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Fibre Rings and Polynomial Rings

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Let R be a noetherian ring. In their joint work [10], Weisfeiler and Dolgachev study the structure of R-algebra A satisfying the following condition: The fibre ring $A \otimes k(\mathfrak{p})$ is always isomorphic to a polynomial ring in n variables over $k(\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of R where $k(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. They investigate such an A and ask what conditions on A guarantee that A must be a polynomial ring. In particular they conjecture that A must be a polynomial ring in the case where A is finitely generated flat over a normal local domain R. This conjecture has been settled affirmatively by Waterhouse [9] provided that A is a ring of functions on a group scheme over a discrete valuation ring R. Without assumption on group structure, the conjecture is true for $n=1$ [1] (See also [5] and [7]). The case $n=2$ has been proved for R a discrete valuation ring by Sathaye after the Kambayashi's contribution [6] with the additional hypothesis that R contains the rational number field $\boldsymbol{Q}$. On the other hand there is a counter example to the conjecture due to Swan and Yanik [11] as is pointed out by Eakin in [4] in the case where R is not a valuation ring.

Now we will discuss the stable structure of A, where the stable structure means the structure of a polynomial ring $A^{[m]} := A[x_1, \cdots, x_m]$ in m variables as an R-algebra for some large integer m. From this point of view we have first the following theorem.

Theorem 1. *Let R be a discrete valuation ring with quotient field K and residue field k. Let A be an integral R-domain such that $A \otimes K \cong K^{[n]}$ and $A \otimes k \cong k^{[n]}$. Then $A^{[n]} \cong R^{[2n]}$.*

Two R-algebras B and C are called stably isomorphic if $B^{[m]} \cong C^{[m]}$ for some positive integer m. So the theorem shows that A is stably isomorphic to $R^{[n]}$ in the case where A is an integral domain and R is a discrete valuation ring. By virtue of the theorem we can delete the hypothesis "finitely generated" from the Sathaye-Kambayashi's result. However we can not delete the additional hypothesis $R \supset \boldsymbol{Q}$ as follows:

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Theorem 2. *Let R be a discrete valuation ring with maximal ideal πR, quotient field $K=R[\pi^{-1}]$ and residue field $k=R/\pi R$. Suppose* char $k=p>0$. *Let e and s be positive integers such that $p^e \nmid sp$ and $sp \nmid p^e$. Given a positive integer m, let us set*

$$A=R[X, Y, Z]/(X^{p^e}+Y+a_1^{p^e}Y^p+\cdots+a_s^{p^e}Y^{sp}+\pi^m Z)$$

where $R[X, Y, Z]=R^{[3]}$, $a_i \in R$ $(i=1, \cdots, s-1)$ and $a_s \in R\backslash \pi R$. Then

(1) $A\otimes K \cong K^{[2]}$,

(2) $A\otimes k \cong k^{[2]}$,

(3) $A^{[1]} \cong R^{[3]}$,

(4) $A \ncong R^{[2]}$.

Note that the assumption that $R \not\supset \boldsymbol{Q}$ always implies char $k=p>0$ in Theorem 2. From the facts (1), (2) and (3), the R-algebra A defined in Theorem 2 clearly satisfies the condition in the conjecture of Weisfeiler and Dolgachev. This shows that the conjecture is not affirmative even if R is a discrete valuation ring.

The following theorem treats the general case where R is not necessarily a valuation ring. Given a B-module M, let us denote by $S_B(M)$ the symmetric B-algebra.

Theorem 3. *Let R be a noetherian ring and let A be a finitely generated flat R-algebra such that $A\otimes k(\mathfrak{p}) \cong k(\mathfrak{p})^{[n]}$ for any prime ideal $\mathfrak{p}$ of R. Then the differential A-module $\Omega_R(A)$ is projective. Furthermore A is an R-subalgebra (up to isomorphisms) of a polynomial ring $R^{[m]}$ for some m such that*

$$A^{[m]} \cong {}_R S_{R^{[m]}}(\Omega_R(A)\otimes_A R^{[m]}).$$

Since A is an R-subalgebra of $R^{[m]}$ in Theorem 3, there is an injection $f\colon A \hookrightarrow R^{[m]}$. So the tensor product $\Omega_R(A)\otimes_A R^{[m]}$ is welldefined through this injection f. Therefore $S_{R^{[m]}}(\Omega_R(A)\otimes_A R^{[m]})$ is a welldefined symmetric $R^{[m]}$-algebra of projective $R^{[m]}$-module $\Omega_R(A)\otimes_A R^{[m]}$. Theorem 3 shows that $A^{[m]}$ and $S_{R^{[m]}}(\Omega_R(A)\otimes_A R^{[m]})$ are isomorphic to each other as R-algebras. In addition, if R is a regular local ring, then $\Omega_R(A)\otimes_A R^{[m]}$ is stably free by Grothendieck (See [3, Chapter XII]). As an easy consequence of this fact we have the following

Corollary. *Let R be a regular local ring and let A be as in Theorem* 3. *Then A is stably isomorphic to $R^{[n]}$.*

For the details we refer to [2].

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