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Compact Rigid Analytic Spaces —With special regard to surfaces—

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§ 0. Introduction

The notion of rigid analytic spaces was introduced by Tate [T]. Since then the theory has been developed considerably and we now have a rather satisfactory general theory (for example, the proper mapping theorem, GAGA, resolution of singularities). Thus it may be worthwhile to see how many of the results known for compact complex manifolds have analogues for rigid analytic spaces.

In the following we shall show that many important results on compact complex manifolds have counterparts in the category of compact smooth rigid analytic spaces. For example, the notions of algebraic dimension and Kodaira dimension will be introduced and the structure theorems of algebraic reductions and pluricanonical mappings will be shown. Using these general results, we shall develop the bimeromorphic geometry of compact smooth rigid analytic spaces of dimension two (rigid analytic surfaces) which is similar to that of the complex analytic surfaces.

A rigid analytic surface appears naturally as the "generic" fibre S_η of a formal lifting of a surface S_0 in characteristic $p > 0$ to characteristic zero. We shall show that the Kodaira dimensions of S_η and S_0 are equal in almost all cases. (See Theorem 5.11, below.) We conjecture that S_η and S_0 have always the same Kodaira dimension. This will be proved, if we know the structure of certain rigid analytic surfaces with algebraic dimension zero. At the moment, we have no satisfactory theory of such surfaces.

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On the other hand, for the case of algebraic dimension one, we have a satisfactory theory as we shall show below.

Since the theory of rigid analytic spaces are not familiar to algebraic geometers, we explain in Sections 1, 2, and 3 basic ideas of the theory without proofs. The proofs can be found mainly in [BGR] and [FP]. In Section 4 a general theory of bimeromorphic geometry of compact rigid analytic spaces will be developed. In Section 5 we deal with the theory of rigid analytic surfaces, and show that the theory is quite analogous to the complex analytic case. Finally in Section 6 examples of rigid analytic analogues of non-Kähler surfaces will be givens.

§ 1. Affinoid algebras and affinoid spaces

In what follows, by k we always mean a complete non-Archimedean valuation field with a multiplicative valuation $|\ \ |$. Namely, the valuation $|\ \ |$ satisfies the following conditions.

U1) $|a| \geqq 0$ for each $a \in k$. $|a|=0$ if and only if $a=0$.

U2) $|a \cdot b|=|a| \cdot |b|$ for all $a, b \in k$.

U3) $|a+b| \leqq \max\{|a|, |b|\}$ for all $a, b \in k$.

We also assume $\operatorname{char} k=0$ and the valuation is non-trivial. The function $d(a, b):=|a-b|$ on k^2 defines a metric on k and k is totally disconnected with respect to the metric.

Put

$$k^\circ := \{a \in k \mid |a| \leqq 1\}, \ k^{\circ\circ} := \{a \in k \mid |a|<1\}.$$

Then k° is a ring and $k^{\circ\circ}$ is its maximal ideal. The field $\bar{k}=k^\circ/k^{\circ\circ}$ is of characteristic $p=\min\{m \in \boldsymbol{Z} \mid m>0, |m|<1\}$.

Put

$$\begin{aligned} T^n &= k\langle z_1, z_2, \cdots, z_n\rangle \\ &:= \{\sum_{\alpha_i \geqslant 0} a_{\alpha_1 \cdots \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n} \mid \lim_{|\alpha| \to +\infty} a_\alpha = 0\}, \end{aligned}$$

where $z_1, \cdots, z_n$ are variables.

Then T_n is a ring consisting of power series which converge on

$$\{(x_1, \cdots, x_n) \in k^n \mid |x_i| \leqq 1, 1 \leqq i \leqq n\}.$$

Define a semi-norm $\|\ \ \|$ on T_n by

$$\|f\| = \max |a_\alpha| \qquad \text{for } f = \sum a_\alpha z^\alpha.$$

Then, $\|\ \ \|$ is indeed a norm and $(T_n, \|\ \ \|)$ is a Banach algebra.

Definition 1.1. An *affinoid algebra* A over k is a k-algebra with a finite k-algebra homomorphism $\rho\colon T_n \to A$. (That is, A is a finite T_n-module through ρ.)

The following theorem is fundamental.

Theorem 1.2. 1) *T_n is Noetherian.*

2) *T_n is a UFD.*

3) *Every ideal in T_n is closed with respect to $\| \ \|$.*

4) *For each ideal I of T_n, there exists a finite injective k-algebra homomorphism $T_d \to T_n/I$, where $d := \text{Krull dim } T_n/I$.*

5) *For every maximal ideal $\mathfrak{m}$ of T_n, $T_n/\mathfrak{m}$ is a finite extension of k.*

6) *Every affinoid algebra A is isomorphic to a certain quotient k-algebra T_n/I, and has a structure of a Banach algebra by the quotient norm on T_n/I induced by T_n. The norms on A given by different expressions T_n/I, T_m/J are equivalent.*

7) *Every k-algebra homomorphism $f\colon A \to B$ of affinoid algebras are continuous with respect to the norm given in* 6).

Definition 1.3. For an affinoid algebra A put

$$\mathrm{Sp}(A) := \{m \mid m \text{ is a maximal ideal of } A\}.$$

The space $\mathrm{Sp}(A)$ is called an *affinoid space*.

For any point $x \in \mathrm{Sp}(A)$, by $f(x)$ we mean the image of $f \in A$ in A/x. Since A/x is finite over k and k is complete, the valuation $| \ |$ can be uniquely extended to that of A/x. Hence $|f(x)|$ is well-defined. The spectral seminorm $\|f\|_{\mathrm{sp}}$ of A is defined by

$$\|f\|_{\mathrm{sp}} := \sup_{x \in \mathrm{Sp}(A)} |f(x)|.$$

The seminorm $\| \ \|_{\mathrm{sp}}$ is a norm if and only if the radical of A is zero. By Theorem 1.2, 6) an affinoid algebra A has a Banach norm $\| \ \|$. Then, we have the inequality

$$\|f\|_{\mathrm{sp}} \leqq \|f\| \qquad \text{for any } f \in A.$$

For T_n, we can easily show that $\|f\|_{\mathrm{sp}} = \|f\|$. More generally we have the following theorem.

Theorem 1.4. *The spectral seminorm of an affinoid algebra A is a norm if and only if A is reduced. In this case, the spectral norm is equivalent to the quotient norm introduced in Theorem* 1.2, 6).

We also have the following theorems.

Theorem 1.5 (Maximum modulus principle). *For an affinoid algebra A and an element $f \in A$, there is a point $x \in \mathrm{Sp}(A)$ such that we have*

$$|f(x)| = \|f\|_{\mathrm{sp}}.$$

Theorem 1.6. *If A is a reduced affinoid algebra, then the intergral closure of A in its total quotient ring is a finitely generated A-module.*

Theorem 1.7 ([BKKN, Satz 3.3.3]). *An affinoid algebra is an excellent ring.*

Let A be an affinoid algebra. Put

$$A^\circ := \{f \in A \mid \|f\|_{\mathrm{sp}} \leqq 1\}, \quad A^{\circ\circ} := \{f \in A \mid \|f\|_{\mathrm{sp}} < 1\}.$$

Then A° is a ring and $A^{\circ\circ}$ is an ideal of A°. The quotient $\bar{A} = A^\circ / A^{\circ\circ}$ is a $\bar{k}$-algebra. Let $\phi: A \to B$ be a k-algebra homomorphism of affinoid algebras. Then we have $\phi(A^\circ) \subset B^\circ$ and $\phi(A^{\circ\circ}) \subset B^{\circ\circ}$. Hence it induces a $\bar{k}$-algebra homomorphism $\bar{\phi}: \bar{A} \to \bar{B}$.

Theorem 1.8. *The following conditions are equivalent.*
1) *$\phi: A \to B$ is finite.*
2) *$\phi: A^\circ \to B^\circ$ is integral.*
3) *$\bar{\phi}: \bar{A} \to \bar{B}$ is finite.*

Any k-algebra homomorphism $\phi: A \to B$ of affinoid algebras defines a map $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ by $x \mapsto \phi^{-1}(x)$. This map is denoted by ϕ^a or $\mathrm{Sp}(\phi)$. A *morphism* of affinoid spaces is, by definition, a map induced by a k-algebra homomorphism.

Now we introduce a topology on $X = \mathrm{Sp}(A)$ by taking $\{x \in X \mid |f(x)| \leqq 1\}$, $f \in A$, as a subbasis of open sets. This topology is the weakest topology such that $X \in x \mapsto f(x) \in k^a :=$ (the algebraic closure of k) is continuous for all $f \in A$. Then we can easily show that $\mathrm{Sp}(T_n)$ is homeomorphic to $D^n := \{(z_1, \cdots, z_n) \in k^n \mid |z_i| \leqq 1, 1 \leqq i \leqq n\}$ and $\mathrm{Sp}(T_n/I)$ with $\sqrt{I} = I$ is homeomorphic to the closed subset $\{(z_1, \cdots, z_n) \in D^n \mid g(z) = 0$ for all $g \in I\}$ of D^n, if k is algebraically closed.

Example 1.7. On $\{(z_1, \cdots, z_n) \in k^n \mid |\pi_i| \leqq |z_i| \leqq 1\}$ with $\pi_i \in k$, $|\pi_i| \leqq 1$, we can introduce a structure of an affinoid space by $\mathrm{Sp}(A)$, where

$$A = k\langle z_1, \cdots, z_n, w_1, \cdots, w_n \rangle / (w_1 z_1 - \pi_1, \cdots, w_n z_n - \pi_n).$$

We often write A as $k\langle z_1, \cdots, z_n, \pi_1/z_1, \cdots, \pi_n/z_n \rangle$. Especially we have

$$\{(z_1, \cdots, z_n) \in k^n \mid |z_i|=1, 1\leqq i\leqq n\} \simeq \mathrm{Sp}\,(k\langle z_1, \cdots, z_n, 1/z_1, \cdots, 1/z_n\rangle),$$

if k is algebraically closed.

§ 2. Rigid analytic spaces

Since the topology on an affinoid space X defined above is totally disconnected, it is necessary to introduce a Grothendieck topology to obtain a good theory of analytic spaces.

Definition 2.1. An open subset $Y \subset X = \mathrm{Sp}\,(A)$ of an affinoid space X is called an *affinoid subdomain*, if there exist an affinoid algebra A_Y and a k-algebra homomorphism ψ: $A \to A_Y$ such that for any morphism ϕ: $\mathrm{Sp}(B) \to X$ with $\phi(\mathrm{Sp}(B)) \subset Y$, there exists a unique k-algebra homomorphism ξ: $A_Y \to B$ with $\phi = \psi^a \cdot \xi^a$.

$$\begin{tikzcd} \mathrm{Sp}\,(B) \arrow[rr, "\phi"] \arrow[dr, "\xi^a"'] & & \mathrm{Sp}\,(A) \\ & \mathrm{Sp}\,(A_Y) \arrow[ur, "\psi^a"'] & \end{tikzcd}$$

The following lemma is an easy consequence of the definition.

Lemma 2.2. *Let $Y \subset X = \mathrm{Sp}\,(A)$ be an affinoid subdomain. Then the canonical morphism ψ^a: $\mathrm{Sp}\,(A_Y) \to Y$ is bijective. Moreover, if $x \in \mathrm{Sp}\,(A_Y)$, $y = \psi^a(x) \in Y \subset \mathrm{Sp}\,(A)$, then there are canonical isomorphisms*

$$A/y^n \xrightarrow{\sim} A_Y/x^n, \qquad n \geqq 1$$

where $y = \psi^{-1}(x)$.

Corollary 2.3. $\text{proj lim}_n\, A/y^n \xrightarrow{\sim} \text{proj lim}_n\, A_Y/x^n$.

By the above lemma, we can regard an affinoid subdomain as an affinoid space $\mathrm{Sp}\,(A_Y)$.

Lemma 2.4. *Let $Y_1 \subset X = \mathrm{Sp}\,(A)$ be an affinoid subdomain and $Y_2 \subset Y_1$ an affinoid subdomain of Y_1. Then Y_2 is an affinoid subdomain of X.*

Lemma 2.5. *Let f: $\mathrm{Sp}\,(B) \to \mathrm{Sp}\,(A)$ be a morphism of affinoid spaces and Y be an affinoid subdomain of $\mathrm{Sp}\,(A)$. Then $\tilde{Y} = f^{-1}(Y)$ is an affinoid subdomain of $\mathrm{Sp}\,(B)$ and $B_{\tilde{Y}}$ is isomorphic to $B \hat{\otimes}_A A_Y$ (the complete tensor product). In particular, if Y_1 and Y_2 are affinoid subdomains, then so is $Y_1 \cap Y_2$.*

Example 2.6. *A rational domain $R \subset X = \mathrm{Sp}\,(A)$* is, by definition, an open subset in X defined by

$$R=\{x \in X \mid |f_i(x)| \leqq |f_0(x)|, 1 \leqq i \leqq n\},$$

where $f_0, f_1, \cdots, f_n$ are elements of A and have no common zeros in X. A rational domain R is an affinoid subdomain. The affinoid algebra A_R is given by

$$A\langle X_1, \cdots, X_n\rangle/(f_1-X_1f_0, \cdots, f_n-X_nf_0\rangle$$

with the canonical injection $\psi: A \to A_R$, where by $A\langle X_1, \cdots, X_n\rangle$ we mean

$$A\langle X_1, \cdots, X_n\rangle=\{\sum_\alpha a_\alpha X^\alpha | a_\alpha \in A, \lim_{|\alpha|\to\infty} \|a_\alpha\|=0\}.$$

If $A=T_n/I$, then

$$A\langle X_1, \cdots, X_n\rangle \simeq k\langle Z_1, \cdots, Z_n, X_1, \cdots, X_n\rangle/Ik\langle \cdots, Z_i, \cdots, X_i, \cdots\rangle.$$

We have the following lemma.

Lemma 2.7. 1) *If Y_1 and Y_2 are rational domains of $X=\mathrm{Sp}(A)$, then so is $Y_3=Y_1 \cap Y_2$ and $A_{Y_3} \simeq A_{Y_1} \hat{\otimes}_A A_{Y_2}$.*

2) *If Y_1 is a rational domain of X and Y_2 is a rational domain of Y_1, then Y_2 is a rational domain of X.*

Now we introduce two Grothendieck topologies on $X=\mathrm{Sp}(A)$. Let $\mathscr{F}$ be the set consisting of all affinoid subdomains (resp. rational domains) of X. We consider the empty set as a rational domain of X. For any $U \in \mathscr{F}$ put

$$\mathrm{Cov}(U):=\{\mathscr{U}=\{U_i\}_{i\in I} | \#(I)<\infty, \bigcup_{i\in I} U_i=U\},$$

where U_i is an affinoid subdomain (resp. a rational domain) of X. Then by virtue of Lemma 2.4 and Lemma 2.5 (resp. Lemma 2.7) $(\mathscr{F}, \mathrm{Cov})$ defines a Grothendieck topology of X. Namely, $(\mathscr{F}, \mathrm{Cov})$ has the following properties.

1) $\phi \in \mathscr{F}$, $X \in \mathscr{F}$. If $U, V \in \mathscr{F}$, then $U \cap V \in \mathscr{F}$.

2) For any $U \in \mathscr{F}$, $\{U\} \in \mathrm{Cov}(U)$.

3) If $\{U_i\}_{i\in I} \in \mathrm{Cov}(U)$ and $V \subset U$, $V \in \mathscr{F}$, then $\{U_i \cap V\}_{i\in I} \in \mathrm{Cov}(V)$.

4) If $\{U_j\}_{j\in J} \in \mathrm{Cov}(U)$ and $\{U_{jk}\}_{k\in K_J} \in \mathrm{Cov}(U_j)$, $j \in J$, then $\{U_{jk}\}_{j\in J, k\in K_J} \in \mathrm{Cov}(U)$.

An element of $\mathscr{F}$ is called an *admissible open set* and an element of Cov(U) is called an *admissible covering* of U. The Grothendieck topology given by affinoid subdomains and the one given by rational domains are essentially the same by virtue of the following theorem.

Theorem 2.8 (Gerritzen-Grauert [GG]). *Any affinoid subdomain of an affinoid space X is a finite union of rational domains of X.*

Let G_X be one of the Grothendieck topologies defined above. Then, we can define a presheaf $\mathcal{O}_X$, called the structure presheaf by associating A_U to each admissible open set U. Similarly for any finite A-module M,

$$U \longmapsto M \underset{A}{\otimes} A_U$$

defines a presheaf $M^{\sim}$.

Theorem 2.9 (Tate [T]). *For any admissible covering $\mathfrak{X}$ of X, we have*

$$\check{H}^p(\mathfrak{X}, M^{\sim}) = \begin{cases} M, & p=0, \\ 0, & p \neq 0. \end{cases}$$

In particular, $\mathcal{O}_X$ and $M^{\sim}$ are actually sheaves for the Grothendieck topology.

Thus, on X we can introduce a ringed space structure (G-ringed space structure) $(X, G_X, \mathcal{O}_X)$ with respect to the Grothendieck topology. Moreover, by Corollary 2.3, for any point $x \in X = \mathrm{Sp}(A)$, we have

$$\hat{\mathcal{O}}_{X,x} = \text{proj lim } A/x^n.$$

Hence, $\mathcal{O}_{X,x}$ is a local ring. Thus, $(X, G_X, \mathcal{O}_X)$ is a G-local ringed space.

Definition 2.10. Let X and Y be affinoid spaces. A *morphism* $f = (\varphi, \psi)$: $(X, G_X, \mathcal{O}_X) \to (Y, G_X, \mathcal{O}_X)$ of G-local ringed spaces is defined as follows.

1) $\varphi: X \to Y$ is a continuous map of the Grothendieck topologies. That is, for any admissible open set U of Y, $\varphi^{-1}(U)$ is admissible in X and for any admissible covering $\mathcal{U}$ of U, $\varphi^{-1}(\mathcal{U})$ is an admissible covering of $\varphi^{-1}(U)$.

2) For any admissible open set U of Y, there is a k-algebra homomorphism ψ_U: $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$, which is compatible with restrictions. Moreover, $\psi = \{\psi_U\}$ induces a local homomorphism ψ_x: $\mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$.

One of the main reasons why we must introduce the Grothendieck topology is the following.

Proposition 2.11. *There is a one-to-one correspondence between the set of morphisms $X \to Y$ of affinoid spaces and the set of morphisms $(X, G_X, \mathcal{O}_X) \to (Y, G_Y, \mathcal{O}_Y)$ of G-local ringed spaces.*

Definition 2.12. A morphism f: $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ of affinoid spaces is a *closed immersion* (resp. *finite*), if the corresponding homomorphism f^*: $A \to B$ is surjective (resp. finite).

Now a rigid analytic space is defined as follows.

Definition 2.13. A G-local ringed space $(X, G_X, \mathcal{O}_X)$ is called a *rigid analytic space* over k, if X has an admissible covering $\{X_i\}_{i \in I}$ ($\#(I)$ may be infinite) such that $(X_i, G_X|X_i, \mathcal{O}_X|X_i)$ is an affinoid space for each i. A morphism f: $(X, G_X, \mathcal{O}_X) \to (Y, G_Y, \mathcal{O}_Y)$ of rigid analytic spaces is a G-local ringed space morphism.

Remark. Later we shall construct rigid analytic spaces X by patching together affinoid spaces X_i. There are many Grothendieck topologies which induce the same Grothendieck topology on X_i. Among them we can find the finest one, if the G-topologies satisfy certain mild conditions. (See, for example, [BGR, 9.1].) Therefore, in the following we do not give explicitly the G-topology.

Definition 2.14. 1) A morphism f: $(X, G_X, \mathcal{O}_X) \to (Y, G_Y, \mathcal{O}_Y)$ of rigid analytic spaces is a *closed immersion* (resp. *finite*), if there exists an admissible affinoid covering $\{U_i\}_{i \in I}$ of X such that the induced morphism $f^{-1}(U_i) \to U_i$ of affinoid spaces is a closed immersion (resp. finite) for each $i \in I$.

2) A morphism f: $X \to Y$ of rigid analytic spaces is *separated* if the diagonal morphism Δ: $X \to X \times_X X$ is a closed immersion. If $Y = \mathrm{Sp}(k)$, X is called a separated rigid analytic space (over k).

If X is separated and $U, V \subset X$ are affinoid subspaces of X, then $U \cap V$ is also an affinoid subspace of X.

As already mentioned above, the topology of an affinoid space $X = \mathrm{Sp}(A)$ given by the Banach norm of A is totally disconnected. Further, if the residue field k is not finite, X is *not* locally compact. But we can define proper morphisms.

Definition 2.15. Let f: $X = \mathrm{Sp}(A) \to Y = \mathrm{Sp}(B)$ be a morphism of affinoid spaces and $U \subset X$ an affinoid subdomain of X. The open set U is said to be *relatively compact* in X over Y (we use the symbol $U \Subset_Y X$), if there exists an affinoid generating system $\{f_1, \cdots, f_n\}$ of A over B (that is, $\|f_i\| < 1$ and $B\langle T_1, \cdots, T_n\rangle \to A$ sending T_i to f_i is a surjective B-algebra homomorphism) such that

$$U \subset \{x \in X \mid |f_1(x)| < 1, \cdots, |f_n(x)| < 1\}.$$

Lemma 2.16. *Let X_1, X_2 be affinoid spaces over an affinoid space Y and $U_i \Subset_{Y_i} X_i$, $i = 1, 2$. Then we have*

$$U_1 \times_Y X_2 \Subset_{X_2} X_1 \times_Y X_2 \quad \textit{and} \quad U_1 \times_Y U_2 \Subset_Y X_1 \times_Y X_2.$$

That is, relative compactness is closed under base change and fibre product.

Definition 2.17. A morphism f: $X \to Y$ of rigid analytic spaces is *proper*, if

1) f is separated;

2) there exist an admissible affinoid covering $\{Y_i\}_{i \in I}$ of Y and a finite admissible affinoid coverings $\{X_{ij}\}_{1 \leqq j \leqq n_i}$ and $\{X'_{ij}\}_{1 \leqq j \leqq n_i}$ of $f^{-1}(Y_i)$ such that $X'_{ij} \Subset_{Y_i} X_{ij}$. If $Y = \mathrm{Sp}\,(k)$ and f is proper, X is called a *compact* rigid analytic space.

Note that the underlying topological space of a rigid analytic space is not necessarily locally compact and a proper morphism is not necessarily closed.

Definition 2.18. Let $(X, G_X, \mathcal{O}_X)$ be a rigid analytic space and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. If there exist an admissible affinoid covering $\{U_i\}_{i \in I}$, $U_i = \mathrm{Sp}\,(A_i)$, and a finite A-module M_i for each $i \in I$ such that $\mathscr{F} | U_i \simeq M_i^{\sim}$, then $\mathscr{F}$ is said to be *coherent*. (If we need to specify the covering $\mathscr{U}$, we say *$\mathscr{U}$-coherent*.)

Theorem 2.19 (Kiehl [KI2]). *Let $\mathscr{F}$ be an $\mathcal{O}_X$-module over an affinoid space $X = \mathrm{Sp}\,(A)$. The following conditions are equivalent.*

1) *$\mathscr{F}$ is $\mathscr{U}$-coherent for an admissible affinoid covering $\mathscr{U}$ of X.*

2) *$\mathscr{F}$ is $\mathscr{U}$-coherent for any admissible affinoid covering $\mathscr{U}$ of X.*

3) *There exists a finite A-module M such that $\mathscr{F} \simeq M^{\sim}$.*

Theorem 2.20 (Kiehl [KI1]). *Let f: $(X, G_X, \mathcal{O}_X) \to (Y, G_Y, \mathcal{O}_Y)$ be a proper morphism of rigid analytic spaces and $\mathscr{F}$ a coherent $\mathcal{O}_X$ module. Then $R^i f_* \mathscr{F}$ is a coherent $\mathcal{O}_X$-module for any integer i.*

Corollary 2.21. *If X is a compact rigid analytic space over k, then*

$$\dim_k H^i(X, \mathscr{F}) < \infty$$

for any coherent $\mathcal{O}_X$-module $\mathscr{F}$ on X and for any integer i.

From the above theorems we infer the existence of the Stein factorization for a proper morphism. If X is a proper algebraic k-scheme, then X carries a structure of a compact rigid analytic space X^{an}. Then GAGA holds for X and X^{an} ([Kö]). (See also Example 3.3 and Example 3.5 below.) Also we have the Serre duality for locally free sheaves on a compact smooth rigid analytic space. (Actually a more general duality theorem holds. For example, one can use the argument in [HA, Appendix].) Moreover, by Theorem 1.7 Hironaka's theorem on resolution of singularities holds for rigid analytic spaces.

§ 3. Elementary examples and analytic reduction

In the following, for simplicity we assume that the field k is *algebraically closed.*

Example 3.1. *The affine space A_k^n.* Take an element $c \in k$ with $|c|>1$. For any interger $m \geqq 0$, put

$$A_m = \{\sum_{i_j \geqq 0} a_{i_1 \cdots i_n} T_1^{i_1} \cdots T_n^{i_n} | \lim_{i_1+\cdots+i_n \to \infty} |a_{i_1 \cdots i_n}||c|^{m(i_1+\cdots+i_n)} = 0\}.$$

Then $A_m \simeq k\langle z_1/c^m, \cdots, z_n/c^m \rangle$ and there are natural inclusions

$$A_0 \subset A_1 \subset A_2 \subset \cdots A_m \subset A_{m+1} \subset \cdots.$$

Moreover, we have

$$\mathrm{Sp}(A_m) \simeq \{z \in A_k^n | \ |z_i| \leqq |c|^m, 1 \leqq i \leqq n\}.$$

Hence we have $A_k^n = \cup_{m=0}^{\infty} \mathrm{Sp}(A_m)$. Taking $\{\mathrm{Sp}(A_m)\}$ as an admissible covering of A_k^n, we can introduce a structure of a rigid analytic space on A_k^n.

Also we have another admissible covering. Put

$$U_0 = \{z \in A_k^n | \ |z_i| \leqq |c|, i=1, \cdots, n\}$$
$$U_m = \{z \in A_k^n | \ |c|^m \leqq |z_i| \leqq |c|^{(m+1)}, i=1, \cdots, n\},$$

where $c \in k$ with $|c|>1$, $m=1, 2, \cdots$. Then we have

$$U_0 = \mathrm{Sp}(k\langle z_1/c, \cdots, z_n/c \rangle),$$
$$U_m = \mathrm{Sp}(k\langle z_1/c^{m+1}, \cdots, z_n/c^{m+1}, c^m/z_1, \cdots, c^m/z_n \rangle)$$
$$\bigcup_{m=0}^{\infty} U_m = A_k^n.$$

Example 3.2. *An affine scheme.* Let X be an affine scheme of finite type over k. Then X is a closed subscheme of an affine space, hence using the rigid analytic structure of the affine space, we regard X as a closed analytic subspace of the affine space.

Example 3.3. *A scheme X of finite type over k.* Since each affine subscheme of X carries the structure of a rigid analytic space, by patching together the rigid analytic structures of open affine subschemes of X, we can introduce on X a rigid analytic structure. If X is a separated scheme, the corresponding rigid analytic space is separated.

Example 3.4. *The projective space $\boldsymbol{P}_k^n$.*

Put

$$X_i := \{(z_0 : z_1 : \cdots : z_n) \in \boldsymbol{P}_k^n \mid |z_k| \leqq |z_i|,\ 0 \leqq k \leqq n\}, \qquad i=0, 1, 2, \cdots, n.$$

Then we have

$$\begin{aligned} X_i \simeq D^n &= \{(w_1, \cdots, w_n) \mid |w_i| \leqq 1,\ i=1, \cdots, n\} \\ &\simeq \mathrm{Sp}\,(k\langle z_0/z_i, \cdots, z_{i-1}/z_i, z_{i+1}/z_i, \cdots, z_n/z_i\rangle). \end{aligned}$$

If we put

$$A_{ij} := k\langle z_0/z_i, \cdots, z_n/z_i, (z_j/z_i)^{-1}\rangle,$$

then we have $X_{ij} = X_i \cap X_j \simeq \mathrm{Sp}\,(A_{ij})$. Moreover, a k-algebra homomorphism $A_{ij} \to A_{ji}$ which sends z_k/z_i to $(z_k/z_j)(z_i/z_j)^{-1}$ induces an isomorphism $X_{ij} \simeq X_{ji}$. In this way, $\boldsymbol{P}_k^n$ has a structure of a rigid analytic space. We claim that $\boldsymbol{P}_k^n$ is compact with respect to this structure. Indeed, let us take an element c of k with $|c| > 1$ and put

$$\tilde{X}_i := \{(z_0 : z_1 : \cdots : z_n) \in \boldsymbol{P}_k^n \mid |z_k| \leqq |c z_i|,\ 0 \leqq k \leqq n\}, \quad i=0, 1, 2, \cdots, n.$$

Then $\tilde{X}_i$ is also an affinoid space and $\{\tilde{X}_i\}_{0 \leqq i \leqq n}$ is also a finite admissible covering of $\boldsymbol{P}_k^n$. By definition, we have $X_i \Subset \tilde{X}_i$.

Example 3.5. Any projective scheme of finite type over k has a natural rigid analytic structure, because it can be regarded as a closed analytic subspace of the (rigid analytic) projective space. (For details, see, for example, [RA], [FP, IV.1].)

Example 3.6. *Tate's elliptic curve.* Take an element $q \in k$ with $0 < |q| < 1$. The map $z \mapsto qz$ generates an infinite cyclic group $\langle q\rangle$ of automorphisms of k^*. Put $\boldsymbol{T} = k^*/\langle q\rangle$. By a method similar to that in Example 3.1, on $k^* = \boldsymbol{A}_k^1 - \{0\}$ we can introduce a structure of a rigid analytic space such that the map $z \mapsto qz$ defines an analytic automorphism of the rigid analytic structure. Take an element π with $q = \pi^2$. We let X_1 (resp. X_2) be the image of $\{z \in k \mid |\pi| \leqq |z| \leqq 1\}$ (resp. $\{z \in k \mid |q| \leqq |z| \leqq |\pi|\}$) in $\boldsymbol{T}$.

Also by U_1 (resp. U_2), we mean the image of $\{z \in k \mid |z|=1\}$ (resp. $\{z \in k \mid |z| = |\pi|\}$) in $\boldsymbol{T}$. Then, we have $X_1 \cap X_2 = U_1 \cup U_2$ (disjoint union). We consider X_1, X_2, U_1, U_2 as affinoid spaces. (For example $X_1 \simeq \mathrm{Sp}(k\langle z, \pi/z\rangle)$, $U_2 \simeq \mathrm{Sp}(k\langle z/\pi, \pi/z\rangle)$.) Then $\boldsymbol{T}$ is a compact rigid analytic space and $\{X_1, X_2\}$ is an admissible covering. Since $\{X_1, X_2\}$ is an acyclic covering, we can easily calculate the cohomology groups $H^i(\boldsymbol{T}, \mathcal{O}_{\boldsymbol{T}})$ and obtain $\dim_k H^i(\boldsymbol{T}, \mathcal{O}_{\boldsymbol{T}}) = 1$ for $i=0, 1$. The rigid analytic space $\boldsymbol{T}$ is an elliptic curve. Put

$$\wp_1(z) = \sum_{n \in \boldsymbol{Z}} q^n z/(1-q^n z)^2, \qquad \wp_1'(z) = \sum_{n \in \boldsymbol{Z}} q^{2n} z^2/(1-q^n z)^3.$$

Then $\wp_1(qz) = \wp_1(z)$, $\wp_1'(qz) = \wp_1'(z)$ and they are meromorphic functions on $\boldsymbol{T}$. Put

$$s_k = \sum_{m \geq 1} m^k q^m/(1-q^m),$$
$$\wp(z) = \wp_1(z) - 2s_1, \quad \wp'(z) = \wp_1'(z) + s_1.$$

Theorem 3.6 (Tate). *$\wp$ and $\wp'$ satisfy the following equation:*

$$(*) \qquad \wp'^2 + \wp\wp' = \wp^3 + B\wp + C,$$

where

$$B = -5s_3, \qquad C = -(5s_3 + 7s_5)/12.$$

Moreover, the map

$$\boldsymbol{T} \longrightarrow \boldsymbol{P}_k^2, \qquad [z] \longmapsto (1 : \wp(z) : \wp'(z))$$

defines an embedding of $\boldsymbol{T}$ as a cubic curve defined by the equation (*).

The j invariant of the cubic curve defined by (*) *has the form*

$$j = (1-48B)^2/\Delta, \qquad \Delta = q \prod_{m \geq 1} (1-q^n)^{24}$$
$$= 1/q + R(q)$$
$$= 1/q + 744 + 196884q + \cdots,$$

where $R(q) \in \boldsymbol{Z}[[q]]$ (resp. $\boldsymbol{F}_p[[q]]$), if $\mathrm{char}\, k = 0$ *(resp.* $\mathrm{char}\, k = p > 0$*).*

The following proposition shows that not all elliptic curves (that is, curves of genus one with rational points) are Tate's curves.

Proposition 3.7. *Let k be a complete non-Archimedean valuation field (not necessarily algebraically closed) and E an elliptic curve defined over k. Then the following conditions are equivalent.*

1) *There exists a finite extension K/k such that*

$$E \underset{k}{\times} K \simeq K^*/\langle q \rangle.$$

2) $|j(E)|>1$.

Next let us consider an analytic reduction of a rigid analytic space and relation to a formal scheme.

Let A be an affinoid algebra. Put

$$A^\circ := \{f \in A \mid \|f\|_{\mathrm{sp}} \leqq 1\},\ A^{\circ\circ} := \{f \in A \mid \|f\|_{\mathrm{sp}} < 1\},$$
$$\bar{A} := A^\circ / A^{\circ\circ}.$$

If $A = T_n = k\langle X_1, \cdots, X_n\rangle$, then $A^\circ = \{\sum_\alpha a_\alpha X^\alpha \in A \mid a_\alpha \in k^\circ\} = k^\circ\langle X_1, \cdots, X_n\rangle$, $A^{\circ\circ} = k^{\circ\circ}\langle X_1, \cdots, X_n\rangle$. For any element $g = \sum b_\alpha X^\alpha$ of A°, we have $b_\alpha \in k^{\circ\circ}$ for sufficiently large $|\alpha|$, hence $\bar{A} \simeq \bar{k}[X_1, \cdots, X_n]$. Since any affinoid algebra is a quotient T_n/I, $\bar{A}$ is a $\bar{k}$-algebra of finite type. (See for example [BSG, 6.3.4, Corollary 3].) The *analytic reduction* $\bar{X}$ of an affinoid space $X = \mathrm{Sp}(A)$ is, by definition, the algebraic k-scheme $\mathrm{Spec}(\bar{A})$.

Lemma 3.8. *There exists a reduction morphism R: $\mathrm{Sp}(A) \to \mathrm{Spec}(\bar{A})$, where for any maximal ideal $\mathfrak{m}$ of A, $R(\mathfrak{m})$ is the image of $\mathfrak{m} \cap A^\circ$ in $\bar{A}$. Moreover, R is surjective onto the set of closed points.*

Definition 3.9. Let $(X, G_X, \mathcal{O}_X)$ be a separated rigid analytic space. An admissible covering $\mathcal{U} = \{U_i\}$ is called a *pure* covering, if $\mathcal{U}$ satisfies the following conditions.

1) For each i, U_i intersects a finite number of U_j.

2) If $U_i \cap U_j \neq \emptyset$, then there exists an affine open set $V_{ij} \subset \bar{U}_i$ with $U_i \cap U_j = R_i^{-1}(V_{ij})$, where R_i: $U_i \to \bar{U}_i$ and R_{ij}: $U_{ij} \to \bar{U}_{ij}$ are the analytic reductions.

From the above definition, if $\{\mathrm{Sp}(A_i)\}$ is a pure covering of a rigid analytic space X, we can define a scheme $\bar{X}$ locally of finite type over $\bar{k}$, by patching together $\mathrm{Spec}(\bar{A}_i)$. The scheme $\bar{X}$ thus obtained is called the *analytic reduction* of X (with respect to the pure covering $\{\mathrm{Sp}(A_i)\}$. Note that $\bar{X}$ depends on the choice of a pure covering.

Example 3.10. $X = \{z \in k \mid |q| \leqq |z| \leqq 1\}$ where $q \in k$, $0 < |q| < 1$. Then we have

$$A = k\langle X, Y\rangle/(XY-q),\ A^\circ = k^\circ\langle X, Y\rangle/(XY-q).$$

Hence, we have

$$\bar{A} \simeq \bar{k}[X, Y]/(XY).$$

Therefore, $\bar{X} = \mathrm{Spec}\,(\bar{A})$ is a join of two affine lines (see Figure 1).

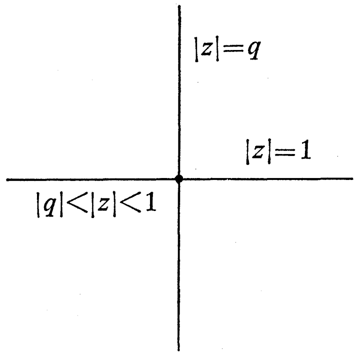

Figure 1

Example 3.11. $X = A_k^n$. The admissible covering $\{U_i\}_{0\leq i}$ given in Example 3.1 is pure and $\bar{X}$ is an infinite union of $A_{\bar{k}}^n$ (cf. Figure 2).

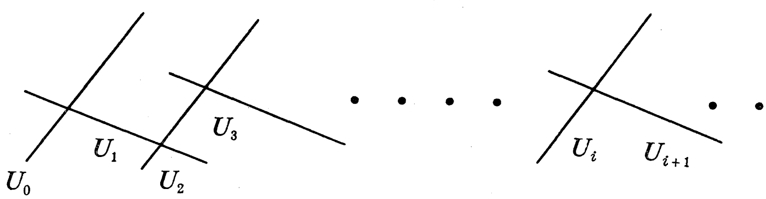

Figure 2

Example 3.12. $\boldsymbol{T} = k^*/\langle q \rangle$. If we use the admissible covering $\{X_1, X_2\}$, $\bar{\boldsymbol{T}}$ is as in Figure 3.

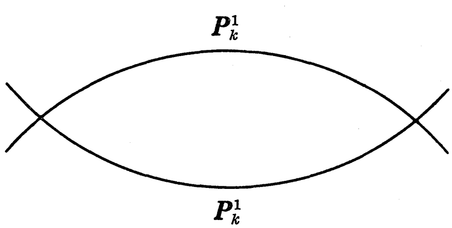

Figure 3

The dependence of the analytic reduction on pure coverings is clarified, if we consider the relationship between rigid analytic spaces and formal shemes. Recall that for $A = T_n$ we have

$$A^\circ = k^\circ\langle X_1, \cdots, X_n\rangle,\ A^{\circ\circ} = k^{\circ\circ}\langle X_1, \cdots, X_n\rangle = \mathfrak{m}A^\circ,$$

where $\mathfrak{m} = k^{\circ\circ}$, the maximal ideal of the valuation ring k°. Hence we have

$$A^\circ = \text{proj lim } A^\circ/\mathfrak{m}^s A^\circ = \text{proj lim } k^\circ[X_1, \cdots, X_n]/\mathfrak{m}^s k^\circ[X_1, \cdots, X_n].$$

Thus, to A° we can associated a formal scheme $\mathrm{Spf}\, A^\circ$ over $\mathrm{Spf}\, k^\circ$. More generally, for any affinoid algebra we have a formal scheme $\mathrm{Spf}\, A^\circ$ over $\mathrm{Spf}\, k^\circ$ and $\mathrm{Sp}\,(A)$ can be regarded as its generic fibre. In general, to any rigid analytic space X of finite type over k (i.e., covered by a finite number of affinoid spaces), we can associate a formal scheme $\mathscr{X}$ over $\mathrm{Spf}\, k^\circ$ and X can be regarded as the generic fibre of $\mathscr{X} \to \mathrm{Spf}\, k^\circ$. Conversely, for a formal scheme $\mathscr{X} \to \mathrm{Spf}\, k^\circ$ locally of finite type, we can construct a rigid analytic space X over k. The correspondence is not one to one, since the formal schemes depend on pure coverings. Moreover, if $\mathscr{X}'$ is obtained from the formal monoidal transformation of $\mathscr{X}$ along a closed subscheme whose ideal sheaf is contained in an ideal generated by an element $q \in k$ with $|q| < 1$, then the rigid analytic spaces corresponding to $\mathscr{X}'$ and $\mathscr{X}$ are the same. The details can be found in [RA].

§ 4. Compact rigid analytic spaces

In the following a compact reduced irreducible rigid analytic space is called a *rigid analytic variety*. Also a non-singular rigid analytic variety is called a *rigid analytic manifold.*

a) Algebraic reduction

A *meromorphic mapping* $\varphi: X \to Y$ of rigid analytic varieties is defined in the same way as in the complex analytic case. (See, for example, [U 2, Definition 2.2].) By [HI] we can always resolve the points of indeterminacy of φ by succession of monoidal transformations of X with non-singular centers. If a meromorphic mapping $\varphi: X \to Y$ has an inverse, we call φ a bimeromorphic mapping and X and Y are said to be bimeromorphically equivalent.

Theorem 4.1 ([BO2]). *The field $\mathscr{M}(X)$ consisting of all meromorphic functions on a rigid analytic variety X (the meromorphic function field of X) is an algebraic function field and we have*

$$\mathrm{tr\ deg}_k\, \mathscr{M}(X) \leqq \dim_k X.$$

The number $a(X) = \mathrm{tr\ deg}_k(X)$ is called the *algebraic dimension* of X. Since char $k=0$, by [HI], there exists a nonsingular projective variety V defined over k whose rational function field is isomorphic to $\mathscr{M}(X)$. Then by GAGA, if we regard V as a rigid analytic manifold, the meromorphic function field $\mathscr{M}(V)$ is equal to the rational function field of V. Therefore, we have a meromorphic mapping

$$\varphi\colon X \longrightarrow V$$

such that φ induces an isomorphism of meromorphic function fields of X and V. Choosing a suitable bimeromorphically equivalent non-singular model $\hat{X}$ of X, we have a proper surjective morphism $\hat{\varphi}\colon \hat{X} \to V$ which induces an isomorphism of meromorphic function fields of $\hat{X}$ and V. The morphism $\hat{\varphi}\colon \hat{X} \to V$ is called an *algebraic reduction* of X. The algebraic reduction is unique up to bimeromorphic equivalence. Moreover, all fibres of $\hat{\varphi}$ are connected.

A slight modification of the proof given in [U2, Section 12, (12.8), (12.9)] gives the following theorem.

Theorem 4.2. 1) *Let $f\colon X \to Y$ be a surjective morphism of rigid analytic varieties with connected fibres. Then there exists a proper rigid analytic subspace Y_1 of Y such that*

$$a(Y) \leqq a(X) \leqq a(Y) + a(X_y) \leqq a(Y) + \dim f,$$

for any $y \in Y_1$, where $X_y = f^{-1}(y)$, the fibre of f over y, is nonsingular and $\dim f = \dim X - \dim Y$.

2) *If M is an irreducible subvariety of a rigid analytic variety N, we have*

$$a(N) \leqq a(M) + \operatorname{codim} M.$$

A rigid analytic variety X is called a *Moishezon variety* if $a(X) = \dim X$.

Corollary 4.3. 1) *An irreducible subvariety of a Moishezon variety is a Moishezon variety.*

2) *If M is a Moishezon variety and $f\colon M \to N$ is a surjective morphism of rigid analytic varieties, then N is also a Moishezon variety.*

b) L-dimension and Kodaira dimension

Let L be a line bundle on a normal rigid analytic variety X. Put

$$N(L) := \{m \geqq 1 \mid H^0(X, L^m) \neq 0\}.$$

Then the *L-dimension* $\kappa(X, L)$ is defined by

$$\kappa(X, L) := \begin{cases} \max\limits_{m \in N(L)} \dim \Phi_{|mL|}(X), & \text{if } N(L) \neq \phi, \\ -\infty \quad , & \text{if } N(L) = \phi, \end{cases}$$

where $\Phi_{|mL|}$ is the meromorphic mapping to a projective space associated with $H^0(X, L^m)$. If X is not normal, $\kappa(X, L)$ is defined by

$$\kappa(X, L)=\kappa(X^*, \iota^*L),$$

where $\iota\colon X^*\to X$ is the normalization of X.

For a Cartier divisor D on X, the *D-dimension* $\kappa(X, D)$ *of* D is defined by

$$\kappa(X, D)=\kappa(X, [D]),$$

where $[D]$ is the line bundle associated with D.

If X is non-singular, the L-dimension $\kappa(X, K_X)$ of the canonical bundle K_X of X is called the *Kodaira dimension* of X and denoted $\kappa(X)$.

Now it is easy to show that the algebraic dimension, the L-dimension and the Kodaira dimension enjoy the same properties as those of complex varieties. Here we only state some important facts. For details consult, for example, [U2, § 5 ~ § 9].

Lemma 4.4. $\kappa(X)\leqq a(X)$.

Theorem 4.5 (Iitaka's fibration theorem). *If $\kappa(X)>0$ for a rigid analytic manifold X, there exist a bimeromorphically equivalent model $\hat{X}$ of X, a non-singular projective variety W and a surjective morphism $\varphi\colon \hat{X}\to W$ such that*

1) $\dim W=\kappa(X)$,
2) *all fibres of φ are connected,*
3) $\kappa(X_y)=0$, *for a general fibre $\tilde{X}_y$ of φ,*
4) *the morphism φ is bimeromorphic to $\Phi_{|mK|}\colon X\to\Phi_{|mK|}(X)$ for a sufficiently large $m\in N(K_X)$.*

Theorem 4.6. *Let $\varphi\colon \hat{X}\to V$ be the algebraic reduction of a rigid analytic variety X. Then we have*

$$\kappa(\hat{X}_y)\leqq 0$$

for a general fibre $\hat{X}_y$ of φ. Moreover, if $a(X)=\dim X-1$, then $\hat{X}_y$ is a non-singular curve of genus one.

§ 5. Rigid analytic surfaces

A smooth rigid analytic variety of dimension 2 is called a *rigid analytic surface*. In this section we shall study the structure of rigid analytic surfaces.

a) Intersection theory and the Riemann-Roch theorem

Let C, D be irreducible curves on a rigid analytic surface S. If $C\neq D$, then C and D intersect at a finite number of points. For an intersection point $x \in C \cap D$, we define the intersection multiplcity $I_x(C, D)$ of C and D at x by

$$I_x(C, D) := \dim_k \mathcal{O}_{X,x}/(f, g),$$

where $f=0$ (resp. $g=0$) is a local equation of C (resp. D) at x. Then the intersection number $C \cdot D$ of C and D is defined by

$$C \cdot D := \sum_{x \in C \cap D} I_x(C, D).$$

It is easy to see that

$$C \cdot D = D \cdot C = \deg [D]|_C = \deg [C]|_D,$$

where $[D]|_C$ is the restriction to C of the line bundle associated with D. Hence, for a line bundle L on S the intersection number $C \cdot L = L \cdot C$ is defined by

$$C \cdot L = L \cdot C = \deg L|_C.$$

Also the self intersection number C^2 of C is defined by

$$C^2 = \deg [C]|_C.$$

Then by linearity we can define the intersection number $D \cdot D'$ of divisors D and D' and the intersection number $D \cdot L = L \cdot D$ of a divisor D and a line bundle L.

If divisors D_1 and D_2 are linearly equivalent, we have

$$D_1 \cdot D' = D_2 \cdot D'.$$

For a divisor D on a surface S, the Riemann-Roch theorem says that

$$\sum_{\nu=0} (-1)^\nu \dim_k H^\nu(S, \mathcal{O}_S(D)) = D \cdot (D - K_X)/2 + \chi(S, \mathcal{O}_S),$$

where

$$\chi(S, \mathcal{O}_S) = p_g(S) - q(S) + 1,$$
$$p_g(S) = \dim_k H^2(S, \mathcal{O}_S),$$
$$q(S) = \dim_k H^1(S, \mathcal{O}_S).$$

For a proof, we can use the argument given in [KO2, I, Section 2]. Note that by the Serre duality we have

$$p_g(S) = \dim_k H^0(S, K_S).$$

If S is algebraic, then by GAGA we have Noether's formula

$$12\chi(S, \mathcal{O}_S) = K_S^2 + c_2(S),$$

but we do not know if this holds for a non-algebraic surface S at the moment. The virtual genus $\pi(C)$ of an irreducible curve C on S is defined by the *adjunction formula*

$$2\pi(C) - 2 = C^2 + C \cdot K_S.$$

Then we have

$$\pi(C) = \dim_k H^1(C, \mathcal{O}_C).$$

b) Exceptional curves of the first kind

An irreducible curve C on a surface S is called an exceptional curve of the first kind, if $C^2 = -1$ and $\pi(C) = 0$, In this case C is isomorphic to $\boldsymbol{P}^1$. If $f: \hat{X} \to X$ is obtained by blowing up a surface X at a point x, then $C = f^{-1}(x)$ is an exceptional curve of the first kind. Here, we shall prove the converse. By GAGA this is true if S is algebraic. In general, we imitate the idea used by [KO2, II, Appendix] in the complex analytic case.

Lemma 5.1. *Let C be a non-singular curve in a rigid analytic surface S with $H^1(C, N_C) = 0$, where N_C is the normal bundle of C in S. Put $n = h^0(C, N_C)$. Then for a sufficiently small $\varepsilon > 0$ there exists a non-singular subvariety $\mathcal{C}$ on $\Delta_\varepsilon^n \times S$ such that the morphism $p: \mathcal{C} \to \Delta_\varepsilon^n$ induced from the projection of $\Delta_\varepsilon^n \times S$ to the first factor is proper and smooth and $p^{-1}(o) = C$, where $\Delta_\varepsilon^n = \{(z_1, \cdots, z_n) \in k^n \mid |z_i| \leqq \varepsilon, i = 1, \cdots, n\}$ and $o = (0, \cdots, 0)$.*

Proof. Let $\{\mathcal{U}_i = \mathrm{Sp}\,(A_i)\}_{i \in I}$ be a finite admissible affinoid covering of C. We assume that there are $f_{i_1}, \cdots, f_{i_s} \in A_i$ such that

$$V_i = \{x \in \mathcal{U}_i \mid |f_{i_j}| < 1\}, \quad j = 1, \cdots, i_s, \quad i \in I$$

form an admissible covering of C. By [KI3], there exists a finite admissible covering $\{\mathrm{Sp}\,(A_{i,j})\}_{j \in J_i}$ of $\mathrm{Sp}\,(A_i)$ such that there exist open immersions $\varphi_{i,j}: \mathrm{Sp}\,(A_{i,j}\langle Z_{i,j}\rangle) \to S$ and that $\{\mathrm{Im}\,\varphi_{i,j}\}_{j \in J_i, i \in I}$ is an admissible covering of a tubular neighbourhood of C in S. Now we can apply the argument of [KO 1] and obtain the desired result. Q.E.D.

Remark 5.2. In the above proof we only need to assume that the curve C is compact. Also we can show the universality of the family constructed above.

Theorem 5.3. *Let C be an exceptional curve of the first kind in a* *rigid analytic surface S. Then there exist a rigid analytic surface $\tilde{S}$ and a surjective morphism λ: $S \to \tilde{S}$ such that $\lambda(C) = x$ is a point and λ is the inverse of the blowing up of S at x.*

Proof. It is enough to show that a neighbourhood of C is isomorphic to the blowing up of a certain non-singular affinoid space. We follow the argument of [KO 2, II, Appendix].

Choose a point $z \in C$. Then we can find an admissible open neighbourhood U of z in S, such that $U \simeq \mathrm{Sp}(k\langle X_1, X_2\rangle)$ and $X_1 = 0$ is a local equation of the curve C in U. In the following, we identify U with $\mathrm{Sp}(k\langle X_1, X_2\rangle)$. Let us consider a homomorphism

$$\varphi\colon k\langle Y_1, Y_2\rangle \longrightarrow k\langle X_1, X_2\rangle,$$

which sends Y_1 (resp. Y_2) to X_1X_2 (resp. X_2). Then we patch together $S - \{y\}$ and $\mathrm{Sp}(k\langle Y_1, Y_2\rangle)$ by means of the morphism $\mathrm{Sp}(\varphi)$ and obtain a non-singular rigid analytic space $\hat{S}$. There is a natural morphism μ: $S \to \hat{S}$ and the image $\hat{C} = \mu(C)$ of C is non-singular with trivial normal bundle $N_{\hat{C}}$ in $\hat{S}$. (Roughly speaking, we obtain $\hat{S}$ by contracting a curve defined by $X_2 = 0$ in U, hence $\deg N_{\hat{C}} = \deg N_C + 1 = 0$.) Hence we can apply the above lemma and obtain a one-parameter family $\mathscr{C} = \{\hat{C}_t\}_{|t| \leqq \varepsilon}$. Since $\hat{C}^2 = 0$, the natural morphism f: $\mathscr{C} \to \hat{S}$ is an open immersion. Hence $p \circ f^{-1}$ is a morphism from a neighbourhood of $\hat{C}$ in $\hat{S}$ to $\Delta_\varepsilon = \{|t| \leqq \varepsilon\}$. Therefore, $g = p \circ f^{-1} \circ \mu$ is a rigid analytic function on a neighbourhood W of C in S, and in this neighbourhood $g = 0$ defines a divisor $C + A$, where A is defined by $X_2 = 0$. Then C and A intersect transversally at z.

Now choose another point $y \in C$. In the same way as above there is a rigid analytic function h in a neighbourhood W of C in S such that $h = 0$ in W defines a divisor $C + B$, where B and C intersect transversally at y. Then it is easy to show that a morphism

$$\lambda\colon W \longrightarrow k^2 \times \boldsymbol{P}^1$$

which sends x to $(g(x), h(x), (g(x)\colon h(x)))$ is an open immersion and $\lambda(C) = (0, 0) \times \boldsymbol{P}^1$. Hence we conclude that there exists a two dimensional smooth affinoid space V such that an admissible open neighbourhood of the curve C in S is isomorphic to the blowing up of V at a point of V. Q.E.D.

A rigid analytic surface S is called a *relatively minimal model*, if S contains no exceptional curves of the first kind. From the above theorem we infer the existence of a relatively minimal model of a rigid analytic surface by contracting exceptional curves of the first kind. Also we can prove the factorization theorem of a bimeromorphic morphism.

Theorem 5.4. *Let $f: X \to Y$ be a bimeromorphic morphism. Then f is decomposed into a finite succession of contractions $f_i: Y_i \to Y_{i+1}$, $i=0, 1, \cdots, n$ of exceptional curves of the first kind in Y_i, where $Y_0 = X$, $Y_{n+1} = Y$, $f = f_n \circ f_{n-1} \circ \cdots \circ f_0$.*

c) General theory

Here we collect together some general facts on rigid analytic surfaces.

Theorem 5.5. *If a rigid analytic surface S is of algebraic dimension two, then S is projective.*

Proof. By the algebraic reduction of S the surface S is bimeromorphic to a non-singular projective surface. Hence, by a finite succession of blowing-ups of S we obtain a projective surface $\hat{S}$. Hence our surface S is obtained from $\hat{S}$ by contracting exceptional curves of the first kind. But contracting a projective surface, we obtain a projective surface. Hence S is projective. Q.E.D.

Theorem 5.6. *If a rigid analytic surface S is of algebraic dimension one, then there exists a surjective morphism $f: S \to C$ from S to a non-singular curve C such that the general fibres of f are non-singular curves of genus one. Thus S has a structure of an elliptic surface (see d) below).*

Proof. By Theorem 4.6, there is a surjective morphism $g: S^* \to C$ from a bimeromorphic model S^* of S to a non-singular curve C such that the general fibres of f are non-singular curves of genus one. By Theorem 5.4 we may assume that S^* is obtained from S by a finite succession of blowing-ups. Now assume that an exceptional curve E of the first kind of S^* is mapped onto C by g. Let F be a general fibre of g. Put $D = E + F$. Since we have $D^2 = C^2 + 2C \cdot E \geqq 1$, by the Riemann-Roch theorem we have $\chi(S^*, \mathcal{O}(mD)) \sim D^2 m^2/2$, For m sufficiently large, we have $H^0(S^*, \mathcal{O}(K_{S^*} - mD)) = 0$. Hence $\kappa(S^*, D) = 2$ and $a(S) = a(S^*) = 2$. This is a contradiction. Hence all exceptional curves are contained in the fibres of g. Therefore, contracting exceptional curves appearing in the blowing-ups of S, we have a surjective morphism $f: S \to C$ such that f is an algebraic reduction of S and the general fibres of f are non-singular curves of genus one. Q.E.D.

The following lemmas will be used later.

Lemma 5.7. *If a rigid analytic surface S contains an effective divisor D with $D^2 > 0$, then $\kappa(S, D) = 2$. Hence, S is projective.*

Proof. By the Riemann-Roch theorem we have $\chi(S, \mathcal{O}(mD)) \sim D^2 m^2/2$, Moreover $h^2(S, \mathcal{O}(mD))=h^0(S, \mathcal{O}(K_S-mD))=0$ for a sufficiently large m. Hence we have $h^0(S, \mathcal{O}(mD))\sim D^2m^2/2$. Q.E.D.

Lemma 5.8. *Let a rigid analytic surface S be relatively minimal and $\kappa(S)\geqq 0$. Then K_S is nef. That is $K_S\cdot C\geqq 0$ for any curve C in S.*

Proof. Assume that there exists a curve C in S with $K_S\cdot C<0$. Since $\kappa(S)\geqq 0$, there is an effective divisor $D\in|nK_S|$ for a positive integer n. Since $D\cdot C<0$ we can write $D=D'+mC$ with $m>0$. This implies $C^2<0$. Then by the adjunction formula, C is an exceptional curve of the first kind. This is a contradiction. Q.E.D.

d) Elliptic surfaces

A surjective morphism f: $S\to C$ of a rigid analytic surface to a non-singular curve is called an elliptic surface, if general fibres of f are non-singular curves of genus one. An elliptic surface f: $S\to C$ is called *minimal*, if every fibre of f contains no exceptional curves of the first kind. In the following we assume that an elliptic surface is minimal, unless otherwise mentioned.

For an elliptic surface f: $S\to C$ we consider a fibre $f^{-1}(x)$ over x as a divisor. If $f^{-1}(x)$ is a singular curve of genus one, $f^{-1}(x)$ is called a singular fibre. A singular fibre $f^{-1}(x)$ is called a multiple fibre mD of multiplicity m, if we have $f^{-1}(x)= mD$, $D=\sum_{i=1}^{s} n_iD_i$, $(n_1, n_2, \cdots, n_s)=1$, $m\geqq 2$. All possible types of singular fibres and multiple fibres are the same as those in the complex analytic case.

Theorem 5.9. *Let f: $S\to C$ be an elliptic surface. Then a canonical divisor K_S of S is written as*

$$K_S=f^*(K_C-F)+\sum_{i=1}^{N}(m_i-1)E_i,$$

where K_C is a canonical divisor of C, F is a divisor on C with

$$-\deg F=\chi(S, \mathcal{O}_S),$$

and m_1E_1, m_2E_2, $\cdots$, m_NE_N are all multiple fibres of f.

This theorem can be proved completely in the same way as in the complex analytic case. (See, for examples, [BH, Part 2, Section 2] or [U1, Theorem 7.4].)

Theorem 5.10. *Let f: $S\to C$ be an elliptic surface with $\kappa(S)=1$. Then for any integer $m\geqq 86$, $\Phi_{|mK|}$: $S\to\Phi_{|mK|}(S)$ is isomorphic to f: $S\to C$.*

For a proof we can use the same argument as in [I, Corollary to Proposition 8]. The number 86 in the above theorem is the best possible as an example in 6, b) shows.

e) Formal lifting of smooth algebraic surfaces in characteristic p

Let S_0 be a non-singular algebraic surface defined over k_0 with char $k_0=p>0$. Assume that S_0 has a formal lifting $f: \mathcal{S}\to \mathrm{Spf}\, R$ to characteristic zero, where R is a complete valuation ring with the residue field $R/\mathfrak{m}=k_0$. Let k be the field of fractions of R. Then as was mentioned above (see Section 3 above), the "generic fibre" S_η of f is defined as a compact rigid analytic surface.

Theorem 5.11. *In the above situation we have*

$$\kappa(S_0)=\kappa(S_\eta),$$

except possibly in the case where $\kappa(S_0)=1$, $\chi(S_0, O_{S_0})=0$, $p_g(S_0)\geqq 1$, $\kappa(S_\eta)=-\infty$ *and* $a(S_\eta)=0$.

Proof. By the upper semi-continuity of cohomology groups ([BO1]) we have

$$P_m(S_0)\geqq P_m(S_\eta), \qquad m\geqq 1.$$

Hence we have always

$$\kappa(S_0)\geqq \kappa(S_\eta). \tag{5.1}$$

Moreover we have

$$\chi(S_0, \mathcal{O}_{S_0})=\chi(S_\eta, \mathcal{O}_{S_\eta}). \tag{5.2}$$

Since for any exceptional curve C_0 of the first kind of S_0 we have $H^0(S_0, N_{C_0})=H^1(S_0, N_{C_0})=0$, C_0 can be uniquely lifted to a smooth formal subscheme of $\mathcal{S}$ and $\mathcal{S}$ has a formal contraction to a smooth formal scheme. Hence we may further assume that the surface S_0 is relatively minimal.

If $p_g(S_0)=0$, then any line bundle on S_0 can be lifted to S_η and S_η is algebraizable. Hence, in this case, the above theorem is true by virtue of [KU, Theorem 9.1]. Therefore, in the following we assume that $p_g(S_0)\geqq 1$, hence $\kappa(S_0)\geqq 0$.

Step I. $\kappa(S_0)=2$ if and only if $\kappa(S_\eta)=2$.

If $\kappa(S_\eta)=2$, then by the above inequality (5.1) we have $\kappa(S_0)=2$. Conversely, if $\kappa(S_0)=2$, then as S_0 is relatively minimal, $K_{S_0}^2>0$. Hence $K_S^2>0$. Therefore, S_η is algebraic and S_η is of general type or rational. If S_η is rational, there is a curve C_η in S_η with $K_S\cdot C_\eta<0$. Then by the analytic reduction we find an effective divisor C_0 in S_0 with $K_{S_0}\cdot C_0 = K_{S_\eta}\cdot C_\eta < 0$. Since S_0 is relatively minimal, by Lemma 5.8, we have $\kappa(S_0) = -\infty$. This is a contradiction. Hence, $\kappa(S_0)=2$.

Step II. $\kappa(S_0)=0$ if and only if $\kappa(S_\eta)=0$.

Since we assume $p_g(S_0)\geqq 1$, S_0 is a $K3$ surface or an abelian surface. If S_0 is a $K3$ surface, then $\chi(S_0, \mathcal{O}_{S_0})=2$. Hence $\chi(S_\eta, \mathcal{O}_{S_\eta})=2$ and $\chi(S, K_S^m)=2$ for any $m\geqq 1$. Since $h^2(S_0, K_{S_0}^m)=1$ for any integer m, we have $h^0\ (S, K_S^m) \geqslant 1$. Hence we have $\kappa(S_\eta)=0$. If S_0 is an abelian surface, the formal lifting of S_0 is well-known and the canonical bundle is trivial. Hence we obtain $\kappa(S_\eta)=0$.

Step III. Assume $\kappa(S_\eta)=1$. Then by (5.1) we have $\kappa(S_0)\geqq 1$. Hence by Step I we have $\kappa(S_0)=1$. Conversely, assume $\kappa(S_0)=1$. Then from the above argument we infer that $\kappa(S_\eta)=1$ or $\kappa(S_\eta)=-\infty$. Since we have $h^2(S_0, K_{S_0}^m)=0$ for $m\geqq 2$ and $x(S_\eta, K_{S_\eta}^m)=\chi(S_\eta, \mathcal{O}_{S_\eta})$, if $\chi(S_0, \mathcal{O}_{S_0})=\chi(S_\eta, \mathcal{O}_{S_\eta})\geqq 1$, then we have $h^0(S_\eta, K_{S_\eta}^m)\geqq 1$. Hence $\kappa(S_\eta)=1$. Next assume $\chi(S_0, \mathcal{O}_{S_0})=0$. If S_η is algebraic, then by [KS, Theorem 9.1] we have $\kappa(S_\eta)=1$. If $a(S_\eta)=1$, then S_η is an elliptic surface. Since $K_{S_\eta}^2 = K_{S_0}^2=0$, S_η is a minimal elliptic surface. Then $\kappa(S_\eta)=-\infty$ implies $h^0(S, K_{S_\eta}^{-m})\geqq 1$ for a suitable positive integer m by Theorem 5.9. This contradicts the fact that $h^2(S_0, K_{S_0}^m)=0$ for any positive integer. Q.E.D.

Remark 5.12. We conjecture that in Theorem 5.11 we have always

$$\kappa(S_\eta)=\kappa(S_0).$$

Further analysis of the last part of the argument in Step III above shows that the conjecture is true if there exists no rigid analytic surface with $a(S)=0$, $b_1(S)=2$. In the complex analytic case, the classification theory tells us that an analytic surface with $a(S)=0$, $b_1(S)=2$ does not exist.

§ 6. Examples

In this section we shall give examples of compact rigid analytic analogues of non-Kähler surfaces.

a) Elliptic surfaces with odd b_1

Let $L\to C$ be a line bundle on a non-singular compact curve C. We consider L as an analytic space. Choose $\alpha\in k$ with $|\alpha|>1$. Then α acts on $L-\{0\text{-section}\}$ fibrewise. Set $S_{L,\alpha} := (L-\{0\text{-section}\})/\langle\alpha\rangle$. Then $S_{L,\alpha}$ carries a structure of a rigid analytic space with a morphism $f\colon S_{L,\alpha}\to C$. All fibres of f are isomorphic to $k^*/\langle\alpha\rangle = E_\alpha$, Tate's elliptic curve. Hence $f\colon S_{L,\alpha}\to C$ is an elliptic surface.

Lemma 6.1. 1) $K_S = f^*K_C$.

2) *If* $\deg(L)=0$, *then we have*

$$b_1(S_{L,\alpha}) = 2+2g(C),\ h^0(\Omega^1_{S_{L,\alpha}}) = 1+g(C),$$
$$q(S_{L,\alpha}) = 1+g(C),$$

where $q(S)=\dim H^1(S, \mathcal{O}_S)$ *and* $g(C)$ *is the genus of* C.

3) *If* $\deg(L)\neq 0$, *then we have*

$$b_1(S_{L,\alpha}) = 1+2g(C),\ h^0(\Omega^1_{S_{L,\alpha}}) = g(C),$$
$$q(S_{L,\alpha}) = 1+g(C).$$

Proof. Let $\{\mathcal{U}_\lambda\}_{\lambda\in\Lambda}$ be an admissible affinoid covering of C and $\{g_{\lambda\mu}\}$ be transition functions of L such that on $\mathcal{U}_\lambda$, L is isomorphic to $\mathcal{U}_\lambda \times A^1_k$. Let w_λ be affine coordinates of A^1_k such that the above transition functions are given with respect to these coordinates. Let τ be a meromorphic 1-form on C. Then $\{dw_\lambda/w_\lambda \wedge \tau\}$ is a global meromorphic 2-form on $S=S_{L,\alpha}$. Hence we have $K_S=f^*K_C$.

Let ω be a regular 1-form on S whose restriction to a general fibre of f is not zero. Then on $f^{-1}(\mathcal{U}_\lambda)$, ω is written as

$$\omega_\lambda = \omega|_{\mathcal{U}_\lambda} = a_\lambda dw_\lambda/w_\lambda + \tau_\lambda,$$

where $a_\lambda \in \Gamma(f^{-1}(\mathcal{U}_\lambda), \mathcal{O}_S) = \Gamma(\mathcal{U}_\lambda, \mathcal{O}_{\mathcal{U}_\lambda})$ and τ_λ is a regular one form on $f^{-1}(\mathcal{U}_\lambda)$ whose restriction to any fibre is zero. Choose $\mu_\lambda \in \Gamma(\mathcal{U}_\lambda, \Omega^1_{\mathcal{U}_\lambda})$ which has no zeros on $\mathcal{U}_\lambda$. (If necessary, we take a refinement of the covering $\{\mathcal{U}_\lambda\}$.) Then we have $\tau_\lambda = b_\lambda \mu_\lambda$, where $b_\lambda \in \Gamma(f^{-1}(\mathcal{U}_\lambda), \mathcal{O}_{\mathcal{U}_\lambda}) = \Gamma(\mathcal{U}_\lambda, \mathcal{O}_{\mathcal{U}_\lambda})$. Hence $\tau_\lambda \in \Gamma(\mathcal{U}_\lambda, \Omega^1_{\mathcal{U}_\lambda})$. On $\mathcal{U}_\lambda \cap \mathcal{U}_\mu$ we have $w_\lambda = g_{\lambda\mu} w_\mu$. Hence we have

$$\omega_\lambda = a_\lambda dw_\mu/w_\mu + a_\lambda dg_{\lambda\mu}/g_{\lambda\mu} + \tau_\lambda = a_\mu dw_\mu/w_\mu + \tau_\mu.$$

Therefore, on $\mathcal{U}_\lambda \cap \mathcal{U}_\mu$ we have

$$\begin{cases} a_\lambda = a_\mu \\ a_\lambda dg_{\lambda\mu}/g_{\lambda\mu} + \tau_\lambda = \tau_\mu. \end{cases}$$

Hence $\{a_\lambda\}$ is constant. Thus we may assume $a_\lambda = 1$. Then we have

$$\tau_\lambda - \tau_\mu = -dg_{\lambda\mu}/g_{\lambda\mu}.$$

This implies $\{dg_{\lambda\mu}/g_{\lambda\mu}\}$ is zero in $H^1(C, \Omega^1_C)$. Thus the existence of a regular 1-form ω whose restriction to a general fibre is not zero implies $\deg(L)=0$. Moreover, in this case we infer from the above expression for ω that $d\omega$

$=0$. Conversely if $\deg(L)=0$, we can reverse the above argument and obtain a regular 1-form ω. Hence we have

$$h^0(\Omega_S^1)=\begin{cases} g(C)+1, & \text{if } \deg(L)=0, \\ g(C) \quad , & \text{if } \deg(L)\neq 0. \end{cases}$$

Next let us consider $H^1(S, \mathcal{O}_S)$. It is easy to show that $R^1 f_* \mathcal{O}_S$ is trivial. Hence by an exact sequence

$$(6.1) \qquad 0 \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(S, \mathcal{O}_S) \longrightarrow H^0(C, R^1 f_* \mathcal{O}_S) \longrightarrow 0,$$

we have

$$h^1(S, \mathcal{O}_S)=g(C)+1.$$

Finally we shall show that $b_1(S)=h^0(S, \Omega_S^1)+h^1(S, \mathcal{O}_S)$. By the Hodge-de Rham spectral sequence, it is enough to show that

$$d\colon H^0(S, \Omega_S^1) \longrightarrow H^0(S, \Omega_S^2),$$
$$d\colon H^1(S, \mathcal{O}_S) \longrightarrow H^1(S, \Omega_S^1)$$

are zero maps. We have already showed that the first map is zero. As for the second map, let us consider an exact sequence

$$0 \longrightarrow f^*\Omega_C^1 \longrightarrow \Omega_S^1 \longrightarrow \Omega_{S/C}^1 \longrightarrow 0.$$

Since f is smooth, there are natural isomorphisms $\Omega_{S/C}^1 \simeq \omega_{S/C} \simeq \mathcal{O}_S$. Since $f\colon S \to C$ is a rigid analytic fibre bundle, we have exact sequences

$$(6.2) \qquad \begin{cases} 0 \longrightarrow \Omega_C^1 \longrightarrow f_*\Omega_S^1 \longrightarrow f_*\Omega_{S/C}^1 \longrightarrow 0, \\ 0 \longrightarrow R^1 f^*(f_*\Omega_C^1) \longrightarrow R^1 f_*\Omega_S^1 \longrightarrow R^1 f_*\Omega_{S/C}^1 \longrightarrow 0. \end{cases}$$

From the first exact sequence of (6.2) and the exact sequence (6.1) we infer that it is enough to show that $d_*\colon H^0(C, R^1 f_* \mathcal{O}_S) \to H^0(C, R^1 f_* \Omega_S^1)$ is the zero map. Since any regular 1-form on a smooth curve is d-closed, $d_{S/C}\colon R^1 f_* \mathcal{O}_S \to R^1 f_* \Omega_S^1$ is the zero map. Hence, by the second exact sequence of (6.2) the homomorphism $d\colon R^1 f_* \mathcal{O}_S \to R^1 f_* \Omega_S^1$ factors through $R^1 f_* \mathcal{O}_S \to R^1 f_*(f_* \Omega_C^1) = R^1 f_* \mathcal{O}_S \otimes \Omega_C^1 \to R^1 f_* \Omega_S^1$. As we have $R^1 f_* \mathcal{O}_S = \mathcal{O}_C$, $H^0(C, R^1 f_* \mathcal{O}_S) \to H^0(C, R^1 f_* \mathcal{O}_S \otimes \Omega_C^1)$ is the zero map. Hence we have the desired result. Q.E.D.

An analytic reduction of $S_{L,\alpha}$ is a fibre bundle over an analytic reduction $\bar{C}$ of C. Its fibre is a cycle of non-singular rational curves which is an analytic reduction of Tate's elliptic curve. If $\deg(L)\neq 0$, the reduction $S_{L,\alpha}$ is not projective.

b) **Logarithmic transformations**

Let C be a non-singular curve, L a line bundle on C and $f\colon S_{L,\alpha}\to C$ be the elliptic surface constructed above. Choose an admissible affinoid neighbourhood $D=\mathrm{Sp}\,(k\langle t\rangle)$ of a point $x\in C$. Put $D^*=\mathrm{Sp}\,(k\langle t, t^{-1}\rangle)=D-\{0\}$. Put $\hat{D}=\mathrm{Sp}\,(k\langle s\rangle)$, $D^*=\mathrm{Sp}\,(k\langle s, s^{-1}\rangle)$. Define a morphism $\varphi\colon \hat{D}^*\to D$ by

$$\varphi^*\colon k\langle t\rangle\longrightarrow k\langle s\rangle, \qquad \varphi^*(t)=s^m,$$

where m is a positive integer. Then φ is an m-sheeted cyclic covering ramified at the origin.

Lemma 6.2. $H^1(D, \mathcal{O}_D^*)=\{1\}$, $H^1(D^*, \mathcal{O}_{D^*}^*)=\{1\}$.

Proof. The cohomology group $H^1(X, \mathcal{O}_X^*)$ corresponds to the set of isomorphism classes of line bundles on X. If X is a smooth affinoid space, a line bundle on X corresponds to a projective $\mathcal{O}(X)$-module of rank one. Now if $X=D$ or D^*, $\mathcal{O}(X)$ is a UFD. Hence any projective module on X is free. Q.E.D.

Let us fix a trivialization of $L|_D$:

$$L|_D\simeq D\times A^1=\mathrm{Sp}\,(k\langle t\rangle)\times A^1.$$

Then we have a trivialization

$$\varphi^*(L|_D)\simeq\hat{D}\times A^1=\mathrm{Sp}\,(k\langle s\rangle)\times A^1.$$

Let us consider an automorphism g of $\hat{D}\times E$ defined by

$$g\colon (s, [\zeta])\longrightarrow(e_m s, [\beta\zeta]),$$

where e_m is a primitive m-th root of unity, $[\zeta]$ is a point of Tate's curve $E_\alpha=(A^1-\{0\})/\langle\alpha\rangle$ corresponding to a point $\zeta\in A^1-\{0\}$, and $\beta^m=\alpha$. The automorphism g genertes a cyclic group G of order m and the group G operates freely on $\hat{D}\times E$. Hence the quotient space $\hat{S}_D=\hat{D}\times E/G$ is a smooth rigid analytic space. There is a natural morphism $\hat{f}_D\colon\hat{S}_D\to D$. By construction we have $\hat{f}_D^{-1}(0)=mF_0$, where F_0 is an elliptic curve isomorphic to $k^*/\langle\beta\rangle$, and other fibres of f_D are isomorphic to E_α.

The group G operates also on $\hat{D}\times A^1$ by

$$g\colon (s, \zeta)\longmapsto(e_m s, \beta\zeta).$$

The action is free on $D^*\times A^1$ and the quotient space $L_D^*=(D^*\times A^1)/G$ is a line bundle on D^*. Then we have an isomorphism

$$\psi\colon \hat{S}_D|_{D^*} \xrightarrow{\sim} L_{D^*}/\langle\beta\rangle$$

over D^* where the action of β on L_D^* is the fibrewise multiplication. On the other hand, by Lemma 6.2, L_D^* is a trivial bundle on D^*. Therefore, there is an isomorphism

$$\varphi\colon L_D^*/\langle\beta\rangle \xrightarrow{\sim} S|_{D^*}$$

over D^*.

Now we patch together $\hat{S}_D$ and $S_{L,\alpha}-f^{-1}(x)$ by $\varphi\circ\psi$ and obtain an elliptic surface $\hat{f}\colon \hat{S}\to C$. We write $\hat{S}=L_x(m,\beta)(S_{L,\alpha})$ and call the above process to construct $\hat{S}$ a *logarithmic transformation*. Note that the elliptic surface $L_x(m,\beta)(S_{L,\alpha})\to C$ has a multiple singular fibre mF_0 over the point x.

Applying logarithmic transformations at a finite number of points on C, we obtain an elliptic surface over S with arbitrarily many multiple fibres. The elliptic surface $h\colon L_{x_1}(2,\beta_1)\circ L_{x_2}(3,\beta_2)\circ L_{x_3}(7,\beta_3)(\boldsymbol{P}^1\times E_\alpha)\to\boldsymbol{P}^1$ has the following property: $\Phi_{|85K|}$ does not give the structure of an elliptic surface but for any $m\geqq 86$, $\Phi_{|mK|}$ gives the structure of an elliptic surface, where $\beta_1^2=\beta_2^3=\beta_3^7=\alpha$. (See Theorem 5.10 above.)

c) **Hopf surfaces** ([GG, p. 182–183])

Choose two elements $\alpha,\beta\in k$ with $0<|\alpha|<|\beta|<1$. Consider an automorphism g of k^2 defined by

$$(z_1, z_2)\longmapsto(\alpha z_1, \beta z_2).$$

Then $H_{\alpha,\beta}=k^2-\{(0,0)\}/\langle g\rangle$ has a structure of a rigid analytic surface. If $\alpha^m=\beta^n$ for suitable positive integers m, n, $H_{\alpha,\beta}$ has a structure of an elliptic surface over $\boldsymbol{P}^1$ and $a(H_{\alpha,\beta})=1$. If α,β are algebraically independent over $\boldsymbol{Q}$, then we have $a(H_{\alpha,\beta})=0$. We always have

$$q(H_{\alpha,\beta})=1,\ h^0(\Omega^1_{H_{\alpha,\beta}})=0.$$

We can construct other examples of rigid analytic surfaces S with $a(S)=0$, $b_1(S)=1$ by using a method of Kato [KA]. This will be discussed elsewhere.

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