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The Hyperadelic Gamma Function: A Précis

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§ 0. Introduction

The point of departure for the results announced here is Y. Ihara's fascinating study [10] of the l-adic Tate module of the jacobian of the Fermat curve of degree l^n in the limit as $n \to \infty$. The object of this paper is to summarize some work of the author (see § 1, § 2 and § 4 below) and some joint work of the author with R. Coleman (see § 3 below), as well as to indicate the connections between our work and that of Ihara (see § 5 below).

The author's interest in these topics was originally sparked by Theorem 10 of [10] and the remark following which suggested the problem of finding a universal power series for Gauss sums into which Ihara's universal power series for Jacobi sums could be factored. We could not see how to carry out this factorization in the context of Ihara's theory, and so we developed a rather different approach to the study of the Fermat tower emphasizing relative homology groups and 1-motives, with the added feature of being adelic rather than purely l-adic. The main concepts of our theory are i) the *adelic beta function*, an object from which Ihara's universal power series for Jacobi sums can be recovered by pro-l specialization, and ii) the *hyperadelic gamma function*, an object which provides an adelic interpolation of Gauss sums and in terms of which the adelic beta function can be factored. While the main results of the theory can be formulated in classfield-theoretic terms and will be so formulated here, ideas from algebraic geometry and topology are needed to give natural proofs. The reader is referred to the forthcoming papers [1, 2] for details.

R. Coleman's interest in these topics arose in connection with a conjecture enunciated by Ihara at the Kyoto conference. Ihara conjectured a striking link between the galois-theoretic properties of the l-power torsion points of the jacobians of the l-power degree Fermat curves and the l-power roots of the numbers of the form $1-\zeta$, $1 \neq \zeta \in \mu_{l^\infty}$. (See [11] for Ihara's conjecture.) Coleman obtained a proof of Ihara's conjecture by employing his explicit reciprocity law [4], Iwasawa's theorem on local units modulo

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circular units [12] and the pro-l specialization of the hyperadelic gamma function. After the conference, Ihara, Kaneko and Yukinari [11] obtained a proof of Ihara's conjecture by a different method of independent interest. Later, the author observed that Ihara's conjecture and Coleman's method could be generalized. Jointly, Coleman and the author arrived at the result enunciated as Theorem 12 of Section 3 below. Still later, the author found another way to prove Theorem 12 that was i) simpler and ii) geometric in spirit. (This sharpening of Theorem 12 appears as Theorem 13 of Section 4 below.) As a result, Coleman's ideas will not be published in their original form, i.e. applied to the proof of Theorem 12, but rather, in the forthcoming paper [3], will be applied to the deduction of consequences of Theorem 13 in cyclotomic number theory, a context in which Coleman's ideas are indispensable.

Coming full circle, we note that the congruences of Miki [13], presented at the Kyoto conference, may be recovered from Theorems 11 and 13 of this paper.

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§ 1. Results concerning the adelic beta function

1.1. We employ the following general notation throughout the paper:

$\bar{\boldsymbol{Q}} \overset{\text{def}}{=}$ the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$,

$G(\boldsymbol{Q}) \overset{\text{def}}{=}$ the Galois group of $\bar{\boldsymbol{Q}}$ over $\boldsymbol{Q}$,

$\rho \overset{\text{def}}{=}$ complex conjugation,

$\hat{\boldsymbol{Z}} \overset{\text{def}}{=}$ the profinite completion of $\boldsymbol{Z}$,

$\chi \overset{\text{def}}{=}$ the cyclotomic character $G(\boldsymbol{Q}) \longrightarrow \hat{\boldsymbol{Z}}^\times$.

Given any subfield K of $\bar{\boldsymbol{Q}}$, we denote the Galois group of $\bar{\boldsymbol{Q}}$ over K by $G(K)$. Given any $x \in \bar{\boldsymbol{Q}}$ and $\sigma \in G(\boldsymbol{Q})$, we denote the image of x under the automorphism σ of $\bar{\boldsymbol{Q}}$ either by x^σ or σx, whichever is more convenient in a given context. But we require the action of $G(\boldsymbol{Q})$ upon $\bar{\boldsymbol{Q}}$ to be a *left* action, and therefore, given also $\tau \in G(\boldsymbol{Q})$, we must have, by convention,

$$(\sigma\tau)x = \sigma(\tau x) = x^{(\sigma\tau)} = (x^\tau)^\sigma.$$

Given $a \in \boldsymbol{Q}/\boldsymbol{Z}$, we denote by $\langle a \rangle$ the least nonnegative rational number in the coset a. Abusing language, we denote the automorphism of $\overline{\boldsymbol{Q}}$ induced by complex conjugation again by ρ.

1.2. For each positive integer m, set

$\boldsymbol{Q}_{[m]} \overset{\mathrm{def}}{=}$ the subfield of $\boldsymbol{C}$ generated by the m^{th} roots of unity,

$\boldsymbol{Q}_{[m]}^{ab} \overset{\mathrm{def}}{=}$ the maximal abelian extension of $\boldsymbol{Q}_{[m]}$ in $\overline{\boldsymbol{Q}}$,

$\boldsymbol{Q}_{[m]}^{abs} \overset{\mathrm{def}}{=}$ the largest subfield of $\boldsymbol{Q}_{[m]}^{ab}$ in which every archimedean place of $\boldsymbol{Q}_{[m]}$ splits completely,

$r_{[m]} \overset{\mathrm{def}}{=}$ the reciprocity homomorphism from the idèle group of $\boldsymbol{Q}_{[m]}$ to $\mathrm{Gal}\,(\boldsymbol{Q}_{[m]}^{ab}/\boldsymbol{Q}_{[m]})$ of global classfield theory.

Let $\bar{r}_{[m]}\colon (\boldsymbol{Q}_{[m]} \otimes \hat{\boldsymbol{Z}})^{\times} \longrightarrow \mathrm{Gal}\,(\boldsymbol{Q}_{[m]}^{abs}/\boldsymbol{Q}_{[m]})$ be the unique homomorphism rendering the diagram

$$\begin{tikzcd}
(\text{idèle group of } \boldsymbol{Q}_{[m]}) \arrow[r, "r_{[M]}"] \arrow[dd, "\text{forget components at } \infty"'] & \mathrm{Gal}\,(\boldsymbol{Q}_{[m]}^{ab}/\boldsymbol{Q}_{[m]}) \arrow[d, "\sigma \mapsto \sigma^{-1}"] \\
 & \mathrm{Gal}\,(\boldsymbol{Q}_{[m]}^{ab}/\boldsymbol{Q}_{[m]}) \arrow[d, "\text{restriction}"] \\
(\boldsymbol{Q}_{[m]} \otimes \hat{\boldsymbol{Z}})^{\times} \arrow[r, "\bar{r}_{[m]}"'] & \mathrm{Gal}\,({}_{[m]}^{abs}/\boldsymbol{Q}_{[m]})
\end{tikzcd}$$

commutative.

1.3. Given a commutative pro-artinian ring R and a profinite abelian group G, set

$$R[[G]] \overset{\mathrm{def}}{=} \varprojlim\, (R/I)[G/U],$$

the limit being extended over all open ideals I of R and open subgroups U of G. The ring $R[[G]]$ is again pro-artinian. We identify R with a subring of $R[[G]]$ and G with a subgroup of $R[[G]]^{\times}$ in the evident fashion. Given $f \in R[[G]]$, we denote by $f^{\dagger}$ the image of f under the unique continuous R-linear homomorphism $R[[G]] \to R[[G]]$ prolonging the group homomorphism $g \mapsto g^{-1}\colon G \to G$. More generally, given a continuous homomorphism $b\colon R \to S$ of pro-artinian rings and a continuous homomorphism $c\colon G \to H$ of profinite abelian groups we denote by $b \hat{\otimes} c$ the unique continuous ring homomorphism $R[[G]] \to S[[H]]$ prolonging both b and c.

1.4. Set

$$\hat{\boldsymbol{Z}}(1) \stackrel{\text{def}}{=} \operatorname{Hom}(\boldsymbol{Q}/\boldsymbol{Z}, \bar{\boldsymbol{Q}}^{\times}),$$

and, more generally, for each positive integer n, set

$$\hat{\boldsymbol{Z}}^n(1) \stackrel{\text{def}}{=} \operatorname{Hom}((\boldsymbol{Q}/\boldsymbol{Z})^n, \bar{\boldsymbol{Q}}^{\times}),$$

where $(\boldsymbol{Q}/\boldsymbol{Z})^n$ denote the product of n copies of $\boldsymbol{Q}/\boldsymbol{Z}$.

Given $a \in (\boldsymbol{Q}/\boldsymbol{Z})^n$ and $f \in \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^n(1)]]$, let $f(a)$ denote the image of f under the unique $\hat{\boldsymbol{Z}}$-linear ring homomorphism $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^n(1)]] \to \bar{\boldsymbol{Q}} \otimes \hat{\boldsymbol{Z}}$ extending the group homomorphism $z \mapsto z(a)$: $\hat{\boldsymbol{Z}}^n(1) \to \bar{\boldsymbol{Q}}^{\times}$, and which, for some positive integer N, factors through the evident surjective homomorphism $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^n(1)]] \to \hat{\boldsymbol{Z}}[\operatorname{Hom}(((1/N)\boldsymbol{Z}/\boldsymbol{Z})^n, \bar{\boldsymbol{Q}}^{\times})]$. Note that an element f of $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^n(1)]]$ vanishes if and only if $f(a)=0$ for all $a \in (\boldsymbol{Q}/\boldsymbol{Z})^n$. In short, f is determined by its values $f(a)$ and may, when convenient, be regarded as a special sort of $\bar{\boldsymbol{Q}} \otimes \hat{\boldsymbol{Z}}$-valued function on $(\boldsymbol{Q}/\boldsymbol{Z})^n$. Given $f \in \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^n(1)]]$ and $a \in (\boldsymbol{Q}/\boldsymbol{Z})^n$, note that for all $\sigma \in G(\boldsymbol{Q})$,

$$((1 \hat{\otimes} \sigma) f)(a) = f(\chi(\sigma) a) = (\sigma \otimes 1)(f(a)).$$

1.5. For each positive integer m, let $W_{[m]}$ denote the set of functions $w: G(\boldsymbol{Q}) \to G(\boldsymbol{Q})$ such that for all $\sigma \in G(\boldsymbol{Q})$ and $\tau \in G(\boldsymbol{Q}_{[m]})$

$$w(\sigma\tau) = w(\sigma) \in \sigma G(\boldsymbol{Q}_{[m]}),\ w(\rho\sigma) w(\sigma)^{-1} \in \{1, \rho\}.$$

For all positive integers m and $\sigma \in G(\boldsymbol{Q})$, let $\sigma_{[m]}$ denote the automorphism of $\boldsymbol{Q}_{[m]}^{abs}$ induced by σ. For each $a, b \in \boldsymbol{Q}/\boldsymbol{Z}$, let $\Phi^{a,b}: G(\boldsymbol{Q}) \to \{0, 1\}$ denote the function defined by the formula

$$\Phi^{a,b}(\sigma) \stackrel{\text{def}}{=} \langle -\chi(\sigma) a \rangle + \langle -\chi(\sigma) b \rangle - \langle -\chi(\sigma)(a+b) \rangle.$$

Proposition/Definition. *For all positive integers m and elements a and b of $\boldsymbol{Q}/\boldsymbol{Z}$ annihilated by m, there exists a unique function* $\mathrm{Ver}_{[m]}^{a,b}: G(\boldsymbol{Q}) \to \operatorname{Gal}(\boldsymbol{Q}_{[m]}^{abs}/\boldsymbol{Q}_{[m]})$ *such that for all $\sigma \in G(\boldsymbol{Q})$ and $w \in W_{[m]}$,*

$$\mathrm{Ver}_{[m]}^{a,b}(\sigma) = \prod_{\tau \in G(\boldsymbol{Q})/G(\boldsymbol{Q}_{[m]})} (w(\sigma\tau)^{-1} \sigma w(\tau))_{[m]}^{\Phi^{a,b}(\sigma\tau)}.$$

Proof. This is a variant of Tate's *half-transfer* [14, 15]. □

1.6. The *adelic beta function B_σ* $\in \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]^{\times}$ associated to an element σ of $G(\boldsymbol{Q})$ is characterized by

Theorem 1. *For all $\sigma \in G(\boldsymbol{Q})$, there exists a unique element B_σ of $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]^{\times}$ enjoying the following four properties:*

(I) *For all* $a \in \boldsymbol{Q}/\boldsymbol{Z}$,

$$B_\sigma(a, 0) = B_\sigma(0, a) = 1 \otimes 1 \in \bar{\boldsymbol{Q}} \otimes \hat{\boldsymbol{Z}}.$$

(II) *For all* $0 \neq a \in \boldsymbol{Q}/\boldsymbol{Z}$,

$$B_\sigma(a, -a) = (\sqrt{-1} \sin \pi a)^{1-\sigma} \otimes \chi(\sigma).$$

(III) *For all* $a, b \in \boldsymbol{Q}/\boldsymbol{Z}$ *such that* $a \neq 0$, $b \neq 0$ *and* $a+b \neq 0$,

$$B_\sigma(a, b) B_\sigma(-a, -b) = \left(\frac{\sqrt{-1} \sin \pi(a+b)}{(\sin \pi a)(\sin \pi b)}\right)^{\sigma-1} \otimes \chi(\sigma).$$

(IV) *For all positive integers* m *and elements* a *and* b *of* $\boldsymbol{Q}/\boldsymbol{Z}$ *annihilated by* m,

$$\bar{r}_{[m]}(B_\sigma(a, b)) = \mathrm{Ver}_{[m]}^{a,b}(\sigma) \in \mathrm{Gal}\,(\boldsymbol{Q}_{[m]}^{abs}/\boldsymbol{Q}_{[m]}). \qquad \square$$

For the proof, see [2], for which the prerequisite is [1].

The adelic beta functions satisfy many functional equations which can be proven by means of the techniques employed in the proof of Theorem 1. Here are some of the most important:

Corollary 2. *For all* $\sigma, \tau \in G(\boldsymbol{Q})$ *and* $a, b, c \in \boldsymbol{Q}/\boldsymbol{Z}$, *the following relations hold*:

(I) $B_\sigma(a, b) = B_\sigma(b, a)$.
(II) $B_\sigma(a, b) B_\sigma(a+b, c) = B_\sigma(a, b+c) B_\sigma(b, c)$.
(III) $B_\sigma(a, b) B_\tau(\chi(\sigma)a, \chi(\sigma)b) = B_{\sigma\tau}(a, b)$.
(IV) $B_\rho(a, b) = 1 \otimes 1$. (*Recall that* $\rho \overset{\mathrm{def}}{=}$ *complex conjugation.*)
(V) B_σ *depends continuously upon* σ. $\square$

Corollary 3. *For all rational primes* $p \neq l$, *elements* a *and* b *of* $\boldsymbol{Q}/\boldsymbol{Z}$ *of order not divisible by* p, *places* w *of* $\bar{\boldsymbol{Q}}$ *dividing* p, *and elements* σ *of the inertia subgroup of* $G(\boldsymbol{Q})$ *at* w,

$$(1 \otimes \pi_l)(B_\sigma(a, b)) = 1 \otimes 1 \in \bar{\boldsymbol{Q}} \otimes \boldsymbol{Z}_l,$$

where $\pi_l \colon \hat{\boldsymbol{Z}} \to \boldsymbol{Z}_l$ *is projection to the l-adic component.* $\square$

Corollary 4. *Let* m *be a positive integer and let* a *and* b *be elements of* $\boldsymbol{Q}/\boldsymbol{Z}$ *annihilated by* m. *Set*

$$\theta \overset{\mathrm{def}}{=} \sum_{\tau \in \mathrm{Gal}\,(\boldsymbol{Q}_{[m]}/\boldsymbol{Q})} \Phi^{a,b}(\tau)\tau^{-1} \in \boldsymbol{Z}[\mathrm{Gal}\,(\boldsymbol{Q}_{[m]}/\boldsymbol{Q})].$$

Then for all $\sigma \in G(\boldsymbol{Q}_{[m]})$ *and* $x \in (\boldsymbol{Q}_{[m]} \otimes \hat{\boldsymbol{Z}})^\times$ *verifying*

$$\sigma_{[m]} = \bar{r}_{[m]}(x) \in \mathrm{Gal}\,(\boldsymbol{Q}^{abs}_{[m]}/\boldsymbol{Q}_{[m]}),$$

one has

$$B_\sigma(a, b)x^{-\theta\otimes 1} \in (\boldsymbol{Q}_{[m]}\otimes 1)^\times \subseteq (\bar{\boldsymbol{Q}}\otimes\hat{\boldsymbol{Z}})^\times. \qquad \square$$

1.7. The classfield theoretic formulation of Theorem 1 and its corollaries belies the geometrical genesis of the adelic beta functions. We shall rectify this omission by formulating an alternate, geometrical characterization of the adelic beta functions. Let m be a positive integer. Set

$U_m \stackrel{\mathrm{def}}{=} \mathrm{Spec}\,(\boldsymbol{Q}[x, y]/(x^m + y^m - 1))$,

$Y_m \stackrel{\mathrm{def}}{=} \mathrm{Spec}\,(\boldsymbol{Q}[x, y]/(x^m + y^m - 1, xy))$,

$V_m \stackrel{\mathrm{def}}{=}$ the first singular homology group of the pair of spaces $(U_m(\boldsymbol{C}), Y_m(\boldsymbol{C}))$,

$SK_m \stackrel{\mathrm{def}}{=}$ the set of continuous maps c: $[0, 1] \to U_m(\boldsymbol{C})$ such that $x(c(t))^m = t$ for all $t \in [0, 1]$.

One can show that V_m is a free abelian group and that the set SK_m of 1-simplices constitutes a basis for V_m. Let

$$\theta_m : \hat{\boldsymbol{Z}}^2(1) \to \mathrm{Aut}\,(U_m \times \mathrm{Spec}(\boldsymbol{C})/\mathrm{Spec}(\boldsymbol{C}))$$

be the unique homomorphism such that

$$\theta_m(z)^* x = z\left(\frac{1}{m}, 0\right)x, \ \theta_m(z)^* y = z\left(0, \frac{1}{m}\right)y.$$

Then the action via θ_m of $\hat{\boldsymbol{Z}}^2(1)$ upon SK_m is transitive and the stabilizer of any element of SK_m is precisely

$$\left\{z \in \hat{\boldsymbol{Z}}^2(1) \,\middle|\, z\left(\frac{1}{m}, 0\right) = z\left(0, \frac{1}{m}\right) = 1\right\}.$$

Let c_m: $[0, 1] \to U_m(\boldsymbol{C})$ be the unique element of SK_m such that $x(c(\frac{1}{2}))$ and $y(c(\frac{1}{2}))$ are positive real numbers. Let β_m denote the element of V_m/mV_m to which c_m gives rise. Let $\mathfrak{A}_m$ denote the kernel of the evident homomorphism from $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]$ to $(\boldsymbol{Z}/m\boldsymbol{Z})[\mathrm{Hom}(((1/m)\boldsymbol{Z}/\boldsymbol{Z})^2, \bar{\boldsymbol{Q}}^\times)]$. Then, via θ_m, V_m/mV_m becomes a module over $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]$ which, viewed as a module over $\hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]/\mathfrak{A}_m$, is free of rank one with basis β_m. Now the module V_m/mV_m admits an interpretation in the context of étale homology theory and therefore is canonically equipped with the structure of a $G(\boldsymbol{Q})$-module. This action of $G(\boldsymbol{Q})$ upon V_m/mV_m is *semilinear*, i.e. for all $f \in \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]$,

$v \in V_m/mV_m$ and $\sigma \in G(\boldsymbol{Q})$,

$$\sigma(fv) = ((1\hat{\otimes}\sigma)f)(\sigma v).$$

It follows, in particular, that there exists $B'_{\sigma,m} \in \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]$, unique modulo $\mathfrak{A}_m$ and a unit modulo $\mathfrak{A}_m$, such that

$$B'_{\sigma,m}\beta_m \overset{\text{def}}{=} \sigma\beta_m.$$

One can easily check that if n is a multiple of m,

$$B'_{\sigma,m} \equiv B'_{\sigma,n} \bmod \mathfrak{A}_m.$$

Passing to the limit, as is evidently permissible, set

$$B'_\sigma \overset{\text{def}}{=} \lim B'_{\sigma,m} \in \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]]^\times.$$

Theorem 5. *For all* $\sigma \in G(\boldsymbol{Q})$,

$$B_\sigma = B'_\sigma. \qquad \square$$

The study of the $G(\boldsymbol{Q})$-module V_m/mV_m is carried out in the paper [1].

§ 2. Results concerning the hyperadelic gamma function

2.1. The functional equations (I) and (II) of Corollary 2 satisfied by the adelic beta function B_σ are analogous to certain functional equations satisfied by the classical beta function

$$B(s, t) \overset{\text{def}}{=} \int_0^1 x^{s-1}(1-x)^{t-1}dx.$$

Specifically, we have

$$B(s, t) = B(t, s),$$

$$B(r, s)B(r+s, t) = B(r, s+t)B(s, t).$$

Now both these functional equations are "explained" by the well known formula

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)},$$

where

$$\Gamma(s) \stackrel{\text{def}}{=} \int_0^\infty x^s e^{-x} \frac{dx}{x}.$$

An analogous "explanation" of (I) and (II) of Corollary 2 can be given in terms of the *hyperadelic gamma function* Γ_σ.

2.2. Let W denote the direct product over all primes p of the rings W_p of Witt vectors of $\bar{F}_p$. The ring W, linearly topologized by taking the collection of ideals $\{nW|n$ a positive integer$\}$ as a fundamental system of neighborhoods of 0, is pro-artinian. Given $a \in (\boldsymbol{Q}/\boldsymbol{Z})^n$ and $f \in W[[\hat{\boldsymbol{Z}}^n(1)]]$, let $f(a)$ denote the image of f under the unique W-linear homomorphism of rings $W[[\hat{\boldsymbol{Z}}^n(1)]] \to \boldsymbol{Q}_N \otimes W$ factoring through $W[\mathrm{Hom}(((1/N)\boldsymbol{Z}/\boldsymbol{Z})^n, \bar{\boldsymbol{Q}}^\times)]$ and prolonging the homomorphism $z \mapsto z(a)$: $\hat{\boldsymbol{Z}}^n(1) \to \bar{\boldsymbol{Q}}_N^\times$, where N is the smallest positive integer annihilating a. Note that for all $f \in W[[\hat{\boldsymbol{Z}}^n(1))]]$, $f(a)=0$ for all $a \in (\boldsymbol{Q}/\boldsymbol{Z})^n$ if and only if $f=0$.

2.3. For each $\sigma \in G(\boldsymbol{Q})$, we say that $\Gamma_\sigma \in W[[\hat{\boldsymbol{Z}}(1)]]^\times$ is a *branch* of the *hyperdelic gamma function* (associated to σ) if for all $a, b \in \boldsymbol{Q}/\boldsymbol{Z}$,

$$B_\sigma(a, b) = \Gamma_\sigma(a)\Gamma_\sigma(b)/\Gamma_\sigma(a+b).$$

We employ the terminology "branch of$\cdots$" because the defining equation above determines Γ_σ only up to a factor belonging to

$$\{E \in W[[\hat{\boldsymbol{Z}}(1)]]^\times \,|\, \forall a, b \in \boldsymbol{Q}/\boldsymbol{Z}, E(a)E(b) = E(a+b)\}.$$

The existence of branches of hyperadelic gamma functions is established by the following general result:

Theorem 6. *Let F be a unit of the ring $W[[\hat{\boldsymbol{Z}}^2(1)]]$ satisfying the following functional equations for all $a, b, c \in \boldsymbol{Q}/\boldsymbol{Z}$:*

(I) $F(a, b) = F(b, a)$.

(II) $F(a, b)F(a+b, c) = F(a, b+c)F(b, c)$.

Then there exists a unit G of the ring $W[[\hat{\boldsymbol{Z}}(1)]]$ such that for all $a, b \in \boldsymbol{Q}/\boldsymbol{Z}$,

(III) $F(a, b) = G(a)G(b)/G(a+b)$.

The theorem above is no more than a mild generalization of the well known fact that over W_p there exist no nontrivial extensions of formal $\boldsymbol{G}_m$ by formal $\boldsymbol{G}_m$. See [1] for the proof "at level N".

2.4. The hyperadelic gamma function is linked to the classical gamma function by reciprocity law which we shall formulate presently. The result is inspired by a conjecture enunciated by Deligne [7, p. 339] linking monomials in gauss sums to monomials in values of the classical gamma function. Let $\boldsymbol{B}$ denote the free abelian group generated by the symbols $[a]$, where $a \in Q/Z$. Let $\boldsymbol{B}^0$ denote the kernel of the unique homomorphism $\boldsymbol{B} \to Q/Z$ under which the symbol $[a]$ maps to a for each $a \in Q/Z$. Let $\sigma \in G(Q)$, $\boldsymbol{a} \in \boldsymbol{B}$ and a branch Γ_σ of the hyperadelic gamma function be arbitrarily specified. We denote by $\sigma\boldsymbol{a}$ the image of $\boldsymbol{a}$ under the unique automorphism of $\boldsymbol{B}$ under which each symbol $[a]$ maps to $[\chi(\sigma)a]$. We denote by $\Gamma_\sigma(\boldsymbol{a})$ the image of $\boldsymbol{a}$ under the unique homomorphism $\boldsymbol{B} \to (\bar{Q} \otimes W)^\times$ under which each symbol [a] maps to $\Gamma_\sigma(a)$. We denote by $\Gamma(\boldsymbol{a})$ the image of $\boldsymbol{a}$ under the unique homomorphism of $\boldsymbol{B}$ to the multiplicative group of the real numbers under which each symbol $[a]$ maps to $\Gamma(\langle a \rangle)$ if $a \neq 0$ and to 1 if $a=0$. Lastly, we denote by $\langle \boldsymbol{a} \rangle$ the image of $\boldsymbol{a}$ under the unique homomorphism $\boldsymbol{B} \to Q$ under which each symbol $[a]$ maps to $\langle a \rangle$. Note that for all $\boldsymbol{a} \in \boldsymbol{B}^0$, $\Gamma_\sigma(\boldsymbol{a}) \in (\bar{Q} \otimes \hat{Z})^\times \subseteq (\bar{Q} \otimes W)^\times$ and is independent of the choice of branch.

Theorem 7. *Let $\boldsymbol{a} \in \boldsymbol{B}^0$ and $w \in Z$ be given, enjoying the following property:*

(I) *For all $\sigma \in G(Q)$, $\langle \sigma\boldsymbol{a} \rangle = w$.*

Then the following relations hold for all $\sigma \in G(Q)$:

(II) $(2\pi\sqrt{-1})^{-w}\Gamma(\boldsymbol{a}) \in \bar{Q}^\times$.

(III) $\chi(\sigma)^{-w}\Gamma_\sigma(\boldsymbol{a}) \in (\bar{Q} \otimes 1)^\times \subseteq (\bar{Q} \otimes W)^\times$.

(IV) $\chi(\sigma)^{-w}\Gamma_\sigma(\boldsymbol{a}) = ((2\pi\sqrt{-1})^{-w}\Gamma(\boldsymbol{a}))^{\sigma^{-1}} \otimes 1$. □

The proof is based upon Deligne's theorem to the effect that "every Hodge cycle on an abelian variety defined over an algebraically closed field of characteristic zero is absolutely Hodge". (See [8].) See [2] for the proof.

Remark. That (I) of Theorem 7 implies (II) of Theorem 7 is known. (See the appendix to [7].)

2.5. At the expense of some redundancy, we recall the basic functional equations satisfied by the classical gamma function and draw their consequences via Theorem 7 for the hyperadelic gamma function, thereby illustrating the heuristic guiding much of our work. From the functional equation

$$\Gamma(n) = (n-1)! \quad (n \in N)$$

follows

Corollary 8. *For all $\sigma \in G(Q)$ and branches Γ_σ of the hyperadelic gamma function,*

$$\Gamma_\sigma(0) = 1 \otimes 1. \qquad \square$$

The corollary above also, clearly, follows from (I) of Theorem 1 together with the definitions. From the functional equation

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin \pi s}$$

follows

Corollary 9. *For all $\sigma \in G(\boldsymbol{Q})$, branches Γ_σ of the hyperadelic gamma function and $0 \neq a \in \boldsymbol{Q}/\boldsymbol{Z}$,*

$$\Gamma_\sigma(a)\Gamma_\sigma(-a)=(\sqrt{-1}\sin \pi a)^{1-\sigma}\otimes\chi(\sigma). \qquad \square$$

The corollary above also, clearly, follows from (II) of Theorem 1 and the definitions. From the functional equation

$$\prod_{i=1}^{n-1}\Gamma\left(\frac{s+i}{n}\right)=\left(\prod_{i=0}^{n-1}\Gamma\left(\frac{i}{n}\right)\right)n^{1-s}\Gamma(s) \qquad (n \in \boldsymbol{N})$$

follows

Corollary 10. *For all $\sigma \in G(\boldsymbol{Q})$, branches Γ_σ of the hyperadelic gamma function, $n \in \boldsymbol{N}$ and $a \in \boldsymbol{Q}/\boldsymbol{Z}$,*

$$\prod_{nb=a}\Gamma_\sigma(b)=\Big(\prod_{nc=0}\Gamma_\sigma(c)\Big)\Gamma_\sigma(a)((n^{\langle -a\rangle})^{\sigma-1}\otimes 1),$$

where for each $x \in \boldsymbol{R}$, n^x denotes the image of x under the unique continuous homomorphism $\boldsymbol{G}_a(\boldsymbol{R})\rightarrow\boldsymbol{G}_m(\boldsymbol{R})$ under which 1 maps to n. $\square$

Unlike Corollaries 8 and 9, Corollary 10 is *not* the consequence in an obvious way of properties of the adelic beta function and constitutes evidence in favor of the "naturalness" of the notion of the hyperadelic gamma function.

2.6. Ihara [10] has shown that his universal power series interpolates the Jacobi sums. We can show that both the adelic beta function and the hyperadelic gamma function interpolate *Gauss* sums. Let p and l be distinct rational primes, w a place of $\bar{\boldsymbol{Q}}$ dividing p, ζ a primitive p^{th} root of unity in $\bar{\boldsymbol{Q}}$. Let k denote the residue field of w. Let σ be an element of the decomposition subgroup of $G(\boldsymbol{Q})$ at w which induces the automorphism $x\rightarrow x^p$ of k *and* which fixes every $(p-1)^{\text{st}}$ root of $1-\zeta$ in $\bar{\boldsymbol{Q}}$. Let $\omega\colon k^\times\rightarrow \mu_\infty(\bar{\boldsymbol{Q}})$ denote the unique homomorphism such that for all $x \in k^\times$,

$$\omega(x)\equiv x \mod w.$$

Let $\pi_l\colon \boldsymbol{W}\rightarrow\boldsymbol{W}_l$ denote the projection to the l-adic factor. Let Γ_σ be a branch of the hyperadelic gamma function. For each positive integer f, let k_f denote the unique subfield of k consisting of exactly p^f elements, and let $\mathrm{tr}_f : k_f \to \boldsymbol{F}_p$ denote the galois-theoretic trace map.

Theorem 11. *Assumptions and notation as above, the following relations hold*:

$$\text{(I)} \qquad (1\otimes\pi_l)\left(\frac{\Gamma_\sigma\left(\dfrac{1}{p}\right)}{\Gamma_\sigma\left(\dfrac{1}{p}+\dfrac{1}{p-1}\right)}\right)^{p-1} = 1\otimes 1 \in \bar{\boldsymbol{Q}}\otimes W_l.$$

(II) *For all positive integers f and nonvanishing elements a of $\boldsymbol{Q}/\boldsymbol{Z}$ annihilated by p^f-1,*

$$\begin{aligned}(-\sum_{x\in k_f^\times}&\omega(x)^{\langle a\rangle(p^f-1)}\zeta^{tr_f(x)})\otimes 1\\ &=(1\otimes\pi_l)\left(\prod_{i=1}^{f} B_\sigma\left(p^i a, \frac{1}{p}\right)\right)\\ &=(1\otimes\pi_l)\left(\left(\frac{\Gamma_\sigma\left(\dfrac{1}{p}\right)}{\Gamma_\sigma\left(\dfrac{1}{p}+\dfrac{1}{p-1}\right)}\right)^{\langle a\rangle(p^f-1)}\prod_{i=1}^{f}\Gamma_\sigma(p^i a)\right) \in (\bar{\boldsymbol{Q}}\otimes W_l)^\times.\end{aligned}$$

The proof of these formulas rests upon the observation (due to Coleman and McCallum [5, 6]) that a suitable twist of the curve $x^m+y^p=1$, where p is prime and $(p, m)=1$, has good reduction at p, and reduces to the curve $y^p-y=x^m$. See [2] for the proof.

§ 3. The logarithmic derivative of the hyperadelic gamma function

3.1. Let G be a profinite group. Let A, B, and C be abelian groups, and let $(a, b)\mapsto ab$: $A\times B\to C$ be a bilinear map. A *measure* μ on G with values in B is a function with domain the set of open compact subsets of G and range B, with the properties

$$\mu(\phi)=0,\ \mu(U)+\mu(V)=\mu(U\cup V)+\mu(U\cap V).$$

Given a locally constant function f: $G\to A$, set

$$\int_G f(g)d\mu(g) \overset{\mathrm{def}}{=} \sum_{g\in G/U} f(g)\mu(gU) \in C,$$

where U is any open normal subgroup of G such that f factors through the quotient G/U. Now suppose further that A, B and C are profinite and that the composition $(a, b)\mapsto ab\colon A\times B\to C$ is continuous. Then for any function $f\colon G\to A$ which is the uniform limit of a sequence of locally constant functions $f_1, f_2, \cdots\colon G\to A$, set

$$\int_G f(g)d\mu(g)\overset{\text{def}}{=}\varinjlim\int_G f_n(g)d\mu(g)\in C,$$

noting that the limit on the right exists and is independent of the presentation of f as a uniform limit.

3.2. Given any abelian group A, set

$$2\pi\sqrt{-1}\otimes A\overset{\text{def}}{=}(\text{kernel of }\exp\colon \boldsymbol{C}\longrightarrow\boldsymbol{C}^\times)\otimes A.$$

Given $x\in 2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}}$ and a positive integer n, set

$$\exp\left(\frac{x}{n}\right)\overset{\text{def}}{=}\exp\left(2\pi\sqrt{-1}\,\frac{k}{n}\right)$$

where k is any integer such that

$$x\equiv 2\pi\sqrt{-1}\,k\otimes 1 \mod 2\pi\sqrt{-1}\,n\otimes\hat{\boldsymbol{Z}}.$$

3.3. Let Ψ be the unique $\boldsymbol{C}$-valued measure on $\hat{\boldsymbol{Z}}(1)$ verifying for all $a\in\boldsymbol{Q}/\boldsymbol{Z}$ the integral formula

$$\int_{\hat{\boldsymbol{Z}}(1)} z(a)d\Psi(z)\overset{\text{def}}{=}\begin{cases}\dfrac{\Gamma'(\langle a\rangle)}{\Gamma(\langle a\rangle)}+\gamma & \text{if } a\neq 0\\ 0 & \text{if } a=0\end{cases}$$

where

$$\gamma\overset{\text{def}}{=}\lim_{N\to\infty} -\log N+\sum_{n=1}^{N}\frac{1}{n}$$
$$=\text{the Euler-Mascheroni constant.}$$

It is a straightforward task to verify that for all positive integers n and complex n^{th} roots of unity ζ,

$$\exp\left(\Psi\left(\left\{z\in\hat{\boldsymbol{Z}}(1)\,\middle|\,z\left(\frac{1}{n}\right)=\zeta\right\}\right)\right)=\begin{cases}1-\zeta^{-1} & \text{if } \zeta\neq 1\\ \dfrac{1}{n} & \text{if } \zeta=1\end{cases}.$$

In particular, the values of Ψ are logarithms of algebraic numbers and the function

$$U \mapsto \exp(\Psi(U)) : \{U \subseteq \hat{\boldsymbol{Z}}(1) \mid U \text{ open compact}\} \longrightarrow \bar{\boldsymbol{Q}}^{\times}$$

is $G(\boldsymbol{Q})$-equivariant. Given $\sigma \in G(\boldsymbol{Q})$, we define a measure Ψ_σ on $\hat{\boldsymbol{Z}}(1)$ with values in $2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}}$ by requiring that for all positive integers n and open compact subsets U of $\hat{\boldsymbol{Z}}(1)$,

$$\exp\left(\frac{\Psi_\sigma(U)}{n}\right) \stackrel{\text{def}}{=} \frac{\exp\left(\frac{\Psi(\sigma^{-1}U)}{n}\right)^{\sigma}}{\exp\left(\frac{\Psi(U)}{n}\right)}.$$

3.4. Let D denote the unique $\boldsymbol{W}$-linear continuous derivation

$$\boldsymbol{W}[[\hat{\boldsymbol{Z}}(1))]] \longrightarrow 2\pi\sqrt{-1}\otimes \boldsymbol{W}[[\hat{\boldsymbol{Z}}(1))]]$$

with the property that

$$D\varepsilon \stackrel{\text{def}}{=} 2\pi\sqrt{-1}\otimes\varepsilon$$

where

$$\varepsilon \stackrel{\text{def}}{=} (a \mapsto \exp(2\pi\sqrt{-1}a)) \in \hat{\boldsymbol{Z}}(1).$$

Note that D is independent of the choice of complex square root of -1 made to define it. Note that for all $\sigma \in G(\boldsymbol{Q})$ and choices Γ_σ of branch of the hyperadelic gamma function, the coset of $2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}}$ in $2\pi\sqrt{-1}\otimes \boldsymbol{W}[[\hat{\boldsymbol{Z}}(1)]]$ to which

$$-\frac{D\Gamma_\sigma}{\Gamma_\sigma} + \int_{\hat{\boldsymbol{Z}}(1)} z\, d\Psi_\sigma(z) \in 2\pi\sqrt{-1}\otimes \boldsymbol{W}[[\hat{\boldsymbol{Z}}(1)]]$$

belongs is independent of the choice of branch of the hyperadelic gamma function. Let

$$\gamma_\sigma \in 2\pi\sqrt{-1}\otimes \boldsymbol{W}[[\hat{\boldsymbol{Z}}(1)]]/2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}}$$

be defined by the relation

$$\frac{D\Gamma_\sigma}{\Gamma_\sigma} + \gamma_\sigma \stackrel{\text{def}}{\equiv} \int_{\hat{\boldsymbol{Z}}(1)} z\, d\Psi_\sigma(z) \mod 2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}}.$$

Theorem 12. *For all* $\sigma \in G(\boldsymbol{Q})$,

$$(\mathrm{id}_{2\pi\sqrt{-1}\,Z} \otimes D)\gamma_\sigma = 0. \qquad \square$$

The theorem above justifies calling γ_σ the *hyperadelic Euler-Mascheroni constant*. The proof of Theorem 12 first found by Coleman and the author exploits the classfield-theoretic properties of and functional equations satisfied by B_σ and Γ_σ combined with Coleman's explicit reciprocity law [4], a generalization of Iwasawa's theorem on local units [12] and some p-adic analysis. This first proof is unlikely to be published since the author subsequently found a simpler proof yielding a stronger result. But see [3] in which the ideas of Coleman entering into the first proof of Theorem 12 find significant application to i) the determination the local behavior of the Jacobi sum Hecke characters of the general type considered in [0] and ii) the generalization of Iwasawa's theorem on local units modulo circular units [12] beyond the pure pro-p setting.

§ 4. Evaluation of the Euler-Mascheroni constant

4.1. The second proof of Theorem 12 mentioned above actually leads to a stronger result, which is formulated as Theorem 13 below.

4.2. For each rational prime p, let $\Phi_p\colon W_p \xrightarrow{\sim} W_p$ denote the arithmetic Frobenius automorphism and set

$$\Phi \overset{\mathrm{def}}{=} \prod_p \Phi_p\colon W \xrightarrow{\sim} W.$$

For each $\sigma \in G(\boldsymbol{Q})$, let $h_\sigma \in 2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}}$ be characterized by the property that for all rational primes p and positive integers n,

$$\exp(p^{-n}h_\sigma) \overset{\mathrm{def}}{=} \exp(p^{-n}\log p)^{\sigma-1}.$$

Theorem 13. *For all* $\sigma \in G(\boldsymbol{Q})$,

$$\gamma_\sigma \in 2\pi\sqrt{-1}\otimes W/2\pi\sqrt{-1}\otimes\hat{\boldsymbol{Z}},$$

and

$$((\mathrm{id}_{2\pi\sqrt{-1}\,Z} \otimes \Phi) - 1)\gamma_\sigma = h_\sigma. \qquad \square$$

That Theorem 13 does indeed uniquely determine γ_σ follows from the well known fact that, for each rational prime p, the sequence

$$0 \longrightarrow Z_p \longrightarrow W_p \xrightarrow{\Phi_p - 1} W_p \longrightarrow 0$$

is exact. The details of the proof of Theorem 13 will appear in [2], for which the prerequisite is [1].

4.3. Let us briefly indicate the geometrical source of the relation between Fermat torsion and roots of circular units asserted by Theorem 13. The reader is referred to the forthcoming paper [1] for details. Let m be a positive integer and set

$$U'_m \overset{\text{def}}{=} \operatorname{Spec}\left(\boldsymbol{Q}\left[u, \left(\sum_{j=0}^{m-1} u^j\right)^{-1}\right]\right),$$

$Y'_m \overset{\text{def}}{=}$ the closed subscheme of U'_m on which $u(1-u)$ vanishes,

$V'_m \overset{\text{def}}{=}$ the relative singular homology group with integral coefficients of the pair $(U'_m(\boldsymbol{C}), Y'_m(\boldsymbol{C}))$.

Then V'_m/mV'_m is naturally equipped with a $G(\boldsymbol{Q})$-module structure because V'_m/mV'_m is canonically isomorphic to a certain relative étale homology group. It turns out that the action of $G(\boldsymbol{Q})$ upon V'_m/mV'_m can be made explicit in terms of the action of $G(\boldsymbol{Q})$ upon the set

$$\{z \in \boldsymbol{C}^\times \mid 1-(1-z^m)^m=0\},$$

and conversely. The most important point of the proof of Theorem 13 is to exhibit the $G(\boldsymbol{Q})$-module V'_m/mV'_m as a subquotient of the $G(\boldsymbol{Q})$-module V_m/mV_m defined in **1.7**. This is done as follows: Set

$U''_m \overset{\text{def}}{=}$ the largest open subscheme of U_m on which $\sum_{j=0}^{m-1} x^j$ is invertible,

$Y''_m \overset{\text{def}}{=}$ the unique reduced closed subscheme of U''_m supported on the points $(x, y)=(1, 0)$ and $(x, y)=(0, 1)$,

$V''_m \overset{\text{def}}{=}$ the relative singular homology group with integral coefficients of the pair $(U''_m(\boldsymbol{C}), Y''_m(\boldsymbol{C}))$.

Then V''_m/mV''_m is likewise naturally equipped with a $G(\boldsymbol{Q})$-module structure and admits maps both to V_m/mV_m and V'_m/mV'_m. Let $\pi\colon U''_m \to U'_m$ denote the unique morphism such that

$$\pi^* u \overset{\text{def}}{=} x,$$

and let $i\colon U''_m \to U_m$ denote the inclusion. Let $\pi_*\colon V''_m/mV''_m \to V'_m/mV'_m$ and $i_*\colon V''_m/mV''_m \to V_m/mV^m$ denote the homomorphisms induced by π and i, respectively. Then one can show that π_* is surjective and that $\ker(i_*) \subseteq \ker(\pi_*)$, thus exhibiting V'_m/mV'_m as a subquotient of V_m/mV_m.

§ 5. Relations with Ihara's theory

5.1. Let l be a rational prime. Ihara [10] has recently initiated the study of the l-adic homology of the $(l^n)^{\text{th}}$ degree Fermat curve in the limit as $n \to \infty$. His investigation provided the inspiration for the work being outlined in this paper. We shall relate Ihara's theory to ours by explaining how to express Ihara's universal power series in terms of the adelic beta function. For simplicity, we shall assume l to be *odd*. (Note that $l=2$ is in fact permitted in Ihara's theory; the exclusion of the case $l=2$ is merely for the author's convenience.)

5.2. For each nonnegative integer n, set

$$X_n \overset{\text{def}}{=} \operatorname{Proj}\,(\boldsymbol{Q}[u, v, w]/(u^{l^n}+v^{l^n}+w^{l^n})),$$
$$J_n \overset{\text{def}}{=} \text{the jacobian variety of } X_n.$$

The curves X_n fit into a tower relative to the transition maps

$$(u, v, w) \longmapsto (u^l, v^l, w^l) : X_{n+1} \longrightarrow X_n$$

and these maps in turn induce maps $J_{n+1} \to J_n$ relative to which the J_n likewise form a tower. Ihara's idea is to study the "big" $G(\boldsymbol{Q})$-module

$$\underset{\boldsymbol{I}}{\prod} \overset{\text{def}}{=} \varprojlim T_l(J_n(\bar{\boldsymbol{Q}})),$$

where, for any abelian group A, we write

$$T_l(A) \overset{\text{def}}{=} \operatorname{Hom}\,(\boldsymbol{Q}_l/\boldsymbol{Z}_l, A).$$

5.3. Set

$$G_n \overset{\text{def}}{=} \operatorname{Aut}\,(X_n \times \operatorname{Spec}\,(\bar{\boldsymbol{Q}})/X_0 \times \operatorname{Spec}\,(\bar{\boldsymbol{Q}})),$$
$$\Lambda \overset{\text{def}}{=} \varprojlim \boldsymbol{Z}_l[G_n].$$

Then Λ is a ring upon which $G(\boldsymbol{Q})$ acts, $\boldsymbol{I}$ a Λ-module, and the action of $G(\boldsymbol{Q})$ upon $\boldsymbol{I}$ semilinear with respect to the action of $\boldsymbol{I}$. Ihara has proven that $\boldsymbol{I}$ is a free Λ-module of rank one; selecting a Λ-basis φ of $\boldsymbol{I}$, one obtains a 1-cocycle

$$\sigma \longmapsto F_{\sigma,\varphi} \overset{\text{def}}{=} (\sigma\varphi/\varphi) : G(\boldsymbol{Q}) \longrightarrow \Lambda^\times.$$

Then, up to an inconsequential change of notation, $F_{\sigma,\varphi}$ is Ihara's universal power series. The term "power series" is used because of the following fact: Set

$$g_n \overset{\text{def}}{=} ((u, v, w) \longmapsto (\exp\,(2\pi\sqrt{-1}/l^n)\,u, v, w)) \in G_n,$$
$$h_n \overset{\text{def}}{=} ((u, v, w) \longmapsto (u, \exp\,(2\pi\sqrt{-1}/l^n)\,v, w)) \in G_n,$$
$$g \overset{\text{def}}{=} \varprojlim g_n \in \Lambda^\times,\ h \overset{\text{def}}{=} \varprojlim h_n \in \Lambda^\times,$$

and let s and t be indeterminates. Then the unique continuous homomorphism $\boldsymbol{Z}_l[[s, t]] \to \Lambda$ of profinite rings under which $s \mapsto g-1$ and $t \mapsto h-1$ is an isomorphism. Now the preceding fact is very important to Ihara's theory, but unimportant for the task of comparing his construction to ours. We shall make no use of it other than to conclude that $g-1$ and $h-1$ are not zero divisors of Λ.

5.4. We fix a particular choice φ of Λ-basis of $\boldsymbol{I}$ as follows. Set

$W_n \overset{\text{def}}{=}$ the first singular homology group of $X_n(\boldsymbol{C})$ with integral coefficients.

Then $W_n \otimes \boldsymbol{Z}_l$ is canonically isomorphic to $T_l(J_n(\overline{\boldsymbol{Q}}))$; in order to exhibit a generator of $\boldsymbol{I}$ it will be enough to exhibit for each n an element of $W_n/l^n W_n$ subject to the evident compatibility as $n \to \infty$. Set

$$b_n \overset{\text{def}}{=} (t \mapsto (\sqrt[l^n]{t}, \sqrt[l^n]{1-t}, -1)) : [0, 1] \longrightarrow X_n(\boldsymbol{C}),$$

thereby defining a singular 1-simplex on $X_n(\boldsymbol{C})$; here the $(l^n)^{\text{th}}$ roots are to be chosen real. Set

$\varphi_n \overset{\text{def}}{=}$ the element of $W_n/l^n W_n$ to which the 1-cycle $(1-g_{n*})(1-h_{n*})b_n$ gives rise.

Now the φ_n are compatible; set

$$\varphi \overset{\text{def}}{=} \varprojlim \varphi_n \in \boldsymbol{I}.$$

Then by Ihara's methods, one can verify that φ generates $\boldsymbol{I}$. That φ generates $\boldsymbol{I}$ follows also, say from Theorem 1 of the appendix to [9]; indeed, for each n, the homology class of the 1-cycle $(1-g_{n*})(1-h_{n*})b_n$ generates W_n over the integral group ring $\boldsymbol{Z}[G_n]$. Having now fixed a generator φ of $\boldsymbol{I}$, we write simply F_σ rather than $F_{\sigma,\varphi}$ to denote Ihara's universal power series.

5.5. We define a pro-l specialization homomorphism

$$\mathrm{sp}_l : \hat{\boldsymbol{Z}}[[\hat{\boldsymbol{Z}}^2(1)]] \longrightarrow \Lambda$$

as follows: Set

$$\varepsilon \overset{\text{def}}{=} (a \longmapsto \exp(2\pi\sqrt{-1}a)) \in \hat{\boldsymbol{Z}}(1),$$

$$\varepsilon \otimes 1 \overset{\text{def}}{=} ((a, b) \longmapsto \varepsilon(a)) \in \hat{\boldsymbol{Z}}^2(1),$$

$$1 \otimes \varepsilon \overset{\text{def}}{=} ((a, b) \longmapsto \varepsilon(b)) \in \hat{\boldsymbol{Z}}^2(1).$$

Let sp_l be the unique continuous ring homomorphism such that

$$\mathrm{sp}_l(\varepsilon\otimes 1)=g,\ \mathrm{sp}_l(1\otimes\varepsilon)=h.$$

5.6. To compare Ihara's construction to ours the essential thing is to exhibit $W_n/l^n W_n$ as a subquotient of $V_{l^n}/l^n V_{l^n}$ for all positive integers n. This is done as follows. Let $j_n\colon U_{l^n}\to X_n$ denote the unique map such that

$$j_n^*(u/w)=-x,\ j_n^*(v/w)=-y.$$

Set

$$W'_n \overset{\mathrm{def}}{=} \text{the singular homology group of } U_{l^n}(\boldsymbol{C}) \text{ with integral coefficients.}$$

Then W'_n may be identified with a subgroup of V_{l^n}, $W'_n/l^n W'_n$ with a $G(\boldsymbol{Q})$-submodule of $V_{l^n}/l^n V_{l^n}$, and we have

$$(1-\varepsilon\otimes 1)(1-1\otimes\varepsilon)\beta_{l^n}\in W'_n/l^n W'_n.$$

Clearly,

$$j_{n*}((1-\varepsilon\otimes 1)(1-1\otimes\varepsilon)\beta_{l^n})=\varphi_n. \tag{5.6.1}$$

It follows that for all $\sigma\in G(\boldsymbol{Q})$,

$$(1-g)(1-h)\mathrm{sp}_l B_\sigma=(1-g^{\chi_l(\sigma)})(1-h^{\chi_l(\sigma)})F_\sigma, \tag{5.6.2}$$

where $\chi_l\colon G(\boldsymbol{Q})\to \boldsymbol{Z}_l^\times$ denotes the l-adic cyclotomic character. Since none of the factors $1-g$, $1-h$, $1-g^{\chi_l(\sigma)}$, $1-h^{\chi_l(\sigma)}$ are zero divisors of Λ, (5.6.2) determines F_σ uniquely in terms of $\mathrm{sp}_l B_\sigma$ and vice versa.

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