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On the Resiliency of Determinantal Ideals

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Abstract

Determinantal ideals associated to "sufficiently general" matrices of linear forms are shown to be *resilient* in the sense that they remain of the "expected" codimension, or prime, even modulo a certain number of linear forms.

This paper is intended to be read as an introduction to the paper Eisenbud [1985], in which a number of further results, and analogues for lower order minors, are treated. We have however included here the material necessary for the construction of Maximal Cohen-Macaulay modules by Herzog and Kühl (elsewhere in these proceedings) and for some other applications to the construction of compressed or nearly compressed algebras and modules.

§ 1. Introduction and main results

It is often of interest to decide whether the determinant of a square matrix of polynomials over some field K is zero, or in geometric terms, whether all matrices in a given algebraic family are singular. More generally, if the matrix is not square, but (say) $a\times b$ with $a\leq b$, then one wants to decide whether the $a\times a$ minors are all zero, but also whether they generate an ideal of the "expected" codimension $b-a+1$, which is the codimension of the family of matrices of rank $<a$ in the family of all $a\times b$ matrices over a field; of course both these questions reduce to our original question if $a=b$. One can also ask similar questions about the lower order minors.

Of course the source of the matrices to be dealt with influences the form of answer desired! For example, Merle and Giusti [1982], motivated by questions about the possible power series in a complete intersection with isolated singularity have studied the questions above (and more) in case L is an $a\times b$ matrix in which the entries L_{ij} are each either a variable x_{ij} or 0; that is, when L is a generic matrix modulo a subset of the variables. They derive a combinatorial formula for the heights of the ideals of various sized minors of L in this case.

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Our investigations were motivated by a family of examples derived from bilinear pairings of vector spaces over K:

$$\mu\colon A \underset{K}{\otimes} B \longrightarrow C$$

given as multiplication maps in integral domains or multiplication maps of sections of line bundles on reduced irreducible varieties. Such pairings have the property that if $a \in A$ and $b \in B$ are nonzero, then $\mu(a\otimes b) \in C$ is non-zero. We will call such a pairing 1-*generic* (more generally k-generic would be the equivalent property for elements of $A\otimes B$ which are sums of $\leq k$ pure tensors). If we choose bases $\{x_i\}_{i=0,\dots,m}$ of C and $\{a_i\}$, $\{b_i\}$ of A and B respectively, then μ corresponds to a matrix $L=(L_{ij})$, where $L_{ij}=\mu(a_i\otimes b_j)$, considered as a linear form in the "variables" x_i. Of course every matrix L of linear forms arises from a pairing as above, and we will call L 1-generic if the associated bilinear form is 1-generic. This condition is equivalent to saying that L has "no generalized entries which are zero" in the sense that even after arbitrary scalar row and column operations, no entry of L is identically 0. Examples of 1-generic matrices include, of course the generic matrix

$$\begin{pmatrix} x_{11} & \cdots & x_{1b} \\ \vdots & & \vdots \\ x_{a1} & \cdots & x_{ab} \end{pmatrix},$$

corresponding to the "identity" pairing $\mu\colon A\otimes B\to C$ with $C=A\otimes B$ and the "catalecticant" or "Hankel" matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 \cdots x_b \\ x_2 & x_3 & \\ x_3 & \ddots & \\ \vdots & & \ddots \\ x_a & & & x_{a+b-1} \end{pmatrix},$$

corresponding to the "multiplication" pairing with A, B, C the spaces of forms in 2 variables of degrees $a-1$, $b-1$, and $a+b-2$ respectively.

There are substantially more 1-generic pairings μ when K is not algebraically closed than when it is. For example, when K is algebraically closed, it is easy to show that if $\mu\colon A\otimes_K B\to C$ is 1-generic then $\dim C\geq \dim A+\dim B-1$, whereas if $K=\boldsymbol{R}$ then, taking $A=B=C=\boldsymbol{C}$ (or the quaternians or Cayley numbers) we get $\dim A=\dim B=\dim C=2$ (or 4 or 8). These "exotic" pairings, for example when $K=\boldsymbol{R}$, have played a role in the topological theory of immersions of real projective spaces, and vector fields in spheres. See Hopf [1940–41] and Ginsburg [1963] for the original accounts of this, and Bott-Gitler-James [1972, pp. 140–144] for a somewhat more recent view.

We will from now on assume that K is algebraically closed; but our techniques also yield something interesting in the general case; see the remark after Theorem 2, below.

It is not difficult to see (and will be proved below) that the ideal $I_a(L)$ generated by the $a\times a$ minors of a 1-generic matrix L has height $b-a+1$, and this implies that

$$K[x_1, \cdots, x_m]/I_a(L)$$

is Cohen-Macaulay. In fact, Kempf's theory [1973] applies, and shows that this ring is a normal domain with only rational singularities.

Our main result is that some of these good properties, at least, persist if one factors out not too many *arbitrary* linear forms:

Theorem 1. *Let L be a 1-generic $a\times b$ matrix of linear forms over $K[X_1, \cdots, X_m]$ with $a\leq b$; let $M_1, \cdots, M_s$ be any s linearly independent forms; and let*

$$R=K[X_1, \cdots, X_m]/(M_1, \cdots, M_s, I_a(L)).$$

i) *If $s\leq a-1$ then R has the "expected" dimension $m-s-(b-a+1)$.*

ii) *If $s\leq a-2$, then R is a domain.*

As is well known, it follows from the dimension statement in i) that R is actually Cohen-Macaulay in this case.

Remark. Theorem 1 is certainly not true as stated over a field which is not algebraically closed, even for $s=0$. For example, taking $K=\boldsymbol{R}$, μ the multiplication pairing on the quaternions, and $s=0$, we see that the corresponding $I_a(L)$ is the principal ideal generated by $(x^2+y_1^2+y_2^2+y_3^2)^2$, the square of the norm polynomial for the quaternions. However, i) does hold for $s=0$ if one considers only K-rational points; see the remark after Theorem 2.

As is easily seen, by induction on s, part ii) of the theorem actually implies part i); but part i) is necessary for our proof of part ii). In this note we will prove only part i), leaving part ii), which is based on a tangent space computation involving lower order minors, to our paper [1985]. One might well hope that if $s\leq a-3$, then R is normal; but except in the case $s=0$ we do not know whether this is true.

Combining Theorem 1, i) with the Principal Ideal Theorem, one immediately obtains:

Corollary. *If L is a 1-generic $a\times b$ matrix of linear forms then $I_a(L)$ connot be contained in an ideal generated by fewer than b linear forms.*

Reading the results above as results in the polynomial ring $K[x_1, \cdots, x_m]/(M_1, \cdots, M_s)$ one obtains criteria of the sort described in the first paragraph of the introduction.

Of course in the case $a=b$ Theorem 1, i) and Corollary 2 coincide, and say (when applied to the generic matrix) that there is no more "efficient" expansion of a determinant in terms of linear forms in its entries than is the well-known "Laplace expansion" along a row or column, and this remains true even for a matrix as degenerate as a Hankel matrix. One could ask similarly, in case $a=b>2$, what the shortest expansion of a determinant in terms of quadratic forms might be; the Laplace expansion along a pair of rows yields one with $\binom{a}{2}$ terms, but we do not know if this is the shortest possible.

It is easy to give examples showing that the above results are sharp:

Example 1, i). To show that Theorem 1 i) is sharp one may take, (aside from the obvious possibility of taking the M_i to be a elements from a single row or column of L):

$$L=\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_1 & x_5 \end{pmatrix}; \quad M_1=x_2, \quad M_2=x_4.$$

In this case one can check that the 2×2 minors of L remain linearly independent (as quadratic forms) modulo (M_1, M_2) so that this example is not "equivalent" to an example when a elements from a single row or column are chosen as the M_i.

Example 1, ii). a) If

$$L=\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}; \quad M_1=x_{22}$$

then $(I_2(L), M_1)$ is reducible but reduced.

b) If

$$L=\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}; \quad M_1=x_3,$$

then $(I_2(L), M_1)$ is irreducible but not reduced.

Example 2. Again there are "obvious" ideals of b linear forms from a single row of L which contain $I_2(L)$. But there may also be others: If

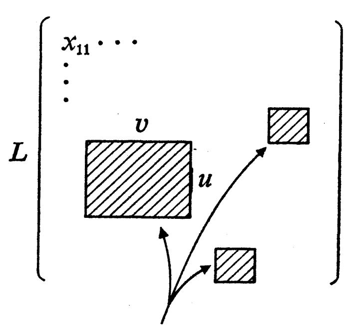
$$L=\begin{pmatrix} y_1 & y_2 & x_1 \\ y_2 & y_3 & x_2 \\ x_5 & x_4 & x_3 \end{pmatrix}, \quad M_1=y_1, \quad M_2=y_2, \quad M_3=y_3,$$

then $(M_1, M_2, M_3) \supset \det L$; but even after scalar row and column operations no column or row of L is contained in (M_1, M_2, M_3). (To see that L is 1-generic, note that L can be specialized to a Catalecticant matrix.)

However, as must be well-known, if L is the generic matrix then any space of b linear forms containing $I_a(L)$ must be (after row and column operations) a row (or if $a=b$ a column); see our [1985] for a proof. But even in this case there are spaces of linear forms of larger dimension, not containing any of these, which contain $I_a(L)$. See Beasley [1987] and Eisenbud-Harris [1987] for more in this direction.

The results outlined above can, with somewhat more complication be generalized to cover the $k \times k$ minors of an "$a-k+1$-generic" matrix of linear forms. Further, it is of considerable interest for the study of ideals defining certain varieties, and their free resolution, such as ideals of canonically embedded curves, to study the ideals of 2×2 minors of 1-generic matrices. For these extensions and some others, see our [1985].

It is interesting to compare Theorem 1 with two other groups of results: (Giusti and Merle [1982] show (among other things) that if $L=(x_{ij})$ is the generic $a\times b$ matrix, then $I_a(L)$ remains of the correct codimension modulo any set Z of the indeterminates x_{ij} so long as the largest semi-perimeter of a contiguous block of elements among those in Z is $\leq a$, or equivalently if no $a\times a$ minor of L is contained in the ideal generated by the elements of Z, and remains prime if this largest semi-perimeter is $<a$.



elements of Z; largest semi-perimeter$=u+v$.

In a different direction, Theorem 1 says geometrically that any plane section of the generic determinantal variety is reduced and irreducible as long as the codimension of the plane does not exceed $a-2$. This is the sense of the word "resiliency" in the title of this paper. The result may be compared with the general theorem of Zak that a hyperplane (that is, codimension 1) section Y of a *smooth* variety in $\boldsymbol{P}^m$ is always reduced and irreducible, and even normal, as long as $2 \dim Y \geq m$ (see Fulton-Lazarsfeld [1981] Section 7 for an exposition). Of course the determinantal varieties are singular, and Zak's result cannot be extended by induction to plane sections of larger codimension because the first hyperplane section may well be singular. It would be quite interesting to extend Zak's result to the singular case, perhaps including the determinantal case treated here.

Finally, partly because it is relevant to the conference in Kyoto that gave rise to this volume of proceedings, I would like to relate the story of my interest in 1-generic matrices of linear forms:

I first became actively involved in algebraic geometry because of lectures in 1978 by Joe Harris at M.I.T. on Rational Normal Scrolls, which are defined by the ideals of 2×2 minors of a 1-generic $2\times b$ matrix and their relation to the problem (then open, but since settled by Mark Green) of whether the ideal of the canonical curve was generated by rank 4 quadrics (determinants of 1-generic 2×2 matrices).

Of course, David Buchsbaum had taught me much about ideals of minors. Afterwards, practically all my work in Algebraic Geometry was related, at least distantly, to such ideals of 2×2 minors of 1-generic $2\times b$ matrices; even my recent work with Harris on such irrational varieties as the moduli space of curves has some of its technical roots in lemmas about rational normal curves, which are defined by such ideals. Because of the connection with conjectures about ideals of canonical and high-degree curves, and their free resolutions, (some of them in the meantime settled by Mark Green, Rob Lazarsfeld, and my student Frank Schreyer) I became interested about 2 years ago in ideals of 2×2 minors of 1-generic $a\times b$ matrices with $a, b>2$; Craig Huneke and I thoroughly analyzed the 3×3 case, "by hand" (and Huneke subsequently simplified the methods, worked out the 3×4 case, and made a conjecture about the general case.) But we were not able to prove anything very general, and we drifted away from the problem. Then in Kyoto, Herzog asked me, in connection with his work with Kühl whether Theorem 1, i) above might be proved for the Catalecticant matrix. He had to be patient with lots of false starts, but maximal minors proved much simpler than 2×2 minors, and after a week of encouragement, I had proved Theorem 1 and a bit more. Emboldened by this, I was able to go back, prove Huneke's conjecture, and do some of what I had wanted about the "linear parts" of the free resolutions of ideals of 2×2 minors as well-for all of this see [1985]. I am very grateful to Herzog, and certainly not less to the others mentioned above, for having introduced me to these topics; and finally,

also, to the organizers of the Kyoto conference and especially to H. Matsumura, for making my encounter with Herzog possible, and for providing such a beautiful and fascinating setting.

§ 2. A more general result

We have already seen that matrices of linear forms and pairings are equivalent constructions. For the proofs it will be more convenient to work with a third construction, also essentially equivalent: a linear space of linear transformations, obtained as an "adjoint" of the pairing or, with respect to chosen bases, as the space of matrices parametrized by the matrix of linear forms. Thus if $\mu\colon A\otimes B\to C$, then we may form the adjoint $m\colon C^*\to A^*\otimes B^*\cong \mathrm{Hom}\,(B, A^*)$, and consider its image M, a linear space of linear transformations.

Equivalently, if $L=(L_{ij}(x_1, \cdots, x_m))$ is a matrix of linear forms, then substituting an element of K for each x_i we get a matrix over K, and this parametrizes a space M of matrices.

Note that the dual space to $H:=\mathrm{Hom}\,(B, A^*)$ is naturally $H^*:=\mathrm{Hom}\,(A, B^*)$, the natural pairing $\langle\ ,\ \rangle$ being given by trace:

$$\phi\in\mathrm{Hom}\,(B, A^*), \quad \psi\in\mathrm{Hom}\,(A, B^*)$$
$$\langle\phi, \psi\rangle:=\mathrm{Trace}\,\phi\psi=\mathrm{Trace}\,\psi\phi.$$

We will write H_k or H_k^* for the linear transformations of rank$\leq k$ in the corresponding space, and we write M_k for $M\cap H_k$ (scheme theoretic intersection).

We must first translate the condition of 1-genericity into this new language. We write $M^\perp\subset H^*$ for the annihilator $\{\psi\in H^*\,|\,\langle\psi, \phi\rangle=0$ for all $\phi\in M\}$ of M in H^*, and set $M_k^\perp=M^\perp\cap H_k^*$. If M corresponds to the pairing $\mu\colon A\otimes B\to C$, then $H^*=A\otimes B$ and $\psi\in M^\perp$ iff $\mu(\psi)=0$. Since the elements of $A\otimes B$ of the form $a\otimes b$ with $a\in A$ and $b\in B$ correspond under the identification $A\otimes B=\mathrm{Hom}\,(A^*, B)$ to the linear transformations of rank≤ 1, we see that μ is 1-generic iff $M_1^\perp:=M^\perp\cap H_1^*=0$; and we will say in this case that M is 1-generic.

Next we translate the statement of Theorem 1. Factoring out the linear forms $M_1, \cdots, M_s$ in $K[x_1, \cdots, x_m]=\mathrm{Symm}\,(C)$ has the same effect as replacing C by its quotient $C/\langle M_1, \cdots, M_s\rangle$, and this replaces M by a subspace M', the image of $(C/\langle M_1, \cdots, M_s\rangle)^*\supset C^*$ in M. This subspace M' has codimension $\leq s$ in M.

For notational convenience we now reverse the roles of M and M', and write $V=B$, $W=A^*$, $v:=\dim V=b\geq w:=\dim W=a$, and Theorem 1 becomes:

Theorem 1′. *Let $M' \subset \operatorname{Hom}(V, W)$ be a 1-generic linear space of linear transformations, and let $M \subset M'$ be a linear subspace of codimension s.*

i) *If $s \leq w-1$ then*

$$\dim M_{w-1} = \dim M - (v - w + 1).$$

ii) *If $s \leq w-2$ then M_{w-1} is reduced and irreducible.*

Theorem 1′, i) follows quickly from the following, much more general result:

Theorem 2. *If $M \subset \operatorname{Hom}(V, W)$ is any linear subspace, then*

$$\dim M_{w-1} \leq \dim M - (v - w + 1) + \max(0, 1 - w + \dim M_1^{\perp}).$$

Remark. Theorem 2 still holds over an arbitrary field K, as long as we interpret dimension as the dimension of the set of K-rational points (it also holds scheme-theoretically without reference to the ground field).

Corollary. *If M as above is codimension s in a 1-generic subspace, then*

$$\dim M_{w-1} \leq \dim M - (v - w + 1) + \max(0, 1 - w + s).$$

Of course Theorem 1′, i) is a special case of the Corollary.

Proof of the Corollary. Let M' be the 1-generic space containing M in codimension s. We have $M'^{\perp} \cap H^* = 0$ by definition, and $M^{\perp}$ contains $M'^{\perp}$ in codimension s, whence $\dim M_1^{\perp} = M^{\perp} \cap H_1^* \leq s$, so that Theorem 2 applies to give the desired result. //

Proof of Theorem 2. Let $\tilde{M}_{w-1} \subset M \times \operatorname{Gr}(w-1, W)$ be "canonical resolution" of M_{w-1} defined by

$$\tilde{M}_{w-1} = \{(\phi, W') \mid \phi \in M,\ W' \in \operatorname{Gr}(w-1, W),\ \phi(V) \subset W'\},$$

where we write $\operatorname{Gr}(w-1, W)$ for the Grassmann variety of $w-1$-planes in W (of course $\operatorname{Gr}(w-1, W)$ is just $w-1$-dimensional projective space). We write

$$\pi_1\colon \tilde{M}_{w-1} \to M_{w-1} \quad \text{and} \quad \pi_2\colon \tilde{M}_{w-1} \to \operatorname{Gr}(w-1, W)$$

for the obvious projections. We will show that

$$\dim \tilde{M}_{w-1} = \dim M - (v - w + 1) + \max(0, 1 - w + \dim M_1^{\perp});$$

since π_1 is onto this implies Theorem 2.

To this end we consider also the variety $M_1^{\perp\sim} \in M^{\perp} \times \mathrm{Gr}(w-1, W)$, which is the "canonical resolution" of $M_1^{\perp}$, defined by

$$M_1^{\perp\sim} = \{(\psi, W') \mid \psi \in M^{\perp},\ W' \in \mathrm{Gr}(w-1, W),\ \psi(W')=0\}.$$

Again we have projections

$$p_1\colon M_1^{\perp\sim} \to M_1^{\perp} \quad \text{and} \quad p_2\colon M_1^{\perp\sim} \to \mathrm{Gr}(w-1, W).$$

Moreover, fixing our attention on $\pi_2^{-1}(W')$ and $p_2^{-1}(W')$ for some fixed $W' \in \mathrm{Gr}(w-1, W)$, and using the fact that

$$\{\phi \mid \phi(V) \subset W'\} = \{\psi \mid \psi(W')=0\}^{\perp},$$

we get an exact sequence of vectorspaces

$$*)\quad 0 \to p_2^{-1}(W') \to \mathrm{Hom}(W/W', V) \oplus M^{\perp} \to \mathrm{Hom}(W, V) \to (\pi_2^{-1}(W'))^* \to 0$$

from which we deduce

$$\begin{aligned}\dim \pi_2^{-1}(W') &= \dim \pi_2^{-1}(W')^* \\ &= \dim M - v + \dim p_2^{-1}(W'),\end{aligned}$$

so that

$$\dim \tilde{M}_{w-1} = \dim M - v + \dim M_1^{\perp\sim}.$$

But the map $p_1\colon M_1^{\perp\sim}$ is an isomorphism except over 0, and the fiber over 0 is $\mathrm{Gr}(w-1, W)$, which has dimension $w-1$; thus

$$\dim M_1^{\perp\sim} = \max(w-1, \dim M_1^{\perp}),$$

and the desired formula follows. //

Remark. The sequence of vectorspaces *) is of course derived from the map (not, unfortunately, of constant rank!) of locally free sheaves over $G = \mathrm{Gr}(w-1, W)$

$$\mathrm{Hom}_G(\mathcal{O}_G(1), V_G) \oplus M_G^{\perp} \to \mathrm{Hom}(W_G, V_G),$$

where W_G, V_G and M_G are the trivial bundles on $\mathrm{Gr}(w-1, W)$.

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