfrak{h}$ ([Hu, Chapter IV]). Hence $\mathfrak{g}'$ is a parabolic subalgebra of $\mathfrak{g}$. In fact $\mathfrak{g}' = \mathfrak{P}(\Delta_0)$ is the standard parabolic subalgebra corresponding to $\Delta_0$. For the subalgebra $\mathfrak{g}_0$, we have

**Proposition 3.6.** $\mathfrak{g}_0$ *is a reductive Lie algebra such that*

(1) *Dimension of the center* $Z(\mathfrak{g}_0)$ *of* $\mathfrak{g}_0$ *is equal to the number of simple roots in* $\Delta \setminus \Delta_0$.

(2) $[\mathfrak{g}_0, \mathfrak{g}_0]$ *is a semisimple Lie algebra with the root system* $\Phi_0$ *and is a Levi subalgebra of* $\mathfrak{g}'$.

*Proof.* Let $\mathfrak{h}_0 = \langle \Delta_0 \rangle_{\mathbb{C}}$ be the linear subspace of $\mathfrak{h}^*$ spanned by elements of $\Delta_0$. Identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ via the Killing form duality, we have an orthogonal decomposition of $\mathfrak{h}$;

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_0^{\perp},$$

which in fact arises from an orthogonal decomposition in $\mathfrak{h}_{\mathbb{R}}$. Then we have $[\mathfrak{h}_0^{\perp}, \mathfrak{g}_0] = \{0\}$ and

$$[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Thus, by Serre's Theorem ([Hu, Theorem 18.3]), $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a semisimple Lie algebra with a simple root system $\Delta_0$. Hence we have $\mathfrak{h}_0^{\perp} = Z(\mathfrak{g}_0)$.

*Remark* 3.7. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a semisimple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is called primitive if $\mathfrak{g}$ is effective and $\mathrm{ad}\colon \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is irreducible. If $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is primitive, then $\mathfrak{g}$ is simple and it follows from Schur's Lemma that $\dim Z(\mathfrak{g}_0) = 1$. Then, by Proposition 3.6, $\mathfrak{g}'$ is a maximal parabolic subalgebra. In fact $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is primitive if and only if $\mathfrak{g}$ is simple and $\mathfrak{g}'$ is a maximal parabolic subalgebra of $\mathfrak{g}$ (cf. the proof of Lemma 3.8. See also §5.4).

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a semisimple graded Lie algebra over $\mathbb{R}$. In the real case, we should start with a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

of $\mathfrak{g}$ such that $E \in \mathfrak{p}$ (cf. [M]). In fact such a Cartan decomposition can be found by Theorem 7.1 of [He, p. 182]. We first take a (complex) Cartan subalgebra $\widehat{\mathfrak{h}}$ of $\mathbb{C}\mathfrak{g}$ such that $E \in \widehat{\mathfrak{h}}$. Moreover we take a compact real form $\mathfrak{u}$ of $\mathbb{C}\mathfrak{g}$ by choosing a Weyl basis of $\mathfrak{g} = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \widehat{\Phi}} \mathfrak{g}_\alpha$. Then we have $E \in \widehat{\mathfrak{h}}_{\mathbb{R}} \subset \sqrt{-1}\mathfrak{u}$. Let $\sigma$ and $\tau$ denote the conjugations of $\mathbb{C}\mathfrak{g}$ with respect to $\mathfrak{g}$ and $\mathfrak{u}$ respectively. Putting $N = \sigma \cdot \tau$, we have $N(E) = -E$. Hence $P(E) = E$ for $P = N^2$. By Theorem 7.1 of [He], a Cartan decomposition of $\mathfrak{g}$ is obtained by putting

$$\mathfrak{k} = \mathfrak{g} \cap \varphi(\mathfrak{u}),$$
$$\mathfrak{p} = \mathfrak{g} \cap \varphi(\sqrt{-1}\mathfrak{u}),$$

where $\varphi = P^{\frac{1}{4}}$. Then, from $\varphi(E) = E$, we see that $E \in \mathfrak{p}$. Here we note that, from $\tau_o(E) = -E$, the conjugation $\tau_o$ with respect to $\varphi(\mathfrak{u})$ reverses the gradation of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, that is, $\tau_o(\mathfrak{g}_p) = \mathfrak{g}_{-p}$.

Let us take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ such that $E \in \mathfrak{a}$. Moreover let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathbb{C}\mathfrak{h}$ is a Cartan subalgebra of $\mathbb{C}\mathfrak{g}$ such that $\mathfrak{a} = (\mathbb{C}\mathfrak{h})_{\mathbb{R}} \cap \mathfrak{g}$ ([He, p. 259, Lemma 3.2]). Hence the root space decomposition $\mathbb{C}\mathfrak{g} = \mathbb{C}\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ of $\mathbb{C}\mathfrak{g}$ or more directly the simultaneous diagonalization of $\operatorname{ad}_{\mathfrak{g}}(\mathfrak{a})$ induces the restricted root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda,$$

where $Z(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and $\Sigma$ is the set of restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ ([He, p. 263]). A restricted root $\lambda \in \mathfrak{a}^*$ is a non-zero linear form on $\mathfrak{a}$ obtained as the restriction of some root $\alpha \in \Phi \subset (\mathbb{C}\mathfrak{h})^*$ to the subspace $\mathfrak{a}$ of $(\mathbb{C}\mathfrak{h})_{\mathbb{R}}$. $\Sigma$ forms a root system in $\mathfrak{a}^*$, which in general is not reduced ([He, Chapter VII]). Thus, by fixing a Weyl chamber $D$ of $\mathfrak{a}$ such that $E \in \bar{D}$, we have a simple root system $\widehat{\Delta} = \{\lambda_1, \ldots, \lambda_p\}$ of $\Sigma$ such that $\lambda_i(E) \geqq 0$ for $\lambda_i \in \widehat{\Delta}$. Then the gradation of $\mathfrak{g}$ can be described as

$$\mathfrak{g}_0 = Z(\mathfrak{a}) \oplus \bigoplus_{\lambda \in \Sigma_0^+} (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda}),$$

$$\mathfrak{g}_k = \bigoplus_{\lambda \in \Sigma_k^+} \mathfrak{g}_\lambda, \quad \mathfrak{g}_{-k} = \bigoplus_{\lambda \in \Sigma_k^+} \mathfrak{g}_{-\lambda} \qquad (k > 0),$$

where $\Sigma_k^+ = \{\, \lambda \in \Sigma^+ \mid \lambda(E) = k \,\}$. For the details, we refer the reader to [K-A].

### 3.4. Generating condition of $\mathfrak{m}$

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$. As in the previous paragraph, let us fix a Cartan subalgebra $\mathfrak{h}$ and a simple root system $\Delta$ such that $E \in \mathfrak{h}$ and $\alpha(E) \geqq 0$ for any $\alpha \in \Delta$. Then, for the generating condition of $\mathfrak{m}$, we have (cf. [K-A, Lemma 2.3])

**Lemma 3.8.** *$\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$, if and only if $\alpha(E) = 0$ or $1$ for any $\alpha \in \Delta$.*

*Proof.* We have $\mathfrak{g}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha}$ for $k > 0$, where $\Phi_k^+ = \{\alpha \in \Phi^+ \mid \alpha(E) = k\}$ and $\Phi^+ = \cup_{k \geqq 0} \Phi_k^+$. Then it follows that $\mathfrak{g}_{-(k+1)} = [\mathfrak{g}_{-k}, \mathfrak{g}_{-1}]$ if and only if each $\alpha \in \Phi_{k+1}^+$ can be written as a sum $\alpha = \beta + \gamma$ of some $\beta \in \Phi_k^+$ and $\gamma \in \Phi_1^+$. Hence $\mathfrak{m}$ satisfies the generating condition if and only if each $\alpha \in \Phi_k^+$ can be written as a sum of $k$ elements of $\Phi_1^+$.

Therefore, if $\mathfrak{m}$ satisfies the generating condition, every simple root must belong to $\Phi_1^+$ or $\Phi_0^+$.

Conversely assume that $\alpha(E) = 0$ or $1$ for any $\alpha \in \Delta$. We start with the following property of roots (cf. [Hu, p. 50, Lemma A]):

*If $\beta \in \Phi$ is positive but not simple, then $\beta - \alpha \in \Phi^+$ for some $\alpha \in \Delta$.*

Hence each $\beta \in \Phi^+$ can be written as $\beta = \alpha_1 + \cdots + \alpha_k$ $(\alpha_i \in \Delta)$ such that $\alpha_1 + \cdots + \alpha_i \in \Phi^+$ for $i = 1, 2, \ldots, k$. This implies a root vector of $\mathfrak{g}_\beta$ can be written as $[x_{\alpha_k}, [\cdots, [x_{\alpha_2}, x_{\alpha_1}]\cdots]]$, where $x_{\alpha_i}$ is a root vector of $\mathfrak{g}_{\alpha_i}$ $(\alpha_i \in \Delta)$. By our assumption, $x_\alpha$ belong to $\mathfrak{g}_0$ or $\mathfrak{g}_1$ for any $\alpha \in \Delta$. Therefore it follows that

(1) $\mathfrak{m} \oplus \mathfrak{g}_0 = \bigoplus_{p \leqq 0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$ and $\mathfrak{g}_0$,

(2) $\widehat{\mathfrak{m}} \oplus \mathfrak{g}_0 = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_1$ and $\mathfrak{g}_0$.

Moreover $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ for some $\mu > 0$ such that $\mathfrak{g}_p \neq \{0\}$ for $p = -1, -2, \ldots, -\mu$.

Now starting from $\mathfrak{a}_\mu = \mathfrak{g}_\mu$, we define a subspace $\mathfrak{a}_p$ of $\mathfrak{g}_p$ for $p < \mu$ inductively by $\mathfrak{a}_p = [\mathfrak{a}_{p+1}, \mathfrak{g}_{-1}]$ and put

$$\mathfrak{a} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{a}_p.$$

Then, as in the proof of Lemma 3.2, we can check that $\mathfrak{a}_p$ is $\mathrm{ad}(\mathfrak{g}_0)$-invariant and satisfies $[\mathfrak{a}_p, \mathfrak{g}_1] \subset \mathfrak{a}_{p+1}$ by (backward) induction on $p$. $\mathfrak{a}$ is $\mathrm{ad}(\mathfrak{g}_{-1})$-invariant by definition. Since $\mathfrak{g}$ is generated by $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$, we conclude $\mathfrak{a}$ is a non-trivial ideal of $\mathfrak{g}$. Then the simplicity of $\mathfrak{g}$ forces $\mathfrak{a} = \mathfrak{g}$. Especially $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$.

Now let $\mathfrak{g}$ be a simple Lie algeba over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of $\Phi$. Take any non-empty subset $\Delta_1$ of $\Delta$ and put

$$\Phi_k^+ = \{\, \alpha = \sum_{i=1}^{\ell} n_i(\alpha)\, \alpha_i \in \Phi^+ \mid \sum_{\alpha_i \in \Delta_1} n_i(\alpha) = k \,\} \quad \text{for } k \geqq 0.$$

Then, by Lemma 3.8, we can construct a (non-trivial) gradation of $\mathfrak{g}$

satisfying the generating condition for $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ by putting

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),$$

$$\mathfrak{g}_k = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha} \qquad (k > 0),$$

or equivalently by defining the characteristic element $E \in \mathfrak{h}$ by

$$\alpha_i(E) = \begin{cases} 1 & \text{if } \alpha_i \in \Delta_1, \\ 0 & \text{if } \alpha_i \in \Delta_0 = \Delta \setminus \Delta_1. \end{cases}$$

We denote the simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ obtained from $\Delta_1$ in this manner by $(X_\ell, \Delta_1)$, when $\mathfrak{g}$ is a simple Lie algebra of type $X_\ell$. Namely $X_\ell$ stands for the Dynkin diagram of $\mathfrak{g}$ representing $\Delta$ and $\Delta_1$ is a subset of vertices of $X_\ell$.

In this case the depth $\mu$ of $(X_\ell, \Delta_1)$ can be computed by means of the heighest root $\theta$ of $\Phi$. In fact we have $\theta \in \Phi_\mu^+$, because $\theta$ is the unique maximal root relative to the partial order $\succ$ of $\Phi$, where $\alpha \succ \beta$ means that $\alpha - \beta$ is a sum of positive roots or $\alpha = \beta$ (cf. [Hu, Lemma 10.4.A]). Thus $\mu$ is given by

$$\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\, \alpha_i$.

As an illustration, let us examine the case of $G_2$. The Dynkin diagram of $G_2$ is given by

$$\underset{\alpha_1}{\circledcirc} \Lleftarrow \underset{\alpha_2}{\circledcirc}$$

and the set $\Phi^+$ of positive roots consists of six elements (cf. [Bu]):

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Here $\theta = 3\alpha_1 + 2\alpha_2$ and we have three choices for $\Delta_1 \subset \Delta = \{\alpha_1, \alpha_2\}$. Namely $\Delta_1 = \{\alpha_1\}$, $\{\alpha_2\}$ or $\{\alpha_1, \alpha_2\}$. Then the structure of each $(G_2, \Delta_1)$ is described as follows.

(1) $(G_2, \{\alpha_1\})$. We have $\mu = 3$ and $\Phi^+$ decomposes as follows;

$$\Phi_3^+ = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}, \qquad \Phi_2^+ = \{2\alpha_1 + \alpha_2\},$$
$$\Phi_1^+ = \{\alpha_1, \alpha_1 + \alpha_2\}, \qquad \Phi_0^+ = \{\alpha_2\}.$$

Thus $\dim \mathfrak{g}_{-3} = \dim \mathfrak{g}_{-1} = 2$, $\dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_0 = 4$. Hence $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is isomorphic with $\mathbb{C}\mathfrak{m}_5$ in §1.3.

(2) $(G_2, \{\alpha_2\})$. We have $\mu = 2$ and $\Phi^+$ decomposes as follows;

$$\Phi_2^+ = \{3\alpha_1 + 2\alpha_2\}, \qquad \Phi_0^+ = \{\alpha_1\},$$
$$\Phi_1^+ = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}.$$

Thus $\dim \mathfrak{g}_{-2} = 1$ and $\dim \mathfrak{g}_{-1} = \dim \mathfrak{g}_0 = 4$. Hence this is a contact gradation (cf. §4.2).

(3) $(G_2, \{\alpha_1, \alpha_2\})$. We have $\mu = 5$ and $\Phi^+$ decomposes as follows;

$$\Phi_5^+ = \{3\alpha_1 + 2\alpha_2\}, \qquad \Phi_4^+ = \{3\alpha_1 + \alpha_2\}, \qquad \Phi_3^+ = \{2\alpha_1 + \alpha_2\},$$
$$\Phi_2^+ = \{\alpha_1 + \alpha_2\}, \qquad \Phi_1^+ = \{\alpha_1, \alpha_2\}, \qquad \Phi_0^+ = \emptyset.$$

Namely $(G_2, \{\alpha_1, \alpha_2\})$ is a gradation according to the height of roots and $\mathfrak{g}' = \bigoplus_{p \geq 0} \mathfrak{g}_p$ is a Borel subalgebra. In this case, by utilizing a Chevalley basis of $\mathfrak{g}$ (cf. [Hu, p. 147]), one can check that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is isomorphic with $\mathbb{C}\mathfrak{m}_6$ in §1.3 (cf. example (3) in §5.3).

We shall see in §5.2 that $G_2$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ in case (2), and is the prolongation of $\mathfrak{m}$ in case (1) and (3).

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ such that $\mathbb{C}\mathfrak{g}$ is simple. In the real case, we can utilize the Satake diagram $S_\ell$ of $\mathfrak{g}$ to describe gradations of $\mathfrak{g}$.

Let us fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. $\mathbb{C}\mathfrak{h}$ is a Cartan subalgebra of $\mathbb{C}\mathfrak{g}$ such that $\mathfrak{a} = (\mathbb{C}\mathfrak{h})_{\mathbb{R}} \cap \mathfrak{g}$. Let $\Phi$ be the root system of $\mathbb{C}\mathfrak{g}$ relative to $\mathbb{C}\mathfrak{h}$ and $\Sigma$ be the restricted root system of $\mathfrak{g}$ relative to $\mathfrak{a}$. Each $\lambda \in \Sigma$ is obtained by restricting some $\alpha \in \Phi$ to $\mathfrak{a} \subset (\mathbb{C}\mathfrak{h})_{\mathbb{R}}$.

Let $\sigma$ denote the conjugation of $\mathbb{C}\mathfrak{g}$ with respect to $\mathfrak{g}$. Let us take a $\sigma$-fundamental system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of $\Phi$ ([Sa]). Namely $\Delta$ is a simple root system of $\Phi$ satisfying the following property:

$$\text{If } \alpha \in \Phi^+ \text{ and } \alpha|_{\mathfrak{a}} \neq 0, \text{ then } \alpha^\sigma \in \Phi^+,$$

where $\alpha^\sigma \in \Phi$ is defined by $\alpha^\sigma(H) = \overline{\alpha(\sigma(H))}$ for $H \in \mathbb{C}\mathfrak{h}$. Put $\Delta^\bullet = \{\, \alpha \in \Delta \mid \alpha|_{\mathfrak{a}} = 0 \,\}$ and $\Delta^\circ = \Delta \setminus \Delta^\bullet$. Then there exists a permutation $\nu$ of order 2 of $\Delta^\circ$ such that

$$\beta^\sigma = \nu(\beta) + \sum_{\alpha_i \in \Delta^\bullet} m_i\, \alpha_i,$$

for $\beta \in \Delta^\circ$ ([Sa, Lemma 1]). The Satake diagram $S_\ell$ of $\mathfrak{g}$ is constructed from the Dynkin diagram $X_\ell$ of $\mathbb{C}\mathfrak{g}$ representing $\Delta$, firstly by marking simple roots of $\Delta^\bullet$ by black vertices and secondly by connecting two white vertices $\alpha_i$ and $\alpha_j$ of $\Delta^\circ$ by an arrow when $\alpha_i|_{\mathfrak{a}} = \alpha_j|_{\mathfrak{a}}$, that is, when $\alpha_i = \nu(\alpha_j)$. A non-compact real form $\mathfrak{g}$ of a simple Lie algebra over $\mathbb{C}$ is determined by its Satake diagram $S_\ell$. For an explicit construction of real form $\mathfrak{g}$ from its Satake diagram $S_\ell$ in terms of root vectors of $\mathbb{C}\mathfrak{g}$, we refer the reader to §4 of [Tk1]. Thus, from a $\sigma$-fundamental system $\Delta$ of $\Phi$, we obtain a simple root system $\widehat{\Delta} = \{\lambda_1, \dots, \lambda_p\}$ of $\Sigma$, by restricting $\alpha_i \in \Delta$ to $\mathfrak{a}$.

Now take any non-empty subset $\widehat{\Delta}_1$ of $\widehat{\Delta}$ and define $E \in \mathfrak{a}$ by

$$\lambda_i(E) = \begin{cases} 1 & \text{if } \lambda_i \in \widehat{\Delta}_1, \\ 0 & \text{if } \lambda_i \in \widehat{\Delta}_0 = \widehat{\Delta} \setminus \widehat{\Delta}_1. \end{cases}$$

Here we note that $\alpha(E) = 0$ or $1$ for any $\alpha \in \Delta$ and that $\Delta_1 = \{\alpha \in \Delta \mid \alpha|_{\mathfrak{a}} \in \widehat{\Delta}_1\}$ is a subset of the Satake diagram $S_\ell$ of $\mathfrak{g}$ which consists of white vertices and is stable under $\nu\colon \Delta^\circ \to \Delta^\circ$, that is, $\Delta_1$ is a $\nu$-invariant subset of $\Delta^\circ$. Then, by Lemma 3.8, $E$ defines a gradation of $\mathfrak{g}$ such that $\mathbb{C}\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathbb{C}\mathfrak{g}_p$ satisfies the generating condition for $\mathbb{C}\mathfrak{m} = \bigoplus_{p<0} \mathbb{C}\mathfrak{g}_p$. Hence $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ is a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Moreover $\Delta_0 = \Delta \setminus \Delta_1$ is a $\sigma$-subsystem containing $\Delta^\bullet$, which corresponds to the parabolic subalgebra $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$. We denote the simple graded Lie algebra over $\mathbb{R}$ obtained in this manner by $(S_\ell, \Delta_1)$. In this case, the depth $\mu$ of $(S_\ell, \Delta_1)$ can be computed by means of the highest root $\theta$ of the $\sigma$-fundamental system $\Delta$;

$$\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\,\alpha_i$.

### 3.5. Conjugacy of simple graded Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$. We denote by $\mathrm{Aut}(\mathfrak{g})$ the group of Lie algebra isomorphisms of $\mathfrak{g}$ over $K$, and by $\mathrm{Int}(\mathfrak{g})$ the adjoint group of $\mathfrak{g}$. $\mathrm{Int}(\mathfrak{g})$ coincides with the identity component of $\mathrm{Aut}(\mathfrak{g})$. We shall consider the conjugacy problems for gradations of $\mathfrak{g}$ satisfying the generating condition for $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ under the group $\mathrm{Aut}(\mathfrak{g})$ or $\mathrm{Int}(\mathfrak{g})$. Two gradations $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ and $\{\widehat{\mathfrak{g}}_p\}_{p\in\mathbb{Z}}$ are called conjugate under $G$ if there exists $\varphi \in G$ such that $\varphi(\mathfrak{g}_p) = \widehat{\mathfrak{g}}_p$ for all $p \in \mathbb{Z}$, where $G = \mathrm{Aut}(\mathfrak{g})$ or $\mathrm{Int}(\mathfrak{g})$.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $K$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. First we consider the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ defined by

$$\mathfrak{f}^p = \bigoplus_{q \geqq p} \mathfrak{g}_q \qquad \text{for } p \in \mathbb{Z}.$$

Then $[\mathfrak{f}^p, \mathfrak{f}^q] \subset \mathfrak{f}^{p+q}$ for $p$, $q \in \mathbb{Z}$ and we have $\mathfrak{f}^p = \mathfrak{g}$ for $p \leqq -\mu$, $\mathfrak{f}^k = \{0\}$ for $k > \mu$ and $\mathfrak{f}^0 = \mathfrak{g}'$. Recall, by the argument in 3.3, that $\mathfrak{g}' = \mathfrak{P}$ is a parabolic subalgebra of $\mathfrak{g}$ (when $K = \mathbb{R}$, a subalgebra $\mathfrak{P}$ of $\mathfrak{g}$ is called parabolic if $\mathbb{C}\mathfrak{P}$ is parabolic in $\mathbb{C}\mathfrak{g}$). Furthermore, by Lemma 3.1, Lemma 3.2 and the generating condition of $\mathfrak{m}$, we have

**Lemma 3.10.** *The filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is determined solely by $\mathfrak{P} = \mathfrak{f}^0$ and given as follows.*

(1) $\mathfrak{f}^1 = \{ X \in \mathfrak{P} \mid B(X, \mathfrak{P}) = 0 \}$ *and is the nilradical of* $\mathfrak{P}$.
(2) $\mathfrak{f}^k = C^k\mathfrak{f}^1 = [\mathfrak{f}^1, C^{k-1}\mathfrak{f}^1]$ *for* $k \geqq 2$, *where* $\mathfrak{f}^1 = C^1\mathfrak{f}^1$ *by convention.*
(3) $\mathfrak{f}^{-1} = \{ X \in \mathfrak{g} \mid [X, \mathfrak{f}^1] \subset \mathfrak{f}^0 \}$.
(4) $\mathfrak{f}^{-k} = C^k\mathfrak{f}^{-1} = [\mathfrak{f}^{-1}, C^{k-1}\mathfrak{f}^{-1}]$ *for* $k \geqq 2$, *where* $\mathfrak{f}^{-1} = C^1\mathfrak{f}^{-1}$ *by convention.*

The last statment in (1) can be obtained by describing the gradation in terms of the root space decomposition of $\mathbb{C}\mathfrak{g}$ as in 3.3.

By Lemma 3.10, we note that, for a simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, the gradation is recovered from the parabolic subalgebra $\mathfrak{P} = \mathfrak{g}'$ firstly by forming the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ given by Lemma 3.10 and secondly by passing to the associated graded Lie algebra of $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$. This observation leads us to the following.

**Proposition 3.11.** *Let $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ and $\{\widehat{\mathfrak{g}}_p\}_{p\in\mathbb{Z}}$ be two gradations of a simple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$. Then $\{\mathfrak{g}_p\}_{p\in\mathbb{Z}}$ and $\{\widehat{\mathfrak{g}}\}_{p\in\mathbb{Z}}$ are conjugate under* $\mathrm{Aut}(\mathfrak{g})$ (*resp.* $\mathrm{Int}(\mathfrak{g})$) *if and only if* $\mathfrak{P} = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ *and* $\widehat{\mathfrak{P}} = \bigoplus_{p\geqq 0} \widehat{\mathfrak{g}}_p$ *are conjugate under* $\mathrm{Aut}(\mathfrak{g})$ (*resp.* $\mathrm{Int}(\mathfrak{g})$).

*Proof.* Only if part is trivial. Let $\varphi$ be an automorphism of $\mathfrak{g}$ such that $\varphi(\mathfrak{P}) = \widehat{\mathfrak{P}}$. Then, by Lemma 3.10, $\varphi$ is an isomorphism as a filtered Lie algebra, that is, $\varphi(\mathfrak{f}^p) = \widehat{\mathfrak{f}}^p$ for all $p \in \mathbb{Z}$. Let $\widehat{\varpi}_p$ be the projection of $\widehat{\mathfrak{f}}^p$ onto $\widehat{\mathfrak{g}}_p$ corresponding to the decomposition $\widehat{\mathfrak{f}}^p = \bigoplus_{q\geqq p} \widehat{\mathfrak{g}}_q$. $\varphi$ induces a graded map $\widehat{\varphi}$ of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ onto $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{g}}_p$ by

$$\widehat{\varphi}(X) = (\widehat{\varpi}_p \cdot \varphi)(X) \qquad \text{for } X \in \mathfrak{g}_p.$$

It is easy to see that $\widehat{\varphi}$ is a graded Lie algebra isomorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ onto $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{g}}_p$. This finishes the proof for $\mathrm{Aut}(\mathfrak{g})$. Furthermore put $\psi = \widehat{\varphi}^{-1} \cdot \varphi$. Then $\psi$ is a filtration preserving automorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Hence, by Lemma 1.7 of [T4], $\psi$ can be written uniquely in the form

$$\psi = \varphi_0 \cdot \exp X_1 \cdots \exp X_\mu,$$

where $\varphi_0 \in G_0$, $X_k \in \mathfrak{g}_k$ and $G_0$ is the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of all gradation preserving automorphisms of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Thus we obtain

$$\varphi = \widehat{\varphi}_0 \cdot \exp X_1 \cdots \exp X_\mu,$$

where $\widehat{\varphi}_0 = \widehat{\varphi} \cdot \varphi_0$ is a graded Lie algebra isomorphism of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ onto $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{g}}_p$, which completes the proof for $\mathrm{Int}(\mathfrak{g})$.

Thus the conjugacy of gradations of a simple Lie algebra $\mathfrak{g}$ over $K = \mathbb{R}$ or $\mathbb{C}$ satisfying the generating condition for $\mathfrak{m}$ is reduced to that of parabolic subalgebras of $\mathfrak{g}$.

The classification of parabolic subalgebras of a simple Lie algebra over $\mathbb{C}$ is achieved by the conjugacy of Borel subalgebras of $\mathfrak{g}$ (cf. [Hu, Chapter IV]): Every parabolic subalgebra in $\mathfrak{g}$ is conjugate to a standard parabolic subalgebra $\mathfrak{P}(\Delta_0)$, where $\Delta_0$ is a subset of $\Delta$. Moreover the conjugacy class of parabolic subalgebras under $\mathrm{Aut}(\mathfrak{g})$ is one to one correspondent to the equivalence class of $(X_\ell, \Delta_0)$ under the diagram automorphisms of $X_\ell$, where $X_\ell$ stands for the Dynkin diagram of $\mathfrak{g}$ and $\Delta_0$ is any subset of $X_\ell$. Similarly, in the real case, we have ([M, p. 431, Theorem 3.1]); the conjugacy class of parabolic subalgebras under $\mathrm{Aut}(\mathfrak{g})$ is one to one correspondent to the equivalence of $(S_\ell, \Delta_0)$ under the diagram automorphisms of $S_\ell$, where $S_\ell$ stands for the Satake diagram of $\mathfrak{g}$ and $\Delta_0$ is any $\sigma$-subsystem containing $\Delta^\bullet$. For the details, we refer the reader to [M].

Summarizing we obtain (cf. [K-A, Theorem 2.7]. For the notation see 3.4.)

**Theorem 3.12.** *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$.*

(1) *The complex case. Let $X_\ell$ be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with a graded Lie algebra $(X_\ell, \Delta_1)$ for some $\Delta_1 \subset \Delta$. Moreover $(X_\ell, \Delta_1)$ and $(X_\ell, \Delta_1')$ are isomorphic if and only if there exists a diagram automorphism $\phi$ of $X_\ell$ such that $\phi(\Delta_1) = \Delta_1'$.*

(2) *The real case. Let $S_\ell$ be the Satake diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with a graded Lie algebra $(S_\ell, \Delta_1)$ for some $\nu$-invariant subset $\Delta_1$ of $\Delta^\circ$. Moreover $(S_\ell, \Delta_1)$ and $(S_\ell, \Delta_1')$ are isomorphic if and only if there exists a diagram automorphism $\phi$ of $S_\ell$ such that $\phi(\Delta_1) = \Delta_1'$.*

## §4. Standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$ of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$

### 4.1. Standard differential system $(M_\mathfrak{g}, D_\mathfrak{g})$

First we shall give general remarks on the model space associated with a simple graded Lie algebra.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. We denote by $\mathrm{Int}(\mathfrak{g})$ the adjoint group of $\mathfrak{g}$. Let $G_0$ be the automorphism group of the graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, that is, the subgroup of $\mathrm{Aut}(\mathfrak{g})$ consisting of elements which preserves the gradation. Then the Lie algebra of $G_0$ coincides with $\mathfrak{g}_0$ ([T3, Lemma 2.4]). Moreover let $G'$ be the automorphism group of the filtered Lie algebra $\mathfrak{g} = \mathfrak{f}^{-\mu}$ (cf. Lemma 3.10). The Lie algebra of $G'$ is $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$.

Now we define an open subgroup $G$ of $\mathrm{Aut}(\mathfrak{g})$ by

$$G = \mathrm{Int}(\mathfrak{g}) \cdot G' = \mathrm{Int}(\mathfrak{g}) \cdot G_0.$$

We consider the homogeneous space $M_\mathfrak{g} = G/G'$. $M_\mathfrak{g}$ is connected and compact (this is a consequence of the Iwasawa decomposition of $G$), on which G acts effectively. $M_\mathfrak{g} = G/G'$ is the model space for the normal Cartan connection of type $\mathfrak{g}$ constructed by N. Tanaka [T4]. Furthermore, when $\mu > 1$, by identifying $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on $G$, $\mathfrak{f}^{-1}$ defines a left invariant subbundle of $T(G)$, which is also preserved by the right action of $G'$ on $G$. Hence $\mathfrak{f}^{-1}$ induces a $G$-invariant differential system $D_\mathfrak{g}$ on $M_\mathfrak{g}$.

Here we remark that, when $\mathfrak{g}$ is a real simple Lie algebra of class A in Proposition 3.3, that is, when $\mathfrak{g}$ is a complex simple Lie algebra regarded as a real simple Lie algebra, the identity component $\mathrm{Int}(\mathfrak{g})$ of $G$ is a complex Lie group. Hence $M_\mathfrak{g} = \mathrm{Int}(\mathfrak{g})/G' \cap \mathrm{Int}(\mathfrak{g})$ is a complex manifold such that $D_\mathfrak{g}$ is a holomorphic differential system on $M_\mathfrak{g}$. However $G$ does not act on $M_\mathfrak{g}$ as a group of holomorphic transformations, although $\mathrm{Int}(\mathfrak{g})$ does. Namely the Lie group $G$ changes depending on whether we regard $\mathfrak{g}$ as a real Lie algebra or as a complex Lie algebra, whereas $M_\mathfrak{g}$ remains the same. In fact the group of all automorphisms of $(M_\mathfrak{g}, D_\mathfrak{g})$, which coincides with $G$ by Theorem 2.7 of [T4], under the assumption that $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$, differs depending on whether we regard $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ as a real or a holomorphic differential differential system, whereas the Lie algebra $\mathcal{A}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of all infinitesimal automorphisms remains the same (cf. Remark at the end of §3.2).

Thus $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ may be called the *standard differential system* of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. In fact, let us fix a reference point $o$ of $M_{\mathfrak{g}}$. Let $\widehat{M}$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$. Then, since $\widehat{M} \subset G \subset GL(\mathfrak{g})$, the unipotent linear subgroup $\widehat{M}$ is simply connected. Moreover, since $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}'$, $\widehat{M}$ has an open orbit through $o$. (This orbit is in fact diffeomorphic with $\widehat{M}$. This follows from the generalized Bruhat decomposition [Tk1, Theorem 8].) Thus the restriction of the projection $G \to M_{\mathfrak{g}} = G/G'$ gives a (local) diffeomorphism $p$ of $\widehat{M} = M(\mathfrak{m})$ into $M_{\mathfrak{g}}$ such that $p(id_{\widehat{M}}) = o$. By the definition of $D_{\mathfrak{g}}$, we see that $p$ is a (local) isomorphism of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ into $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$. We shall see in §5.2 that, in many cases, $G$ coincides with the group $\mathrm{Aut}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of all automorphisms of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$.

By the previous argument in §3, we know that $M_{\mathfrak{g}}$ is in fact a $R$-space, that is, $M_{\mathfrak{g}} = G/G'$ is a quotient space of a simple algebraic group $G$ by a parabolic subgroup $G'$ (cf. [Tt1], [Tk1]). Especially, when $\mathfrak{g}$ is complex simple, we know that $M_{\mathfrak{g}}$ is a compact simply connected projective algebraic manifold (cf. [Wa], [Se], [Tt1], [Tk1]). Hence, in this case, starting from any connected complex Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$, we can construct $M_{\mathfrak{g}}$ as $\widetilde{G}/\widetilde{G}'$, where $\widetilde{G}'$ is the analytic subgroup of $\widetilde{G}$ with Lie algebra $\mathfrak{g}'$.

Now let $\widehat{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $(\rho, V)$ be an irreducible representation of $\widehat{G}$ with the highest weight $\Lambda$, which is strongly associated to $\Phi_0$ in the sense of Borel-Weil [Se]. Namely $\Lambda$ is a dominant weight of $\mathfrak{g}$ such that $(\Lambda, \alpha) = 0$ for $\alpha \in \Delta_0$ and $(\Lambda, \alpha) > 0$ for $\alpha \in \Delta_1$. Then we obtain a $\widehat{G}$-equivariant projective imbedding of $M_{\mathfrak{g}}$ by taking a $\widehat{G}$-orbit passing through $[v_\Lambda]$ in the projective space $P(V)$ consisting of all lines in $V$, where $v_\Lambda$ is a maximal vector in $V$ of the heighest weight $\Lambda$. For the discussion of the real case, we refer the reader to [Tk1].

In the following, we shall give an example of this construction and also discuss explicit examples of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ for simple Lie algebras of the classical type. Our emphasis will be on the differential system $D_{\mathfrak{g}}$.

*Remark* 4.0. In the complex case, since $M_{\mathfrak{g}}$ is a compact complex manifold, the group $\mathrm{Aut}(M_{\mathfrak{g}})$ of all holomorphic transformations of $M_{\mathfrak{g}}$ is a Lie transformation group acting on $M_{\mathfrak{g}}$. It is known ([On]) that

$\mathrm{Int}(\mathfrak{g})$ coincides with the identity component of $\mathrm{Aut}(M_{\mathfrak{g}})$ except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(C_\ell, \{\alpha_1\})$ ($\ell \geqq 2$), $(B_\ell, \{\alpha_\ell\})$ ($\ell \geqq 3$) or $(G_2, \{\alpha_2\})$. In these exceptions, $M_{\mathfrak{g}}$ is biholomorphic with $P^{2\ell-1}(\mathbb{C})$, $SO(2\ell+2)/U(\ell+1)$ or $Q^5(\mathbb{C})$ (complex quadric) and the Lie algebra of $\mathrm{Aut}(M_{\mathfrak{g}})$ is of type $A_{2\ell}$, $D_{\ell+1}$ or $B_3$ respectively. These facts are pointed out to us by the referee (see also Remark 4.3 (1)).

### 4.2. Contact gradation

For each simple Lie algebera over $\mathbb{C}$, we shall show the existence of a complex contact gradation which is unique up to conjugacy (cf. [Bo], [Wo], [Ch], [Tk2]).

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. First assume that $\mathfrak{g}$ has a contact gradation, that is, $\mathfrak{g}$ admits a gradation of depth 2 such that $\mathfrak{g}_{-1} \neq \{0\}$ and $\dim \mathfrak{g}_{-2} = 1$;

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

By Lemma 3.2 (1), the bracket operation $[\,,\,]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ is non-degenerate. Let us fix a Cartan subalgebra $\mathfrak{h}$ and a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ such that $E \in \mathfrak{h}$ and $\alpha(E) \geqq 0$ for $\alpha \in \Delta$. We have a partition of positive roots $\Phi^+$;

$$\Phi^+ = \Phi_2^+ \cup \Phi_1^+ \cup \Phi_0^+.$$

Then, since $\dim \mathfrak{g}_{-2} = 1$, we have $\Phi_2^+ = \{\theta\}$, where $\theta$ is the highest root. Moreover, from the non-degeneracy of $[\,,\,]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$, we see that, for each $\alpha \in \Phi_1^+$, there exists $\beta \in \Phi_1^+$ such that $\alpha + \beta = \theta$. Hence $\Phi_1^+ = \{\, \alpha \in \Phi^+ \mid \theta - \alpha \text{ is a root}\,\}$. Since $\Phi^+ = \Phi_2^+ \cup \Phi_1^+ \cup \Phi_0^+$ is a partition, we get $\Phi_0^+ = \{\, \alpha \in \Phi^+ \setminus \{\theta\} \mid \theta - \alpha \text{ is not a root}\,\}$. On the other hand, since $\theta$ is a long root and $\theta + \alpha$ is not a root for any $\alpha \in \Phi^+$, we have (cf. [Hu, 9.4])

$$\langle \alpha, \theta \rangle = 0 \quad \text{or} \quad 1 \qquad \text{for any } \alpha \in \Phi^+ \setminus \{\theta\},$$

where $\langle \alpha, \theta \rangle = \dfrac{2(\alpha,\theta)}{(\theta,\theta)}$ is a Cartan integer. Moreover, by considering the $\alpha$-string through $\theta$, we see that $\langle \theta, \alpha \rangle = 0$ if and only if $\theta - \alpha$ is not a root. Therefore we obtain

$$\Phi_k^+ = \{\, \alpha \in \Phi^+ \mid \langle \alpha, \theta \rangle = k \,\} \qquad \text{for } k = 0,\ 1,\ 2.$$

This implies that the characteristic element $E$ of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is given by $h_\theta = \dfrac{2\,t_\theta}{(\theta,\theta)} \in \mathfrak{h}$, where $t_\theta \in \mathfrak{h}$ is defined by $B(t_\theta, H) = \theta(H)$ for

$H \in \mathfrak{h}$. Conversely the above argument shows that $h_\theta \in \mathfrak{h}$ indeed defines a contact gradation of $\mathfrak{g}$.

Thus a contact graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ is isomorphic with $(X_\ell, \Delta_\theta)$, where $\Delta_\theta = \{\, \alpha \in \Delta \mid \langle \alpha, \theta \rangle = 1 \,\}$. Here we note that, since depth of $(X_\ell, \Delta_\theta)$ is two, $\Delta_\theta$ should consist of two elements $\{\alpha_i, \alpha_j\}$ of $\Delta$ satisfying $n_i(\theta) = n_j(\theta) = 1$ or consist of a single element $\alpha_i$ of $\Delta$ satisfying $n_i(\theta) = 2$. In fact the information of $\Delta_\theta$ is expressed in the *extended Dynkin diagram* of $\mathfrak{g}$ and the former case can occur only when $\mathfrak{g}$ is of type $A_\ell$ ($\ell \geqq 2$). Thus $\Delta_\theta$ is the subset of $\Delta$ consisting of simple roots which are connected to $-\theta$ in the extended Dynkin diagram of $\mathfrak{g}$.

Summarizing, we obtain (cf. [Wo, Theorem 4.2], [Ch], [Tk2, §1])

**Theorem 4.1.** *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ such that $\operatorname{rank} \mathfrak{g} \geqq 2$. Then $\mathfrak{g}$ admits a unique complex contact gradation up to conjugacy. This gradation is isomorphic with $(X_\ell, \Delta_\theta)$, where $\Delta_\theta = \{\, \alpha \in \Delta \mid \langle \alpha, \theta \rangle = 1 \,\}$ and $\theta$ is the highest root. Furthermore the characteristic element of $(X_\ell, \Delta_\theta)$ is given by $E = h_\theta \in \mathfrak{h}$.*

In the next page we show the extended Dynkin diagrams with the coefficient of the highest root.

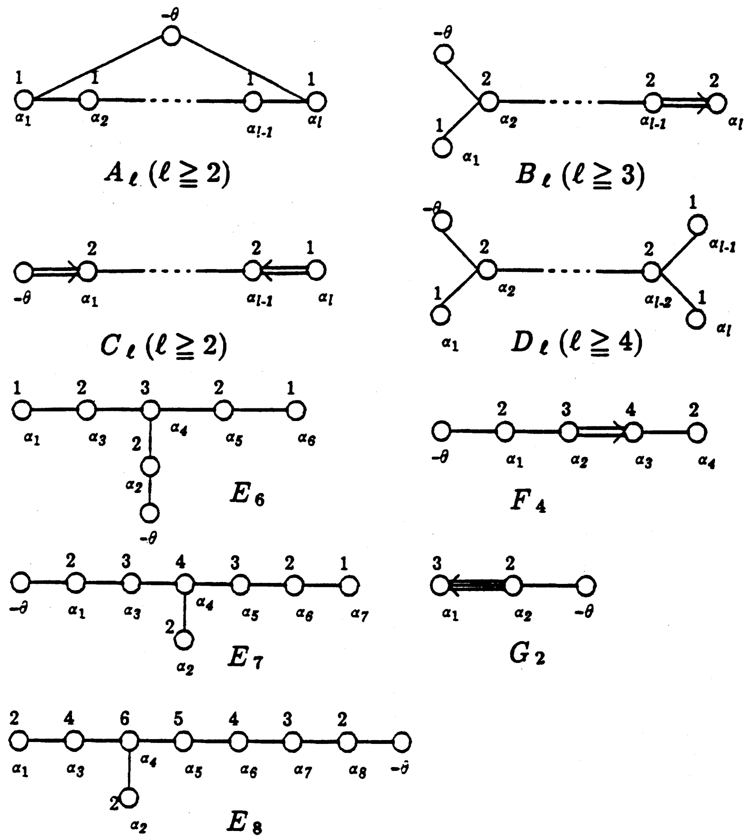
*Remark* 4.2. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{R}$ such that $\mathbb{C}\mathfrak{g}$ is simple and $\operatorname{rank} \mathbb{C}\mathfrak{g} \geqq 2$. By Theorem 4.1, to seek a real contact gradation of $\mathfrak{g}$, we need only to check whether $\Delta_\theta$ is a $\nu$-invariant subset of $\Delta^o$ in its Satake diagram or whether the lowest root $-\theta$ is not connected to any black vertex in the extended Satake diagram of $\mathfrak{g}$ ([Tk2, §3]). In this way, we obtain (cf. [Ch, Theorem 3])

*A real simple Lie algebra $\mathfrak{g}$ of class B admits a unique real contact gradation $(S_\ell, \Delta_\theta)$ up to conjugacy except for the cases when $S_\ell$ is of type*AI ($\ell = 1$), AII, BII, CII, DII, EIV *or* FII *in the list of table VI in* [He, Chapter X, p. 532]. *In the latter cases, they do not admit a contact gradation.*

For the details, we refer the reader to [Ch] and [Tk2, §1].

### 4.3. Standard contact manifolds

We shall discuss the standard differential sytem $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ associated with a contact gradation of a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ as an illustration of the method, mentioned in 4.1, of constructing the model space via a certain representation. Here we note that, by Theorem 4.1, the heighest root $\theta$ is a dominant weight strongly associated to $\Delta \setminus \Delta_\theta$, and $\theta$ is the heighest weight of the adjoint representation $\operatorname{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. In




The extended Dynkin diagrams

fact the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $(X_\ell, \Delta_\theta)$ can be constructed via the adjoint representation as follows.

Let $\mathfrak{g}$ be a simple Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ and let $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ be a contact gradation over $K$. Let $G = \mathrm{Int}(\mathfrak{g})$ be the adjoint group of $\mathfrak{g}$. Let us fix a non-zero vector $X_o \in \mathfrak{g}_2$. First we consider the adjoint orbit $S$ of $G$ passing through $X_o$. Since the adjoint representation and coadjoint representation of $G$ are equivalent via the Killing form duality, it is well-known (cf. [A]) that $S$ has a symplectic structure (over $K$). The symplectic structure on $S$ is given as follows: Let $\omega_o$ be the covector corresponding to $X_o$, that is, $\omega_o \in \mathfrak{g}^*$ is defined by $\omega_o(X) = B(X_o, X)$ for $X \in \mathfrak{g}$. Then we have

$$(4.1) \qquad \mathfrak{f}^{-1} = \bigoplus_{p \geqq -1} \mathfrak{g}_p = \{\, X \in \mathfrak{g} \mid \omega_o(X) = 0 \,\}.$$

Let $\bar{G}$ be the isotropy subgroup of $G$ at $X_o \in \mathfrak{g}$;

$$\bar{G} = \{\, g \in G \mid \mathrm{Ad}(g)(X_o) = X_o \,\} = \{\, g \in G \mid \mathrm{Ad}^*(g)(\omega_o) = \omega_o \,\}.$$

Then the Lie algebra $\bar{\mathfrak{g}}$ of $\bar{G}$ is given by

$$\bar{\mathfrak{g}} = \{\, X \in \mathfrak{g} \mid [X, X_o] = 0 \,\} = Z_{\mathfrak{g}}(\mathfrak{g}_2).$$

On the other hand we see from the root space description of the contact gradation in 4.1 that $Z_{\mathfrak{g}}(\mathfrak{g}_2)$ is an ideal of $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ of codimension 1 such that $\mathfrak{g}' = \langle E \rangle_K \oplus Z_{\mathfrak{g}}(\mathfrak{g}_2)$ and that $\mathfrak{g}'$ is the normalizer $N_{\mathfrak{g}}(\mathfrak{g}_2)$ of $\mathfrak{g}_2$ in $\mathfrak{g}$. In particular $\bar{\mathfrak{g}} \subset \mathfrak{f}^{-1} = \{\, X \in \mathfrak{g} \mid \omega_o(X) = 0 \,\}$. Then, for a left invariant 1-form $\omega_o$ on $G$, we have $R_g^* \omega_o = \mathrm{Ad}^*(g^{-1})\, \omega_o = \omega_o$ for $g \in \bar{G}$ and $\omega_o(X) = 0$ for $X \in \bar{\mathfrak{g}}$, which implies that $\omega_o$ is projectable to $S = G/\bar{G}$. Namely there exists a $G$-invariant 1-form $\alpha$ on $S$ such that $\pi^*\alpha = \omega_o$, where $\pi \colon G \to S = G/\bar{G}$ is the projection. Moreover, since $\iota(X) d\omega_o = L_X \omega_o$ for a left invariant vector field $X \in \mathfrak{g}$, we see from (4.2) that $X \in \bar{\mathfrak{g}}$ if and only if $\iota(X) d\omega_o = 0$. Therefore $d\alpha$ is a symplectic form on $S$. (For an arbitrary coadjoint orbit $S_\omega$ passing through $\omega \in \mathfrak{g}^*$, only $d\omega$ is projectable to $S_\omega$.)

Now let us take a $G$-orbit $J_{\mathfrak{g}}$ passing through $[X_o] = \mathfrak{g}_2$ in the projective space $P(\mathfrak{g})$ over $K$. Let $G'$ be the isotropy subgroup of $G$ at $[X_o] \in P(\mathfrak{g})$:

$$\begin{aligned} G' &= \{\, g \in G \mid \mathrm{Ad}(g)(X_o) = \rho(g) \cdot X_o \,\} \\ &= \{\, g \in G \mid \mathrm{Ad}^*(g^{-1})(\omega_o) = \rho(g) \cdot \omega_o \,\}, \end{aligned}$$

where $\rho \colon G' \to K^\times$ defines a 1-dimensional representation of $G'$. From the existence of the characteristic element $E$, we see that $\rho$ is not trivial. Hence we get $\mathrm{Ker}\,\rho = \bar{G}$, $G'/\bar{G}$ is isomorphic with $K^*$ and the Lie algebra of $G'$ coincides with $\mathfrak{g}' = N_{\mathfrak{g}}(\mathfrak{g}_2)$, where $K^* = \mathbb{C}^*$ when $K = \mathbb{C}$ and $K^* = \mathbb{R}^+$ or $\mathbb{R}^\times$ when $K = \mathbb{R}$ (see Remark 4.3 below). In particular $S$ is stable under the $K^*$(scalar)-action of the ambient vector space $\mathfrak{g}$. Let $p$ be the projection of $S$ onto $J_{\mathfrak{g}}$, which is the restriction of the projection $p \colon \mathfrak{g} \setminus \{0\} \to P(\mathfrak{g})$. Then $(S, J_{\mathfrak{g}}, p)$ is a principal $K^*$-bundle over $J_{\mathfrak{g}}$. From $R_g^* \omega_o = \rho(g) \cdot \omega_o$ for $g \in G'$ and $\omega_o(X) = 0$ for $X \in \mathfrak{g}'$, we have $R_a^* \alpha = a \cdot \alpha$ for $a \in K^*$ and $\mathrm{Ker}\, p_* \subset \mathrm{Ker}\,\alpha = \{\, X \in T(S) \mid \alpha(X) = 0 \,\}$, where $\operatorname{Ker} p_*$ is the vertical subbundle of $T(S)$ of the projection $p\colon S \to J_{\mathfrak{g}}$. Hence a $G$-invariant 1-form $\alpha$ on $S$ defines a $G$-invariant differential system $C_{\mathfrak{g}}$ on $J_{\mathfrak{g}}$ of codimension 1 by

$$C_{\mathfrak{g}}(u) = p_*(\operatorname{Ker} \alpha(x)) \qquad \text{at each } u = p(x) \in J_{\mathfrak{g}}.$$

From (4.1), we see that $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ is a standard differential system of type $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$. $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ is called the standard contact manifold of type $\mathfrak{g}$.

Furthermore we have an imbedding $\gamma$ of $S$ into $T^*(J_{\mathfrak{g}})$, which commutes with $K^*$-actions of $S$ and $T^*(J_{\mathfrak{g}})$. In fact, since $\operatorname{Ker} p_* \subset \operatorname{Ker} \alpha$, for each $x \in S$, $\alpha$ determines a covector $\gamma(x) \in T^*_u(J_{\mathfrak{g}})$ at $u = p(x)$ such that $\gamma(x)(p_*(X)) = \alpha(X)$ for $X \in T_x(S)$. Then, via $\gamma$, $(S, d\alpha)$ is identified with the symplectification of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ when $K = \mathbb{C}$ and with a connected component of the symplectification $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ when $K = \mathbb{R}$ (cf. [A], [K1]).

Standard contact manifolds associated with simple Lie algebras over $\mathbb{C}$ were first found by Boothby [Bo] as compact simply connected homogeneous complex contact manifolds. The above construction was also given in [Wo]. The advantage of this construction is a clarification of the contact structure on $M_{\mathfrak{g}}$ in a unified manner. We shall give a more explicit picture of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ for the classical type in 4.5.

*Remark* 4.3. (1) In the complex case, it is known ([Wo]) that $\operatorname{Int}(\mathfrak{g})$ coincides with the identity component of the group $\operatorname{Aut}(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ of all holomorphic contact transformations of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$.

(2) In the real case, $G'/\bar{G}$ is not necessarily connected. In fact $G'/\bar{G}$ is connected if and only if $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ admits a global contact form, or equivalently, if and only if the symplectification of $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ has two connected components. For example, $G'/\bar{G}$ is connected when $\mathfrak{g} = \mathfrak{su}(r+1, \ell - r)$ $(0 \leqq r \leqq [\frac{n-1}{2}])$ and is not connected when $\mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{R})$ or $\mathfrak{sp}(\ell, \mathbb{R})$.

### 4.4. Gradation and matrices

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of the classical type. We shall describe gradations of $\mathfrak{g}$ in terms of matrices. Here we reproduce the matrices description of the root space decomposition of $\mathfrak{g}$ from §7 of [Tk1] (cf. [K-A], [V, Chapter 4.4]), which gives us explicit pictures of $M_{\mathfrak{g}}$.

(1) $A_\ell$ type $(\ell \geqq 1)$. $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$. We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of $\mathfrak{sl}(\ell + 1, \mathbb{C})$, whose member we denote by $\operatorname{diag}(a_1, \ldots, a_{\ell+1})$. Let $\lambda_1, \ldots, \lambda_{\ell+1}$ be the linear form on

$\mathfrak{h}$ defined by $\lambda_i$: $\operatorname{diag}(a_1, \ldots, a_{\ell+1}) \mapsto a_i$. We write $E_{ij}$ $(1 \leqq i, j \leqq \ell+1)$ for the matrix whose $(i,j)$-component is 1 and all of whose other components are 0. Then we have

$$[H, E_{ij}] = (\lambda_i - \lambda_j)(H)\, E_{ij} \qquad \text{for } H \in \mathfrak{h}.$$

Hence $\Phi = \{\lambda_i - \lambda_j \in \mathfrak{h}^* \; (1 \leqq i, j \leqq \ell+1,\ i \neq j)\}$ and $E_{ij}$ spans the root subspace for $\lambda_i - \lambda_j \in \Phi$. Let us choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ by putting

$$\alpha_i = \lambda_i - \lambda_{i+1}.$$

We have $\lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1}$ when $i < j$. Hence $\theta = \alpha_1 + \cdots + \alpha_\ell$. Then we see that the gradation of $(A_\ell, \{\alpha_i\})$ is given by $\mathfrak{sl}(\ell+1, \mathbb{C}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$;

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \mid C \in M(j,i) \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \mid D \in M(i,j) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M(i,i),\ B \in M(j,j) \text{ and } \operatorname{tr} A + \operatorname{tr} B = 0 \right\},$$

where $j = \ell - i + 1$ and $M(p,q)$ denotes the set of $p \times q$ matrices. This decomposition can be described schematically by the following diagram;

$$\begin{array}{c|c|c|} & i & j \\ \cline{2-3} i & 0 & 1 \\ \cline{2-3} j & -1 & 0 \\ \cline{2-3} \end{array}$$

where the vertical (resp. horizontal) line stands for the $i$-th vertical (resp. horizontal) intermediate line of a matrix in $\mathfrak{sl}(\ell+1, \mathbb{C})$. Then, for example, the diagram of $(A_\ell, \{\alpha_i, \alpha_j\})$ $(i < j)$ is obtained by superposing the diagrams of $(A_\ell, \{\alpha_i\})$ and $(A_\ell, \{\alpha_j\})$;

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \qquad \Rightarrow \qquad \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 & 1 \\ \hline -2 & -1 & 0 \\ \hline \end{array}$$

In general the diagram of $(A_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ is obtained by superposing the $k$ diagrams of $(A_\ell, \{\alpha_{i_1}\})$, $\ldots$ , $(A_\ell, \{\alpha_{i_k}\})$. Namely the gradation of $(A_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ is obtained by subdividing matrices by both vertical and horizontal $k$ lines. Here $i$-th intermediate line corresponds to the simple root $\alpha_i$.

By this description of gradations, we see that the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_i\})$ is the complex Grassmann manifold $Gr(i, V)$ consisting of all $i$-dimensional subspaces of $V = \mathbb{C}^{\ell+1}$. Furthermore the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$ is the flag manifold $F(i_1, \ldots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in $V$ such that $\dim V_j = i_j$ for $j = 1, \ldots, k$ (cf. [Tt1]).

(2) $C_\ell$ type $(\ell \geqq 2)$. Let $(V, \langle\ ,\ \rangle)$ be a symplectic vector space over $\mathbb{C}$ of dimension $2\ell$, that is, $\langle\ ,\ \rangle$ is a non-degenerate skew symmetric bilinear form on $V$. Then $\mathfrak{g} = \mathfrak{sp}(V)$. Let us take a symlectic basis $\{e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\}$ of $V$ such that $\langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0$ and $\langle f_i, e_{\ell+1-j}\rangle = \delta_{ij}$ for $i, j = 1, \ldots, \ell$. Thus we have a matrix representation

$$\mathfrak{g} = \{\, X \in \mathfrak{gl}(2\ell, \mathbb{C}) \mid {}^tXJ + JX = 0 \,\}, \quad \text{where } J = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix},$$

and $K$ is the $\ell \times \ell$ matrix whose $(i,j)$-component is $\delta_{i,\ell+1-j}$. We put $A' = KAK$ for $A \in \mathfrak{gl}(\ell, \mathbb{C})$. Namely $A'$ is the "transposed" matrix of $A$ with respect to the anti-diagonal line. Each $X \in \mathfrak{g}$ is expressed as a matrix of the following form;

$$X = \begin{pmatrix} A & B \\ C & -A' \end{pmatrix},$$

where $A$, $B$, $C$ are $\ell \times \ell$ matrices such that $B$ and $C$ satisfy $B = B'$ and $C = C'$. Namely both $B$ and $C$ are symmetric with respect to the anti-diagonal line. Thus we see that $X$ is determined by its upper anti-diagonal part. In the following we write $X = (A, B, C)$ in short.

We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of the form $H = (\mathrm{diag}(a_1, \ldots, a_\ell), 0, 0)$. Let $\lambda_1, \ldots, \lambda_\ell$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i \colon H \mapsto a_i$. We put $F_{ij} = E_{ij} + E'_{ij}$, where $E'_{ij} = E_{\ell+1-j, \ell+1-i}$. Then we have

$$\begin{aligned} [H, (E_{ij}, 0, 0)] &= (\lambda_i - \lambda_j)(H)(E_{ij}, 0, 0), \\ [H, (0, F_{ij}, 0)] &= (\lambda_i + \lambda_{\ell+1-j})(H)(0, F_{ij}, 0), \\ [H, (0, 0, F_{ij})] &= -(\lambda_{\ell+1-i} + \lambda_j)(H)(0, 0, F_{ij}). \end{aligned}$$

Hence $\Phi = \{\lambda_i - \lambda_j \ (i \neq j),\ \pm(\lambda_i + \lambda_j) \ (1 \leqq i \leqq j \leqq \ell)\}$ and $(E_{ij}, 0, 0)$, $(0, F_{i,\ell+1-j}, 0)$, $(0, 0, F_{\ell+1-i,j})$ are root vectors for $\lambda_i - \lambda_j$,

$\lambda_i + \lambda_j$, $-(\lambda_i + \lambda_j) \in \Phi$ respectively. Let us choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ by putting

$$\begin{cases} \alpha_i = \lambda_i - \lambda_{i+1} & \text{for } i = 1, \ldots, \ell - 1, \\ \alpha_\ell = 2\,\lambda_\ell. & \end{cases}$$

We have

$$\begin{cases} \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1} \quad (1 \leqq i < j \leqq \ell), \\ \lambda_i + \lambda_j = (\alpha_i + \cdots + \alpha_{\ell-1}) + (\alpha_j + \cdots + \alpha_\ell) \quad (1 \leqq i \leqq j \leqq \ell). \end{cases}$$

Hence $\theta = 2\,\alpha_1 + \cdots + 2\,\alpha_{\ell-1} + \alpha_\ell$. Then we see that the gradation of $(C_\ell, \{\alpha_i\})$ is given by the following diagram;

$$\begin{array}{c|c|c|c|} & i & & i \\ \hline i & 0 & 1 & 2 \\ \hline & -1 & 0 & 1 \\ \hline i & -2 & -1 & 0 \\ \hline \end{array} \quad (1 \leqq i < \ell) \qquad\qquad \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline \end{array} \quad (i = \ell)$$

Then the diagram of $(C_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ is obtained by superposing the $k$ diagrams of $(C_\ell, \{\alpha_{i_1}\})$, ... , $(C_\ell, \{\alpha_{i_k}\})$. Here two intermediate lines ($i$-th and $(2\ell - i)$-th lines) correspond to the simple root $\{\alpha_i\}$ for $i = 1$, ... , $\ell - 1$ and the center line corresponds to $\{\alpha_\ell\}$.

By this description of gradation, we see that the model space $M_{\mathfrak{g}}$ of $(C_\ell, \{\alpha_i\})$ is the Grassmann manifold $Sp\text{-}Gr(i, V)$ consisting of all $i$-dimensional isotropic subspaces of $(V, \langle\,,\,\rangle)$. Furthermore the model space $M_{\mathfrak{g}}$ of $(C_\ell, \{\alpha_{i_1}, \ldots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$ is the flag manifold $Sp\text{-}F(i_1, \ldots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in $V$ such that $V_j$ is an $i_j$ dimensional isotropic subspace of $(V, \langle\,,\,\rangle)$ (cf. [Tt1]).

(3) $B_\ell$ $(\ell \geqq 3)$, $D_\ell$ $(\ell \geqq 4)$ type. Let $(V, (\,|\,))$ be an inner product space over $\mathbb{C}$ of dimension $2\ell$ or $2\ell + 1$, that is, $(\,|\,)$ is a non-degenerate symmetric bilinear form on $V$. Then $\mathfrak{g} = \mathfrak{o}(V)$. Let us take a basis $\{e_1, \ldots, e_\ell, e_{\ell+1}, f_1, \ldots, f_\ell\}$ of $V$ such that $(e_i|e_j) = (e_{\ell+1}|e_i) = (e_{\ell+1}|f_i) = (f_i|f_j) = 0$, $(e_{\ell+1}|e_{\ell+1}) = 1$ and $(e_i|f_{\ell+1-j}) = \delta_{ij}$ for $i$, $j = 1$, ... , $\ell$. Here we neglect $e_{\ell+1}$, when $\dim V = 2\ell$. Then we have a matrices representation

$$\mathfrak{g} = \{\, X \in \mathfrak{gl}(n, \mathbb{C}) \mid {}^tXS + SX = 0 \,\}, \quad \text{where } S = \begin{pmatrix} 0 & 0 & K \\ 0 & 1 & 0 \\ K & 0 & 0 \end{pmatrix}$$

and $n = 2\ell$ or $2\ell + 1$. Each $X \in \mathfrak{g}$ is expressed as a matrix of the form

$$X = \begin{pmatrix} A & a & B \\ \xi & 0 & -a' \\ C & -\xi' & -A' \end{pmatrix}$$

where $A$, $B$, $C$ are $\ell \times \ell$ matrices such that $B = -B'$, $C = -C'$ and $a$, $\xi$ are column and row $\ell$-vector respectively such that $a'$ and $\xi'$ are given by $a' = (a_\ell, \dots, a_1)$, $\xi' = {}^t(\xi_\ell, \dots, \xi_1)$ for $a = {}^t(a_1, \dots, a_\ell)$, $\xi = (\xi_1, \dots, \xi_\ell)$ respectively. Here the center column and the center row of $X$ should be deleted when $\dim V = 2\ell$. Both $B$ and $C$ are skew symmetric with respect to the anti-diagonal line. In particular all the anti-diagonal components $x_{i,n+1-i}$ of $X$ are 0. Thus $X$ is determined by its upper anti-diagonal part. We write $X = (A, B, C, a, \xi)$, in short.

We take a Cartan subalgebra $\mathfrak{h}$ consisting of all diagonal elements of the form $H = (\mathrm{diag}(a_1, \dots, a_\ell), 0, 0, 0, 0)$. Let $\lambda_1, \dots, \lambda_\ell$ be the linear form on $\mathfrak{h}$ defined by $\lambda_i \colon H \mapsto a_i$. We put $G_{ij} = E_{ij} - E'_{ij}$ and $E_i = (\delta_{1i}, \dots, \delta_{\ell i}) \in \mathbb{C}^\ell$. Then we have

$$\begin{aligned}
[H, (E_{ij}, 0, 0, 0, 0)] &= (\lambda_i - \lambda_j)(H)(E_{ij}, 0, 0, 0, 0), \\
[H, (0, G_{ij}, 0, 0, 0)] &= (\lambda_i + \lambda_{\ell+1-j})(H)(0, G_{ij}, 0, 0, 0), \\
[H, (0, 0, G_{ij}, 0, 0)] &= -(\lambda_{\ell+1-i} + \lambda_j)(H)(0, 0, G_{ij}, 0, 0), \\
[H, (0, 0, 0, E_i, 0)] &= \lambda_i(H)(0, 0, 0, E_i, 0), \\
[H, (0, 0, 0, 0, E_i)] &= -\lambda_i(H)(0, 0, 0, 0, E_i).
\end{aligned}$$

Hence we have

$$\Phi = \begin{cases} \{\lambda_i - \lambda_j \ (i \neq j),\ \pm(\lambda_i + \lambda_j) \ (1 \leqq i < j \leqq \ell)\} & \text{if } n = 2\ell, \\ \{\pm\lambda_i \ (1 \leqq i \leqq \ell),\ \lambda_i - \lambda_j \ (i \neq j), & \\ \qquad\qquad \pm(\lambda_i + \lambda_j) \ (1 \leqq i < j \leqq \ell)\} & \text{if } n = 2\ell + 1. \end{cases}$$

$(E_{ij}, 0, 0, 0, 0)$, $(0, G_{i,\ell+1-j}, 0, 0, 0)$, $(0, 0, G_{\ell+1-i,j}, 0, 0)$, $(0, 0, 0, E_i, 0)$ and $(0, 0, 0, 0, E_i)$ are root vectors for $\lambda_i - \lambda_j$, $\lambda_i + \lambda_j$, $-(\lambda_i + \lambda_j)$, $\lambda_i$ and $-\lambda_i \in \Phi$ respectively. Let us choose a simple root system $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ by putting

(i) $B_\ell$ type $\quad \begin{cases} \alpha_i = \lambda_i - \lambda_{i+1} & \text{for } i = 1, \dots, \ell - 1, \\ \alpha_\ell = \lambda_\ell. \end{cases}$

(ii) $D_\ell$ type $\quad \begin{cases} \alpha_i = \lambda_i - \lambda_{i+1} & \text{for } i = 1, \dots, \ell - 1, \\ \alpha_\ell = \lambda_{\ell-1} + \lambda_\ell. \end{cases}$

Then we have

(i) $B_\ell$ type

$$\begin{cases} \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1} & (1 \leqq i < j \leqq \ell), \\ \qquad \lambda_i = \alpha_i + \cdots + \alpha_\ell & (1 \leqq i \leqq \ell), \\ \lambda_i + \lambda_j = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_\ell & (1 \leqq i < j \leqq \ell). \end{cases}$$

Hence $\theta = \alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_\ell$.

(ii) $D_\ell$ type

$$\begin{cases} \lambda_i - \lambda_j = \alpha_i + \cdots + \alpha_{j-1} & (1 \leqq i < j \leqq \ell), \\ \lambda_i + \lambda_\ell = \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell & (1 \leqq i \leqq \ell - 2), \\ \lambda_{\ell-1} + \lambda_\ell = \alpha_\ell & \\ \lambda_i + \lambda_{\ell-1} = \alpha_i + \cdots + \alpha_{\ell-1} + \alpha_\ell & (1 \leqq i \leqq \ell - 2), \\ \lambda_i + \lambda_j = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell & \\ & (1 \leqq i < j \leqq \ell - 2). \end{cases}$$

Hence $\theta = \alpha_1 + 2\,\alpha_2 + \cdots + 2\,\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$.

Then we see that the gradation of $(B_\ell, \{\alpha_i\})$ is given by the following diagram;

$$\begin{array}{ccc} 1 & n-2 & 1 \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{*} \\ \hline \multicolumn{1}{|c|}{-1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{*} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \quad (i = 1) \qquad \begin{array}{ccc} i & n-2i & i \\ \hline \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \multicolumn{1}{|c|}{-1} & \multicolumn{1}{c|}{0} & \multicolumn{1}{c|}{1} \\ \hline \multicolumn{1}{|c|}{-2} & \multicolumn{1}{c|}{-1} & \multicolumn{1}{c|}{0} \\ \hline \end{array} \quad (1 < i \leqq \ell)$$

The gradation of $(D_\ell, \{\alpha_i\})$ is given by the same diagram as above for $i = 1, \ldots, \ell - 2$ and the above diagram with $i = \ell - 1$ is that of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$. Moreover the diagrams of $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ are given as follows

$$\begin{array}{c|c|c|c|c|} \cline{2-5} \ell-1 & 0 & 1 & 0 & 1 \\ \cline{2-5} 1 & -1 & 0 & * & 0 \\ \cline{2-5} 1 & 0 & * & 0 & 1 \\ \cline{2-5} \ell-1 & -1 & 0 & -1 & 0 \\ \cline{2-5} \end{array} \quad (i = \ell - 1) \qquad \begin{array}{c|c|c|} \cline{2-3} \ell & 0 & 1 \\ \cline{2-3} \ell & -1 & 0 \\ \cline{2-3} \end{array} \quad (i = \ell)$$

Clearly, by interchanging $e_\ell$ and $f_1$, matrices representations of $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ transforms each other, i.e., $(D_\ell, \{\alpha_{\ell-1}\})$ and $(D_\ell, \{\alpha_\ell\})$ are conjugate. The other gradations of $B_\ell$ or $D_\ell$ type can be obtained by the principle of superposition as in the previous cases. Here two intermediate lines ($i$-th and $(n-i)$-th lines) correspond to the simple root $\{\alpha_i\}$ for $i = 1, \dots, \ell$ in case of type $B_\ell$ and for $i = 1, \dots, \ell - 2$ in case of type $D_\ell$. Moreover in case of type $D_\ell$, $(\ell-1)$-th and $(\ell+1)$-th intermediate lines correspond to the pair $\{\alpha_{\ell-1}, \alpha_\ell\}$ and the center line corresponds to $\{\alpha_\ell\}$.

By this description of gradations, we see that the Grassmann manifold $O\text{-}Gr(i, V)$ consisting of all $i$-dimensional isotropic subspaces of $(V, (\,|\,))$ is the model space $M_\mathfrak{g}$ of $(B_\ell, \{\alpha_i\})$ or $(D_\ell, \{\alpha_i\})$ according as $\dim V = 2\ell + 1$ or $2\ell$, except for the case when $i = \ell - 1$ and $\dim V = 2\ell$. In the latter case $O\text{-}Gr(\ell-1, V)$ is the model space $M_\mathfrak{g}$ of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$, where $\dim V = 2\ell$. Thus, for $D_\ell$ type, we make a following convention for a subset $\Delta_1$ of $\Delta$: If $\alpha_{\ell-1} \in \Delta_1$ and $\alpha_\ell \notin \Delta_1$, we replace $\alpha_{\ell-1}$ by $\alpha_\ell$ (the conjugacy class of $(D_\ell, \Delta_1)$ does not change by this replacement), and if both $\alpha_{\ell-1}$ and $\alpha_\ell \in \Delta_1$, we write $\alpha^*_{\ell-1} = \{\alpha_{\ell-1}, \alpha_\ell\}$. Under this convention, we see that the model space $M_\mathfrak{g}$ of $(B_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ or $(D_\ell, \{\alpha_{i_1}, \dots, \alpha_{i_k}\})$ $(1 \leqq i_1 < \cdots < i_k \leqq \ell)$ is the flag manifold $O\text{-}F(i_1, \dots, i_k; V)$ consisting of all flags $\{V_1 \subset \cdots \subset V_k\}$ in $V$ such that $V_j$ is an $i_j$-dimensional isotropic subspace of $(V, (\,|\,))$, according as $\dim V = 2\ell + 1$ or $2\ell$ (cf. [Tt1]).

## 4.5. Canonical systems on Grassmann bundles

First we recall the notion of canonical systems on Grassmann bundles ([Y1], [Y2]). Let $M$ be a (real or complex) manifold of dimension $m + n$. We consider the Grassmann bundle $J(M, n)$ over $M$ consisting of all $n$-dimensional contact elements to $M$;

$$J(M, n) = \bigcup_{x \in M} J_x(M, n),$$

where $J_x(M, n) = Gr(n, T_x(M))$ is the Grassmann manifold of all $n$-dimensional subspaces of the tangent space $T_x(M)$ to $M$ at $x$. Let $\pi$ be the projection of $J(M, n)$ onto $M$. Each element $u \in J(M, n)$ is a linear subspace of $T_x(M)$ of codimension $m$, where $x = \pi(u)$. Hence we have a differential system $C$ of codimension $m$ on $J(M, n)$ by putting

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M, n)) \qquad \text{at } u \in J(M, n).$$

$C$ is called the canonical system on $J(M, n)$. $J(M, n)$ is the (geometrical) 1-jet space for $n$-dimensional submanifolds in $M$ and $C$ is the contact system on this jet space (cf. [Y2, §1]). In fact let us fix $u_o \in J(M,n)$ and take an inhomogeneous Grassmann coordinate system $(x^1,\dots,x^n,y^1,\dots,y^m,p_i^\alpha)$ $(1 \leqq i \leqq n,\ 1 \leqq \alpha \leqq m)$ of $J(M,n)$ in a neighborhood $U$ of $u_o$, that is, $(x^1,\dots,x^n,y^1,\dots,y^m)$ is a pull back of a coordinate system on $M$ around $x_o = \pi(u_o)$ such that $dx^1\wedge\cdots\wedge dx^n\ |_u \neq 0$ for $u \in U$ and $p_i^\alpha(u)$ is defined by $dy^\alpha|_u = \sum_{i=1}^n p_i^\alpha(u)dx^i|_u$. Then the canonical system $C$ is given in this coordinate system by

$$C = \{\,\varpi^1 = \cdots = \varpi^m = 0\,\},$$

where $\varpi^\alpha = dy^\alpha - \sum_{i=1}^n p_i^\alpha dx^i$ $(1 \leqq \alpha \leqq m)$.

Furthermore, starting from a contact manifold $(J,C)$ of dimension $2n+1$, which can be regarded locally as a space of 1-jets for one unknown function by Darboux's theorem, we can construct a geometric second order jet space $(L(J),E)$ as follows. We consider the Lagrange-Grassmann bundle $L(J)$ over $J$ consisting of all $n$-dimensional integral elements of $(J,C)$;

$$L(J) = \bigcup_{u\in J} L_u(J),$$

where $L_u(J) = \text{Sp-Gr}(n, C(u))$ is the Grassmann manifold of all lagrangian (or legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here $\varpi$ is a local contact form on $J$. Let $\pi$ be the projection of $L(J)$ onto $J$. Then the canonical system $E$ on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \qquad \text{at } v \in L(J).$$

Starting from a canonical coordinate system $(x^1,\dots,x^n,z,p_1,\dots,p_n)$ of $(J,C)$, we can introduce a coordinate system $(x^i,z,p_j,p_{ij})$ $(1 \leqq i \leqq j \leqq n)$ of $L(J)$ such that $p_{ij} = p_{ji}$ and $E$ is defined by

$$E = \{\,\varpi = \varpi_1 = \cdots = \varpi_n = 0\,\},$$

where $\varpi = dz - \sum_{i=1}^n p_i dx^i$ and $\varpi_i = dp_i - \sum_{j=1}^n p_{ij}dx^j$ $(1 \leqq i \leqq n)$. For the details, we refer the reader to [Y1].

These canonical systems appear among our standard differential systems in the following cases.

(1) $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ $(1 \leqq i < \ell)$. Let $V$ be a complex vector space of dimension $\ell+1$. By the discussion in 4.4, we know that the model space $M_{\mathfrak{g}}$ of $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ is given by

$$M_{\mathfrak{g}} = \{\,([v], W) \in P(V) \times Gr(i+1, V) \mid [v] \subset W\,\}.$$

Let $p$ be the projection of $M_{\mathfrak{g}}$ onto $P(V)$. Each fibre of $p\colon M_{\mathfrak{g}} \to P(V)$ is a Grassmann manifold $Gr(i, V/[v])$. At each $x = [v] \in P(V)$, we can naturally identify $T_x(P(V))$ with the quotient space $V/[v]$. With this identification, we have a fibre-preserving diffeomorphism $\varphi$ of $M_{\mathfrak{g}}$ onto $J(P(V), i)$ defined by

$$\varphi(u) = W/[v] \subset V/[v] \cong T_x(P(V)) \qquad \text{for } u = ([v], W) \in M_{\mathfrak{g}}.$$

Moreover let us fix a basis $\{e_0, \ldots, e_\ell\}$ of $V$ and put $x_0 = [e_0]$ and $u_0 = ([e_0], W_0)$, where $W_0 = \langle e_0, \ldots, e_i\rangle$. Let $\pi^1$ and $\pi^2$ denote the projection of $G = SL(V)$ onto $P(V)$ and $M_{\mathfrak{g}}$ defined by $\pi^1(g) = g(x_0)$ and $\pi^2(g) = g(u_0)$ for $g \in G$ respectively. Then, from the matrices description of $(A_\ell, \{\alpha_1, \alpha_{i+1}\})$ in 4.4, we see that

$$(\pi^1_*)^{-1}(W_0/x_0) = \mathfrak{f}^{-1}.$$

Hence it follows from $p \cdot \pi^2 = \pi^1$ that $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is isomorphic with the canonical differential system $(J(P(V), i), C)$ via $\varphi$. Especially $(J(P(V), \ell - 1), C)$ is the standard contact manifold of type $A_\ell$, which is also naturally identified with the projective cotangent bundle $PT^*(P(V))$ over $P(V)$ with its contact structure induced from the symplectic structure on $T^*(P(V))$ (cf. [Bo], [A]). Here we note that the above argument is valid also for the normal real form $\mathfrak{sl}(\ell+1, \mathbb{R})$ of $\mathfrak{sl}(\ell+1, \mathbb{C})$.

(2) $(C_\ell, \{\alpha_1, \alpha_\ell\})$. Let us start with the contact gradation $(C_\ell, \{\alpha_1\})$. From 4.4, we see that the model space of $(C_\ell, \{\alpha_1\})$ is the projective space $P(V)$, where $(V, \langle\ ,\ \rangle)$ is a symplectic vector space over $\mathbb{C}$ of dimension $2\ell$. Let us take a symplectic basis $\{e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\}$ as in 4.4 and let $\pi^1$ denote the projection of $G = Sp(V)$ onto $P(V)$ given by $\pi^1(g) = g([e_1])$ for $g \in G$. Then, under the identification $V/[e_1] \cong T_{x_0}(P(V))$, $x_0 = [e_1]$, we see from the matrices description in 4.4 that $(\pi^1_*)^{-1}([e_1]^\perp/[e_1]) = \mathfrak{f}^{-1}$, where $[e_1]^\perp = \{\, v \in V \mid \langle v, e_1\rangle = 0\,\}$. Thus we see that the contact structure $C$ on $P(V)$ is given by (cf. [K1])

$$C(x) = \operatorname{Ker}\alpha/[v] \subset V/[v] \cong T_x(P(V)) \quad \text{at each } x = [v] \in P(V),$$

where $\alpha$ is the linear symplectic form defined on $V$ by $\alpha_v(w) = \langle v, w\rangle$ for $v$, $w \in V$.

The model space of $(C_\ell, \{\alpha_1, \alpha_\ell\})$ is given by

$$M_{\mathfrak{g}} = \{\, ([v], L) \in P(V) \times \text{Sp-Gr}(\ell, V) \mid [v] \subset L\,\}.$$

Let $p$ be the projection of $M_{\mathfrak{g}}$ onto $P(V)$. We have a fibre-preserving diffeomorphism $\varphi$ of $M_{\mathfrak{g}}$ onto the Lagrange-Grassmann bundle $L(P(V))$

over $P(V)$ defined by

$$\varphi(u) = L/[v] \subset \operatorname{Ker} \alpha/[v] \cong C(x) \quad \text{for } u = ([v], L) \text{ and } x = [v].$$

Let $\pi^2$ denote the projection of $G$ onto $M_{\mathfrak{g}}$ given by $\pi^2(g) = g(u_o)$ for $g \in G$, where $u_o = ([e_1], L_o)$ and $L_o = \langle e_1, \ldots, e_\ell \rangle$. Then we have

$$(\pi^1_*)^{-1}(L_o/[e_1]) = \mathfrak{f}^{-1}$$

from the following diagram for $(C_\ell, \{\alpha_1, \alpha_\ell\})$;

| | | | | |
|---|---|---|---|---|
| $1$ | $0$ | $1$ | $2$ | $3$ |
| $\ell - 1$ | $-1$ | $0$ | $1$ | $2$ |
| $\ell - 1$ | $-2$ | $-1$ | $0$ | $1$ |
| $1$ | $-3$ | $-2$ | $-1$ | $0$ |

Hence it follows from $p \cdot \pi^2 = \pi^1$, that $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is isomorphic with the canonical differential system $(L(P(V)), E)$ via $\varphi$. We here note that the above argument is valid also for the normal real form $\mathfrak{sp}(\ell, \mathbb{R})$ of $\mathfrak{sp}(\ell, \mathbb{C})$.

Finally we shall add another construction of standard contact manifolds of type $B_\ell$ or $D_\ell$ and those of their real forms. Let $(V, (\,|\,))$ be an inner product space over $K = \mathbb{R}$ or $\mathbb{C}$, that is, $(\,|\,)$ is a non-degenerate symmetric bilinear form over $K$ on $V$. In the real case, we assume that $(V, (\,|\,))$ is indefinite and admits 2-dimensional isotropic subspaces.

Let $W = V \oplus V$ be the direct sum of two copies of $V$. The inner product $(\,|\,)$ induces a skew symmetric bilinear form $\langle\, , \rangle$ on $W$ by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = (x_1 | y_2) - (x_2 | y_1).$$

Then $(W, \langle\, , \rangle)$ is a symplectic vector space. Let $\omega$ be the 1-form on $W$ defined by $\omega = (x|dy) - (y|dx)$, where $(x, y)$ is the linear coordinate system of $W$. Put $\alpha = \frac{1}{2}\omega$. Thus $(W, d\alpha)$ is a symplectic manifold.

$GL(2, K)$ acts on $W$ on the right as follows;

$$(x, y)\sigma = (ax + cy, bx + dy) \qquad \text{for } \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, K).$$

We have $R^*_\sigma \alpha = (\det \sigma)\, \alpha$ for $\sigma \in GL(2, K)$. Hence $SL(2, K)$ acts on $(W, d\alpha)$ as a group of symplectic transformations. Let $\{X, H, Y\}$ be the basis of $\mathfrak{sl}(2, K)$ given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $SL(2,K)$-action on $W$ induces hamiltonian vector fields

$$X^* = \sum_{i=1}^n x_i \frac{\partial}{\partial y_i}, \quad H^* = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}), \quad Y^* = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i},$$

with hamiltonians $\alpha(X^*) = \frac{1}{2}(x|x)$, $\alpha(H^*) = -(x|y)$, $\alpha(Y^*) = -\frac{1}{2}(y|y)$ respectively. Thus we have a momentum mapping $f\colon W \to \mathfrak{sl}(2,K)^*$ given by

$$f(w)(Z) = \alpha_w(Z^*) \qquad \text{for } w \in W \text{ and } Z \in \mathfrak{sl}(2,K).$$

Then we have

$$f^{-1}(0) = \{\, (x,y) \in W \mid (x|x) = (x|y) = (y|y) = 0 \,\}.$$

By our assumption on $(\,|\,)$, $f^{-1}(0)$ is a non-empty variety in $W$. Let $F$ be the regular part of $f^{-1}(0)$;

$$F = \{\, (x,y) \in W \mid (x|x) = (x|y) = (y|y) = 0, x \wedge y \neq 0 \,\}.$$

$GL(2,K)$ acts freely on $F$ on the right. Moreover the orthogonal group $O(V)$ of $(V, (\,|\,))$ acts on $F$ in the obvious way. As is well-known (cf. [A, Appendix 5]), the reduced phase space $S = F/SL(2,K)$ is a symplectic manifold over $K$. In fact, since $\alpha$ is $SL(2,K)$-invariant, the restriction $\theta = \alpha|_F$ of $\alpha$ to $F$ projects to $S = F/SL(2,K)$ so that $d\theta$ is a symplectic form on $S$. Furthermore the quotient space $J = F/GL(2,K)$ is naturally identified with $O\text{-}Gr(2,V)$. Thus $F$ is the total space of the universal 2-frame bundle over $J = O\text{-}Gr(2,V)$ and $(S,J,p)$ is a principal $K^\times$-bundle over $J$, where $p\colon S \to J$ denotes the natural projection. Then as in 4.3, the contact structure $C$ on $J$ is defined by $\theta = 0$ so that $(S, d\theta)$ is the symplectification of $(J,C)$. From the equivalence of the adjoint representation and the exterior representation on $\wedge^2 V$ for $O(V)$, it follows that $(S, d\theta)$ is isomorphic with the adjoint orbit constructed in 4.3, which implies that $(J,C)$ is isomorphic with the standard contact manifold of type $B_\ell$, $D_\ell$ or one of their real forms.

## §5. Infinitesimal automorphisms of $(M_\mathfrak{g}, D_\mathfrak{g})$

### 5.1. Review of harmonic theory (Kostant's Theorem)

We here review the harmonic theory of Kostant [K] for the Lie algebra cohomology, which enables us to compute the generalized Spencer cohomology groups $H^q(\mathfrak{m}, \mathfrak{g})$ (cf. [O2]).

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Let us fix a Cartan subalgebra $\mathfrak{h}$ containing $E$. In accordance with Kostant's paper [K], let us fix a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ such that $\alpha(E) = 0$ or $-1$ for $\alpha \in \Delta$ throughout this section. Thus, by putting $\widehat{\Phi}^+ = \Phi^+ \setminus \Phi_0^+$, $\mathfrak{m}$ is a direct sum of positive root subspaces:

$$\mathfrak{m} = \bigoplus_{\alpha\in\widehat{\Phi}^+} \mathfrak{g}_\alpha.$$

Let us take a compact real form $\mathfrak{u}$ of $\mathfrak{g}$ by choosing a Weyl basis of the root space decomposition relative to $\mathfrak{h}$ (cf. [He, p. 421]). Let $\tau$ denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}$. Then $E \in \mathfrak{h}_\mathbb{R} \subset \sqrt{-1}\mathfrak{u}$ and we have a hermitian inner product $\{\,,\,\}$ of $\mathfrak{g}$, which is given by

$$\{X, Y\} = -B(X, \tau(Y)) \qquad \text{for } X, Y \in \mathfrak{g}.$$

By our choice of $\mathfrak{u}$, we have $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Phi$. For a linear subspace $\mathfrak{a}$ of $\mathfrak{g}$, we put $\widehat{\mathfrak{a}} = \tau(\mathfrak{a})$ and $\mathfrak{a}^o = \{\, X \in \mathfrak{g} \mid B(X, \mathfrak{a}) = 0\,\}$. Then the orthogonal complement $\mathfrak{a}^\perp$ of $\mathfrak{a}$ with respect to $\{\,,\,\}$ coincides with $\widehat{\mathfrak{a}}^o$. By definition of $\{\,,\,\}$, it follows that the Killing form $B$ gives a non-degenerate pairing of $\mathfrak{a}$ and $\widehat{\mathfrak{a}}$ (cf. Lemma 3.1). Especially we have

$$\widehat{\mathfrak{m}} = \bigoplus_{p>0} \mathfrak{g}_p, \qquad \mathfrak{h} = \mathfrak{B}(\Delta) \cap \widehat{\mathfrak{B}(\Delta)},$$

where $\mathfrak{B}(\Delta) = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ is a standard Borel subalgebra relative to $\mathfrak{h}$. Moreover, starting from a parabolic subalgebra $\mathfrak{P}_- = \bigoplus_{p\leqq 0} \mathfrak{g}_p$ containing $\mathfrak{B}(\Delta)$, we have

$$\mathfrak{m} = \mathfrak{P}_-^o, \qquad \mathfrak{g}_0 = \mathfrak{P}_- \cap \widehat{\mathfrak{P}}_-,$$

and the orthogonal decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0 \oplus \widehat{\mathfrak{m}}.$$

Thus $\mathfrak{m}$ is a nilpotent Lie summand of $\mathfrak{g}$ in the sense of §5 in [K] and the argument in [K] is thoroughly applicable to our situation.

We shall summarize the argument in [K] in the following. Let $(C(\mathfrak{m}, \mathfrak{g}), \partial)$ be the cochain complex (the generalized Spencer complex) associated with the representation $\mathrm{ad}\colon \mathfrak{m} \to \mathfrak{gl}(\mathfrak{g})$. Namely $C(\mathfrak{m}, \mathfrak{g}) = \mathfrak{g} \otimes \wedge\mathfrak{m}^*$ and $\partial\colon C^q(\mathfrak{m}, \mathfrak{g}) \to C^{q+1}(\mathfrak{m}, \mathfrak{g})$ is given as in §2.4. The hermitian inner product $\{\,,\,\}$ of $\mathfrak{g}$ induces the hermitian inner product of

$C(\mathfrak{m},\mathfrak{g}) = \mathfrak{g}\otimes\wedge\mathfrak{m}^*$ and $\mathfrak{g}\otimes\wedge\mathfrak{g}$ in a natural manner. Let $\{e_1,\dots,e_{n_1}\}$ be a (orthnormal) basis of $\mathfrak{m}$ and let $\{e_1^*,\dots,e_{n_1}^*\}$ be the dual basis of $\widehat{\mathfrak{m}}$ under the Killing form duality. Then the adjoint operator $\partial^*\colon C^{q+1}(\mathfrak{m},\mathfrak{g}) \to C^q(\mathfrak{m},\mathfrak{g})$ of $\partial$ with respect to this inner product is given by the following formula ([T4, Lemma 1.10], [K, Lemma 4.2]);

$$(\partial^* p)(X_1,\dots,X_q) = \sum_j [e_j^*, p(e_j, X_1,\dots,X_q)] + \tfrac{1}{2}\sum_{i,j}(-1)^{i+1} p([e_j^*, X_i]_-, e_j, X_1,\dots,\check{X}_i,\dots,X_q),$$

for $p\in C^{q+1}(\mathfrak{m},\mathfrak{g})$ and $X_1, \dots, X_q\in\mathfrak{m}$, where $[e_j^*,X_i]_-$ denotes the $\mathfrak{m}$-component of $[e_j^*,X_i]$ with respect to the decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{g}'$. Here we note that $\partial^*$ sends $C^{p,q+1}(\mathfrak{m},\mathfrak{g})$ into $C^{p+1,q}(\mathfrak{m},\mathfrak{g})$ and $\partial^* p$ does not depend on the choice of the basis $\{e_1,\dots,e_{n_1}\}$, hence, nor on the choice of the compact real form $\mathfrak{u}$.

Now, in order to describe the harmonic space $\mathcal{H} = \operatorname{Ker}\Box$ for the Laplacian $\Box = \partial\partial^* + \partial^*\partial$, we shall utilize the natural representation of $\mathfrak{g}_0$ on the cochain space $C(\mathfrak{m},\mathfrak{g}) = \mathfrak{g}\otimes\wedge\mathfrak{m}^*$. In fact $\mathfrak{m} = \bigoplus_{p<0}\mathfrak{g}_p$ is $\operatorname{ad}(\mathfrak{g}_0)$-invariant. Hence, from the $\mathfrak{g}_0$-module $\mathfrak{m}$, we have the $\mathfrak{g}_0$-module $\mathfrak{m}^*$ contragradient to $\mathfrak{m}$. Let

$$\rho\colon \mathfrak{g}_0 \to \mathfrak{gl}(C(\mathfrak{m},\mathfrak{g}))$$

be the representation of $\mathfrak{g}_0$ on $C(\mathfrak{m},\mathfrak{g})$ formed by taking the tensor product of $\operatorname{ad}\colon \mathfrak{g}_0\to\mathfrak{gl}(\mathfrak{g})$ and the exterior representation of $\mathfrak{g}_0$ on $\wedge\mathfrak{m}^*$. $\rho$ is a completely reducible representation of the reductive Lie algebra $\mathfrak{g}_0$ (cf. Proposition 3.6). Here we note that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Let $\widehat{G}$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\widehat{G}_0$ be the analytic subgroup of $\widehat{G}$ with Lie algebra $\mathfrak{g}_0$. Then an irreducible representation of $\mathfrak{g}_0$, which is induced from a representation of $\widehat{G}_0$, is described as a standard cyclic module with heighest weight $\xi\in D_0$ ([K, §5.5], [Hu, Chapter VI]). Here $\xi$ is a dominant integral weight in

$$D_0 = \{\,\mu\in\Lambda \mid \langle\mu,\alpha\rangle \geqq 0 \quad \text{for each } \alpha\in\Phi_0^+\,\},$$

where $\Lambda = \{\,\mu\in\mathfrak{h}^\sharp \mid \langle\mu,\alpha\rangle\in\mathbb{Z} \text{ for each } \alpha\in\Phi\,\}$. Moreover, under the identification of $\mathfrak{g}\otimes\wedge\mathfrak{m}^*$ with $\mathfrak{g}\otimes\wedge\widehat{\mathfrak{m}}$ via the Killing form duality between $\mathfrak{m}$ and $\widehat{\mathfrak{m}}$, the representation $\rho$ is equivalent to the subrepresentation $\widehat{\rho} = \widehat{\operatorname{ad}}|_{\mathfrak{g}\otimes\wedge\widehat{\mathfrak{m}}}$ on $\mathfrak{g}\otimes\wedge\widehat{\mathfrak{m}}$ of the tensor representation $\widehat{\operatorname{ad}}$ of $\mathfrak{g}_0$ on $\mathfrak{g}\otimes\wedge\mathfrak{g}$ induced from $\operatorname{ad}\colon \mathfrak{g}_0\to\mathfrak{gl}(\mathfrak{g})$. Hence the weight space decomposition of $\mathfrak{g}_0$-module $\mathfrak{g} \otimes \wedge \widehat{\mathfrak{m}}$ is provided by the root space decomposition of $\mathfrak{g}$. More precisely, let $\Lambda^{\hat{\rho}}$ be the set of weights of $\widehat{\rho}$. Then we have

$$\Lambda^{\hat{\rho}} = \{\, \xi = \alpha - \langle A \rangle \in \Lambda \mid \alpha \in \Phi \cup \{0\}, \quad A \subset \widehat{\Phi}^+ \,\},$$

where $\langle A \rangle = \sum_{\alpha \in A} \alpha$.

By Lemma 3.1, we know that the restriction of the Killing form $B$ to $\mathfrak{g}_0$ is non-degenerate. Let $C_\rho \in \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$ be the Casimir operator corresponding to the restriction of $B$ to $\mathfrak{g}_0$, that is,

$$C_\rho = \sum_{i=1}^{n_0} \rho(X_i) \cdot \rho(Y_i),$$

where $\{X_1, \ldots, X_{n_0}\}$ and $\{Y_1, \ldots, Y_{n_0}\}$ are basis of $\mathfrak{g}_0$ such that $B(X_i, Y_i) = \delta_{ij}$. We put

$$\delta = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \quad \delta_1 = \tfrac{1}{2} \sum_{\alpha \in \widehat{\Phi}^+} \alpha \quad \text{and} \quad F = t_{\delta_1} \in \mathfrak{h},$$

where $t_{\delta_1}$ is defined by $B(t_{\delta_1}, H) = \delta_1(H)$ for $H \in \mathfrak{h}$. Let $\sigma_\alpha$ denote the reflection in $\mathfrak{h}^\sharp = \langle \Phi \rangle_{\mathbb{R}}$ corresponding to $\alpha \in \Phi$, that is,

$$\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle\, \alpha \qquad \text{for } \beta \in \mathfrak{h}^\sharp.$$

From $\sigma_\alpha(\delta_1) = \delta_1$ for $\alpha \in \Phi_0$, we have $(\delta_1, \alpha) = 0$ for $\alpha \in \Phi_0$, which implies that $F \in Z(\mathfrak{g}_0)$ by Proposition 3.6. Then we have the following expression of the Laplacian $\Box$ on $C(\mathfrak{m}, \mathfrak{g})$ ([K, Theorem 5.7]);

$$\Box = \tfrac{1}{2}(|\delta + \theta|^2 - |\delta|^2) \cdot id - (\rho(F) + \tfrac{1}{2} C_\rho), \tag{5.1}$$

where $|\alpha|$ denotes the length of $\alpha \in \mathfrak{h}^\sharp$ and $\theta$ is the highest root. This expression of $\Box$ can be obtained by expressing the operators $\partial$ and $\partial^*$ in terms of elementary operations in $\mathfrak{g} \otimes \wedge \mathfrak{g}$ under the identification of $\mathfrak{g} \otimes \wedge \mathfrak{m}^*$ with $\mathfrak{g} \otimes \wedge \widehat{\mathfrak{m}} \subset \mathfrak{g} \otimes \wedge \mathfrak{g}$. For the details, we refer the reader to the discussion in §§3 and 4 of [K].

The important fact on the representation $\rho \colon \mathfrak{g}_0 \to \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$ is that each $\rho(Z) \in \mathfrak{gl}(C(\mathfrak{m}, \mathfrak{g}))$, $Z \in \mathfrak{g}_0$, commutes with both operators $\partial$ and $\partial^*$, which can be easily checked by utilizing the above expression of $\partial$ and $\partial^*$ ([K, §5], [T4, Lemma 1.11]). Thus the orthogonal decomposition of $C(\mathfrak{m}, \mathfrak{g})$;

$$C(\mathfrak{m}, \mathfrak{g}) = \operatorname{Im} \partial \oplus \operatorname{Im} \partial^* \oplus \mathcal{H}$$

is stable under $\rho(Z)$ for all $Z \in \mathfrak{g}_0$ and $C(\mathfrak{m},\mathfrak{g})$ has the isotypic decomposition as a $\mathfrak{g}_0$-module;

$$C(\mathfrak{m},\mathfrak{g}) = \bigoplus_{\xi \in D_0} C^{\xi},$$

where $C^{\xi}$ is the isotypic component of $C(\mathfrak{m},\mathfrak{g})$ with highest weight $\xi \in D_0$. Namely $C^{\xi}$ is the sum of irreducible components in $C(\mathfrak{m},\mathfrak{g})$ with highest weight $\xi$. Then, by (5.1) and the Schur's Lemma, the Laplacian $\square$ reduces to a scalar on each isotypic component $C^{\xi}$ and this scalar is given by ([K, Theorem 5.7])

$$\tfrac{1}{2}(|\delta+\theta|^2 - |\delta+\xi|^2).$$

Hence $\mathcal{H}$ consists of isotypic components $C^{\xi}$ of $C(\mathfrak{m},\mathfrak{g})$ such that $|\delta+\theta| = |\delta+\xi|$.

Thus, to describe the harmonic space $\mathcal{H}$, we need to find $\xi \in \Lambda^{\hat{\rho}}$ such that $|\delta+\theta| = |\delta+\xi|$. This is accomplished by the Weyl group $W$ of the root system $\Phi$ as follows. For an element $\sigma \in W$, we put $\Phi^- = -\Phi^+$, $\Phi_\sigma = \sigma(\Phi^-) \cap \Phi^+$ and define the subset $W^0$ of $W$ by putting

$$W^0 = \{\, \sigma \in W \mid \Phi_\sigma \subset \widehat{\Phi}^+ \,\}.$$

Put $\xi_\sigma = \sigma(\delta+\theta) - \delta$ for $\sigma \in W^0$. Then, from $\sigma(\delta) = \delta - \langle \Phi_\sigma \rangle$, we obtain $\xi_\sigma = \sigma(\theta) - \langle \Phi_\sigma \rangle \in \Lambda^{\hat{\rho}}$ and $|\delta+\theta| = |\delta+\xi_\sigma|$. Since $\delta+\theta$ is a strongly dominant weight, the mapping $\sigma \mapsto \xi_\sigma$ of $W^0$ into $\Lambda^{\hat{\rho}}$ is one to one. In fact ([K, Lemma 5.12], [Cr]), this mapping gives a bijection of $W^0$ onto the set of highest weights in $\Lambda^{\hat{\rho}}$ appearing in the isotypic decomposition of $\mathcal{H}$ and $\dim V_{\xi_\sigma} = 1$, where $V_{\xi_\sigma}$ is the weight space of weight $\xi_\sigma$ in $\mathfrak{g} \otimes \wedge\widehat{\mathfrak{m}} \cong C(\mathfrak{m},\mathfrak{g})$. Furthermore we put

$$W(q) = \{\, \sigma \in W \mid n(\sigma) = q \,\} \quad \text{and} \quad W^0(q) = W^0 \cap W(q),$$

where $n(\sigma)$ is the number of roots in $\Phi_\sigma$. For an element $\sigma \in W^0(q)$, we put $\hat{x}_{\Phi_\sigma} = x_{-\beta_1} \wedge \cdots \wedge x_{-\beta_q}$, where $\Phi_\sigma = \{\beta_1, \ldots, \beta_q\} \subset \widehat{\Phi}^+$ and $x_{-\beta_i}$ is a root vector for the root $-\beta_i \in \widehat{\Phi}^- = -\widehat{\Phi}^+$. Then we have

$$V_{\xi_\sigma} = \langle x_{\sigma(\theta)} \otimes \hat{x}_{\Phi_\sigma} \rangle_{\mathbb{C}} \subset \mathfrak{g} \otimes \wedge^q \widehat{\mathfrak{m}},$$

which implies $C^{\xi_\sigma} \subset \mathfrak{g} \otimes \wedge^q \mathfrak{m}^*$ for $\sigma \in W^0(q)$. We denote by $x_{\Phi_\sigma}$ the element in $\wedge^q \mathfrak{m}^*$ which corresponds to $\hat{x}_{\Phi_\sigma}$ under the identification of $\wedge\mathfrak{m}^*$ with $\wedge\widehat{\mathfrak{m}}$ via the Killing form duality.

Summarizing we can state

**Theorem** (Kostant). *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then the irreducible decomposition of the harmonic space $\mathcal{H} = \operatorname{Ker}\Box$ of the generalized Spencer complex, as a $\mathfrak{g}_0$-module, is given by*

$$\mathcal{H} = \bigoplus_{\sigma\in W^0} \mathcal{H}^\sigma,$$

*where $\mathcal{H}^\sigma$ is the irreducible $\mathfrak{g}_0$-module with highest weight $\xi_\sigma = \sigma(\theta+\delta)-\delta$ generated by the highest weight vector $x_{\sigma(\theta)}\otimes x_{\Phi_\sigma} \in \mathfrak{g}\otimes\wedge\mathfrak{m}^*$. Moreover degree-wise, for any non-negative integer $q$,*

$$\mathcal{H}^q = \bigoplus_{\sigma\in W^0(q)} \mathcal{H}^\sigma.$$

Utilizing this theorem, we shall compute $H^{p,1}(\mathfrak{m},\mathfrak{g})$ and $H^{p,2}(\mathfrak{m},\mathfrak{g})$ for $p \geqq 0$ in the following paragraph.

### 5.2. Theorem on infinitesimal automorphisms of $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$

First we shall compute $H^{p,1}(\mathfrak{m},\mathfrak{g})$ for $p \geqq 0$ by virtue of Kostant's theorem. In order to apply the theorem to our computation, we note here that each $\rho(Z) \in \mathfrak{gl}(C(\mathfrak{m},\mathfrak{g}))$, $Z \in \mathfrak{g}_0$, preserves the bigradation of $C(\mathfrak{m},\mathfrak{g})$ given by $C^{p,q}(\mathfrak{m},\mathfrak{g}) = \bigoplus_{j\leqq -q} \mathfrak{g}_{j+p+q-1}\otimes\wedge^q_j\mathfrak{m}^*$. Hence each irreducible component $\mathcal{H}^\sigma$ of the $\mathfrak{g}_0$-module $\mathcal{H}$ is a subspace of some $C^{p,q}(\mathfrak{m},\mathfrak{g})$. Recall that $\mathfrak{g}_p$ is a direct sum of the root subspaces $\mathfrak{g}_\beta$ satisfying $p = \beta(E)$ and that $\mathfrak{g}_p^*$ is identified with $\mathfrak{g}_{-p}$ by the Killing form. Then, for the generator $x_{\sigma(\theta)}\otimes x_{\Phi_\sigma}$ of $\mathcal{H}^\sigma$, we have $x_{\sigma(\theta)} \in \mathfrak{g}_{\sigma(\theta)(E)}$ and $x_{\Phi_\sigma} \in \wedge^q_j\mathfrak{m}^*$, where $q = n(\sigma)$ and $j = \sum_{i=1}^q \beta_i(E)$ for $\Phi_\sigma = \{\beta_1,\dots,\beta_q\}$. Hence we have

$$\mathcal{H}^\sigma \subset C^{p,q}(\mathfrak{m},\mathfrak{g}),$$

where $q = n(\sigma)$ and $p$ can be computed from the following equality;

$$\sigma(\theta)(E) = \sum_{i=1}^q \beta_i(E) + p + q - 1, \tag{5.2}$$

One important consequence of Kostant's theorem is that $H^q(\mathfrak{m},\mathfrak{g})$ never vanishes for $q = 1$. Thus our task is to find $\Delta_1 \subset \Delta$ and $\sigma \in W^0(1)$ so that $\mathcal{H}^\sigma \subset C^{p,1}(\mathfrak{m},\mathfrak{g})$ for some $p \geqq 0$.

In the following we denote by $\sigma_i = \sigma_{\alpha_i}$ the reflection in $\mathfrak{h}^\sharp$ corresponding to the simple root $\alpha_i \in \Delta$. Then $W(1) = \{\sigma_i \in W \mid i = 1,$

$\ldots, \ell\}$ and $\Phi_{\sigma_i} = \{\alpha_i\}$ (cf. [Hu, Lemma 10.3.A]). Thus we have

$$W^0(1) = \{\,\sigma_i \in W \mid \alpha_i \in \Delta_1\,\}.$$

Recall that the depth $\mu$ of $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ is given by $\mu = \sum_{\alpha_i \in \Delta_1} n_i(\theta)$, where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\,\alpha_i$ (see §3.4). Then, by our choice of the simple root system $\Delta$ in 5.1, (5.2) reduces to

$$-\mu + \langle \theta, \alpha_i \rangle = p_i - 1 \qquad \text{for } \sigma_i \in W^0(1).$$

Hence we obtain

$$\mathcal{H}^{\sigma_i} \subset C^{p_i,1}(\mathfrak{m}, \mathfrak{g}) \qquad \text{for } \sigma_i \in W^0(1),$$

where $p_i = \langle \theta, \alpha_i \rangle - \mu + 1$.

On the other hand, from the extended Dynkin diagram in §4.2, we know that $\langle \theta, \alpha_i \rangle = 0$, 1 or 2, which implies that $p_i \geqq 0$ occurs only when $\mu \leqq 3$. More precisely $\langle \theta, \alpha_i \rangle = 2$ if and only if $\mathfrak{g}$ is of type $C_\ell$ or $A_1$ and $\alpha_i = \alpha_1$, and $\langle \theta, \alpha_i \rangle = 1$ if and only if $\mathfrak{g}$ is not of type $C_\ell$ nor $A_1$ and $\alpha_i \in \Delta_\theta$ (see 4.2). Especially if $\Delta_1 \cap \Delta_\theta = \emptyset$, we have $p_i = 1 - \mu$ for each $\sigma_i \in W^0(1)$. Hence $p_i \geqq 0$ occurs if and only if $\mu = 1$. Namely $\Delta_1 = \{\alpha_{i_o}\}$ such that $n_{i_o}(\theta) = 1$. In this case (cf. [O2]) we have $W^0(1) = \{\sigma_{i_o}\}$ and $p_{i_o} = 0$, that is,

$$\mathcal{H}^1 = \mathcal{H}^{\sigma_{i_o}} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}).$$

Now assume that $\Delta_1 \cap \Delta_\theta \neq \emptyset$. If $\mathfrak{g}$ is of type $C_\ell$, we have $p_i = 3 - \mu$ and $\alpha_1 \in \Delta_1$. Then $p_i \geqq 0$ occurs only when $\mu = 2$ or 3, which forces $\Delta_1 = \{\alpha_1\}$ or $\{\alpha_1, \alpha_\ell\}$. In these cases we have

(1) $\Delta_1 = \{\alpha_1\}$ $\qquad \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \subset C^{1,1}(\mathfrak{m}, \mathfrak{g})$,

(2) $\Delta_1 = \{\alpha_1, \alpha_\ell\}$ $\qquad \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_\ell} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}) \oplus C^{-2,1}(\mathfrak{m}, \mathfrak{g})$

In the other cases we have $p_i = 2 - \mu$. Moreover, except for type $A_\ell$, we have $\Delta_\theta \subset \Delta_1$ and $\mu \geqq 2$. Hence, in these cases, $p_i \geqq 0$ occurs only when $\Delta_1 = \Delta_\theta$. Namely $p_i \geqq 0$ occurs only if $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a contact gradation. In these cases we have

$$\mathcal{H}^1 = \mathcal{H}^{\sigma_{i_o}} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}),$$

where $\Delta_\theta = \{\alpha_{i_o}\}$. Finally, if $\mathfrak{g}$ is of type $A_\ell$, we may assume $\alpha_1 \in \Delta_1$, up to conjugacy. Then $p_i \geqq 0$ occurs only when $\mu = 1$ or 2, which forces

$\Delta_1 = \{\alpha_1\}$, $\{\alpha_1, \alpha_j\}$ $(1 < j < \ell)$ or $\{\alpha_1, \alpha_\ell\}$. In these cases we have

$$\begin{array}{lll} (1) & \Delta_1 = \{\alpha_1\} & \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \subset C^{1,1}(\mathfrak{m}, \mathfrak{g}) \ (\ell \geqq 2), \\ & & \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \subset C^{2,1}(\mathfrak{m}, \mathfrak{g}) \ (\ell = 1). \\ (2) & \Delta_1 = \{\alpha_1, \alpha_j\} & \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_j} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}) \oplus C^{-1,1}(\mathfrak{m}, \mathfrak{g}). \\ (3) & \Delta_1 = \{\alpha_1, \alpha_\ell\} & \mathcal{H}^1 = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_\ell} \subset C^{0,1}(\mathfrak{m}, \mathfrak{g}). \end{array}$$

Summarizing we have (here we follow [Bu] for the numbering of simple roots)

**Proposition 5.1.** *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $H^{p,1}(\mathfrak{m}, \mathfrak{g}) \neq \{0\}$ for some $p \geqq 0$ occurs only in the following cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1 (*cf.* [O2]), *that is, it is isomorphic with $(A_\ell, \{\alpha_i\})$ $(1 \leqq i \leqq [\frac{\ell+1}{2}])$, $(B_\ell, \{\alpha_1\})$, $(C_\ell, \{\alpha_\ell\})$, $(D_\ell, \{\alpha_1\})$, $(D_\ell, \{\alpha_\ell\})$, $(E_6, \{\alpha_1\})$ or $(E_7, \{\alpha_7\})$. In these cases*

$$\begin{array}{lll} \text{(i)} & (A_\ell, \{\alpha_1\}) & H^{2,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1} \ (\ell = 1), \\ & & H^{1,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1} \ (\ell \geqq 2), \\ \text{(ii)} & \textit{otherwise} & H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_{i_o}}. \end{array}$$

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a contact gradation, that is, it is isomorphic with $(A_\ell, \{\alpha_1, \alpha_\ell\})$, $(B_\ell, \{\alpha_2\})$, $(C_\ell, \{\alpha_1\})$, $(D_\ell, \{\alpha_2\})$, $(E_6, \{\alpha_2\})$, $(E_7, \{\alpha_1\})$, $(E_8, \{\alpha_8\})$, $(F_4, \{\alpha_1\})$ or $(G_2, \{\alpha_2\})$. In these cases*

$$\begin{array}{lll} \text{(i)} & (A_\ell, \{\alpha_1, \alpha_\ell\}) & H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_\ell}, \\ \text{(ii)} & (C_\ell, \{\alpha_1\}) & H^{1,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1}, \\ \text{(iii)} & \textit{otherwise} & H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_{i_o}}. \end{array}$$

(3) $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ or $(C_\ell, \{\alpha_1, \alpha_\ell\})$. In these cases*

$$H^{0,1}(\mathfrak{m}, \mathfrak{g}) \cong \mathcal{H}^{\sigma_1}.$$

Combined with Lemma 2.1 in §2.4, we obtain

**Theorem 5.2.** *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation (over $\mathbb{C}$) of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1.

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a (complex) contact gradation.*

(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ *or* $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

*Furthermore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1\})$ or $(C_\ell, \{\alpha_1\})$.*

Here $(A_\ell, \{\alpha_1\})$ is the graded Lie algebra $\mathfrak{g} = V \oplus \mathfrak{gl}(V) \oplus V^*$ of depth 1 associated with the (complex) projective structure (cf. [K2, Chapter IV]) and $(C_\ell, \{\alpha_1\})$ is known as the projective contact algebra (cf. [T2, p. 29]).

Now, by Lemmas 3.4, 3.5 and their proof, we have the real version of Theorem 5.2, which answers the question posed in §2.3. Here we note that, in the Satake diagram of type $A$ (resp. $C$), $\Delta_1 = \{\alpha_1\}$ or $\{\alpha_1, \alpha_i\}$ $(1 < i < \ell)$ (resp. $\Delta_1 = \{\alpha_1\}$ or $\{\alpha_1, \alpha_\ell\}$) is $\nu$-invariant subset of $\Delta^0$ only for the normal real form $AI$ (resp. $CI$).

**Theorem 5.3.** *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}]$ for $p < -1$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1.

(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a real or complex contact gradation.*

(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic with* $(A_\ell, \{\alpha_1, \alpha_i\})$, $(C_\ell, \{\alpha_1, \alpha_\ell\})$ *or their normal real forms* $(AI, \{\alpha_1, \alpha_i\})$, $(CI, \{\alpha_1, \alpha_\ell\})$ $(1 < i < \ell)$.

*Furthermore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1\})$, $(C_\ell, \{\alpha_1\})$ or their normal real forms $(AI, \{\alpha_1\})$, $(CI, \{\alpha_1\})$.*

Real simple graded Lie algebra of depth 1 were classified by Kobayashi and Nagano [K-N]. In this case $M_\mathfrak{g}$ is a symmetric $R$-space (cf. [K-N], [Tk1]). The exceptional cases (2) and (3) are already discussed in §4. In these cases $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a symbol algebra of canonical systems on real or complex jet spaces.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ with $\mu > 1$. Let $\mathcal{A}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ denote the Lie algebra sheaf of all infinitesimal automorphisms (in the real or complex analytic category) of the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. We denote by $\mathcal{A}_x(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ the stalk of $\mathcal{A}(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ at $x \in M_{\mathfrak{g}}$. Then we have

**Corollary 5.4.** *Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ be a simple graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ with $\mu > 1$. Then the following holds either in the real or complex analytic category.*

*$\mathcal{A}_x(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is isomorphic with $\mathfrak{g}$ at each $x \in M_{\mathfrak{g}}$ except when $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is locally isomorphic with a canonical system on a real or complex jet space. The latter case occurs if and only if $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is one of the standard contact manifolds $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ over $K$, the canonical system $(J(P(V), i), C)$ $(1 \leqq i < \ell - 1)$ on the Grassmann bundle over the $\ell$-dimensional projective space $P(V)$ over $K$ or the canonical system $(L(P(V)), E)$ on the Lagrange-Grassmann bundle over the odd dimensional (contact) projective space $P(V)$ over $K$, where $K = \mathbb{C}$ in the complex category and $K = \mathbb{R}$ or $\mathbb{C}$ in the real category.*

### 5.3. Calculation of $H^{p,2}(\mathfrak{m}, \mathfrak{g})$

First we shall compute $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ for $p \geqq 0$, which is important to know the fundamental invariants of the normal Cartan connection for the geometric structures subordinate to regular differential system of type $\mathfrak{m}$ (cf. [T4, §2]).

For simple reflections $\sigma_i = \sigma_{\alpha_i}$, $\alpha_i \in \Delta$, we put $\sigma_{ij} = \sigma_i \cdot \sigma_j$ for $i \neq j$. Then we see that $\sigma_{ij} = \sigma_{ji}$ if and only if $\langle \alpha_i, \alpha_j \rangle = 0$ and that

$$\Phi_{\sigma_{ij}} = \{\alpha_i, \alpha_j - \langle \alpha_j, \alpha_i \rangle \, \alpha_i\}.$$

Thus $W^0(2)$ consists of $\sigma_{ij} \in W(2)$ such that one of the following holds:

(a) Both $\alpha_i$ and $\alpha_j$ belong to $\Delta_1$.

(b) $\alpha_i \in \Delta_1$ and $\alpha_j \in \Delta_0$ such that $\langle \alpha_i, \alpha_j \rangle \neq 0$.

Then, by (5.2), we have

$$\mathcal{H}^{\sigma_{ij}} \subset C^{p_{ij},2}(\mathfrak{m}, \mathfrak{g}),$$

where

$$p_{ij} = \begin{cases} 1 - \mu + \langle \theta, \alpha_i \rangle + \langle \theta, \alpha_j \rangle - (\langle \theta, \alpha_j \rangle + 1) \, \langle \alpha_j, \alpha_i \rangle & \text{in case (a),} \\ -\mu + \langle \theta, \alpha_i \rangle - (\langle \theta, \alpha_j \rangle + 1) \, \langle \alpha_j, \alpha_i \rangle & \text{in case (b).} \end{cases}$$

First assume that $\langle \alpha_j, \alpha_i \rangle = 0$ for $\sigma_{ij} \in W^0(2)$. Then we have $\sigma_{ij} = \sigma_{ji}$, $\{\alpha_i, \alpha_j\} \subset \Delta_1$ and $p_{ij} = 1 - \mu + \langle \theta, \alpha_i \rangle + \langle \theta, \alpha_j \rangle$. Especially we have $\mu \geqq n_i(\theta) + n_j(\theta) \geqq 2$. Hence $p_{ij} < 0$ if $\Delta_1 \cap \Delta_\theta = \emptyset$, that is, if $\langle \theta, \alpha_i \rangle = \langle \theta, \alpha_j \rangle = 0$. If $\Delta_1 \cap \Delta_\theta \neq \emptyset$, from the diagram in §4.2, we know that $\mu \geqq 3$ except for $A_\ell$-type. Thus $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(C_\ell, \{\alpha_1, \alpha_\ell\})$ $(\ell \geqq 3)$, $(A_\ell, \{\alpha_1, \alpha_j\})$ $(2 < j < \ell)$, $(A_\ell, \{\alpha_1, \alpha_\ell\})$ or $(A_\ell, \{\alpha_1, \alpha_j, \alpha_\ell\})$ $(1 < j \leqq [\frac{\ell}{2}])$. In fact we have $p_{1\ell} = 0$, $p_{1j} = 0$ $(2 < j < \ell)$, $p_{1\ell} = 1$ and $p_{1\ell} = 0$ $(1 < j \leqq [\frac{\ell}{2}])$ in each case.

Secondly assume that $\langle \alpha_j, \alpha_i \rangle \neq 0$ and $\{\alpha_i, \alpha_j\} \cap \Delta_\theta = \emptyset$ for $\sigma_{ij} \in W^0(2)$. Then we have

$$p_{ij} = \begin{cases} 1 - \mu - \langle \alpha_j, \alpha_i \rangle & \text{in case (a)}, \\ -\mu - \langle \alpha_j, \alpha_i \rangle & \text{in case (b)}. \end{cases}$$

Moreover $\langle \alpha_j, \alpha_i \rangle = -2$ or $-1$ and $\langle \alpha_j, \alpha_i \rangle = -2$ occurs only for $\langle \alpha_{\ell-1}, \alpha_\ell \rangle$ in type $B_\ell$ $(\ell \geqq 4)$, $\langle \alpha_\ell, \alpha_{\ell-1} \rangle$ in type $C_\ell$ $(\ell \geqq 3)$ and $\langle \alpha_2, \alpha_3 \rangle$ in type $F_4$. Thus, if $\langle \alpha_j, \alpha_i \rangle = -2$, we see, from the diagram in §4.2, that $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(C_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$ in case (a) and isomorphic with $(B_\ell, \{\alpha_\ell\})$ or $(C_\ell, \{\alpha_{\ell-1}\})$ in case (b). In fact we have $p_{\ell-1\,\ell} = 0$, $p_{\ell\,\ell-1} = 0$ and $p_{\ell-1\,\ell} = 0$ in each case. If $\langle \alpha_j, \alpha_i \rangle = -1$, $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_i, \alpha_{i+1}\})$ $(1 < i \leqq [\frac{\ell}{2}])$ in case (a) and isomorphic with $(A_\ell, \{\alpha_2\})$, $(A_\ell, \{\alpha_i\})$ $(2 < i \leqq [\frac{\ell+1}{2}])$, $(C_\ell, \{\alpha_\ell\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_\ell\})$ $(\ell \geqq 5)$, $(E_6, \{\alpha_1\})$ or $(E_7, \{\alpha_7\})$ in case (b). In fact we have $p_{i\,i+1} = p_{i+1\,i} = 0$ $(1 < i \leqq [\frac{\ell}{2}])$, $p_{23} = 0$, $p_{i\,i-1} = p_{i\,i+1} = 0$ $(2 < i \leqq [\frac{\ell+1}{2}])$, $p_{\ell\,\ell-1} = 0$, $p_{\ell\,\ell-2} = 0$, $p_{13} = 0$ and $p_{76} = 0$ in each case.

Thirdly assume that $\langle \alpha_j, \alpha_i \rangle \neq 0$ and $\{\alpha_i, \alpha_j\} \cap \Delta_\theta \neq \emptyset$ for $\sigma_{ij} \in W^0(2)$. Then, from the diagram in §4.2, $\{\alpha_i, \alpha_j\}$ equals to $\{\alpha_1, \alpha_2\}$ or $\{\alpha_{\ell-1}, \alpha_\ell\}$ in type $A_\ell$, $\{\alpha_1, \alpha_2\}$ or $\{\alpha_2, \alpha_3\}$ in type $B_\ell$, $\{\alpha_1, \alpha_2\}$ in type $C_\ell$, $\{\alpha_1, \alpha_2\}$ or $\{\alpha_2, \alpha_3\}$ in type $D_\ell$, $\{\alpha_2, \alpha_4\}$ in type $E_6$, $\{\alpha_1, \alpha_3\}$ in type $E_7$, $\{\alpha_7, \alpha_8\}$ in type $E_8$, $\{\alpha_1, \alpha_2\}$ in type $F_4$ or $\{\alpha_1, \alpha_2\}$ in type $G_2$. Now assume further $\langle \alpha_j, \alpha_i \rangle = -1$ and $\operatorname{rank} \mathfrak{g} \geqq 3$. (In fact $\langle \alpha_j, \alpha_i \rangle < -1$ occurs only for $\langle \alpha_2, \alpha_1 \rangle$ in type $C_2$ $(\cong B_2)$, $\langle \alpha_2, \alpha_1 \rangle$ in type $G_2$ and $\langle \alpha_2, \alpha_3 \rangle$ in type $B_3$.) Then $\{\alpha_i, \alpha_j\} \cap \Delta_\theta$ consists of a single element.

In case (a), we have $\{\alpha_i, \alpha_j\} \subset \Delta_1$ and

$$p_{ij} = 2 - \mu + \langle \theta, \alpha_i \rangle + 2\,\langle \theta, \alpha_j \rangle.$$

More precisely, if $\mathfrak{g}$ is of type $C_\ell$, $p_{ij} = 4 - \mu$ or $6 - \mu$ according to $\alpha_i \in \Delta_\theta$ or $\alpha_j \in \Delta_\theta$. In other cases $p_{ij} = 3 - \mu$ or $4 - \mu$ according to $\alpha_i \in \Delta_\theta$ or $\alpha_j \in \Delta_\theta$. For the exceptional types, from the diagram in §4.2, we observe that $\mu \geqq n_i(\theta) + n_j(\theta) = 5$. Hence

$p_{ij} < 0$ if $\mathfrak{g}$ is of type $E_6$, $E_7$, $E_8$ or $F_4$. For the classical types, $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1, \alpha_2\})$, $(A_\ell, \{\alpha_1, \alpha_2, \alpha_k\})$ $(2 < k \leqq \ell)$, $(A_\ell, \{\alpha_1, \alpha_2, \alpha_k, \alpha_m\})$ $(2 < k < m \leqq \ell)$, $(B_\ell, \{\alpha_1, \alpha_2\})$, $(B_\ell, \{\alpha_2, \alpha_3\})$, $(C_\ell, \{\alpha_1, \alpha_2\})$, $(C_\ell, \{\alpha_1, \alpha_2, \alpha_k\})$ $(2 < k \leqq \ell)$, $(D_\ell, \{\alpha_1, \alpha_2\})$, $(D_\ell, \{\alpha_1, \alpha_2, \alpha_\ell\})$ or $(D_\ell, \{\alpha_2, \alpha_3\})$.

In case (b), we have $p_{ij} = 3 - \mu$ if $\mathfrak{g}$ is of type $C_\ell$ and $p_{ij} = 2 - \mu$ otherwise. Hence $p_{ij} \geqq 0$ occurs only when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic with $(A_\ell, \{\alpha_1\})$, $(A_\ell, \{\alpha_1, \alpha_j\})$ $(2 < j < \ell)$, $(A_\ell, \{\alpha_2\})$, $(A_\ell, \{\alpha_2, \alpha_j\})$ $(2 < j \leqq \ell)$, $(B_\ell, \{\alpha_1\})$, $(B_\ell, \{\alpha_3\})$, $(C_\ell, \{\alpha_1, \alpha_\ell\})$, $(C_\ell, \{\alpha_2\})$, $(C_\ell, \{\alpha_2, \alpha_\ell\})$, $(D_\ell, \{\alpha_1\})$, $(D_\ell, \{\alpha_1, \alpha_\ell\})$, $(D_\ell, \{\alpha_3\})$ or contact gradations of each type.

We leave it to the reader to check the remaining cases, that is, the cases $\mathfrak{g}$ is of type $A_2$, $B_2 = C_2$, $G_2$ or $B_3$.

Summarizing we obtain

**Proposition 5.5.** *Let $(X_\ell, \Delta_1)$ be a simple graded Lie algebra over $\mathbb{C}$ described in §3.4. Then the following are the list of $(X_\ell, \Delta_1)$ and $p_{ij}$ such that $p_{ij} \geqq 0$ holds for the irreducible component $\mathcal{H}^{\sigma_{ij}} \subset C^{p_{ij},2}(\mathfrak{m}, \mathfrak{g})$ of the harmonic space $\mathcal{H}^2 \cong H^2(\mathfrak{m}, \mathfrak{g})$ corresponding to $\sigma_{ij} \in W^0(2)$ in Kostant's theorem.*

(I) *$A_\ell$-type $(\ell \geqq 2)$.*

(1) $\{\alpha_1\}$ $\quad p_{12} = 2 \quad (\ell = 2)$,
$p_{12} = 1 \quad (\ell \geqq 3)$.

(2) $\{\alpha_2\}$ $\quad p_{21} = 1, \quad p_{23} = 0$.

(3) $\{\alpha_i\}$ $\quad p_{i\,i-1} = p_{i\,i+1} = 0 \quad (2 < i \leqq [\frac{\ell+1}{2}])$.

(4) $\{\alpha_1, \alpha_2\}$ $\quad p_{12} = p_{21} = 3 \quad (\ell = 2)$,
$p_{12} = 1, \quad p_{21} = 2 \quad (\ell \geqq 3)$.

(5) $\{\alpha_1, \alpha_i\}$ $\quad p_{12} = p_{1i} = 0 \quad (2 < i < \ell - 1)$.

(6) $\{\alpha_1, \alpha_{\ell-1}\}$ $\quad p_{12} = p_{1\,\ell-1} = p_{\ell-1\,\ell} = 0 \quad (\ell \geqq 4)$.

(7) $\{\alpha_1, \alpha_\ell\}$ $\quad p_{12} = p_{\ell\,\ell-1} = 0, \quad p_{1\ell} = 1 \quad (\ell \geqq 3)$.

(8) $\{\alpha_2, \alpha_3\}$ $\quad p_{21} = p_{23} = p_{32} = p_{34} = 0 \quad (\ell = 4)$,
$p_{21} = p_{23} = p_{32} = 0 \quad (\ell \geqq 5)$.

(9) $\{\alpha_2, \alpha_i\}$ $\quad p_{21} = 0 \quad (3 < i < \ell - 1)$.

(10) $\{\alpha_2, \alpha_{\ell-1}\}$ $\quad p_{21} = p_{\ell-1\,\ell} = 0 \quad (\ell \geqq 5)$.

(11) $\{\alpha_i, \alpha_{i+1}\}$ $\quad p_{i\,i+1} = p_{i+1\,i} = 0 \quad (2 < i \leqq [\frac{\ell}{2}])$.

(12) $\{\alpha_1, \alpha_2, \alpha_\ell\}$ $\quad p_{13} = p_{12} = p_{32} = 0, \quad p_{21} = p_{23} = 1 \quad (\ell = 3)$,
$p_{1\ell} = p_{12} = 0, \quad p_{21} = 1 \quad (\ell \geqq 4)$.

(13) $\{\alpha_1, \alpha_i, \alpha_\ell\}$ $\quad p_{1\ell} = 0 \quad (2 < i \leqq [\frac{\ell}{2}])$.

(14) $\{\alpha_1, \alpha_2, \alpha_i, \alpha_j\}$ $\quad p_{21} = 0 \quad (2 < i < j \leqq \ell)$.

(15) $\{\alpha_1, \alpha_2, \alpha_{\ell-1}, \alpha_\ell\}$ $p_{21} = p_{\ell-1\,\ell} = 0$.

(II) *$B_\ell$-type* $(\ell \geqq 3)$.

| | | | |
|---|---|---|---|
| (1) | $\{\alpha_1\}$ | $\mu = 1$ | $p_{12} = 1$. |
| (2) | $\{\alpha_2\}$ | $\mu = 2$ | $p_{21} = p_{23} = 0$. |
| (3) | $\{\alpha_3\}$ | $\mu = 2$ | $p_{32} = 2 \quad (\ell = 3)$,<br>$p_{32} = 0 \quad (\ell \geqq 4)$. |
| (4) | $\{\alpha_\ell\}$ | $\mu = 2$ | $p_{\ell\,\ell-1} = 0 \quad (\ell \geqq 4)$. |
| (5) | $\{\alpha_1, \alpha_2\}$ | $\mu = 3$ | $p_{21} = 0, \quad p_{12} = 1$. |
| (6) | $\{\alpha_1, \alpha_3\}$ | $\mu = 3$ | $p_{32} = 1 \quad (\ell = 3)$. |
| (7) | $\{\alpha_2, \alpha_3\}$ | $\mu = 3$ | $p_{32} = 2 \quad (\ell = 3)$,<br>$p_{32} = 0 \quad (\ell \geqq 4)$. |
| (8) | $\{\alpha_1, \alpha_2, \alpha_3\}$ | $\mu = 5$ | $p_{32} = 1 \quad (\ell = 3)$. |

(III) *$C_\ell$-type* $(\ell \geqq 2)$.

| | | | |
|---|---|---|---|
| (1) | $\{\alpha_\ell\}$ | $\mu = 1$ | $p_{21} = 2 \quad (\ell = 2), \qquad p_{\ell\,\ell-1} = 0 \quad (\ell \geqq 3)$. |
| (2) | $\{\alpha_1\}$ | $\mu = 2$ | $p_{12} = 2 \quad (\ell = 2), \qquad p_{12} = 1 \quad (\ell \geqq 3)$. |
| (3) | $\{\alpha_2\}$ | $\mu = 2$ | $p_{21} = 2 \quad (\ell = 2), \qquad p_{21} = 1 \quad (\ell \geqq 4)$,<br>$p_{21} = 1, \quad p_{23} = 0 \quad (\ell = 3)$. |
| (4) | $\{\alpha_{\ell-1}\}$ | $\mu = 2$ | $p_{\ell-1\,\ell} = 0 \quad (\ell \geqq 4)$. |
| (5) | $\{\alpha_1, \alpha_\ell\}$ | $\mu = 3$ | $p_{12} = 2, \quad p_{21} = 3 \quad (\ell = 2)$,<br>$p_{1\ell} = p_{12} = 0 \quad (\ell \geqq 3)$. |
| (6) | $\{\alpha_2, \alpha_\ell\}$ | $\mu = 3$ | $p_{21} = p_{23} = 0 \quad (\ell = 3)$,<br>$p_{21} = 0 \quad (\ell \geqq 4)$. |
| (7) | $\{\alpha_{\ell-1}, \alpha_\ell\}$ | $\mu = 3$ | $p_{\ell-1\,\ell} = 0 \quad (\ell \geqq 4)$. |
| (8) | $\{\alpha_1, \alpha_2\}$ | $\mu = 4$ | $p_{12} = 0, \quad p_{21} = 2 \quad (\ell \geqq 3)$. |
| (9) | $\{\alpha_1, \alpha_2, \alpha_\ell\}$ | $\mu = 5$ | $p_{21} = 1$. |
| (10) | $\{\alpha_1, \alpha_2, \alpha_i\}$ | $\mu = 6$ | $p_{21} = 0 \quad (2 < i < \ell)$. |

(IV) *$D_\ell$-type* $(\ell \geqq 4)$.

| | | | |
|---|---|---|---|
| (1) | $\{\alpha_1\}$ | $\mu = 1$ | $p_{12} = 1$. |
| (2) | $\{\alpha_\ell\}$ | $\mu = 1$ | $p_{\ell\,\ell-2} = 0 \quad (\ell \geqq 5)$. |
| (3) | $\{\alpha_2\}$ | $\mu = 2$ | $p_{21} = p_{23} = 0$. |
| (4) | $\{\alpha_3\}$ | $\mu = 2$ | $p_{32} = 0 \quad (\ell \geqq 5)$. |
| (5) | $\{\alpha_1, \alpha_\ell\}$ | $\mu = 2$ | $p_{12} = 0$. |
| (6) | $\{\alpha_1, \alpha_2\}$ | $\mu = 3$ | $p_{12} = 1, \quad p_{21} = 0$. |
| (7) | $\{\alpha_1, \alpha_2, \alpha_\ell\}$ | $\mu = 4$ | $p_{12} = 0$. |
| (8) | $\{\alpha_2, \alpha_3\}$ | $\mu = 4$ | $p_{32} = 0 \quad (\ell \geqq 5)$. |

(V) *Exceptional types.*

(1) $(E_6, \{\alpha_1\})$, $(E_7, \{\alpha_7\})$ $\mu = 1$ $p_{ij} = 0$, *where* $\{\alpha_i\} = \Delta_1$ *and* $\langle \alpha_i, \alpha_j \rangle \neq 0$.

(2) $(E_6, \{\alpha_2\})$, $(E_7, \{\alpha_1\})$, $(E_8, \{\alpha_8\})$, $(F_4, \{\alpha_1\})$ *and* $(G_2, \{\alpha_2\})$.
*Contact gradations*: $\mu = 2$ $p_{ij} = 0$, *where* $\{\alpha_i\} = \Delta_\theta$ *and* $\langle \alpha_i, \alpha_j \rangle \neq 0$.

(3) $(G_2, \{\alpha_1\})$ $\mu = 3$ $p_{12} = 3$.

(4) $(G_2, \{\alpha_1, \alpha_2\})$ $\mu = 5$ $p_{12} = 3$.

Now we shall give some remarks on regular differential systems of type $\mathfrak{m}$.

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{m}$ is fundamental. Let $M$ be a manifold with a $G_0^\sharp$-structure of type $\mathfrak{m}$ in the sense of [T4] (for the precise definition, see §2 of [T4]). In [T4], under the assumption that $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$, N. Tanaka constructed a normal Cartan connection $(P, \omega)$ of type $\mathfrak{g}$ over $M$, which settles the equivalence problem for the $G_0^\sharp$-structure of type $\mathfrak{m}$ in the following sense: Let $M$ and $\widehat{M}$ be two manifolds with $G_0^\sharp$-structures of type $\mathfrak{m}$. Let $(P, \omega)$ and $(\widehat{P}, \widehat{\omega})$ be the normal connections of type $\mathfrak{g}$ over $M$ and $\widehat{M}$ respectively. Then a diffeomorphism $\varphi$ of $M$ onto $\widehat{M}$ preserving the $G_0^\sharp$-structures lifts uniquely to an isomorphism $\varphi^\sharp$ of $(P, \omega)$ onto $(\widehat{P}, \widehat{\omega})$ and vice versa ([T4, Theorem 2.7]).

Here we note that, if $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$, a $G_0^\sharp$-structure on $M$ is nothing but a regular differential system of type $\mathfrak{m}$ (see [T4, §2.2]). Moreover let $K$ be the curvature of the normal connection $(P, \omega)$, which can be regarded as a $C^2(\mathfrak{m}, \mathfrak{g})$-valued function on $P$ ([T4, Lemma 2.2]). Then, by the normality condition for $K$: $K^p = 0$ for $p < 0$ and $\partial^* K^p = 0$ for $p \geqq 0$, where $K^p$ is the $C^{p,2}(\mathfrak{m}, \mathfrak{g})$-component of $K$, and the Bianchi identity, it is further shown ([T4, Theorem 2.9]) that the harmonic part $H(K)$ of $K$, with respect to the orthogonal decomposition $C^2(\mathfrak{m}, \mathfrak{g}) = \operatorname{Im} \partial \oplus \operatorname{Im} \partial^* \oplus \mathcal{H}$, gives a fundamental system of invariants of the connection $(P, \omega)$. Namely $K$ vanishes if and only if $H(K)$ vanishes. Hence, as a corollary to Theorems 2.7 and 2.9 of [T4], we have

*Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental. Assume that $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ and $H^{p,2}(\mathfrak{m}, \mathfrak{g}) = \{0\}$ for $p \geqq 0$. Then every regular differential system $(M, D)$ of type $\mathfrak{m}$ is locally isomorphic with the standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$.*

Thus, by Proposition 5.5, we can find many examples of regular differential systems $(M, D)$ of type $\mathfrak{m}$ with no local invariants, whose Lie algebra $\mathcal{A}(M, D)$ of all infinitesimal automorphisms are isomorphic with simple Lie algebras over $\mathbb{R}$.

We shall give below some examples of fundamental graded algebras $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of the second kind whose prolongation $\mathfrak{g}(\mathfrak{m})$ become finite dimensional and simple. Namely we shall describe the structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of several simple graded Lie algebras $(X_\ell, \Delta_1)$ over $\mathbb{C}$ and their normal real forms such that $H^{p,1}(\mathfrak{m}, \mathfrak{g})$ vanishes for $p \geqq 0$ and $\mu = 2$. In the following we shall discuss in the complex analytic or the real $C^\infty$ category depending on whether we treat complex simple graded Lie algebras $(X_\ell, \Delta_1)$ or their normal real forms.

(1) $(B_\ell, \{\alpha_\ell\})$ $(\ell \geqq 3)$. First we have (see §4.4)

$$\Phi_2^+ = \{\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_\ell \quad (1 \leqq i < j \leqq \ell)\},$$
$$\Phi_1^+ = \{\beta_i = \alpha_i + \cdots + \alpha_\ell \quad (1 \leqq i \leqq \ell)\}.$$

Each $\alpha_{ij} \in \Phi_2^+$ is uniquely written as a sum $\alpha_{ij} = \beta_i + \beta_j$ of roots in $\Phi_1^+$. We have $\dim \mathfrak{g}_{-1} = \ell$ and $\dim \mathfrak{g}_{-2} = \frac{1}{2}\ell(\ell-1)$. Hence the structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is described by

$$\mathfrak{g}_{-2} = \wedge^2 V,$$

where we put $\mathfrak{g}_{-1} = V$. Namely $\mathfrak{m}$ is the universal fundamental graded algebra of second kind such that $\dim \mathfrak{g}_{-1} = \ell \geqq 3$. In this case, it is easy to see that $\mathfrak{g}_0$ is naturally identified with $\mathfrak{gl}(\mathfrak{g}_{-1})$ (see also the matrix representation of $(B_\ell, \{\alpha_\ell\})$ in §4.4). This example was first found by Tanaka [T1, p. 245]. The standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ is given as follows: Let $(x_1, \ldots, x_\ell, x_{ij})$ $(1 \leqq i < j \leqq \ell)$ be a coordinate system of $M(\mathfrak{m}) = K^{\frac{1}{2}\ell(\ell+1)}$. Then $D_\mathfrak{m}$ is defined by the following $\frac{1}{2}\ell(\ell-1)$ forms

$$\varpi_{ij} = dx_{ij} - \tfrac{1}{2}(x_i\,dx_j - x_j\,dx_i) \quad (1 \leqq i < j \leqq \ell).$$

$\mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ is isomorphic with $\mathfrak{o}(2\ell+1, \mathbb{C})$ or $\mathfrak{o}(\ell+1, \ell)$ depending on $K = \mathbb{C}$ or $\mathbb{R}$.

(2) $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$ $(\ell \geqq 4)$. First we have

$$\begin{aligned}\Phi_2^+ = \{\alpha_{i\,\ell-1} &= \alpha_i + \cdots + \alpha_{\ell-1} + \alpha_\ell \quad (1 \leqq i \leqq \ell - 2),\\ \alpha_{ij} &= \alpha_i + \cdots + \alpha_{j-1} + 2\,\alpha_j + \cdots + 2\,\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell\\ &\qquad\qquad (1 \leqq i < j \leqq \ell - 2)\},\\ \Phi_1^+ = \{\beta_i &= \alpha_i + \cdots + \alpha_{\ell-1}\ (1 \leqq i \leqq \ell - 1), \quad \gamma_{\ell-1} = \alpha_\ell,\\ \gamma_i &= \alpha_i + \cdots + \alpha_{\ell-2} + \alpha_\ell\ (1 \leqq i \leqq \ell - 2)\}.\end{aligned}$$

Each $\alpha_{ij} \in \Phi_2^+$ $(1 \leqq i < j \leqq \ell - 1)$ is written as a sum

$$\alpha_{ij} = \beta_i + \gamma_j = \beta_j + \gamma_i$$

of roots in $\Phi_1^+$ in two ways. We have $\dim \mathfrak{g}_{-1} = 2(\ell - 1)$ and $\dim \mathfrak{g}_{-2} = \frac{1}{2}(\ell-1)(\ell-2)$. By the explicit matrix representation of $(D_\ell, \{\alpha_{\ell-1}, \alpha_\ell\})$ in §4.4, we can describe the structure of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ as follows: There exist basis $\{X_1, \ldots, X_{\ell-1}, Y_1, \ldots, Y_{\ell-1}\}$ of $\mathfrak{g}_{-1}$ and $\{Z_{ij}\ (1 \leqq i < j \leqq \ell - 1)\}$ of $\mathfrak{g}_{-2}$ such that

$$\begin{aligned}Z_{ij} = [X_i, Y_j] = [Y_i, X_j] \quad (1 \leqq i < j \leqq \ell - 1),\\ [X_i, X_j] = [Y_i, Y_j] = 0.\end{aligned}$$

Thus the standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ is given as follows: Let $(x_1, \ldots, x_{\ell-1}, y_1, \ldots, y_{\ell-1}, z_{ij})$ $(1 \leqq i < j \leqq \ell - 1)$ be a coordinate system of $M(\mathfrak{m}) = K^{\frac{1}{2}(\ell-1)(\ell+2)}$. Then $D_\mathfrak{m}$ is defined by the following $\frac{1}{2}(\ell - 1)(\ell - 2)$ forms

$$\varpi_{ij} = dz_{ij} - (x_i\,dy_j + y_i\,dx_j) \quad (1 \leqq i < j \leqq \ell - 1).$$

$\mathcal{A}(M(\mathfrak{m}), D_\mathfrak{m})$ is isomorphic with $\mathfrak{o}(2\ell, \mathbb{C})$ or $\mathfrak{o}(\ell, \ell)$ depending on $K = \mathbb{C}$ or $\mathbb{R}$.

Furthermore, by Proposition 5.5 (IV), we see that $H^{p,2}(\mathfrak{m}, \mathfrak{g})$ vanishes for $p \geqq 0$ when $\ell \geqq 5$. Hence, in this case $(\ell \geqq 5)$, every regular differential system $(M, D)$ of type $\mathfrak{m}$ is locally isomorphic with $(M(\mathfrak{m}), D_\mathfrak{m})$ given above. Namely assume that $(M, D)$ is a differential system which has local defining 1-forms $\varpi_{ij}$;

$$D = \{\,\varpi_{ij} = 0 \quad (1 \leqq i < j \leqq \ell - 1)\,\},$$

satisfying the following structure equation, for $1 \leqq i < j \leqq \ell - 1$

$$d\varpi_{ij} \equiv \omega_i \wedge \varpi_j + \varpi_i \wedge \omega_j \quad (\mathrm{mod}\ \varpi_{rs} \quad (1 \leqq r < s \leqq \ell - 1)),$$

where $\{\varpi_{ij}\ (1 \leqq i < j \leqq \ell-1), \omega_1, \ldots, \omega_{\ell-1}, \varpi_1, \ldots, \varpi_{\ell-1}\}$ is a local (free) basis of 1-forms on $M$. Then there exists a local coordinate system $(x_1, \ldots, x_{\ell-1}, y_1, \ldots, y_{\ell-1}, z_{ij})$ $(1 \leqq i < j \leqq \ell-1)$ of $M$ such that

$$D = \{\, dz_{ij} - (x_i\, dy_j + y_i\, dx_j) = 0 \quad (1 \leqq i < j \leqq \ell-1)\,\}.$$

(3) $(F_4, \{\alpha_4\})$. Here we shall show that the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case has a following description, which was discovered by E. Cartan [C1]: Let $(z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, x_{ij})$ $(1 \leqq i < j \leqq 4)$ be a coordinate system on $M_F = K^{15}$. Let $D_F$ be a differential system on $M_F$ defined by the following 7 forms;

$$\begin{cases} \varpi = dz - y_1\, dx_1 - y_2\, dx_2 - y_3\, dx_3 - y_4\, dx_4, \\ \varpi_{ij} = dx_{ij} - (x_i\, dx_j - x_j\, dx_i + y_h\, dy_k - y_k\, dy_h) \quad (1 \leqq i < j \leqq 4) \end{cases}$$

where $(h, k)$ is determined by the requirement that $(i, j, h, k)$ is an even permutation of $(1, 2, 3, 4)$. By taking the dual vector fields $Z$, $X_{ij}$, $X_i$, $Y_j$ of the basis $\{\varpi, \varpi_{ij}, dx_i, dy_j\}$ of 1-forms on $M_F$, we have

$$(*) \qquad \begin{cases} Z = [Y_i, X_i] \quad (i = 1,\, 2,\, 3,\, 4), \\ 2\, X_{ij} = [X_i, X_j] = [Y_h, Y_k] \quad (1 \leqq i < j \leqq 4), \end{cases}$$

where $(i, j, h, k)$ is an even permutation of $(1, 2, 3, 4)$. Namely $(M_F, D_F)$ is the standard differential system of type $\mathfrak{m}_F$. Here $\mathfrak{m}_F = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is the fundamental graded algebra of the second kind such that there exist bases $\{Z, X_{12}, X_{13}, X_{14}, X_{23}, X_{24}, X_{34}\}$ of $\mathfrak{g}_{-2}$ and $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4\}$ of $\mathfrak{g}_{-1}$ satisfying $(*)$ above. Thus our aim here is to show that $\mathfrak{m}_F$ is isomorphic with the negative part $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ of the simple graded Lie algebra $(F_4, \{\alpha_4\})$ or its normal real form.

For $(F_4, \{\alpha_4\})$, we have (cf. [Bu, p. 272, Planche VIII])

$$\begin{aligned} \Phi_2^+ = \{&\alpha_{14} = 0122,\ \alpha_{13} = 1122,\ \alpha_{12} = 1222, \\ &\alpha = 1232,\ \alpha_{34} = 1242,\ \alpha_{24} = 1342,\ \alpha_{23} = 2342\}, \\ \Phi_1^+ = \{&\beta_1 = 0001,\ \gamma_2 = 0011,\ \gamma_3 = 0111,\ \beta_4 = 0121, \\ &\gamma_1 = 1231,\ \beta_2 = 1221,\ \beta_3 = 1121,\ \gamma_4 = 1111\}. \end{aligned}$$

where $a_1\, a_2\, a_3\, a_4$ stands for the coefficients of the positive root with respect to the simple roots $\alpha_1$, $\alpha_2$, $\alpha_3$ and $\alpha_4$. Each root in $\Phi_2^+$ is written as a sum of roots in $\Phi_1^+$ as follows.

$$\begin{cases} \alpha = \beta_i + \gamma_i \quad (i = 1,\, 2,\, 3,\, 4), \\ \alpha_{ij} = \beta_i + \beta_j = \gamma_h + \gamma_k \quad (1 \leqq i < j \leqq 4), \end{cases}$$

where $\{i, j, h, k\} = \{1, 2, 3, 4\}$.

Let us take a Chevalley basis $\{x_\alpha\ (\alpha \in \Phi); h_i\ (1 \leqq i \leqq 4)\}$ of $F_4$ and put $y_\beta = x_{-\beta}$ for $\beta \in \Phi^+$ (cf. [Hu, Chapter VII]). We consider the structure of the negative part $\mathfrak{m}$ of $(F_4, \{\alpha_4\})$ in terms of $\{y_\beta\}_{\beta \in \Phi_1^+ \cup \Phi_2^+}$. Here we note that $\alpha \in \Phi_2^+$ and all roots in $\Phi_1^+$ are short roots in $\Phi$, whereas the other roots in $\Phi_2^+$ are long roots in $\Phi$ (see [Bu, Planche VIII]). Moreover, in the root system $\widehat{\Phi}$ of type $A_2$ or $C_2 = B_2$, we observe that, if $\alpha + \beta \in \widehat{\Phi}$, the $\alpha$-string through $\beta$ starts from $\beta - \alpha$ when $\alpha$, $\beta$ are short and $\alpha + \beta$ is a long root, and starts from $\beta$ otherwise. (See [Hu, p. 44].) These observations readily show that $\mathfrak{m}$ satisfies $(*)$ above up to signs of the structure constants. However the question of signs is a subtle point of the Chevalley basis (cf. [Tt2]). We are obliged to check the question of signs as follows: First let us choose signs of $y_i = y_{\alpha_i}$ ($i = 1, 2, 3, 4$) corresponding to the simple roots $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$ by fixing the root vectors $y_i \in \mathfrak{g}_{-\alpha_i}$. We fix the signs of $y_\beta$ for $\beta \in \Phi_1^+ \cup \Phi_2^+$ by the following;

$$\begin{aligned}
y_{\beta_1} &= y_4, & y_{\gamma_2} &= [y_3, y_{\beta_1}], & y_{\gamma_3} &= [y_2, y_{\gamma_2}],\\
y_{\gamma_4} &= [y_1, y_{\gamma_3}], & y_{\beta_3} &= [y_3, y_{\gamma_4}], & y_{\beta_2} &= [y_2, y_{\beta_3}],\\
y_{\beta_4} &= [y_3, y_{\gamma_3}], & y_{\gamma_1} &= [y_3, y_{\beta_2}], & y_\alpha &= [y_4, y_{\gamma_1}],\\
2\, y_{\alpha_{14}} &= [y_4, y_{\beta_4}], & 2\, y_{\alpha_{13}} &= [y_4, y_{\beta_3}], & 2\, y_{\alpha_{12}} &= [y_4, y_{\beta_2}],\\
2\, y_{\alpha_{34}} &= [y_3, y_\alpha], & y_{\alpha_{24}} &= [y_2, y_{\alpha_{34}}], & y_{\alpha_{23}} &= [y_1, y_{\alpha_{24}}].
\end{aligned}$$

Then, by the repeated application of Jacobi identity, one can check that, by putting

$$\begin{aligned}
X_i &= y_{\beta_i}, \quad & Y_i &= (-1)^i\, y_{\gamma_i} \quad (i = 1,\ 2,\ 3,\ 4),\\
Z &= y_\alpha, & Z_{ij} &= y_{\alpha_{ij}} \quad (1 \leqq i < j \leqq 4),
\end{aligned}$$

$\{Z, Z_{ij}, X_i, Y_j\}$ satisfies $(*)$ above, that is, $\mathfrak{m}$ is isomorphic with $\mathfrak{m}_F$.

Finally we remark that, by Proposition 5.5 (V) and Tanaka's Theorem [T4], every regular differential system $(M, D)$ of type $\mathfrak{m}_F$ is locally isomorphic with $(M_F, D_F)$.

## 5.4. Reducible primitive actions

We shall characterize the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, whose isotropy subalgeras $\mathfrak{g}' = \bigoplus_{p \geqq 0} \mathfrak{g}_p$ are maximal parabolic, as homogeneous differential systems which have nonlinear reducible primitive actions of Lie groups (cf. [O1], [Go]).

We shall consider reducible primitive actions of finite dimensional Lie groups, following the arguments in [Go], [K-N, I and II] and [Gu]. We shall discuss in either real or complex category.

Let $L$ be a connected Lie group acting transitively and effectively on a manifold $M$. Let $L'$ be the isotropy subgroup of $L$ at a point $o$ of $M$ so that $M = L/L'$. We denote by $\mathcal{L}$ and $\mathcal{L}'$ the Lie algebras of $L$ and $L'$ respectively. Let $\gamma\colon L' \to GL(T_o(M))$ be the linear isotropy representation of $L'$ given by

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\mathrm{Ad}(g)} & \mathcal{L} \\ \pi_*\downarrow & & \downarrow\pi_* \\ T_o(M) & \xrightarrow{\gamma(g)} & T_o(M) \end{array}$$

for $g \in L'$, where $\pi\colon L \to M$ is the projection defined by $\pi(g) = g(o)$. Then a $\gamma(L')$-invariant subspace $D_o$ of $T_o(M)$ corresponds to an $\mathrm{Ad}(L')$-invariant subspace $\mathcal{L}^{-1}$ of $\mathcal{L}$ containing $\mathcal{L}'$, which further corresponds to a $L$-invariant differential system $D$ on $M$ such that $D(o) = D_o$. We say that $L$ acts primitively on $M$ if $L$ leaves invariant no completely integrable differential systems on $M$ (cf. [Go, Definition 1.3]). From the above diagram, it follows that $L$ acts primitively on $M$ if and only if $\mathcal{L}'$ is a maximally $\mathrm{Ad}(L')$-invariant subalgebra of $\mathcal{L}$. Namely (cf. [Go, Theorem 2.1])

*If $\mathfrak{h}$ is a subalgebra of $\mathcal{L}$ satisfying $\mathfrak{h} \supset \mathcal{L}'$ and $\mathrm{Ad}(L')(\mathfrak{h}) = \mathfrak{h}$, then either $\mathfrak{h} = \mathcal{L}$ or $\mathfrak{h} = \mathcal{L}'$.*

Here we note that $\mathcal{L}'$ is self-normalizing in $\mathcal{L}$. In fact the normalizer $N(\mathcal{L}')$ of $\mathcal{L}'$ in $\mathcal{L}$ is obviously preserved by $\mathrm{Ad}(L')$. Hence we have $N(\mathcal{L}') = \mathcal{L}'$ or $\mathcal{L}$. However $N(\mathcal{L}') = \mathcal{L}$ implies $\mathcal{L}'$ is an ideal of $\mathcal{L}$, which contradicts to the assumption that $L$ acts effectively on $M$. Thus $N(\mathcal{L}') = \mathcal{L}'$.

Now we consider the following situation: Assume that $L$ acts primitively on $M$ and the linear isotropy representation $\gamma\colon L' \to GL(T_o(M))$ is reducible. Namely $L$ acts primitively on $M$ and leaves invariant a differential system $D$ on $M$ (which is, of course, non-integrable). Let us take $D$ to be minimal, that is, $D_o = D(o)$ is a $\gamma(L')$-irreducible subspace of $T_o(M)$. $\mathcal{L}$ is naturally identified with the Lie algebra of vector fields on $M$ induced by the $L$-action. We introduce a filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ of $\mathcal{L}$ induced from the $L$-invariant differential system $D$ as follows ([T2, §6], [We], [Gu, §7], [Go, §4]), which will be the main tool in our argument.

Put $\mathcal{L}^{-1} = \pi_*^{-1}(D_o)$ or equivalently

$$\mathcal{L}^{-1} = \{ X \in \mathcal{L} \mid X_o \in D(o) \},$$

under the above identification. Starting from $(\mathcal{L}, \mathcal{L}^{-1}, \mathcal{L}')$, we first define $\mathcal{L}^p$ for $p < -1$ inductively by

$$\mathcal{L}^p = \mathcal{L}^{p+1} + [\mathcal{L}^{p+1}, \mathcal{L}^{-1}].$$

We put $\mathcal{L}^0 = \mathcal{L}'$ and define $\mathcal{L}^k$ for $k > 0$ inductively by

$$\mathcal{L}^k = \{ X \in \mathcal{L}^{k-1} \mid [X, \mathcal{L}^{-1}] \subset \mathcal{L}^{k-1} \}.$$

Here we note that, since $\mathcal{L}^0$ is self-normalizing, $\mathcal{L}^1$ is properly contained in $\mathcal{L}^0$. Obviously $\mathcal{L}^p$ is $\mathrm{Ad}(L')$-invariant for all $p \in \mathbb{Z}$. It is easy to check that $\{\mathcal{L}^p\}_{p \in \mathbb{Z}}$ satisfies

$$[\mathcal{L}^p, \mathcal{L}^q] \subset \mathcal{L}^{p+q} \qquad \text{for all } p,\ q \in \mathbb{Z}.$$

Since $\mathcal{L}$ is finite dimensional, there exist integers $\mu > 1$ and $\nu \geqq 0$ such that

$$\mathcal{L}^p = \mathcal{L}^{-\mu} \supsetneqq \mathcal{L}^{-\mu+1} \quad \text{for } p \leqq -\mu, \quad \mathcal{L}^\nu \supsetneqq \mathcal{L}^{\nu+1} = \mathcal{L}^k \quad \text{for } k \geqq \nu + 1.$$

Then $\mathcal{L}^{-\mu}$ is a $\mathrm{Ad}(L')$-invariant subalgebra of $\mathcal{L}$ properly containing $\mathcal{L}^0$ and $\mathcal{L}^{\nu+1}$ is an ideal of $\mathcal{L}$ properly contained in $\mathcal{L}^0$. Hence, by our assumption that $L$ acts primitively and effectively on $M$, we obtain $\mathcal{L} = \mathcal{L}^{-\mu}$ and $\mathcal{L}^{\nu+1} = \{0\}$. Thus $\mathcal{L} = \{\mathcal{L}^p\}_{p \in \mathbb{Z}}$ becomes a (transitive) filtered Lie algebra. This filtration $\{\mathcal{L}^p\}_{p \in \mathbb{Z}}$ is called the Weisfeiler filtration of $(\mathcal{L}, \mathcal{L}^0)$ in §7 of [Gu] and §4 of [Go].

We now consider the associated graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of $\mathcal{L} = \{\mathcal{L}^p\}_{p \in \mathbb{Z}}$. Namely we put $\mathfrak{g}_p = \mathcal{L}^p / \mathcal{L}^{p+1}$ for $p \in \mathbb{Z}$ and put

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p.$$

Let $\varpi_p$ be the projection of $\mathcal{L}^p$ onto $\mathfrak{g}_p = \mathcal{L}^p / \mathcal{L}^{p+1}$. Then, for $X \in \mathfrak{g}_p$ and $Y \in \mathfrak{g}_q$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}$ is defined by

$$[X, Y] = \varpi_{p+q}([\tilde{X}, \tilde{Y}]),$$

where $\tilde{X} \in \mathcal{L}^p$ and $\tilde{Y} \in \mathcal{L}^q$ are any element such that $\varpi_p(\tilde{X}) = X$ and $\varpi_q(\tilde{Y}) = Y$ (cf. §1.2). For each $g \in L'$, the graded map $\hat{\varphi}_g$ of $\mathrm{Ad}(g)$ is a graded Lie algebra automorphism of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ (cf. Proposition

3.11). Thus we have a representation $\beta\colon L' \to \mathrm{Aut}_g(\mathfrak{g})$ by $\beta(g) = \hat{\varphi}_g$, where $\mathrm{Aut}_g(\mathfrak{g})$ is the group of all graded Lie algebra automorphisms of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. $G'_0 = \beta(L')$ is a Lie subgroup of $\mathrm{Aut}_g(\mathfrak{g})$ with Lie algebra isomorphic with $\mathfrak{g}_0 = \mathcal{L}^0/\mathcal{L}^1$.

Then, by our choice of $\mathcal{L}^{-1}$ and the construction of $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$, we have

$$(5.3)\quad \begin{cases} \text{(i)} & \mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \text{ for } p < -1, \\ \text{(ii)} & \text{For } k \geqq 0, \text{ if } X \in \mathfrak{g}_k \text{ and } [X, \mathfrak{g}_{-1}] = \{0\}, \text{ then } X = 0, \\ \text{(iii)} & G'_0 \text{ acts irreducibly on } \mathfrak{g}_{-1}. \end{cases}$$

Here we note that, from the structure equation of $L$, it follows that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ gives the symbol algebra of $(M, D)$ (cf. [T2, §6]). Any subalgebra $\mathfrak{a}$ of $\mathcal{L}$ becomes a filtered subalgebra of $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ with the filtration $\{\mathfrak{a}^p\}_{p\in\mathbb{Z}}$ given by $\mathfrak{a}^p = \mathfrak{a} \cap \mathcal{L}^p$ for $p \in \mathbb{Z}$. Its associated graded Lie algrebra $\widehat{\mathfrak{a}} = \bigoplus_{p\in\mathbb{Z}} \widehat{\mathfrak{a}}_p$ is a graded subalgebra of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ satisfying $\dim \mathfrak{a} = \dim \widehat{\mathfrak{a}}$. Especially $\widehat{\mathcal{L}^0} = \bigoplus_{p\geqq 0} \mathfrak{g}_p$. Moreover $\widehat{\mathfrak{a}}$ is an ideal of $\mathfrak{g}$ if $\mathfrak{a}$ is an ideal of $\mathcal{L}$. With these preparation, we have ([K-N, I, p. 878, Lemmas 1 and 2])

**Lemma 5.6.** *$\mathcal{L}$ is simple.*

*Proof.* Let $\mathfrak{c}$ be an $\mathrm{Ad}(L')$-invariant ideal of $\mathcal{L}$. Since $\mathcal{L}' = \mathcal{L}^0$ is a maximally $\mathrm{Ad}(L')$-invariant subalgebra and contains no ideal of $\mathcal{L}$, we have $\mathcal{L} = \mathfrak{c} + \mathcal{L}^0$. Then we have $\mathfrak{g} = \widehat{\mathfrak{c}} + \mathfrak{g}'$, where $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$. Hence $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p \subset \widehat{\mathfrak{c}}$. Here we note that $\widehat{\mathfrak{c}}$ is abelian if $\mathfrak{c}$ is so. On the other hand, by our assumption; $\mu > 1$, $\mathfrak{m}$ is not abelian. Hence $\mathcal{L}$ has no abelian ideals, which is $\mathrm{Ad}(L')$-invariant. However if the radical $\mathfrak{r}$ of $\mathcal{L}$ is non-trivial, the last ideal in the derived series of $\mathfrak{r}$ is a non-trivial abelian ideal, which is obviously invariant by $\mathrm{Ad}(L')$. Therefore $\mathcal{L}$ is semisimple. Then, since $\mathrm{Ad}(L')$ is a subgroup of the adjoint group $\mathrm{Int}(\mathcal{L}) = \mathrm{Ad}(L)$, each simple ideal of $\mathcal{L}$ is $\mathrm{Ad}(L')$-invariant. For two simple ideals of $\mathfrak{c}_1$ and $\mathfrak{c}_2$ of $\mathcal{L}$, we have $\widehat{\mathfrak{c}}_1 \cap \widehat{\mathfrak{c}}_2 \supset \mathfrak{m}$. Thus $[\widehat{\mathfrak{c}}_1, \widehat{\mathfrak{c}}_2] \neq \{0\}$, which implies $\mathfrak{c}_1 = \mathfrak{c}_2$. Therefore $\mathcal{L}$ is simple.

*Remark* 5.7. When the linear isotropy representation $\gamma\colon L' \to GL(T_o(M))$ is irreducible, the nonlinearity of the action; $\mathrm{Ker}\,\gamma$ is non-discrete, is necessary to conclude that $\mathcal{L}$ is simple (see [K-N, I, Lemma 2]). In fact when $\mathcal{L}$ is not simple, the structure of the pair $(\mathcal{L}, \mathcal{L}')$ is determined by Morosov and Golubitsky (see [Go, Proposition 2.3]). Especially $\mathcal{L}/\mathcal{L}'$ is $\mathcal{L}'$-irreducible in this case. Lemma 5.6 follows also from this fact.

The structure of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is determined by the following Lemma due to Weisfeiler and Golubitsky ([We], [Go, Theorem 4.3]).

**Lemma 5.8.** *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$ satisfying conditions in* (5.3). *Then*

(1) *If $\mathfrak{g}_1 \neq \{0\}$, $\mathfrak{g}$ is semisimple.*
(2) *If $\mathfrak{g}_1 = \{0\}$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0$, that is, $\mathfrak{g}_k = \{0\}$ for $k \geqq 1$, and $\mathfrak{g}_0$ is reductive.*

*Proof.* We reproduce the proof from Lemma 8.1 of [Gu] and Lemma 4.2 of [Go]. Let $\delta_o$ be the derivation of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ defined by $\delta_o(X) = pX$ for $X \in \mathfrak{g}_p$. We consider the radical $\mathfrak{r}$ of $\mathfrak{g}$. $\mathfrak{r}$ is preserved by any Lie algebra automorphism of $\mathfrak{g}$. Hence $\mathfrak{r}$ is invariant by $G'_0$ and by $\delta_o$ as well. Thus $\mathfrak{r}$ is a graded ideal of $\mathfrak{g}$, that is, $\mathfrak{r} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{r}_p$, where $\mathfrak{r}_p = \mathfrak{r} \cap \mathfrak{g}_p$. Then $\mathfrak{r}_{-1}$ is a $G'_0$-invariant subspace of $\mathfrak{g}_{-1}$. Hence, by (iii) of (5.3), we have two cases to distinguish; (1) $\mathfrak{r}_{-1} = \{0\}$ or (2) $\mathfrak{r}_{-1} = \mathfrak{g}_{-1}$.

In case (1), by (ii) of (5.3), we get $\mathfrak{r}_k = \{0\}$ for $k \geqq 0$ by induction on $k \geqq 0$. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be a Levi decomposition of $\mathfrak{g}$. With respect to the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$, $\mathfrak{f}^p = \bigoplus_{j\geqq p} \mathfrak{g}_j$, of $\mathfrak{g}$, we take the associated graded Lie algebras of both sides of $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. Then, since $\mathfrak{r}$ is graded, we get $\mathfrak{g} = \mathfrak{r} + \widehat{\mathfrak{s}}$. Hence, from $\mathfrak{r}_k = \{0\}$ for $k \geqq -1$, $\widehat{\mathfrak{s}} \supset \mathfrak{g}_{-1} \oplus \mathfrak{f}^0$. Thus, by (i) of (5.3), we obtain $\widehat{\mathfrak{s}} = \mathfrak{g}$. From $\dim \mathfrak{s} = \dim \widehat{\mathfrak{s}}$, it follows that $\mathfrak{g} = \mathfrak{s}$ and $\mathfrak{r} = \{0\}$. Hence $\mathfrak{g}$ is semisimple in this case. In particular $\mathfrak{g}_1 \neq \{0\}$.

In case (2), $\mathfrak{r}$ is a $G'_0$-invariant graded ideal of $\mathfrak{g}$ containing $\mathfrak{g}_{-1}$. First we shall show that $\mathfrak{g}_1 = \{0\}$ in this case, which implies $\mathfrak{g}_k = \{0\}$ for $k > 1$ by (ii) of (5.3) and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_0$. Assume the contrary; $\mathfrak{g}_1 \neq \{0\}$. Then we claim

*If $\mathfrak{c}$ is a $G'_0$-invariant graded ideal of $\mathfrak{g}$ containing $\mathfrak{g}_{-1}$, then $[\mathfrak{c}, \mathfrak{c}]$ is also a $G'_0$-invariant graded ideal of $\mathfrak{g}$ containing $\mathfrak{g}_{-1}$.*

In fact, obviously, $[\mathfrak{c}, \mathfrak{c}]$ is a $G'_0$-invariant graded ideal of $\mathfrak{g}$. By (ii) of (5.3), $[\mathfrak{g}_{-1}, \mathfrak{g}_1] \neq \{0\}$ if $\mathfrak{g}_1 \neq \{0\}$. Since $\mathfrak{c}$ is an ideal satisfying $\mathfrak{c}_{-1} = \mathfrak{g}_{-1}$, we get $\mathfrak{c}_0 \neq \{0\}$. Then, again by (ii) of (5.3), $[\mathfrak{c}_{-1}, \mathfrak{c}_0] \neq \{0\}$. Since $\mathfrak{c}$ is $G'_0$-invariant, we obtain $[\mathfrak{c}_{-1}, \mathfrak{c}_0] = \mathfrak{g}_{-1}$ by (iii) of (5.3). The above claim implies that $\mathfrak{c}$ cannot be solvable. Therefore $\mathfrak{g}_1 = \{0\}$ in case (2).

Finally we shall show that $\mathfrak{g}_0$ is reductive following Lemma 4.2 of [Go]. We consider the representation ad: $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$. Let us take a nonzero $ad(\mathfrak{g}_0)$-irreducible subspace $V$ of $\mathfrak{g}_{-1}$. For a graded Lie algebra automorphism $\varphi \in G'_0$, $\varphi(V)$ is also ad$(\mathfrak{g}_0)$-irreducible and is isomorphic with $V$ as a $\mathfrak{g}_0$-module. Put $W = \sum_{\varphi\in G'_0} \varphi(V)$. Then $W$ is a non-trivial $G'_0$-invariant subspace of $\mathfrak{g}_{-1}$. Hence, by (iii) of (5.3), we get $\mathfrak{g}_{-1} = W$.

Thus $\mathfrak{g}_{-1}$ can be written as a direct sum of ad($\mathfrak{g}_0$)-irreducible subspaces. Hence ad: $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ is completely reducible and also faithful by (ii) of (5.3), which shows that $\mathfrak{g}_0$ is reductive (cf. [V, Theorem 3.16.3]).

Next we recall the following Lemma ([K-N, IV, Theorem 4.1], [Gu, Proposition 7.2]), which enables us to determine the structure of the filtered Lie algebra $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ in case (1) of Lemma 5.8.

**Lemma 5.9.** *Let $\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ be a filtered Lie algebra over $K = \mathbb{R}$ or $\mathbb{C}$, whose associated graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-\mu}^{\nu} \mathfrak{g}_p$ satisfies conditions* (i) *and* (ii) *of* (5.3). *Then if $\mathfrak{g}_0$ contains an element $E$ such that*

$$[E, X] = -X \qquad \text{for } X \in \mathfrak{g}_{-1},$$

*then $\mathcal{L}$ is isomorphic with $\mathfrak{g}$ as a filtered Lie algebra, where the filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of $\mathfrak{g}$ is given by $\mathfrak{f}^p = \bigoplus_{j\geqq p} \mathfrak{g}_j$ for $p \in \mathbb{Z}$.*

*Proof.* First we note that, for all $p \in \mathbb{Z}$,

$$(5.4) \qquad [E, X] = pX \qquad \text{for } X \in \mathfrak{g}_p.$$

In fact, for $p < 0$, this follows from the generating condition (i) of (5.3). For $p \geqq 0$, we have

$$[Y, [E, X]] = [Y, X] + [E, [Y, X]] \qquad \text{for } Y \in \mathfrak{g}_{-1} \text{ and } X \in \mathfrak{g}_p.$$

Then, for $X \in \mathfrak{g}_0$, we get $[Y, [E, X]] = 0$ for all $Y \in \mathfrak{g}_{-1}$. Hence, by (ii) of (5.3), we get $E \in Z(\mathfrak{g}_0)$. Thus, for $p \geqq 0$, (5.4) follows from (ii) of (5.3) by induction on $p \geqq 0$.

Let us take an element $\widehat{E}$ of $\mathcal{L}^0$ such that $\varpi_0(\widehat{E}) = E$. Then, by (5.4), we see that the eigenvalues of $\mathrm{ad}(\widehat{E})$ are $-\mu, \ldots, \nu$ and $\mathcal{L}^p$ is the direct sum of the primary components $\mathcal{L}_j = \mathrm{Ker}(\mathrm{ad}(\widehat{E}) - j \cdot \mathrm{id})^{n_j}$ of $\mathrm{ad}(\widehat{E})$ for the eigenvalues $j = p, p+1, \ldots, \nu$. Moreover $[\mathcal{L}_p, \mathcal{L}_q] \subset \mathcal{L}_{p+q}$ (cf. [Hu, §15.1]). Namely the primary decomposition $\mathcal{L} = \bigoplus_{p=-\mu}^{\nu} \mathcal{L}_p$ with respect to $\mathrm{ad}(\widehat{E})$ gives a gradation of $\mathcal{L}$ such that $\mathcal{L}^p = \bigoplus_{j=p}^{\nu} \mathcal{L}_j$. By definition of the associated graded Lie algebra, it follows that $\mathcal{L} = \bigoplus_{p\in\mathbb{Z}} \mathcal{L}_p$ is isomorphic with $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ as a graded Lie algebra.

Now we have

**Theorem 5.10.** *Let $L$ be a connected real* (*or complex*) *Lie group acting transitively and effectively on a real* (*or complex*) *manifold $M$ and $L'$ be the isotropy subgroup of $L$ at a point $o$ of $M$ so that $M = L/L'$. Let $\mathcal{L}$ and $\mathcal{L}'$ be the Lie algebras of $L$ and $L'$ respectively. Assume that*

*$L$ acts primitively on $M$ and leaves invariant a differential system $D$ on $M$. Then $L$ is simple. Moreover let us take $D$ to be minimal and introduce a filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ of $\mathcal{L}$ induced from $D$. Assume further $\mathcal{L}^1 \neq \{0\}$. Then the following holds in either real or complex category.*

(1) *$\mathcal{L}'$ is a maximal parabolic subalgebra of $\mathcal{L}$.*

(2) *$\mathcal{L} = \{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ is naturally isomorphic with the associated graded Lie algebra $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ as a filtered Lie algebra. In particular $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a simple graded Lie algebra such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is fundamental, and the filtration $\{\mathcal{L}^p\}_{p\in\mathbb{Z}}$ of $\mathcal{L}$ is the one uniquely determined by $\mathcal{L}'$ as in* Lemma 3.10.

(3) *$M$ is a covering space over $M_{\mathfrak{g}}$ such that $D$ is the lift of $D_{\mathfrak{g}}$, where $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ is the standard differential system of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Especially $(M, D)$ is isomorphic with $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ always in the complex category and when $\mathcal{L}$ is complex simple in the real category.*

(4) *Except when $(M, D)$ is locally isomorphic with a real or complex standard contact manifold, $\mathcal{A}_x(M, D)$ is isomorphic with $\mathcal{L}$ at each $x \in M$, where $\mathcal{A}_x(M, D)$ denotes the stalk at $x$ of the Lie algebra sheaf $\mathcal{A}(M, D)$ of all infinitesimal automorphisms of $(M, D)$.*

*Proof.* By Lemma 5.6, $\mathcal{L}$ is simple over $K = \mathbb{R}$ or $\mathbb{C}$, depending on whether we work in the real or complex category. Put $G = \mathrm{Int}(\mathcal{L})$ and let $G'$ be the normalizer of $\mathcal{L}'$ in $G$:

$$G' = \{\, g \in G \mid \mathrm{Ad}(g)(\mathcal{L}') = \mathcal{L}' \,\}.$$

Since $\mathcal{L}'$ is self-normalizing, $G'$ is the largest Lie subgroup of $G$ with Lie algebra $\mathcal{L}'$. Then, for the adjoint representation $\mathrm{Ad}\colon L \to GL(\mathcal{L})$ (in the category we are working), we have $\mathrm{Ad}(L) = G$ and $\mathrm{Ad}\colon L \to G$ is a covering homomorphism such that $\mathrm{Ad}(L') \subset G'$. Put $\widehat{L}' = \mathrm{Ad}^{-1}(G')$. Then $\widehat{L}'$ is a closed subgroup of $L$ containing $L'$ such that $L/\widehat{L}'$ is diffeomorphic with $G/G'$. Thus we see that the projection $p\colon M = L/L' \to G/G'$, defined by the following commutative diagram, is a covering map;

$$\begin{array}{ccc} L & \xrightarrow{\mathrm{Ad}} & G \\ \downarrow & & \downarrow \\ M & \xrightarrow{p} & G/G' \end{array} \tag{5.5}$$

Now assume that $\mathcal{L}^1 \neq \{0\}$. Then the assertion (2) follows from Lemmas 5.8 and 5.9 and $\mathcal{L}'$ is a parabolic subalgebra of $\mathcal{L}$. The last statement of (2) is a consequence of (i) of (5.3) and Lemma 3.10. Then, by the construction of the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (see §4.1) and (5.5), we see that $G/G' = M_{\mathfrak{g}}$ and $D = p_*^{-1}(D_{\mathfrak{g}})$, which shows the first assertion in (3).

Next let us show the second assertion in (3) and the assertion (1). First we treat the case when $\mathcal{L}$ is a simple Lie algebra over $\mathbb{C}$. In this case, $G$ and $G'$ are complex Lie groups. It is well-known (cf. [Wa], [Tt1], [Tk1]) that the complex $R$-space $M_{\mathfrak{g}} = G/G'$ is simply connected, which implies the second assertion in (3) and that $G'$ is connected. Hence $\mathrm{Ad}(L') = G'$ in this case. Then, by (iii) of (5.3), $\mathfrak{g}_{-1}$ is $\mathrm{ad}(\mathfrak{g}_0)$-irreducible, which implies $\mathcal{L}'$ is maximal parabolic (see Remark 3.7). Moreover, from the assumption that $L$ acts effectively on $M = L/L'$, it is easy to see that $\mathrm{Ad}\colon L \to G$ is an isomorphism such that $\mathrm{Ad}(L') = G'$ (in the category we are working) in this case.

Now we treat the case when $\mathcal{L}$ is a simple Lie algebra over $\mathbb{R}$ such that $\mathbb{C}\mathcal{L}$ is complex simple. We put $\mathbb{C}G = \mathrm{Int}(\mathbb{C}\mathcal{L})$ and

$$\mathbb{C}G' = \{\, g \in \mathbb{C}G \mid \mathrm{Ad}(g)(\mathbb{C}\mathcal{L}') = \mathbb{C}\mathcal{L}' \,\}.$$

Then $G$ is identified with the identity component of the closed real Lie subgroup of $\mathbb{C}G$ consisting of all elements of $\mathbb{C}G$ which commutes with the conjugation with respect to the real form $\mathcal{L}$ of $\mathbb{C}\mathcal{L}$ (cf. [He, Chapter III, Lemma 6.2]). We have $G' = G \cap \mathbb{C}G'$ and $\mathbb{C}G'$ is connected. If there exists a proper subalgebra $\mathfrak{h}$ of $\mathcal{L}$ containing $\mathcal{L}'$ properly, $\mathbb{C}\mathfrak{h}$ is $\mathbb{C}G'$-invariant by the connectivity of $\mathbb{C}G'$. Hence $\mathfrak{h}$ is $G'$-invariant and also $\mathrm{Ad}(L')$-invariant from $\mathrm{Ad}(L') \subset G'$, which contradicts the assumption that $L$ acts primitively on $M = L/L'$. Therefore $\mathcal{L}'$ is maximal parabolic, which completes the proof of (1).

Finally, observing that $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ is not maximal parabolic in case (3) of Theorem 5.3, the assertion (4) follows from (3) and Corollary 5.4.

*Remark* 5.11. (1) Since the Lie algebera of $\mathrm{Ker}\,\gamma$ coincides with $\mathcal{L}^{\mu}$ in case (1) and vanishes in case (2) of Lemma 5.8, the condition $\mathcal{L}^1 \neq \{0\}$ is equivalent to the nonlinearity of the action: $\mathrm{Ker}\,\gamma$ is nondiscrete. The finite dimensional nonlinear primitive Lie algebras $(\mathcal{L}, \mathcal{L}')$ were first classified by Ochiai [O1], where a primitive subalgebra $\mathcal{L}'$ of $\mathcal{L}$ is, by definition, a maximal subalgebra of $\mathcal{L}$. In the present article, we follow the definition given in [Go] for the primitive action of a connected Lie group $L$. Fixing a Lie algebra pair $(\mathcal{L}, \mathcal{L}')$, where $\mathcal{L}$ is the Lie algebra of $L$, this notion of primitivity depends on the choice of $L'$, although if $\mathcal{L}'$ is maximal, $L$ acts primitively on $L/L'$ for any choice of $L'$. In fact Golubitsky [Go] has shown many examples of $(\mathcal{L}, \mathcal{L}')$ such that $\mathcal{L}'$ is nonmaximal and $L = \mathrm{Int}(\mathcal{L})$ acts primitively on $L/L'$, where $L'$ is the normalizer of $\mathcal{L}'$ in $L$. Moreover he has shown that this phenomenon (nonmaximality of $\mathcal{L}'$) occurs only when $\mathcal{L}$ is simple and $\mathcal{L}'$ is reductive. For the details, we refer the reader to the original paper [Go].

(2) The nonlinearity of the action: $\mathcal{L}^1 \neq \{0\}$ is necessary in Theorem 5.10 as the following example shows (cf. [D]): We consider the simple Lie algebra $\mathcal{L}$ of type $G_2$. Let us fix a Cartan subalgebra $\mathfrak{h}$ and simple root system $\Delta = \{\alpha_1, \alpha_2\}$ as in §3.4. Let $\mathcal{L}(\alpha_1)$ be the subalgebra of $\mathcal{L}$ generated by the root vectors for the roots $\alpha_2$, $-\theta$, $-\alpha_2$ and $\theta$, where $\theta = 3\,\alpha_1 + 2\,\alpha_2$ is the highest root. Then we have

$$\mathcal{L}(\alpha_1) = \mathfrak{g}_{\theta-\alpha_2} \oplus \mathfrak{g}_{\theta} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{-\theta} \oplus \mathfrak{g}_{\alpha_2-\theta}.$$

$\mathcal{L}(\alpha_1)$ is a maximal simple subalgebra of type $A_2$. This is an example of the construction of regular semisimple subalgebras due to Dynkin [D]. Moreover we have an $\mathrm{ad}(\mathcal{L}(\alpha_1))$-irreducible decomposition of $\mathcal{L}$;

$$\mathcal{L} = V_1 \oplus V_2 \oplus \mathcal{L}(\alpha_1),$$

where

$$\begin{cases} V_1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{-(2\alpha_1+\alpha_2)}, \\ V_2 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-(\alpha_1+\alpha_2)} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2}. \end{cases}$$

In fact we have $[\mathcal{L}(\alpha_1), V_i] = V_i$ $(i = 1,\ 2)$, $[V_1, V_1] = V_2$, $[V_2, V_2] = V_1$ and $[V_1, V_2] = \mathcal{L}(\alpha_1)$. Put $L = \mathrm{Int}(\mathcal{L})$ and let $L'$ be the analytic subgroup of $L$ with Lie algebra $\mathcal{L}(\alpha_1)$. Then, since $\mathcal{L}(\alpha_1)$ is a maximal subalgebra, $L$ acts primitively and effectively on $L/L'$ such that the linear isotropy representation is reducible. Since $V_1$ and $V_2$ are isomorphic as an $\mathcal{L}(\alpha_1)$-module, there are many minimal $\mathrm{Ad}(L')$-invariant subspaces $\mathcal{L}^{-1}$ containing $\mathcal{L}(\alpha_1)$. However, for any choice of $\mathcal{L}^{-1}$, we see that the associated graded Lie algebra $\mathfrak{g}$ has a following description;

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

such that $\mathfrak{g}_{-2} = \wedge^2 V$ and $\mathfrak{g}_0 = \mathfrak{sl}(V)$ by putting $V = \mathfrak{g}_{-1}$. Namely $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is isomorphic with the universal fundamental graded algebra of second kind with $\dim \mathfrak{g}_{-1} = 3$ (cf. [T2, §3]) and $\mathfrak{g}_0 = \mathfrak{sl}(\mathfrak{g}_{-1}) \subset \mathfrak{gl}(\mathfrak{g}_{-1})$, where $\mathfrak{gl}(\mathfrak{g}_{-1})$ is naturally identified with the Lie algebra of all gradation preserving derivations of $\mathfrak{m}$.

Finally we note that, if we take $L'$ to be the normalizer of $\mathcal{L}(\alpha_1)$ in $L$, $L$ acts primitively and effectively on $L/L'$ such that the linear isotropy representation is irreducible.

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