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K. Saito's Period Map for Holomorphic Functions with Isolated Critical Points

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Dedicated to Professor Masayoshi Nagata on his sixtieth birthday

Introduction

The purpose of this survey is to explain the theory of Kyoji Saito on the period map for the universal unfolding of a holomorphic function with only isolated critical points, which he has been developing in a series of published and unpublished papers [SK1] through [SK19]. We can take advantage of the added perspective of algebraic analysis developed by Sato, Kashiwara, Kawai and Mebkhout as well as results obtained by Morihiko Saito.

For a positive integer n which we eventually assume to be even, let $F_1(x, t')$ be a holomorphic function on a neighborhood X of the origin in $\boldsymbol{C}^{n+1}\times\boldsymbol{C}^{\mu-1}$ with coordinates $(x, t')=(x_0, \cdots, x_n, t_2, \cdots, t_\mu)$ such that $F_1(0, 0)=0$ and that the function $F_1(x, t')$ in $x\in\boldsymbol{C}^{n+1}$ for each fixed t' has at most isolated critical points. Consider the holomorphic map

$$\varphi\colon X\longrightarrow S\subset\boldsymbol{C}\times T\subset\boldsymbol{C}\times\boldsymbol{C}^{\mu-1},$$

defined by $\varphi(x, t')=(F_1(x, t'), t')$ for a neighborhood S (resp. T) of the origin in $\boldsymbol{C}\times\boldsymbol{C}^{\mu-1}$ (resp. $\boldsymbol{C}^{\mu-1}$) with coordinates $s=(t_1, t')$ (resp. $t'=(t_2, \cdots, t_\mu)$). Thus φ gives rise to a family, parametrized by $s=(t_1, t')\in S$, of germs of n-dimensional hypersurfaces

$$X_s:=\varphi^{-1}(s)=\{(x, t')\in X;\ t_1-F_1(x, t')=0\}$$

in $\boldsymbol{C}^{n+1}$ with at most isolated singular points.

The *critical locus*

$$C:=\mathrm{Specan}\,(\mathcal{O}_X/(\partial F_1/\partial x_0, \cdots, \partial F_1/\partial x_n))$$

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of φ is a closed analytic subspace of X and is finite over S. Its image $D:=\varphi(C)$ is called the *discriminant locus* of φ. Thus X_s for $s \in S$ is smooth if and only if s is not in D. X_s is then called a *Milnor fiber* for φ.

Consider the holomorphic function

$$F(x, t_1, t'):=t_1-F_1(x, t')$$

on the neighborhood $Z:=X\times_T S$ of the origin in $C^{n+1}\times C^{\mu}$ with coordinates (x, t_1, t'). Let $p: Z\to S$ be the projection $p(x, t_1, t'):=(t_1, t')$ and denote by Θ_S the holomorphic tangent sheaf of S, i.e., the sheaf of germs of derivations with respect to the variables $t_1, \cdots, t_\mu$. Let

$$\hat{C}:=\operatorname{Specan}(\mathcal{O}_Z/(\partial F/\partial x_0, \cdots, \partial F/\partial x_n)).$$

$\varphi: X\to S$ is said to be the *universal unfolding* of the holomorphic function $f(x):=-F_1(x, 0)$ if the $\mathcal{O}_S$-linear map sending $\theta \in \Theta_S$ to the image $\overline{\theta F}$ of θF under the canonical surjection $p_*\mathcal{O}_Z\to p_*\mathcal{O}_{\hat{C}}$ is an isomorphism

$$\Theta_S \xrightarrow{\sim} p_*\mathcal{O}_{\hat{C}}.$$

In this case, $\mu=\dim S$ coincides with the so-called *Milnor number* of $f(x)$. Moreover, Θ_S then has a commutative and associative $\mathcal{O}_S$-algebra structure, with $\partial_1:=\partial/\partial t_1$ as the identity, induced from that on $p_*\mathcal{O}_{\hat{C}}$ by the above isomorphism. We denote the product on Θ_S so defined by $\theta*\theta' \in \Theta_S$ for θ and θ' in Θ_S. Thus $\overline{(\theta*\theta')F}=\overline{(\theta F)}\overline{(\theta' F)}$. We also have an $\mathcal{O}_S$-linear endomorphism w of Θ_S which corresponds to that on $p_*\mathcal{O}_{\hat{C}}$ induced by the multiplication of F on $\mathcal{O}_{\hat{C}}$. Thus $\overline{w(\theta)F}=\overline{F}\overline{(\theta F)}$. The determinant Δ of w turns out to be the defining equation for the discriminant locus $D\subset S$ with the reduced structure.

$\varphi: X\to S$ is neither smooth nor proper and may not be algebraizable, either. Nevertheless, we can construct the *Gauss-Manin system* Ξ for φ, a left module over the sheaf $\mathcal{D}_S$ of germs of holomorphic linear differential operators on S. Its restriction to $S\setminus D$ is

$$\Xi|_{S\setminus D}=(\mathcal{O}_S\otimes_{C_S}R^n\varphi_*C_X)|_{S\setminus D}$$

and coincides with the n-dimensional relative de Rham cohomology sheaf for the smooth morphism $X\setminus\varphi^{-1}(D)\to S\setminus D$, endowed with the usual Gauss-Manin connection. It is a locally free $\mathcal{O}_{S\setminus D}$-module of rank μ with the subsheaf of the horizontal sections equal to $(R^n\varphi_*C_X)|_{S\setminus D}$, whose stalk at $s \in S\setminus D$ is the n-dimensional cohomology group $H^n(X_s, C)$ of the Milnor fiber X_s.

In the terminology of algebraic analysis, Ξ turns out to coincide with the n-th integration $\int_{\varphi}^{n} \mathcal{O}_X$ of the $\mathcal{D}_X$-module $\mathcal{O}_X$ along the fibers of φ and is a holonomic left $\mathscr{D}_S$-module with regular singularities. (Note, however, that we shift the degree of the conventional one for convenience. In the literature, Ξ is the zero-th integration along the fiber.) Ξ also has an increasing filtration $\{\Xi^{(k)};\ k \in \boldsymbol{Z}\}$ by $\mathscr{O}_S$-submodules such that

$$\Xi^{(k)}/\Xi^{(k-1)} \cong \varphi_* \Omega_{X/S}^{n+1} \qquad k \in \boldsymbol{Z},$$

where $\Omega_{X/S}^{n+1}$ is the sheaf of germs of relative $(n+1)$-forms for φ and is $\mathscr{O}_C$-invertible.

$\Xi^{(0)}$ is called the *Brieskorn lattice* for φ and was shown to be $\mathscr{O}_S$-locally free of rank μ by Brieskorn [B1], Sebastiani [S1], Greuel [G], Malgrange [M1] and Saito [Sk1]. [B1] showed that $\Xi^{(0)}$ has a Gauss-Manin connection with regular singularities along D and algebraically describes the monodromy of the Milnor fibers for φ.

When φ: $X \to S$ is a universal unfolding, Saito describes the structure of Ξ in terms of interesting linear algebra on Θ_S and then constructs a period map from a ramified covering $\tilde{S}$ of S to a μ-dimensional complex affine space E. For that purpose, he proposed to choose a certain global section

$$\zeta \in H^0(S, \Xi^{(-1)})$$

called a *primitive form.* It was M. Saito [Sm7] who actually showed the local existence of a *good* primitive form.

A good primitive form ζ determines a torsion-free integrable connection

$$\nabla: \Theta_S \times \Theta_S \longrightarrow \Theta_S \qquad \text{sending } (\theta, \theta') \text{ to } \nabla_\theta \theta'$$

with $\nabla_\theta(\partial_1)=0$ for all $\theta \in \Theta_S$, an $\mathscr{O}_S$-linear endomorphism N: $\Theta_S \to \Theta_S$ and a nondegenerate symmetric $\mathscr{O}_S$-bilinear form J: $\Theta_S \times \Theta_S \to \mathscr{O}_S$, which together satisfy nice properties described in Section 4. ζ generates Ξ as a left $\mathscr{D}_S$-module and satisfies the following system of holomorphic linear differential equations:

$$\begin{aligned} &\theta\theta'\zeta = \{(\theta * \theta')\partial_1 + \nabla_\theta \theta'\}\zeta && \text{for all } \theta, \theta' \in \Theta_S \\ &w(\theta)\partial_1\zeta = (N(\theta) - \theta)\zeta && \text{for all } \theta \in \Theta_S \\ &w(\partial_1)\zeta = r\zeta && \text{for a rational number } r, \end{aligned}$$

where $\partial_1 = \partial/\partial t_1$ and where the $\mathscr{O}_S$-linear endomorphism w of Θ_S and the product $\theta * \theta'$ on Θ_S are those defined by the universality of the unfolding.

By the Poincaré duality, the solution sheaf of Ξ turns out to be

$$\mathscr{H}om_{\mathscr{D}_S}(\Xi, \mathscr{O}_S) = R^n \varphi_! \boldsymbol{C}_X,$$

the n-th direct image with proper support of C_X by φ and is a constructible C_S-module. Its restriction to $S \setminus D$ is $C_{S\setminus D}$-locally free of rank μ with the stalk at each $s \in S \setminus D$ canonically isomorphic to the n-dimensional homology group $H_n(X_s, C)$ of the Milnor fiber X_s. When n is even, Saito describes the intersection form on $H_n(X_s, C)$, up to a nonzero constant multiple, in terms of an amazing algebraic formula involving ζ, J and w above. (cf. Theorem 4.6 and Corollary 4.7.)

Note that a section γ of $R^n\varphi_! C_X$ over $S \setminus D$ can be thought of as a family, parametrized by $S \setminus D$, of n-dimensional homology classes $\gamma(s)$ of the Milnor fibers X_S. The evaluation at the section ζ of Ξ gives rise to a $C_{S\setminus D}$-linear map

$$(1/2\pi i)^n \int \zeta \quad : (R^n\varphi_! C_X)|_{S\setminus D} \longrightarrow \mathcal{O}_{S\setminus D}$$

in Saito's notation, which sends γ to $(1/2\pi i)^n \int_\gamma \zeta$.

Modifying the above system of holomorphic linear differential equations, Saito introduces a new holonomic left $\mathcal{D}_S$-module with regular singularities

$$\mathcal{M}^{(k)} := \mathcal{D}_S m_k \qquad \text{for each } k \in C$$

via the *defining relations*

$$(*_k) \qquad \begin{aligned} &\left\{\frac{\partial^2}{\partial t_i \partial t_j} - \left(\frac{\partial}{\partial t_i} * \frac{\partial}{\partial t_j}\right)\frac{\partial}{\partial t_1} - \nabla_{\partial/\partial t_i}\left(\frac{\partial}{\partial t_j}\right)\right\} m_k = 0 \\ &\left\{w\left(\frac{\partial}{\partial t_i}\right)\frac{\partial}{\partial t_1} - N\left(\frac{\partial}{\partial t_i}\right) + (k+1)\frac{\partial}{\partial t_i}\right\} m_k = 0 \end{aligned}$$

for all i and j between 1 and μ. Sending m_{k+1} to $\partial_1 m_k$ (resp. m_0 to ζ), we have a $\mathcal{D}_S$-homomorphism ε_k: $\mathcal{M}^{(k+1)} \to \mathcal{M}^{(k)}$ (resp. ε: $\mathcal{M}^{(0)} \to \Xi$).

The solution sheaf

$$M_k := \mathcal{H}om_{\mathcal{D}_S}(\mathcal{M}^{(k)}, \mathcal{O}_S)$$

is a constructible C_S-module such that $(M_k)|_{S\setminus D}$ is $C_{S\setminus D}$-locally free of rank $\mu+1$. We have the induced C_S-linear maps

$$\varepsilon_k^*: M_k \longrightarrow M_{k+1} \quad \text{and} \quad \varepsilon^*: R^n\varphi_! C_X \longrightarrow M_0.$$

The evaluation at m_k gives a C_S-linear map $M_k \to \mathcal{O}_S$ whose image contains the constant sheaf C_S and consists of the "holomorphic solution functions" for the system of differential equations $(*_k)$.

If we choose a $\boldsymbol{C}_S$-basis $\{\partial_1, \cdots, \partial_\mu\}$ with $\partial_1 = \partial/\partial t_1$ of the subsheaf Θ_S^{∇} of horizontal sections of Θ_S with respect to the connection ∇, then the defining relations $(*_k)$ above have a much simpler form so that the image of M_k in $\mathcal{O}_S$ under the evaluation map at m_k consists of elements u in $\mathcal{O}_S$ satisfying the system of holomorphic linear differential equations

$$(**_k) \qquad \begin{aligned} &\partial_i\partial_j u = (\partial_i * \partial_j)\partial_1 u && \text{for } 1 \leq i, j \leq \mu \\ &w(\partial_i)\partial_1 u = (N(\partial_i) - (k+1)\partial_i)u && \text{for } 1 \leq i \leq \mu. \end{aligned}$$

A coordinate system $(v_1, \cdots, v_\mu)$ of S is said to be *flat* if $\partial/\partial v_1 = \partial_1$ and if $\{\partial/\partial v_1, \cdots, \partial/\partial v_\mu\}$ is a $\boldsymbol{C}_S$-basis of Θ_S^{∇}. It then gives a "flat" open embedding $v\colon S \to \boldsymbol{C}^\mu$ sending $s \in S$ to $v(s) := (v_1(s), \cdots, v_\mu(s))$.

Saito then shows that the quotient of $(M_{n/2})|_{S\setminus D}$ by a canonical submodule $\boldsymbol{C}_{S\setminus D}$ is a self dual locally free $\boldsymbol{C}_{S\setminus D}$-module of rank μ, using the same formula as the one he used to describe the intersection pairing in terms of ζ, J and w.

The situation is particularly nice, when n is even. In this case, we have a composite $\boldsymbol{C}_S$-linear map

$$R^n\varphi_!\boldsymbol{C}_X \xrightarrow{\varepsilon^*} M_0 \xrightarrow{\varepsilon_0^*} M_1 \longrightarrow \cdots \longrightarrow M_{n/2}$$

which induces $(\partial_1)^{n/2}$ on the solution functions $(1/2\pi i)^n \int_\gamma \zeta$ for $\gamma \in R^n\varphi_!\boldsymbol{C}_X$. The smallest (possibly infinite) unramified covering $Y \to S\setminus D$, such that the pull-back to Y of $(M_{n/2})|_{S\setminus D}$ is $\boldsymbol{C}_Y$-free, turns out to have an extension $\tilde{S} \to S$ as a ramified covering with $\tilde{S}$ nonsingular and with the image equal to $\{s \in S;\ X_s$ has at most simple singularities$\}$. If we choose independent holomorphic functions $u_0 = 1, u_1, \cdots, u_\mu$ on $\tilde{S}$ which satisfy the system of differential equations $(**_{n/2})$ above, we get a local isomorphism

$$u\colon \tilde{S} \to E := \{(y_0, \cdots, y_\mu) \in \boldsymbol{C}^{\mu+1};\ y_0 = 1\}$$

into a μ-dimensional complex affine space by sending $\tilde{s}$ to $u(\tilde{s}) := (u_0(\tilde{s}), \cdots, u_\mu(\tilde{s}))$. This is the *period map* of Saito. Together with the "flat" open embedding v above, we have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{u} & E \\ \downarrow & & \downarrow \\ S & \xrightarrow{v} & \boldsymbol{C}^\mu \end{array}$$

with $\Delta^{1/2}$ as the Jacobian determinant of the map $E \to \boldsymbol{C}^\mu$. Except when $f(x) := -F_1(x, 0)$ has a simple (i.e., rational double when $n=2$) or simple elliptic critical point at the origin, it is yet unknown if $u: \tilde{S} \to E$ is injective. Nevertheless, u seems to have properties much nicer than $u': \tilde{S}' \to E'$ which we obtain using $R^n\varphi_! C_X = \mathscr{H}om_{\mathscr{D}_S}(\mathcal{Z}, \mathcal{O}_S)$ instead of $M_{n/2}$. We refer the reader, for instance, to Looijenga [L1]–[L4], Saito [Sk4], [Sk15]–[Sk19], Saito-Yano-Sekiguchi [SYS], Ishiura-Noumi [IN], Noumi [N2], Varchenko [V1]–[V3] and Varchenko-Givental [VG].

In Section 1, we define the Gauss-Manin system $\mathcal{Z}$ and give an indication for an elementary proof of its basic properties. Those readers familiar with algebraic analysis can just read this section up to the statement of Theorem 1.2 and then turn to Appendix for the idea of the proof.

In Section 2, we prove the Poincaré duality for the Gauss-Manin system. This duality is known to hold in a more general context as we see in Appendix. Here we give a more elementary proof by means of the relative Serre duality due to Ramis-Ruget [RR]. This Poincaré duality is equivalent to Saito's algebraic description in Section 4 of the intersection pairing.

In Section 3, we deal with the formal microlocalization

$$\hat{\mathcal{Z}} := \text{proj} \lim_{k \to -\infty} \mathcal{Z}/\mathcal{Z}^{(k)}$$

with respect to the variable t_1, which has the structure much simpler than that of $\mathcal{Z}$. We prove the microlocal version of the Poincaré duality which is equivalent to Saito's higher residue pairing, as Kashiwara pointed out (cf. [Sm7]).

Section 4 is the main part of this survey. We here deal with universal unfoldings and explain Saito's interesting linear algebra on Θ_S arising from the choice of a good primitive form. We explain in great detail the system of differential equations mentioned above, which plays a key role in the construction of the period map.

In Appendix, we overview the part of algebraic analysis relevant to Saito's work surveyed here.

Unless otherwise mentioned explicitly, the results surveyed here are due to Kyoji Saito, although we do not indicate where in his papers each of the results can be found. The readers are also referred to his papers for relevant results due to people other than Saito himself. The references given in this survey are quite incomplete.

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§ 1. The Gauss-Manin systems

For a positive integer m, let T be an open set containing the origin O of the complex affine space $\boldsymbol{C}^{m-1}$ with coordinates $t'=(t_2, \cdots, t_m)$. Our main concern here is a family $\varphi: X \to S$, parametrized by T, of holomorphic functions of the following form: We fix an integer $n \geq 2$. (We need this assumption in our spectral sequence arguments below, although most of the results also hold for $n=1$.) $X \subset \boldsymbol{C}^{n+1} \times T \subset \boldsymbol{C}^{n+1} \times \boldsymbol{C}^{m-1}$ is an open set containing the origin O and with coordinates $(x, t')=(x_0, \cdots, x_n, t_2, \cdots, t_m)$, while $S \subset \boldsymbol{C} \times T \subset \boldsymbol{C} \times \boldsymbol{C}^{m-1}$ is an open set containing the origin O and with coordinates $s:=(t_1, t')=(t_1, t_2, \cdots, t_m)$. Then $\varphi: X \to S$ is a holomorphic map which sends (x, t') to $(F_1(x, t'), t')$ for a holomorphic function $F_1(x, t')$ on X such that $F_1(0, 0)=0$.

Furthermore, we assume that for each fixed $t' \in T$, the function $F_1(x, t')$ in $x \in \boldsymbol{C}^{n+1}$ has at most isolated critical points. Thus $\varphi: X \to S$ is necessarily flat and can also be thought of as a family of germs of n-dimensional hypersurfaces in $\boldsymbol{C}^{n+1}$ with at most isolated singular points.

For technical reasons, we assume that the following are satisfied:

Assumptions. (i) φ is a Stein morphism and each fiber is either smooth or is contractible with isolated singular points. Moreover, the restriction of φ to the critical locus C defined below is finite (i.e., proper with finite fibers) over S.

(ii) The projection $q: X \to T$ sending (x, t') to t' is a Stein morphism with contractible fibers.

(iii) The projection $\pi: S \to T$ sending $s=(t_1, t')$ to t' is again a Stein morphism with contractible fibers.

We can always attain these conditions by replacing T, X and S by smaller open sets, e.g., small open balls centered at the origin.

The *critical locus*

$$C:=\operatorname{Specan}(\mathcal{O}_X/(\partial F_1/\partial x_0, \cdots, \partial F_1/\partial x_n))$$

is a closed analytic subspace of X, and the restriction of φ to C is finite over S by assumption. We define the *discriminant locus* $D:=\varphi(C) \subset S$ to be the image with reduced analytic structure. Thus for s in S the fiber $\varphi^{-1}(s)$ is smooth if s is not in D, while it has isolated singular points if s is in D. Obviously by Sard's theorem, we have $D \neq S$. (D is in fact a divisor on S. See Lemma 2.5).

Remark. Eventually in Section 4, we assume $\varphi: X \to S$ to be a universal unfolding of the function $\boldsymbol{C}^{n+1} \to \boldsymbol{C}$ sending x to $f(x):=-F_1(x, 0)$ so that the dimension m of S coincides with the so-called *Milnor number*

$$\mu := \dim_C \mathcal{O}_{C^{n+1}}/(\partial f/\partial x_0, \cdots, \partial f/\partial x_n)$$

of $f(x)$.

As usual, we denote by $\mathcal{D}_S$ the sheaf of germs of holomorphic linear differential operators of finite order on S. It contains the structure sheaf $\mathcal{O}_S$ as the subring of zero-th order operators and contains the holomorphic tangent sheaf Θ_S as the subsheaf of pure first order operators. $\mathcal{D}_S$ is generated as a ring sheaf by $\mathcal{O}_S$ and Θ_S.

The following complex $(K^{\boldsymbol{\cdot}}, \delta)$ on X will play a key role in this paper. To define it, we denote by

$$(\Omega^{\boldsymbol{\cdot}}_{X/T}, d) \quad \text{with} \quad d = d_{X/T}$$

the relative de Rham complex on X with respect to the smooth morphism $q\colon X \to T$.

Definition. $(K^{\boldsymbol{\cdot}}, \delta)$ is the complex

$$0 \longrightarrow K^{-1} \xrightarrow{\delta} K^0 \xrightarrow{\delta} K^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} K^n \longrightarrow 0$$

on X defined for each integer j by

$$K^j := \Omega^{j+1}_{X/T}[\partial_1] := \Omega^{j+1}_{X/T} \otimes_{q^{-1}(\mathcal{O}_T)} q^{-1}(\mathcal{O}_T)[\partial_1],$$

where $q^{-1}(\mathcal{O}_T)[\partial_1]$ is the polynomial ring over $q^{-1}(\mathcal{O}_T)$ in a variable ∂_1, and the $q^{-1}(\mathcal{O}_T)[\partial_1]$-linear map $\delta\colon K^j \to K^{j+1}$ sends $\omega\partial_1^\nu$ in K^j with $\omega \in \Omega^{j+1}_{X/T}$ and $\nu \geq 0$ to

$$\delta(\omega\partial_1^\nu) := (d\omega)\partial_1^\nu - (dF_1 \wedge \omega)\partial_1^{\nu+1},$$

where $F_1 := F_1(x, t')$. We define a left $\varphi^{-1}(\mathcal{D}_S)$-module structure on each K^j as follows so that δ becomes $\varphi^{-1}(\mathcal{D}_S)$-linear: Let ω be in $\Omega^{j+1}_{X/T}$ and let ν be a nonnegative integer.

(a) The subring $q^{-1}(\mathcal{O}_T)$ of $\varphi^{-1}(\mathcal{O}_S)$ acts through the usual $q^{-1}(\mathcal{O}_T)$-module structure on $\Omega^{j+1}_{X/T}$ and $q^{-1}(\mathcal{O}_T)[\partial_1]$, i.e.,

$$a(\omega\partial_1^\nu) := (a\omega)\partial_1^\nu \qquad \text{for } a \text{ in } q^{-1}(\mathcal{O}_T).$$

(b) The multiplication by the element t_1 in $\varphi^{-1}(\mathcal{O}_S)$ is defined by

$$t_1(\omega\partial_1^\nu) := (F_1\omega)\partial_1^\nu - \nu\omega\partial_1^{\nu-1}.$$

(c) The element $\partial/\partial t_1$ of $\varphi^{-1}(\Theta_S)$ acts as multiplication by ∂_1, i.e.,

$$(\partial/\partial t_1)(\omega\partial_1^\nu) := \omega\partial_1^{\nu+1}.$$

(d) For $2 \leq j \leq m$, the element $\partial/\partial t_j$ of $q^{-1}(\Theta_T) \subset \varphi^{-1}(\Theta_S)$ acts by

$$(\partial/\partial t_j)(\omega\partial_1^\nu) := (\partial\omega/\partial t_j)\partial_1^\nu - \{(\partial F_1/\partial t_j)\omega\}\partial_1^{\nu+1},$$

where $\partial\omega/\partial t_j$ is the partial derivation of the coefficients in $\mathcal{O}_X$ of ω with respect to the wedge products of $dx_0, \cdots, dx_n$.

Finally, we introduce an increasing filtration by $\varphi^{-1}(\mathcal{O}_S)$-submodules

$$\mathcal{F}^{(0)}K^\cdot \subset \mathcal{F}^{(1)}K^\cdot \subset \cdots \subset \mathcal{F}^{(k)}K^\cdot \subset \cdots\cdots$$

on $K^\cdot$ defined for each j and nonnegative integer k by

$$\mathcal{F}^{(k)}K^j := \sum\nolimits_{\nu \leq j+k-n} \Omega_{X/T}^{j+1} \cdot \partial_1^\nu.$$

It is straightforward to check that the above formulas indeed define a left $\varphi^{-1}(\mathcal{D}_S)$-module structure on K^j and that δ: $K^j \to K^{j+1}$ is $\varphi^{-1}(\mathcal{D}_S)$-linear satisfying

$$\delta(\mathcal{F}^{(k)}K^\cdot) \subset \mathcal{F}^{(k)}K^\cdot.$$

For instance, the actions of t_1 and $\partial/\partial t_1$ satisfy the relation $[\partial/\partial t_1, t_1] = 1$.

The above definition might look artificial and unnatural. As we see in Appendix, however, it is an explicit rendition of a complex arising naturally in connection with the integrations along the fibers $\int_\varphi \mathcal{O}_X$ and $\int_\varphi^{\mathrm{pr}} \mathcal{O}_X$ in algebraic analysis.

Remark. Since φ: $X \to S$ is a holomorphic map, we have a canonical homomorphism $\varphi^{-1}(\mathcal{O}_S) \to \mathcal{O}_X$. This induces a $\varphi^{-1}(\mathcal{O}_S)$-module structure on $\Omega_{X/T}^{j+1}$, hence on K^j. With respect to this latter structure, t_1 acts as multiplication by F_1, but we should forget this action. Instead, in our definition above, it is convenient to regard t_1 as acting on $\Omega_{X/T}^{j+1}$ as multiplication by F_1, while on $q^{-1}(\mathcal{O}_T)[\partial_1]$ as the derivation $-\partial/\partial(\partial_1)$. Moreover, if we introduce a new function

$$F(x, s) := t_1 - F_1(x, t')$$

as we do in later sections, the above (c) and (d) can be unified so that the action of $\theta \in \Theta_S$ becomes

$$\theta(\omega\partial_1^\nu) := (\theta\omega)\partial_1^\nu + (\theta F)\omega\partial_1^{\nu+1}.$$

Taking the n-th hyperdirect image with respect to φ, and that with proper support, of this complex $(K^\cdot, \delta)$, we get the Gauss-Manin systems Ξ and $\Xi_!$ on S, which are the main objects of our study. In this section, we indicate the steps necessary to give elementary proofs for the following theorems.

Theorem 1.1 (The Gauss-Manin system for φ).

(1) *For each integer j, we have a canonical isomorphism*

$$\boldsymbol{R}^j\varphi_*(K^{\cdot}, \delta)=\mathscr{H}^j(\varphi_*K^{\cdot}, \varphi_*\delta),$$

where the left hand side is the hyperdirect image, while the right hand side is the cohomology sheaf of the complex $(\varphi_*K^{\cdot}, \varphi_*\delta)$ *of left* $\mathscr{D}_S$*-modules obtained as the ordinary direct image of* $(K^{\cdot}, \delta)$.

(2) *We have*

$$\boldsymbol{R}^j\varphi_*(K^{\cdot}, \delta)=\begin{cases}\mathcal{O}_S & j=0\\ 0 & j\neq 0, n.\end{cases}$$

(3) *The n-th hyperdirect image*

$$\Xi:=\boldsymbol{R}^n\varphi_*(K^{\cdot}, \delta)=\mathscr{H}^n(\varphi_*K^{\cdot}, \varphi_*\delta)$$

is a left $\mathscr{D}_S$*-module and we have an exact sequence*

$$0\longrightarrow R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S))\longrightarrow\Xi\longrightarrow\varphi_*\mathscr{H}^n(K^{\cdot}, \delta)\longrightarrow 0.$$

The right hand term, having support on D, is the direct image of the n-th cohomology sheaf $\mathscr{H}^n(K^{\cdot}, \delta)$ *on X of the complex* $(K^{\cdot}, \delta)$*, while the left hand term is the n-th ordinary direct image of the sheaf* $\varphi^{-1}(\mathcal{O}_S)$ *so that its restriction to* $S\setminus D$ *is a locally free* $\mathcal{O}_{S\setminus D}$*-module*

$$R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S))|_{S\setminus D}=\{\mathcal{O}_S\otimes_{C_S}R^n\varphi_*\boldsymbol{C}_X\}|_{S\setminus D},$$

where $\boldsymbol{C}_X$ *(resp.* $\boldsymbol{C}_S$*) is the constant sheaf on X (resp. on S).*

(4) *We have an increasing filtration by* $\mathcal{O}_S$*-submodules*

$$\cdots\subset\Xi^{(-2)}\subset\Xi^{(-1)}\subset\Xi^{(0)}\subset\Xi^{(1)}\subset\cdots$$

on Ξ *defined for each nonnegative integer k by*

$$\Xi^{(k)}:=\mathrm{Image}\,[\boldsymbol{R}^n\varphi_*(\mathscr{F}^{(k)}K^{\cdot}, \delta)\longrightarrow\boldsymbol{R}^n\varphi_*(K^{\cdot}, \delta)]$$
$$\Xi^{(-k)}:=\{\xi\in\Xi^{(0)};\ \partial_1^k\xi\in\Xi^{(0)}\}$$

satisfying the following:

(i) $\Xi^{(k)}$ *for each integer k is* $\mathcal{O}_S$*-coherent with* $\bigcup_k\Xi^{(k)}=\Xi$. *Moreover,* $\Xi^{(k)}$ *is* $\mathcal{O}_S$*-locally free of rank equal to the Milnor number* μ *for each* $k\leq 0$.

(ii) *The multiplication by* ∂_1 *is surjective on* Ξ *and* $\Xi^{(k)}=\{\xi\in\Xi;\ \partial_1\xi\in\Xi^{(k+1)}\}$ *for each k.*

(iii) *For each k,*

$$\Xi^{(k)}/\Xi^{(k-1)}\cong\varphi_*\Omega^{n+1}_{X/S}$$

has support on D, where $\Omega^{n+1}_{X/S}=\Omega^{n+1}_{X/T}/(dF_1\wedge\Omega^n_{X/T})$ *is the sheaf on X of relative* $(n+1)$*-forms for* X/S*, and is* $\mathcal{O}_C$*-invertible.*

(iv) *We have a canonical* $\mathcal{O}_S$*-isomorphism*

$$\Xi^{(0)}=\varphi_*\Omega^{n+1}_{X/T}/(dF_1\wedge d\varphi_*\Omega^{n-1}_{X/T}),$$

the so-called Brieskorn lattice. Moreover, the wedge product with dF_1 *induces* $\mathcal{O}_S$*-isomorphisms*

$$dF_1\wedge\quad:\varphi_*\Omega^n_{X/T}/d\varphi_*\Omega^{n-1}_{X/T}\xrightarrow{\ \sim\ }\Xi^{(-1)}$$
$$dF_1\wedge\quad:\boldsymbol{R}^n\varphi_*(\Omega^{\bullet}_{X/S},d_{X/S})\xrightarrow{\ \sim\ }\Xi^{(-2)}.$$

Remark. As we see in Appendix, we have

$$\boldsymbol{R}\varphi_*(K^{\bullet},\delta)=\boldsymbol{R}\varphi_*(\mathscr{D}_{S\leftarrow X}\otimes^{L}_{\mathscr{D}_X}\mathcal{O}_X)[-n]=:\int_{\varphi}\mathcal{O}_X,$$

the integration along the fibers but with the degree shifted from the usual one. The same remark applies to Theorem 1.2 below, where we have

$$\boldsymbol{R}\varphi_!(K^{\bullet},\delta)=\boldsymbol{R}\varphi_!(\mathscr{D}_{S\leftarrow X}\otimes^{L}_{\mathscr{D}_X}\mathcal{O}_X)[-n]=:\int_{\varphi}^{\mathrm{pr}}\mathcal{O}_X,$$

the integration with proper support along the fibers.

For $k\leq 0$, our $\Xi^{(k)}$ coincides with $\mathscr{H}^{(k)}_F$ appearing in Saito's papers. The filtration we obtain gives information equivalent to the decreasing *Hodge filtration* $\{\Phi^l\Xi;\ l\in\boldsymbol{Z}\}$ with

$$\Phi^l\Xi:=\Xi^{(n-l)}\qquad\text{for } l\in\boldsymbol{Z}$$

which appears in a more recent theory due to Brylinski [B2], [B3], M. Saito [Sm3], [Sm4], Kashiwara and others, which deals with the derived category of filtered complexes of holonomic $\mathscr{D}_S$-modules with regular singularities.

Theorem 1.2 (The Gauss-Manin system with proper support).

(1) *For each integer j, we have a canonical isomorphism*

$$\boldsymbol{R}^j\varphi_!(K^{\bullet},\delta)=\mathscr{H}^{j-n}(R^n\varphi_!K^{\bullet},R^n\varphi_!\delta),$$

where the left hand side is the hyperdirect image with proper support, while *the right hand side is the cohomology sheaf of the complex of left $\mathscr{D}_S$-modules obtained as the n-th ordinary direct image with proper support of the complex $(K^\cdot, \delta)$.*

(2) *We have*

$$\boldsymbol{R}^j\varphi_!(K^\cdot, \delta) = \begin{cases} 0 & j \neq n, 2n \\ \mathcal{O}_S & j = 2n. \end{cases}$$

(3) *The n-th hyperdirect image with proper support*

$$\Xi_! := \boldsymbol{R}^n\varphi_!(K^\cdot, \delta) = \mathscr{H}^0(R^n\varphi_! K^\cdot, R^n\varphi_!\delta)$$

is a left $\mathscr{D}_S$-module and we have an exact sequence

$$0 \longrightarrow R^n\varphi_!(\varphi^{-1}(\mathcal{O}_S)) \longrightarrow \Xi_! \longrightarrow \varphi_*\mathscr{H}^n(K^\cdot, \delta) \longrightarrow 0.$$

The right hand term has support on D, while the left hand term is the n-th ordinary direct image with proper support of the sheaf $\varphi^{-1}(\mathcal{O}_S)$ so that its restriction to $S \setminus D$ is a locally free $\mathcal{O}_{S\setminus D}$-module

$$R^n\varphi_!(\varphi^{-1}(\mathcal{O}_S))|_{S\setminus D} = \{\mathcal{O}_S \otimes_{\boldsymbol{C}_S} R^n\varphi_! \boldsymbol{C}_X\}|_{S\setminus D}.$$

(4) *We have an increasing filtration by $\mathcal{O}_S$-submodules*

$$\cdots \subset \Xi_!^{(n-1)} \subset \Xi_!^{(n)} \subset \Xi_!^{(n+1)} \subset \cdots$$

on $\Xi_!$ defined for each nonnegative integer k by

$$\Xi_!^{(n+k)} := \text{Image}\,[\boldsymbol{R}^n\varphi_!(\mathscr{F}^{(n+k)}K^\cdot, \delta) \longrightarrow \boldsymbol{R}^n\varphi_!(K^\cdot, \delta)]$$
$$\Xi_!^{(n-k)} := \{\xi \in \Xi_!^{(n)};\ \partial_1^k\xi \in \Xi_!^{(n)}\}$$

satisfying the following:

(i) *$\Xi_!^{(k)}$ for each integer k is $\mathcal{O}_S$-coherent with $\bigcup_k \Xi_!^{(k)} = \Xi_!$ (cf. Corollary 2.2).*

(ii) *The multiplication by ∂_1 is surjective on $\Xi_!$ and $\Xi_!^{(k)} = \{\xi \in \Xi_!;\ \partial_1\xi \in \Xi_!^{(k+1)}\}$ for each k.*

(iii) *For each k, we have an $\mathcal{O}_S$-isomorphism*

$$\Xi_!^{(k)}/\Xi_!^{(k-1)} \cong \varphi_*\Omega^{n+1}_{X/S}.$$

(iv) *We have a canonical $\mathcal{O}_S$-isomorphism*

$$\Xi_!^{(n)} = \{\eta \in dR^n\varphi_!\mathcal{O}_X;\ dF_1 \wedge \eta = 0\}.$$

We now give a sketch of elementary proofs for these theorems.

The hyperdirect image is the common abutment of two well-known spectral sequences

$$\mathrm{I}_1^{j,k} := R^k\varphi_* K^j \Longrightarrow \boldsymbol{R}^{j+k}\varphi_*(K^{\bullet}, \delta)$$
$$\mathrm{II}_2^{j,k} := R^j\varphi_*(\mathscr{H}^k(K^{\bullet}, \delta)) \Longrightarrow \boldsymbol{R}^{j+k}\varphi_*(K^{\bullet}, \delta).$$

By our assumption, φ is a Stein morphism. Thus

$$R^k\varphi_* M = 0 \qquad \text{for } k \neq 0 \text{ and an } \mathcal{O}_X\text{-coherent module } M$$

(see, for instance, [BS]). However, $K^j = \Omega^{j+1}_{X/T}[\partial_1]$ is an *infinite* direct sum of locally free $\mathcal{O}_X$-modules $\Omega^{j+1}_{X/T} \cdot \partial_1^{\nu}$ of finite rank and is not $\mathcal{O}_X$-coherent. As M. Saito pointed out (cf. Scherk-Steenbrink [SS, § 3]), $R^k\varphi_*$ does *not* commute with inductive limits so that we cannot immediately conclude $\mathrm{I}_1^{j,k} = R^k\varphi_* K^j = 0$ for $k \neq 0$ and $\mathrm{I}_2^{j,0} = \boldsymbol{R}^j\varphi_*(K^{\bullet}, \delta)$. Thus Theorem 1.1, (1) is not so trivial. We come back to its proof after we prove most of the other statements of the theorem.

Similarly, the hyperdirect image with proper support is the common abutment of two spectral sequences

$$\mathrm{I}_1^{j,k} := R^k\varphi_! K^j \Longrightarrow \boldsymbol{R}^{j+k}\varphi_!(K^{\bullet}, \delta)$$
$$\mathrm{II}_2^{j,k} := R^j\varphi_!(\mathscr{H}^k(K^{\bullet}, \delta)) \Longrightarrow \boldsymbol{R}^{j+k}\varphi_!(K^{\bullet}, \delta).$$

Since φ is a Stein morphism of relative dimension n, we know that

$$R^k\varphi_! M = 0 \qquad \text{for } k \neq n,$$

if M is $\mathcal{O}_X$-locally free of finite rank by Ramis-Ruget [RR, Théorème 1], Fujiki [F] and Ohsawa [O2], [O3]. Since K^j is an *infinite* direct sum of locally free $\mathcal{O}_X$-modules of finite rank, we cannot immediately conclude that $\mathrm{I}_1^{j,k} = R^k\varphi_! K^j = 0$ for $k \neq n$ and $\mathrm{I}_2^{j,n} = \boldsymbol{R}^{j+n}\varphi_!(K^{\bullet}, \delta)$, either. Again we come back to this later.

As for the second spectral sequences, we need more information on the cohomology sheaves $\mathscr{H}^k(K^{\bullet}, \delta)$ on X.

Since $q: X \to T$ is a smooth morphism of relative dimension $n+1$, the following is well-known:

Lemma 1.3 (The Poincaré lemma). *The relative de Rham complex $(\Omega^{\bullet}_{X/T}, d)$ with $q^{-1}(\mathcal{O}_T)$-linear $d = d_{X/T}$ satisfies*

$$\mathscr{H}^j(\Omega^{\bullet}_{X/T}, d) = \begin{cases} q^{-1}(\mathcal{O}_T) & j = 0 \\ 0 & j \neq 0. \end{cases}$$

On the other hand, the wedge product

$$dF_1 \wedge \quad : \Omega^j_{X/T} \longrightarrow \Omega^{j+1}_{X/T}$$

with dF_1 is $\mathcal{O}_X$-linear and obviously gives rise to a complex $(\Omega^{\cdot}_{X/T}, dF_1 \wedge)$. Then we have the following:

Lemma 1.4 (The generalized de Rham lemma. cf. Saito [Sk3]).

$$\mathcal{H}^j(\Omega^{\cdot}_{X/T}, dF_1 \wedge) = \begin{cases} 0 & j \neq n+1 \\ \Omega^{n+1}_{X/S} & j = n+1. \end{cases}$$

Since $\Omega^j_{X/S} = \Omega^j_{X/T}/(dF_1 \wedge \Omega^{j-1}_{X/T})$ for each j, we see by Lemma 1.4 above that the wedge product with $(-1)^j dF_1$ induces an isomorphism

$$(-1)^j dF_1 \wedge \quad : \Omega^j_{X/S} \xrightarrow{\sim} dF_1 \wedge \Omega^j_{X/T} \subset \Omega^{j+1}_{X/T} \subset \mathcal{F}^{(n)} K^j \subset K^j$$

for $0 \leq j \leq n$. Since $\delta((-1)^j dF_1 \wedge \omega) = (-1)^{j+1} dF_1 \wedge d\omega$ for ω in $\Omega^j_{X/T}$, we thus have a homomorphism of complexes

$$\rho\colon (\Omega^{\cdot}_{X/S}, d_{X/S}) \longrightarrow (\mathcal{F}^{(n)} K^{\cdot}, \delta) \subset (K^{\cdot}, \delta).$$

The cohomology sheaves $\mathcal{H}^j(\Omega^{\cdot}_{X/S}, d_{X/S})$ are left $\varphi^{-1}(\mathcal{D}_S)$-modules for which $t_1 \in \varphi^{-1}(\mathcal{O}_S)$ acts as multiplication by F_1, $q^{-1}(\Theta_T)$ acts through the differentiation on the coefficients of differential forms and $\partial_1 = \partial/\partial t_1 \in \varphi^{-1}(\Theta_S)$ acts as follows: If the image $\bar{\xi}$ in $\Omega^j_{X/S}$ of $\xi \in \Omega^j_{X/T}$ satisfies $d_{X/S}\bar{\xi} = 0$, so that $d\xi = dF_1 \wedge \eta$ for $\eta \in \Omega^j_{X/T}$, then $\partial_1 = \partial/\partial t_1$ sends $\bar{\xi}$ to the image $\bar{\eta}$ of η in $\mathcal{H}^j(\Omega^{\cdot}_{X/S}, d_{X/S})$. The action thus defined is the usual Gauss-Manin connection. The following can be found in Pham [P, p. 159f] and is an easy consequence of Lemmas 1.3 and 1.4:

Lemma 1.5. (i) *The above homomorphism of complexes induces a $\varphi^{-1}(\mathcal{D}_S)$-linear homomorphism*

$$\rho_*\colon \mathcal{H}^j(\Omega^{\cdot}_{X/S}, d_{X/S}) \longrightarrow \mathcal{H}^j(K^{\cdot}, \delta),$$

which is bijective for $j \neq n$ and is injective for $j = n$ with the image equal to the $\varphi^{-1}(\mathcal{O}_S)$-submodule

$$\mathcal{F}^{(-2)}\mathcal{H}^n(K^{\cdot}, \delta) := dF_1 \wedge \{\eta \in \Omega^n_{X/T};\, d\eta \in dF_1 \wedge \Omega^n_{X/T}\}/(dF_1 \wedge d\Omega^{n-1}_{X/T})$$

of the $\varphi^{-1}(\mathcal{O}_S)$-submodule

$$\mathcal{F}^{(-1)}\mathcal{H}^n(K^{\cdot}, \delta) := (dF_1 \wedge \Omega^n_{X/T})/(dF_1 \wedge d\Omega^{n-1}_{X/T})$$

of $\mathcal{F}^{(0)}\mathcal{H}^n(K^{\cdot}, \delta) := \Omega^{n+1}_{X/T}/(dF_1 \wedge d\Omega^{n-1}_{X/T})$. More generally for each non-negative integer k, the $\varphi^{-1}(\mathcal{O}_S)$-linear map $\mathcal{H}^n(\mathcal{F}^{(k)}K^{\cdot}, \delta) \to \mathcal{H}^n(K^{\cdot}, \delta)$ is injective and we define $\mathcal{F}^{(k)}\mathcal{H}^n(K^{\cdot}, \delta)$ to be its image.

(ii) *For* $j \neq 0$, *the supports of* $\mathcal{H}^j(\Omega^{\cdot}_{X/S}, d_{X/S})$ *and* $\mathcal{H}^j(K^{\cdot}, \delta)$ *are contained in the critical locus* C.

(iii) *The multiplication by* ∂_1 *is surjective for* $\mathcal{H}^0(k^{\cdot}, \delta)$ *and is bijective for* $\mathcal{H}^j(K^{\cdot}, \delta)$ *with* $j \neq 0$.

The following is known:

Proposition 1.6 (Brieskorn [B1], Sebastiani [S1], Greuel [G], Malgrange [M1] and Saito [Sk1]). *We have*

$$\boldsymbol{R}^j\varphi_*(\Omega^{\cdot}_{X/S}, d_{X/S}) = \begin{cases} \mathcal{O}_S & j=0 \\ 0 & j\neq 0, n \end{cases}$$

and an exact sequence $0 \to R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S)) \to \boldsymbol{R}^n\varphi_*(\Omega^{\cdot}_{X/S}, d_{X/S}) \to \varphi_*\mathcal{H}^n(\Omega^{\cdot}_{X/S}, d_{X/S}) \to 0$ *with* $\boldsymbol{R}^n\varphi_*(\Omega^{\cdot}_{X/S}, d_{X/S})$ *being* $\mathcal{O}_S$*-locally free of rank equal to the Milnor number* μ. *Moreover,*

$$\mathcal{H}^j(\Omega^{\cdot}_{X/S}, d_{X/S}) = \begin{cases} \varphi^{-1}(\mathcal{O}_S) & j=0 \\ 0 & j\neq 0, n \end{cases}$$

and the support of $\mathcal{H}^n(\Omega^{\cdot}_{X/S}, d_{X/S})$ *is contained in the critical locus* C.

As we have just seen in Lemma 1.5, $\mathcal{H}^n(K^{\cdot}, \delta)$ has an increasing sequence of $\varphi^{-1}(\mathcal{O}_S)$-submodules $\mathcal{F}^{(k)}\mathcal{H}^n(K^{\cdot}, \delta)$ for $k \geq -2$. Since the multiplication by ∂_1 is bijective on $\mathcal{H}^n(K^{\cdot}, \delta)$, we easily see that

$$\mathcal{F}^{(k)}\mathcal{H}^n(K^{\cdot}, \delta) = \partial_1^k \mathcal{F}^{(0)}\mathcal{H}^n(K^{\cdot}, \delta)$$

for $k \geq -2$. We can thus define $\mathcal{F}^{(k)}\mathcal{H}^n(K^{\cdot}, \delta)$ for $k \leq -3$ as well by this formula and get an increasing filtration $\{\mathcal{F}^{(k)}\mathcal{H}^n(K^{\cdot}, \delta) : k \in \boldsymbol{Z}\}$ of $\mathcal{H}^n(K^{\cdot}, \delta)$ by $\varphi^{-1}(\mathcal{O}_S)$-submodules. Using Lemmas 1.3 and 1.4, we can easily show that

$$\mathcal{F}^{(k)}\mathcal{H}^{(n)}(K^{\cdot}, \delta)/\mathcal{F}^{(k-1)}\mathcal{H}^n(K^{\cdot}, \delta) \cong \Omega^{n+1}_{X/S}$$

holds also for $k \leq -2$, hence for all integers k.

In view of this, we get the following, using spectral sequences, our assumption at the beginning, Lemma 1.5, Proposition 1.6 and the result $(R^j\varphi_* C_X)|_{S \setminus D} = 0$ for $j \neq 0$, n due to Milnor [M3]:

Corollary 1.7. *We have*

$$\boldsymbol{R}^j\varphi_*(K^{\cdot}, \delta) = \begin{cases} \mathcal{O}_S & j=0 \\ 0 & j\neq 0, n \end{cases}$$

and an exact sequence $0 \to R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S)) \to \Xi \to \varphi_*\mathcal{H}^n(K^\cdot, \delta) \to 0$ *with* $\Xi^{(k)}$ *being* $\mathcal{O}_S$*-locally free of rank equal to the Milnor number* μ *for* $k \leq 0$. *Moreover,*

$$\mathcal{H}^j(K^\cdot, \delta) = \begin{cases} \varphi^{-1}(\mathcal{O}_S) & j=0 \\ 0 & j \neq 0, n. \end{cases}$$

The support of $\mathcal{H}^n(K^\cdot, \delta)$ *is contained in the critical locus* C. *The multiplication by* ∂_1 *on* $\mathcal{H}^n(K^\cdot, \delta)$ *is bijective and* $\mathcal{H}^n(K^\cdot, \delta)$ *has an increasing filtration* $\{\mathcal{F}^{(k)}\mathcal{H}^n(K^\cdot, \delta);\ k \in \boldsymbol{Z}\}$ *by* $\varphi^{-1}(\mathcal{O}_S)$*-submodules such that*

$$\mathcal{F}^{(k)}\mathcal{H}^n(K^\cdot, \delta) = \partial_1^k \mathcal{F}^{(0)}\mathcal{H}^n(K^\cdot, \delta)$$

with

$$\mathcal{F}^{(0)}\mathcal{H}^n(K^\cdot, \delta) = \Omega_{X/T}^{n+1}/(dF_1 \wedge d\Omega_{X/T}^{n-1}),$$
$$\mathcal{F}^{(k)}\mathcal{H}^n(K^\cdot, \delta)/\mathcal{F}^{(k-1)}\mathcal{H}^n(K^\cdot, \delta) \cong \Omega_{X/S}^{n+1} \qquad \text{for } k \in \boldsymbol{Z}$$

and the wedge product with $(-1)^n dF_1$ *induces an isomorphism*

$$\mathcal{H}^n(\Omega_{X/S}^\cdot, d_{X/S}) \xrightarrow{\sim} \mathcal{F}^{(-2)}\mathcal{H}^n(K^\cdot, \delta).$$

Remark. By Lemma 1.4, results similar to those in Corollary 1.7 hold if we replace $(K^\cdot, \delta)$ by its subcomplex $(\mathcal{F}^{(l)}K^\cdot, \delta)$ for $l \geq n$. Namely,

$$\boldsymbol{R}^j\varphi_*(\mathcal{F}^{(l)}K^\cdot, \delta) = \begin{cases} \mathcal{O}_S & j=0 \\ 0 & j \neq 0, n, \end{cases}$$

$$\mathcal{H}^j(\mathcal{F}^{(l)}K^\cdot, \delta) = \begin{cases} \varphi^{-1}(\mathcal{O}_S) & j=0 \\ 0 & j \neq 0, n \end{cases}$$

and we get an exact sequence $0 \to R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S)) \to \boldsymbol{R}^n\varphi_*(\mathcal{F}^{(l)}K^\cdot, \delta) \to \varphi_*\mathcal{H}^n(\mathcal{F}^{(l)}K^\cdot, \delta) \to 0$. Moreover, since $\mathcal{F}^{(l)}K^j$ is $\mathcal{O}_X$-coherent for all j this time, we have $\mathrm{I}_1^{j,k} = R^k\varphi_*(\mathcal{F}^{(l)}K^j) = 0$ for $k \neq 0$, hence $\mathrm{I}_2^{j,0} = \boldsymbol{R}^j(\mathcal{F}^{(l)}K^\cdot, \delta)$.

In view of the remark above, Lemma 1.5 and Corollary 1.7, we have an exact sequence

$$0 \longrightarrow R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S)) \longrightarrow \Xi^{(l)} \longrightarrow \varphi_*(\mathcal{F}^{(l)}\mathcal{H}^n(K^\cdot, \delta)) \longrightarrow 0$$

for $l \geq n$, hence for all l, since the multiplication by ∂_1 on $R^n\varphi_*(\varphi^{-1}(\mathcal{O}_S))$ is surjective. Thus

$$\Xi^{(l)}/\Xi^{(l-1)} = \varphi_*\Omega_{X/S}^{n+1} \quad \text{and} \quad \partial_1\Xi^{(l)} = \Xi^{(l+1)}$$

for all l.

Similarly, for the spectral sequence

$$\mathrm{II}_2^{j,k} = R^j\varphi_!\mathscr{H}^k(K^\cdot, \delta) \Longrightarrow \boldsymbol{R}^{j+k}\varphi_!(K^\cdot, \delta)$$

we have $\mathrm{II}_2^{j,k}=0$ if $k\neq 0$, n and j arbitrary; $\mathrm{II}_2^{j,n}=0$ for $j\neq 0$; and $\mathrm{II}_2^{0,n}=\varphi_*\mathscr{H}^n(K^\cdot, \delta)$. Moreover, $\mathrm{II}_2^{j,0}=R^j\varphi_!(\varphi^{-1}(\mathcal{O}_S))$ vanishes for $j\neq n$, $2n$ by Milnor [M3] and by our assumption on φ. Since $n\geq 2$ by assumption, we get Theorem 1.2, (2) and an exact sequence

$$0 \longrightarrow \mathrm{II}_2^{n,0} \longrightarrow \varXI_! \longrightarrow \mathrm{II}_2^{0,n} \longrightarrow 0.$$

As before, we more generally have an exact sequence

$$0 \longrightarrow \mathrm{II}_2^{n,0} \longrightarrow \varXI_!^{(l)} \longrightarrow \varphi_*(\mathscr{F}^{(l)}\mathscr{H}^n(K^\cdot, \delta)) \longrightarrow 0$$

so that

$$\varXI_!^{(l)}/\varXI_!^{(l-1)} = \varphi_*\Omega_{X/S}^{n+1} \qquad \text{and} \qquad \partial_1\varXI_!^{(l)} = \varXI_!^{(l+1)}$$

for all l. The $\mathcal{O}_S$-coherence of $\varXI_!^{(l)}$ for all l will be shown in Corollary 2.2.

The proofs of Theorem 1.1, (1) and Theorem 1.2, (2) due to Scherk-Steenbrink [SS] proceed as follows:

The quotient complex $\mathscr{Q}^\cdot := K^\cdot/\mathscr{F}^{(n)}K^\cdot$ has a filtration by subcomplexes $\mathscr{F}^{(l)}\mathscr{Q}^\cdot := \mathscr{F}^{(l)}K^\cdot/\mathscr{F}^{(n)}K^\cdot$ for $l\geq n$. By Lemma 1.5, (i), Corollary 1.7 and the remark after it, we see that $\mathscr{H}^j(\mathscr{Q}^\cdot)=0$ and $\mathscr{H}^j(\mathscr{F}^{(l)}\mathscr{Q}^\cdot)=0$ for $j\neq n$. Moreover, $\mathscr{H}^n(\mathscr{Q}^\cdot)$ and $\mathscr{H}^n(\mathscr{F}^{(l)}\mathscr{Q}^\cdot)$ have support on C.

Using the second spectral sequences, we thus see that $\boldsymbol{R}^j\varphi_*(\mathscr{Q}^\cdot) = \varphi_*\mathscr{H}^j(\mathscr{Q}^\cdot)$ and $\boldsymbol{R}^j\varphi_!(\mathscr{Q}^\cdot)=\varphi_*\mathscr{H}^j(\mathscr{Q}^\cdot)$ vanish for $j\neq n$ and have support on D for $j=n$. The same result holds when we replace $\mathscr{Q}^\cdot$ by $\mathscr{F}^{(l)}\mathscr{Q}^\cdot$.

Since φ is a Stein morphism and $\mathscr{F}^{(n)}K^j$ is $\mathcal{O}_X$-coherent, we have $R^1\varphi_*\mathscr{F}^{(n)}K^j=0$ and hence an exact sequence of complexes

$$0 \longrightarrow \varphi_*\mathscr{F}^{(n)}K^\cdot \longrightarrow \varphi_*K^\cdot \longrightarrow \varphi_*\mathscr{Q}^\cdot \longrightarrow 0.$$

We have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{H}^j(\varphi_*\mathscr{F}^{(n)}K^\cdot) \arrow[r] \arrow[d, "\alpha_j"] & \mathscr{H}^j(\varphi_*K^\cdot) \arrow[r] \arrow[d, "\beta_j"] & \mathscr{H}^j(\varphi_*\mathscr{Q}^\cdot) \arrow[r] \arrow[d, "\gamma_j"] & \cdots \\
\cdots \arrow[r] & \boldsymbol{R}^j\varphi_*(\mathscr{F}^{(n)}K^\cdot) \arrow[r] & \boldsymbol{R}^j\varphi_*(K^\cdot) \arrow[r] & \boldsymbol{R}^j\varphi_*(\mathscr{Q}^\cdot) \arrow[r] & \cdots,
\end{tikzcd}$$

where the first (resp. second) row is the long exact sequence of cohomology sheaves (resp. hyperdirect images) and the arrows α_j, β_j and γ_j are the edge homomorphisms for the first spectral sequences.

As we pointed out in the remark after Corollary 1.7, we see that α_j's are isomorphisms. To show β_j's to be isomorphisms, it thus suffices to show that so are γ_j's. We have

$$\varphi_*\mathcal{Q}^\cdot=\varphi_*(\text{ind lim } \mathcal{F}^{(l)}\mathcal{Q}^\cdot)=\text{ind lim } \varphi_*(\mathcal{F}^{(l)}\mathcal{Q}^\cdot),$$

since $\mathcal{F}^{(l)}\mathcal{Q}^j$ is $\mathcal{O}_X$-locally free. Hence $\mathcal{H}^j(\varphi_*\mathcal{Q}^\cdot)=\text{ind lim } \mathcal{H}^j(\varphi_*\mathcal{F}^{(l)}\mathcal{Q}^\cdot)$. Moreover,

$$\boldsymbol{R}^j\varphi_*(\mathcal{Q}^\cdot)=\varphi_*\mathcal{H}^j(\mathcal{Q}^\cdot)=\varphi_*(\text{ind lim } \mathcal{H}^j(\mathcal{F}^{(l)}\mathcal{Q}^\cdot))=\text{ind lim } \varphi_*\mathcal{H}^j(\mathcal{F}^{(l)}\mathcal{Q}^\cdot),$$

since the restriction of φ to C is proper over S. We are done, since $\mathcal{F}^{(l)}\mathcal{Q}^j$ is $\mathcal{O}_X$-coherent, hence

$$\mathcal{H}^j(\varphi_*\mathcal{F}^{(l)}\mathcal{Q}^\cdot)=\boldsymbol{R}^j\varphi_*(\mathcal{F}^{(l)}\mathcal{Q}^\cdot)=\varphi_*\mathcal{H}^j(\mathcal{F}^{(l)}\mathcal{Q}^\cdot).$$

Thus we have Theorem 1.1, (1).

Since $\mathcal{F}^{(n)}K^j$ and $\mathcal{F}^{(l)}\mathcal{Q}^j$ are $\mathcal{O}_X$-locally free of finite rank and since φ is a Stein morphism of relative dimension n, we have $R^k\varphi_!(\mathcal{F}^{(n)}K^j)=0$ and $R^k\varphi_!(\mathcal{F}^{(l)}\mathcal{Q}^j)=0$ for $k\neq n$. Thus the same argument as above shows Theorem 1.2, (1).

Finally, to conclude the proof of Theorems 1.1 and 1.2, we need to show (4), (iv). We analyze Ξ and $\Xi_!$ through the first spectral sequences in greater detail than necessary here, since we need the result in the next section, where we sketch the proof of the fact that Ξ and $\Xi_!$ are holonomic left $\mathcal{D}_S$-modules with regular singularities and prove that the Poincaré duality holds.

We need the following consequences of Lemmas 1.3 and 1.4, which we can show again using spectral sequences, our assumption at the beginning and Milnor's result in [M3]:

Lemma 1.8. (i) $\mathcal{H}^j(\varphi_*\Omega^\cdot_{X/T}, \varphi_*d)=0$ *for* $j\neq 0, n$.

(ii) $\mathcal{H}^j(\varphi_*\Omega^\cdot_{X/T}, \varphi_*(dF_1\wedge))=\begin{cases}0 & \text{for } j\neq n+1,\\ \varphi_*\Omega^{n+1}_{X/S} & \text{for } j=n+1.\end{cases}$

(iii) $\mathcal{H}^j(R^n\varphi_!\Omega^\cdot_{X/T}, R^n\varphi_!d)=0$ *for* $j\neq 0, n$.

(iv) $\mathcal{H}^j(R^n\varphi_!\Omega^\cdot_{X/T}, R^n\varphi_!(dF_1\wedge))=\begin{cases}\varphi_*\Omega^{n+1}_{X/S} & \text{for } j=1,\\ 0 & \text{for } j\neq 1.\end{cases}$

We have already seen that

$$\mathcal{H}^j(\varphi_*K^\cdot, \varphi_*\delta)=\boldsymbol{R}^j\varphi_*(K^\cdot, \delta)=\begin{cases}\mathcal{O}_S & j=0,\\ 0 & j\neq 0, n,\\ \Xi & j=n,\end{cases}$$

$$\mathscr{H}^{j-n}(R^n\varphi_! K^\bullet, R^n\varphi_!\delta)=\boldsymbol{R}^j\varphi_!(K^\bullet,\delta)=\begin{cases}\Xi_1 & j=n,\\ 0 & j\neq n,\ 2n,\\ \mathcal{O}_S & j=2n.\end{cases}$$

In fact by Lemma 1.8, (ii), we easily see that the wedge product with dF_1 induces an $\mathcal{O}_S$-isomorphism

$$\begin{aligned}\mathcal{O}_S&=\{y\in\varphi_*\mathcal{O}_X;\ dF_1\wedge dy=0\}\\ &\xrightarrow{\sim}\{\eta\in\varphi_*\Omega^1_{X/T};\ d\eta=0,\ dF_1\wedge\eta=0\}=\mathscr{H}^0(\varphi_*K^\bullet,\varphi_*\delta).\end{aligned}$$

Similarly by Lemma 1.8, (iv), we see that

$$\mathscr{H}^n(R^n\varphi_! K^\bullet, R^n\varphi_!\delta)=R^n\varphi_!\Omega^{n+1}_{X/T}/(dF_1\wedge dR^n\varphi_!\Omega^{n-1}_{X/T}),$$

which is $\mathcal{O}_S$-isomorphic to $\boldsymbol{R}^{2n}\varphi_!(K^\bullet,\delta)=\mathcal{O}_S$.

By Lemma 1.8, (i), (ii), we know that φ_*d: $I:=\varphi_*\Omega^n_{X/T}\to\varphi_*\Omega^{n+1}_{X/T}$ is surjective and that the kernel of $\varphi_*(dF_1\wedge)$: $I\to\varphi_*\Omega^{n+1}_{X/T}$ coincides with $dF_1\wedge\varphi_*\Omega^{n-1}_{X/T}$. For simplicity, let us denote φ_*d and $\varphi_*(dF_1\wedge)$ by d and $dF_1\wedge$, respectively. We then have the following easy result for Ξ:

Proposition 1.9. (i) *$I:=\varphi_*\Omega^n_{X/T}$ has the following sequence of $\pi^{-1}(\mathcal{O}_T)$-submodules*

$$I=:I^{(0)}\supset I^{(-1)}\supset I^{(-2)}\supset\cdots\supset I_2\supset I_1\supset I_0\supset I'$$

defined inductively by

$$\begin{aligned}&I^{(-k)}:=d^{-1}(dF_1\wedge I^{(-k+1)}) \qquad \text{for } k\geq 1,\\ &I':=(dF_1\wedge)^{-1}(dF_1\wedge d\varphi_*\Omega^{n-1}_{X/T})=d\varphi_*\Omega^{n-1}_{X/T}+dF_1\wedge\varphi_*\Omega^{n-1}_{X/T},\\ &I_0:=d^{-1}(dF_1\wedge d\varphi_*\Omega^{n-1}_{X/T})=(\ker d)+dF_1\wedge\varphi_*\Omega^{n-1}_{X/T},\\ &I_k:=d^{-1}(dF_1\wedge I_{k-1}) \qquad \text{for } k\geq 1.\end{aligned}$$

I' and $I^{(-k)}$ are $\mathcal{O}_S$-submodules of I. Moreover, $dI^{(0)}=dI=\varphi_*\Omega^{n+1}_{X/T}$, $dI_0=dF_1\wedge I'=dI'=dF_1\wedge d\varphi_*\Omega^{n-1}_{X/T}$ and

$$dI^{(-k)}=dF_1\wedge I^{(-k+1)},\qquad dI_k=dF_1\wedge I_{k-1}\qquad \text{for } k\geq 1$$

are all $\mathcal{O}_S$-submodules of $\varphi_*\Omega^{n+1}_{X/T}$.

(ii) *For $k\geq 0$, the $\pi^{-1}(\mathcal{O}_T)$-homomorphism $\partial_1^k d$: $I\to\varphi_*\Omega^{n+1}_{X/T}\cdot\partial_1^k\subset\varphi_*K^n$ induces*

$$(dI)\partial_1^k/(dI_k)\partial_1^k=\Xi^{(k)} \qquad \text{for } k\geq 0.$$

(iii) *The wedge product with dF_1 induces an $\mathcal{O}_S$-isomorphism*

$$dF_1\wedge \quad : I^{(-k+1)}/I' \xrightarrow{\sim} dI^{(-k)}/dI_0 = \Xi^{(-k)} \qquad \text{for } k\geq 1.$$

As for $\Xi_!$, we have the following: By Lemma 1.8, (iii), (iv), we know that $R^n\varphi_!(dF_1\wedge)$: $R^n\varphi_!\mathcal{O}_X \to R^n\varphi_!\Omega^1_{X/T}$ is injective and that the kernel of $R^n\varphi_!d$: $R^n\varphi_!\Omega^1_{X/T} \to R^n\varphi_!\Omega^2_{X/T}$ coincides with $dR^n\varphi_!\mathcal{O}_X$, since $n\geq 2$. We again denote $R^n\varphi_!d$ and $R^n\varphi_!(dF_1\wedge)$ simply by d and $dF_1\wedge$, respectively.

Proposition 1.10. (i) *Define the $\mathcal{O}_S$-submodule J of $R^n\varphi_!\mathcal{O}_X$ by*

$$J := \{\beta \in R^n\varphi_!\mathcal{O}_X;\ dF_1\wedge d\beta = 0\} = d^{-1}((dR^n\varphi_!\mathcal{O}_X)\cap(\ker(dF_1\wedge))).$$

We then have a surjection d: $J\to dJ = (dR^n\varphi_!\mathcal{O}_X)\cap(\ker(dF_1\wedge))$ and a bijection $dF_1\wedge \ : J \xrightarrow{\sim} dF_1\wedge J \subset dJ$. J has a sequence of $\pi^{-1}(\mathcal{O}_T)$-submodules

$$J =: J^{(0)} \supset J^{(-1)} \supset J^{(-2)} \supset \cdots \supset J_2 \supset J_1 \supset J_0$$

defined inductively by

$$\begin{aligned} &J^{(-k)} := d^{-1}(dF_1\wedge J^{(-k+1)}) \qquad \text{for } k\geq 1,\\ &J_0 := \ker[d\colon R^n\varphi_!\mathcal{O}_X \to R^n\varphi_!\Omega^1_{X/T}],\\ &J_k := d^{-1}(dF_1\wedge J_{k-1}) \qquad \text{for } k\geq 1. \end{aligned}$$

$J^{(-k)}$ are $\mathcal{O}_S$-submodules of J. Moreover,

$$dJ^{(-k)} = dF_1\wedge J^{(-k+1)}, \qquad dJ_k = dF_1\wedge J_{k-1} \qquad \text{for } k\geq 1$$

are all $\mathcal{O}_S$-submodules of $R^n\varphi_!(\Omega^1_{X/T})$.

(ii) *For $k\geq 0$, the homomorphism $\partial_1^k d$: $J\to (R^n\varphi_!\Omega^1_{X/T})\cdot\partial_1^k \subset R^n\varphi_!K^0$ induces*

$$(dJ)\partial_1^k/(dJ_k)\partial_1^k = \Xi_!^{(n+k)} \qquad \text{for } k\geq 0.$$

(iii) *The wedge product with dF_1 induces an $\mathcal{O}_S$-isomorphism*

$$dF_1\wedge \quad : J^{(-k+1)} \xrightarrow{\sim} dJ^{(-k)} = \Xi_!^{(n-k)} \qquad \text{for } k\geq 1.$$

§ 2. Regularity and the Poincaré duality

By the very definition, we have a canonical homomorphism of left $\mathcal{D}_S$-modules

$$\kappa\colon \Xi_! := \boldsymbol{R}^n\varphi_!(K^\cdot, \delta) \longrightarrow \Xi := \boldsymbol{R}^n\varphi_*(K^\cdot, \delta),$$

which sends $\Xi_!^{(k)}$ to $\Xi^{(k)}$ for all k. By Theorems 1.1 and 1.2, the stalks of $\Xi_!$ and Ξ at $s\in S\setminus D$ are $\Xi_{!s} = \mathcal{O}_{S,s}\otimes_{\boldsymbol{C}}(R^n\varphi_!\boldsymbol{C}_X)_s$ and $\Xi_s = \mathcal{O}_{S,s}\otimes_{\boldsymbol{C}}(R^n\varphi_*\boldsymbol{C}_X)_s$. It is well-known that $(R^n\varphi_*\boldsymbol{C}_X)_s = H^n(X_s, \boldsymbol{C})$, where $X_s := \varphi^{-1}(s)$ is the

Milnor fiber, while $(R^n\varphi_! C_X)_s$ is canonically isomorphic to the n-dimensional homology group $H_n(X_s, \boldsymbol{C})$. Thus the above homomorphism κ contains information on the canonical $\boldsymbol{C}$-linear map

$$\kappa_s\colon H_n(X_s, \boldsymbol{C}) = (R^n\varphi_! \boldsymbol{C}_X)_s \longrightarrow (R^n\varphi_* \boldsymbol{C}_X)_s = H^n(X_s, \boldsymbol{C})$$

for the Milnor fibers. By the Poincaré duality for the Milnor fibers, we have canonically

$$H^n(X_s, \boldsymbol{C}) = \mathrm{Hom}_{\boldsymbol{C}}(H_n(X_s, \boldsymbol{C}), \boldsymbol{C}),$$

which, together with κ_s gives $(-1)^{n(n+1)/2}$ times the intersection pairing $H_n(X_s, \boldsymbol{C}) \times H_n(X_s, \boldsymbol{C}) \to \boldsymbol{C}$.

In this section, we extend these to the entire S in the context of $\mathscr{D}_S$-modules. In Saito's papers, the Poincaré duality is not explicitly dealt with. As we see in Section 3, however, he deals with its formal microlocal version. As we also see in Section 4, he gives a description of the Poincaré duality in terms of primitive forms when $\varphi\colon X \to S$ is a universal unfolding.

We also give a sketch of the proof that Ξ and $\Xi_!$ are holonomic left $\mathscr{D}_S$-modules with regular singularities.

Since $\Xi^{(-k)}/\Xi^{(-k-1)} \simeq \varphi_* \Omega^{n+1}_{X/S}$ has support on D for each integer k by Theorem 1.1, we get a canonical injective $\mathcal{O}_S$-linear map

$$\mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k)}, \mathcal{O}_S) \longrightarrow \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k-1)}, \mathcal{O}_S).$$

We can define on the resulting inductive limit

$$\text{ind}\lim_{k\to\infty} \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k)}, \mathcal{O}_S)$$

a canonical left $\mathscr{D}_S$-module structure by letting $\theta \in \Theta_S$ act by

$$(\theta u)(\xi) := \theta(u(\xi)) - u(\theta\xi)$$

for $u \in \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k)}, \mathcal{O}_S)$ and $\xi \in \Xi^{(-k-1)}$. We also endow the inductive limit with the increasing filtration $\{\Psi^{(k)}; k \in \boldsymbol{Z}\}$ given by $\Psi^{(k)} = \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k)}, \mathcal{O}_S)$.

Both (i) and (ii) in the following are due to Kashiwara.

Theorem 2.1 (The Poincaré duality). (i) *For each integer k, we have a canonical $\mathcal{O}_S$-homomorphism*

$$\Xi_!^{(n+k)} \longrightarrow \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k-1)}, \mathcal{O}_S),$$

which is an isomorphism for $k \geq 0$ and which gives rise to a left $\mathscr{D}_S$-isomorphism

$$\Xi_! \xrightarrow{\sim} \text{ind lim}_{k\to\infty} \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(-k-1)}, \mathcal{O}_S)$$

decreasing the filtration by $n-1$.

(ii) $\Xi_!$ *is the left* $\mathscr{D}_S$*-module adjoint to* Ξ, *i.e.*,

$$\Xi_! = \Xi^* := \mathscr{E}xt^m_{\text{left } \mathscr{D}_S}(\Xi, \mathscr{D}_S) \otimes_{\mathcal{O}_S} \omega_S^{-1}.$$

As a consequence of (i), we first get the following:

Corollary 2.2. $\Xi_!^{(k)}$ *is* $\mathcal{O}_S$*-locally free of rank* μ *for all integers* k. *In particular,* $\Xi_!$ *is* $\mathcal{O}_S$*-flat.*

Proof. $\Xi_!^{(k)}$ for $k \geq n$ is $\mathcal{O}_S$-locally free of rank μ by Theorem 2.1, (i), since so is $\Xi^{(l)}$ for $l \leq -1$ by Theorem 1.1, (4), (i). We are done as in Corollary 1.7.

Remark. As Kashiwara pointed out, $\Xi^{(k)}$ for $k \geq 1$ may not be $\mathcal{O}_S$-locally free. Thus the canonical homomorphism $\Xi \to \Xi \otimes_{\mathcal{O}_S} \mathcal{O}_S(*D)$, given in (2) below immediately before Theorem 2.6, may not be an isomorphism.

We also sketch the proof of the following regularity in this section (cf. Appendix).

Theorem 2.3. Ξ *and* $\Xi_!$ *are holonomic left* $\mathscr{D}_S$*-modules with regular singularities. Their characteristic varieties in the cotangent bundle* T^*S *of* S *are both equal to the union* $(T^*_S S) \cup (T^*_D S)$ *of the zero section* $T^*_S S$ *of* T^*S *and the closure* $T^*_D S$ *in* T^*S *of the conormal bundle in* S *of the smooth locus of the discriminant* D.

As the first step in proving these results, we recall the following from Saito's papers, which reappear in Section 4 and in Appendix:

The fiber product $Z := X \times_T S$ is obviously an open set in $\boldsymbol{C}^{n+1} \times \boldsymbol{C}^m$ containing the origin with coordinates $(x, s) = (x_0, \cdots, x_n, t_1, \cdots, t_m)$. We denote by $p: Z \to S$ and $\Pi: Z \to X$ the projections. $\varphi: X \to S$ gives rise to the section $\sigma: X \to Z$ of Π defined by $\sigma(x, t') := (x, F_1(x, t'), t')$. Thus σ is an isomorphism onto the nonsingular hypersurface $\sigma(X) = \{(x, s) \in Z;\ F(x, s) = 0\}$, where $F(x, s) := t_1 - F_1(x, t')$.

$$\begin{tikzcd} Z \arrow[r, shift left, "\Pi"] \arrow[d, "p"'] & X \arrow[l, shift left, "\sigma"] \arrow[d, "q"] \\ S \arrow[r, "\pi"'] & T \end{tikzcd}$$

We have

$$\hat{C} := \Pi^{-1}(C) = \text{Specan}\,(\mathcal{O}_Z/(\partial F/\partial x_0, \cdots, \partial F/\partial x_n))$$

so that $p\colon \hat{C} \to S$ is the base extension of $q\colon C \to T$ by $\pi\colon S \to T$. Since the fibers of φ have at most isolated singularities, $\hat{C}$ (resp. C) is finite over S (resp. T) and is a complete intersection in Z (resp. X), hence is Cohen-Macaulay. Thus $p_*\mathcal{O}_{\hat{C}}$ is $\mathcal{O}_S$-locally free of rank equal to the Milnor number μ. Similarly, $q_*\mathcal{O}_C$ is $\mathcal{O}_T$-locally free of rank μ.

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\hat{C}} \xrightarrow{\bar{F}} \mathcal{O}_{\hat{C}} \longrightarrow \sigma_*\mathcal{O}_C \longrightarrow 0$$

and $\hat{C} \cap \sigma(X) = \sigma(C)$, where the arrow with $\bar{F}$ denotes the multiplication by the restriction $\bar{F}$ to $\hat{C}$ of $F := F(x, s)$ regarded as an element of $H^0(Z, \mathcal{O}_Z)$. We thus have the following:

Lemma 2.4. (i) *The $\mathcal{O}_S$-module $\varphi_*\mathcal{O}_C$ has an $\mathcal{O}_S$-locally free resolution*

$$0 \longrightarrow p_*\mathcal{O}_{\hat{C}} \xrightarrow{p_*(\bar{F})} p_*\mathcal{O}_{\hat{C}} \longrightarrow \varphi_*\mathcal{O}_C \longrightarrow 0.$$

(ii) *Let $\Delta := \det(p_*(\bar{F}))$ be the determinant of the $\mathcal{O}_S$-endomorphism $p_*(\bar{F})$ of $p_*\mathcal{O}_{\hat{C}}$. Then the divisor* Specan $(\mathcal{O}_S/\Delta\mathcal{O}_S)$ *is set-theoretically equal to the discriminant D of φ.*

Let $\Omega^{\cdot}_{Z/S}$ with $d := d_{Z/S}$ be the complex of relative Kähler differential forms for the smooth morphism $p\colon Z \to S$ of relative dimension $n+1$. The wedge product with $dF = -dF_1$ gives rise to another complex $(\Omega^{\cdot}_{Z/S}, dF\wedge)$ of $\mathcal{O}_Z$-modules as in Section 1. Then we have the following, the first assertion of which is again due to Saito [Sk3].

Lemma 2.5. (i) (The generalized de Rham lemma)

$$\mathcal{H}^j(\Omega^{\cdot}_{Z/S}, dF\wedge) = \begin{cases} 0 & \textit{for } j \neq n+1 \\ \Omega^{n+1}_{Z/S}/(dF \wedge \Omega^n_{Z/S}) & \textit{for } j = n+1. \end{cases}$$

(ii) (The Poincaré lemma)

$$\mathcal{H}^j(\Omega^{\cdot}_{Z/S}, d) = \begin{cases} p^{-1}(\mathcal{O}_S) & \textit{for } j = 0 \\ 0 & \textit{for } j \neq 0. \end{cases}$$

(iii) *$\Omega^{n+1}_{Z/S}/(dF \wedge \Omega^n_{Z/S})$ is $\mathcal{O}_{\hat{C}}$-invertible and*

$$0 \longrightarrow \Omega^{n+1}_{Z/S}/(dF\wedge\Omega^n_{Z/S}) \xrightarrow{\bar{F}} \Omega^{n+1}_{Z/S}/(dF\wedge\Omega^n_{Z/S}) \longrightarrow \sigma_*\Omega^{n+1}_{X/S} \longrightarrow 0$$

is exact. Taking its direct image under p, we get an $\mathcal{O}_S$-locally free resolution of $\varphi_*\Omega^{n+1}_{X/S}$.

Consider the left $\mathcal{D}_S$-module

$$\mathcal{O}_S(*D) := \mathcal{O}_S[\Delta^{-1}] = \text{ind}\lim_{\nu\to\infty}\mathcal{O}_S(\nu D),$$

the sheaf of germs of meromorphic functions on S whose poles have support on D. Since $\mathrm{gr}^k(\Xi)$ and $\mathrm{gr}^k(\Xi_!)$ for each $k \in \boldsymbol{Z}$ are isomorphic to $\varphi_*\Omega^{n+1}_{X/S}$ by Theorems 1.1 and 1.2, we have by Lemmas 2.4 and 2.5 the equalities of left $\mathcal{D}_S$-modules for each k:

$$\Xi\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D) = \Xi^{(k)}\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D), \qquad \Xi_!\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D) = \Xi_!^{(k)}\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D)$$

as well as canonical left $\mathcal{D}_S$-isomorphisms

$$\begin{aligned}&\text{ind}\lim_{k\to\infty}\mathcal{H}om_{\mathcal{O}_S}(\Xi^{(-k-1)}, \mathcal{O}_S)\\ &\quad = \mathcal{H}om_{\mathcal{O}_S}(\Xi, \mathcal{O}_S(*D)) = \mathcal{H}om_{\mathcal{O}_S}(\Xi\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D), \mathcal{O}_S(*D)).\end{aligned}$$

Thus Theorem 2.1, (i) will be the consequence of the following:

(1) There exists a perfect $\mathcal{O}_S$-bilinear pairing

$$\langle\ ,\ \rangle : \Xi_!^{(n)} \times \Xi^{(-1)} \longrightarrow \mathcal{O}_S.$$

Its scalar extension

$$\langle\ ,\ \rangle : (\Xi_!\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D)) \times (\Xi\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D)) \longrightarrow \mathcal{O}_S(*D)$$

is a perfect $\mathcal{O}_S$-bilinear pairing satisfying

$$\langle\theta\eta, \xi\rangle + \langle\eta, \theta\xi\rangle = \theta\langle\eta, \xi\rangle$$

for all $\theta \in \Theta_S$, $\eta \in \Xi_!\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D)$ and $\xi \in \Xi\otimes_{\mathcal{O}_S}(*D)$.

(2) The canonical $\mathcal{D}_S$-homomorphism $\Xi_! \to \Xi_!\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D)$ is an isomorphism.

The following is a particular case of the relative Serre duality due to Ramis-Ruget [RR].

Theorem 2.6. *For each j, the composite*

$$R^n\varphi_!\Omega^j_{X/T} \times \varphi_*\Omega^{n+1-j}_{X/T} \longrightarrow R^n\varphi_!\Omega^{n+1}_{X/T} \longrightarrow \mathcal{O}_S$$

of the cup product with the trace map is a perfect topological $\mathcal{O}_S$-bilinear pairing, where $\varphi_*\Omega^{n+1-j}_{X/T}$ (resp. $R^n\varphi_!\Omega^j_{X/T}$) is a Fréchet nuclear space (resp. the strong dual of a Fréchet nuclear space). Similarly, we have a perfect topological $\mathcal{O}_S$-bilinear pairing

$$R^{n+1}p_!\Omega^j_{Z/S} \times p_*\Omega^{n+1-j}_{Z/S} \longrightarrow R^{n+1}p_!\Omega^{n+1}_{Z/S} \to \mathcal{O}_S.$$

Lemma 2.7. *There exist canonical perfect $\mathcal{O}_S$-bilinear pairings*

$$\langle\ ,\ \rangle : \Xi_!^{(n)} \times \Xi^{(-1)} \longrightarrow \mathcal{O}_S,$$
$$\langle\!\langle\ ,\ \rangle\!\rangle : p_*(\Omega_{Z/S}^{n+1}/(dF\wedge\Omega_{Z/S}^n)) \times p_*(\Omega_{Z/S}^{n+1}/(dF\wedge\Omega_{Z/S}^n)) \longrightarrow \mathcal{O}_S$$

such that for each $k\geq 0$ the $\mathcal{O}_S$-bilinear pairing

$$\mathrm{gr}^{n+k}(\Xi_!) \times \mathrm{gr}^{-k}(\Xi) \longrightarrow \mathcal{O}_S(*D)/\mathcal{O}_S$$

induced by the former is perfect and coincides with the perfect $\mathcal{O}_S$-bilinear pairing

$$\varphi_*\Omega_{X/S}^{n+1} \times \varphi_*\Omega_{X/S}^{n+1} \longrightarrow \mathcal{O}_S(*D)/\mathcal{O}_S$$

induced by the latter under suitable identifications $\mathrm{gr}^{n+k}(\Xi_!) = \varphi_*\Omega_{X/S}^{n+1} = \mathrm{gr}^{-k}(\Xi)$.

Proof. On the one hand, $\Xi_!^{(n)} = dJ = \{d\beta \in dR^n\varphi_!\mathcal{O}_X;\ dF_1\wedge d\beta = 0\}$ by Proposition 1.10, (ii). On the other hand, we have an isomorphism $(dF_1\wedge)$: $I/I' \simeq dI^{(-1)}/dI_0 = \Xi^{(-1)}$ by Proposition 1.9, (iii), where $I = \varphi_*\Omega_{X/T}^n \supset I' = d\varphi_*\Omega_{X/T}^{n-1} + dF_1\wedge\varphi_*\Omega_{X/T}^{n-1}$.

The canonical surjection $R^n\varphi_!\Omega_{X/T}^{n+1} \to \boldsymbol{R}^n\varphi_!(K^\cdot, \delta) = \mathcal{O}_S$ with kernel $dF_1\wedge dR^n\varphi_!\Omega_{X/T}^{n-1}$ coincides with the trace map. By Theorem 2.6, we have a perfect topological pairing

$$R^n\varphi_!\Omega_{X/T}^1 \times I \longrightarrow R^n\varphi_!\Omega_{X/T}^{n+1} \longrightarrow \mathcal{O}_S$$

which sends (γ, α) to the image $[\gamma\cup\alpha] \in \mathcal{O}_S$ of $\gamma\cup\alpha \in R^n\varphi_!\Omega_{X/T}^{n+1}$.

It induces a perfect pairing $dJ\times(I/I') \to \mathcal{O}_S$. Indeed, $[\gamma\cup I'] = 0$ means

$$(*)\quad 0 = [\gamma\cup(dF_1\wedge\varphi_*\Omega_{X/T}^{n-1})] = [(dF_1\wedge\gamma)\cup\varphi_*\Omega_{X/T}^{n-1}] = [dF_1\wedge(\gamma\cup\varphi_*\Omega_{X/T}^{n-1})]$$

and

$$(**)\qquad [\gamma\cup d\omega] = 0 \qquad \text{for all } \omega\in\varphi_*\Omega_{X/T}^{n-1}.$$

$(*)$ is equivalent to $dF_1\wedge\gamma = 0$ by the relative Serre duality in Theorem 2.6. $(*)$ also implies $\gamma\cup\varphi_*\Omega_{X/T}^{n-1} \subset dR^n\varphi_!\Omega_{X/T}^{n-1} + dF_1\wedge R^n\varphi_!\Omega_{X/T}^{n-1}$ by Lemma 1.8, (iv), hence $d(\gamma\cup\omega) \in dF_1\wedge dR^n\varphi_!\Omega_{X/T}^{n-1}$ for all $\omega\in\varphi_*\Omega_{X/T}^{n-1}$. Thus $(**)$ is equivalent under $(*)$ to $[(d\gamma)\cup\omega] = [d(\gamma\cup\omega)] - [\gamma\cup d\omega] = 0$ for all $\omega\in\varphi_*\Omega_{X/T}^{n-1}$ again by the relative Serre duality. Hence $[\gamma\cup I'] = 0$ is equivalent to $d\gamma = 0 = dF_1\wedge\gamma$, which is nothing but $\gamma\in dJ$ by Lemma 1.8, (iii).

Thus we have a perfect $\mathcal{O}_S$-bilinear pairing

$$\langle d\beta, [d\alpha_{-1}]\rangle := [(d\beta)\cup\alpha_0]$$

for $d\beta \in dJ = \Xi_!^{(n)}$ and $[d\alpha_{-1}] \in dI^{(-1)}/dI_0 = \Xi^{(-1)}$, where $[d\alpha_{-1}]$ is the coset of $d\alpha_{-1} \in dI^{(-1)}$ modulo dI_0 and $\alpha_0 \in I$ is determined uniquely modulo I' by $d\alpha_{-1} = dF_1 \wedge \alpha_0$. By its scalar extension, we have an $\mathcal{O}_S$-bilinear pairing

$$\Xi_!^{(n+k)} \times \Xi^{(-k)} \longrightarrow \mathcal{O}_S(*D)$$

whose restrictions to $\Xi_!^{(n+k-1)} \times \Xi^{(-k)}$ and $\Xi_!^{(n+k)} \times \Xi^{(-k-1)}$ have values in $\mathcal{O}_S$. Hence we get an $\mathcal{O}_S$-bilinear pairing

$$\mathrm{gr}^{n+k}(\Xi_!) \times \mathrm{gr}^{-k}(\Xi) \longrightarrow \mathcal{O}_S(*D)/\mathcal{O}_S.$$

On the other hand, we clearly have an exact sequence of complexes

$$0 \longrightarrow (\Omega^{\cdot}_{Z/S}, dF\wedge) \xrightarrow{F} (\Omega^{\cdot}_{Z/S}, dF\wedge) \longrightarrow (\sigma_* \Omega^{\cdot}_{X/T}, dF_1\wedge) \longrightarrow 0,$$

where the arrow with F denotes the multiplication by F. It induces exact sequences of complexes

$$0 \longrightarrow (p_* \Omega^{\cdot}_{Z/S}, dF\wedge) \longrightarrow (p_* \Omega^{\cdot}_{Z/S}, dF\wedge) \longrightarrow (\varphi_* \Omega^{\cdot}_{X/T}, dF_1\wedge) \longrightarrow 0$$

and

$$0 \rightarrow (R^n \varphi_! \Omega^{\cdot}_{X/T}, dF_1\wedge) \longrightarrow (R^{n+1} p_! \Omega^{\cdot}_{Z/S}, dF\wedge) \longrightarrow (R^{n+1} p_! \Omega^{\cdot}_{Z/S}, dF\wedge) \rightarrow 0.$$

By Lemma 1.8, (ii), (iv) and by similar results obtained from Lemma 2.5, the exact sequences of cohomology sheaves arising from these exact sequences are reduced to

$$0 \longrightarrow \mathscr{K} \xrightarrow{R^{n+1}p_!(\bar{F})} \mathscr{K} \longrightarrow \varphi_* \Omega^{n+1}_{X/S} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{C} \xrightarrow{p_*(\bar{F})} \mathscr{C} \longrightarrow \varphi_* \Omega^{n+1}_{X/S} \longrightarrow 0,$$

where

$$\mathscr{K} := \ker [dF\wedge \quad : R^{n+1} p_! \mathcal{O}_Z \longrightarrow R^{n+1} p_! \Omega^1_{Z/S}]$$

$$\mathscr{C} := \operatorname{coker} [dF\wedge \quad : p_* \Omega^n_{Z/S} \longrightarrow p_* \Omega^{n+1}_{Z/S}].$$

By Lemma 2.5, both $\mathscr{K}$ and $\mathscr{C}$ are canonically isomorphic to the locally free $\mathcal{O}_S$-module $p_*(\Omega^{n+1}_{Z/S}/(dF\wedge \Omega^n_{Z/S}))$ of rank μ.

By Theorem 2.6, we have a perfect topological $\mathcal{O}_S$-bilinear pairing

$$R^{n+1} p_! \Omega^j_{Z/S} \times p_* \Omega^{n+1-j}_{Z/S} \longrightarrow \mathcal{O}_S$$

for each j, sending (b, a) to $[b \cup a]$, where the bracket denotes the trace map $R^{n+1} p_! \Omega^{n+1}_{Z/S} \rightarrow \mathcal{O}_S$ which is surjective and has kernel $dR^{n+1} p_! \Omega^n_{Z/S}$ by Lemma 2.5, (ii). These pairings for different j's are obviously compatible with $dF\wedge$ so that we have a perfect $\mathcal{O}_S$-bilinear pairing $\langle\!\langle\ ,\ \rangle\!\rangle: \mathcal{K}\times\mathcal{C}\to\mathcal{O}_S$ which sends $(b, [da])$ to

$$\langle\!\langle b, [da]\rangle\!\rangle := -[b\cup(da)] = [(db)\cup a],$$

where b is in $\mathcal{K}\subset R^{n+1}p_!\mathcal{O}_Z$ and $[da]$ is the image in $\mathcal{C}$ of $da\in dp_*\Omega^n_{Z/S}=p_*\Omega^{n+1}_{Z/S}$.

d induces $\mathcal{O}_S$-linear isomorphisms $R^{n+1}p_!(d)$: $\mathcal{K}\overset{\sim}{\to} d\mathcal{K}\subset R^{n+1}p_!\Omega^1_{Z/S}$ and $p_*(d)$: $\tilde{I}/\tilde{I}'\overset{\sim}{\to}\mathcal{C}$, which obviously commute with $R^{n+1}p_!(\bar{F})$ and $p_*(\bar{F})$, respectively, where $\tilde{I}:=p_*\Omega^n_{Z/S}$ and $\tilde{I}':=\{a\in\tilde{I};\ da\in dF\wedge p_*\Omega^n_{Z/S}\}$. Hence we have a perfect $\mathcal{O}_S$-bilinear pairing

$$\langle\!\langle\ ,\ \rangle\!\rangle: (d\mathcal{K})\times\tilde{I}/\tilde{I}'\longrightarrow\mathcal{O}_S.$$

The $\mathcal{O}_S$-bilinear pairing which it induces on the cokernels of $R^{n+1}p_!(\bar{F})$ and $p_*(\bar{F})$ and which has values in $\mathcal{O}_S(*D)/\mathcal{O}_S$ is perfect and coincides with the pairing $\mathrm{gr}^{n+k}(\Xi_!)\times\mathrm{gr}^{-k}(\Xi)\to\mathcal{O}_S(*D)/\mathcal{O}_S$ above induced by $\langle\ ,\ \rangle$ for $k\geq 0$, since $\Xi^{(-k)}$ is then $\mathcal{O}_S$-locally free by Theorem 1.1, (4), (i). Thus we conclude that this latter is also perfect.

Proof of Theorem 2.1. We show (1) and (2) stated immediately before Theorem 2.6.

By Lemma 2.7, we already have the perfect $\mathcal{O}_S$-bilinear pairing required in (1). Its scalar extension to $\mathcal{O}_S(*D)$ satisfies $\langle\theta\eta, \xi\rangle+\langle\eta, \theta\xi\rangle=\theta\langle\eta, \xi\rangle$, since $\Delta^\nu\theta$ for ν large enough preserves $\Xi_!^{(k)}$ and $\Xi^{(k)}$ for $k\in\mathbf{Z}$ and for $\theta\in\Theta_S$ by the definition of the action of θ on $K^\cdot$.

Using Lemma 2.7, we can show by induction on $k\geq 0$ that the induced $\mathcal{O}_S$-bilinear pairing

$$\langle\ ,\ \rangle: \Xi_!^{(n+k)}\times\Xi^{(-k-1)}\longrightarrow\mathcal{O}_S$$

is perfect for $k\geq 0$. Hence we have canonical isomorphisms

$$\begin{aligned}\Xi_! = \mathrm{ind\ lim}_{k\to\infty}\Xi_!^{(n+k)} &\xrightarrow{\ \sim\ } \mathrm{ind\ lim}_{k\to\infty}\mathcal{H}om_{\mathcal{O}_S}(\Xi^{(-k-1)}, \mathcal{O}_S)\\ &= \mathcal{H}om_{\mathcal{O}_S}(\Xi, \mathcal{O}_S(*D)) \xleftarrow{\ \sim\ } \Xi_!\otimes_{\mathcal{O}_S}\mathcal{O}_S(*D).\end{aligned}$$

Thus we have (2).

$\Xi_!=\Xi^*$ in (ii) was shown in a more general context by Kashiwara-Schapira [KS] as we see in Appendix. q.e.d.

Remark. Kashiwara also constructed the $\mathcal{D}_S$-isomorphism

$$\Xi^*\xrightarrow{\ \sim\ }\mathrm{ind\ lim}_{k\to-\infty}\mathcal{H}om_{\mathcal{O}_S}(\Xi^{(k)}, \mathcal{O}_S)$$

using the *first Spencer sequence*

$$0 \longrightarrow \mathscr{D}_S \otimes_{\mathcal{O}_S} \wedge^m \Theta_S \otimes_{\mathcal{O}_S} \Xi^{(k-m)} \longrightarrow \cdots \longrightarrow \mathscr{D}_S \otimes_{\mathcal{O}_S} \Xi^{(k)} \longrightarrow \Xi \longrightarrow 0,$$

which is exact for k large enough at each point of S.

A sketch of the proof of Theorem 2.3. By Theorem 2.1, (ii), it suffices to check the assertion only for Ξ.

To compute the characteristic variety of Ξ, we look at the non-negative part of the filtration

$$0 \subset \Xi^{(0)} \subset \Xi^{(1)} \subset \cdots\cdots$$

by $\mathcal{O}_S$-coherent submodules. By the definition of our $\varphi^{-1}(\mathscr{D}_S)$-module structure on $K^\cdot$, the associated graded $\mathrm{gr}(\mathscr{D}_S)$-module $\Xi^{(0)} \oplus (\oplus_{k \geq 1} \Xi^{(k)}/\Xi^{(k-1)})$ is the fiber product of

(i) $\Xi^{(0)}$ which is annihilated by $\mathrm{gr}^1(\mathscr{D}_S) \cong \Theta_S$ and

(ii) $(\varphi_* \Omega_{X/S}^{n+1}) \otimes_{\mathcal{O}_S} \mathcal{O}_S[\bar{\partial}_1]$ on which each $\bar{\theta} \in \mathrm{gr}^1(\mathscr{D}_S)$ acts as $\bar{\partial}_1 \cdot \varphi_*([\theta F])$, where $\bar{\theta}$, $\bar{\partial}_1 \in \mathrm{gr}^1(\mathscr{D}_S)$ are the images of θ, $\partial_1 \in \Theta_S$ and where $\varphi_*([\theta F])$ is the image of $\theta F \in \mathcal{O}_Z$ in $\varphi_* \mathcal{O}_C$ acting as multiplication on $\varphi_* \Omega_{X/S}^{n+1}$

(iii) with respect to the surjective homomorphisms

$$\Xi^{(0)} \longrightarrow \varphi_* \Omega_{X/S}^{n+1} \longleftarrow (\varphi_* \Omega_{X/S}^{n+1}) \otimes_{\mathcal{O}_S} \mathcal{O}_S[\bar{\partial}_1],$$

the first one being the canonical surjection $\Xi^{(0)} \to \Xi^{(0)}/\Xi^{(-1)}$ while the second being the reduction modulo $\bar{\partial}_1$.

Thus the characteristic variety is $(T_S^* S) \cup (T_D^* S)$ and Ξ is holonomic.

Unfortunately, the graded $\mathrm{gr}(\mathscr{D}_S)$-module above is not reduced, hence we cannot immediately conclude that Ξ has regular singularities. Instead, Saito uses Deligne's criterion

$$\Theta_S(-\log D) \cdot \Xi^{(k)} \subset \Xi^{(k)} \qquad \text{for } k \in \boldsymbol{Z}$$

for regularity, where $\Theta_S(-\log D) := \{\theta \in \Theta_S : \theta \Delta \in \Delta \mathcal{O}_S\}$.

By the definition of the action of θ on $K^\cdot$, we have $\theta \Xi^{(k)} \subset \Xi^{(k)}$ for all $k \in \boldsymbol{Z}$ if and only if θ belongs to the kernel of $\Theta_S \to \varphi_* \mathcal{O}_C$ which sends θ to $\varphi_*([\theta F])$. As we see in Proposition 4.1, this kernel coincides with $\Theta_S(-\log D)$ when $\varphi \colon X \to S$ is a *universal unfolding*, and we are done in this case. The general case is a pull-back of a universal unfolding, hence the regularity also follows. q.e.d.

§ 3. Formal microlocalization and the higher residue pairing

In this section, we consider the formal microlocalizations $\hat{\Xi}$, $\hat{\Xi}_!$ of Ξ, $\Xi_!$ with respect to the variable t_1 as well as their direct images $\pi_* \hat{\Xi}$, $\pi_* \hat{\Xi}_!$. They are much easier to handle than Ξ, $\Xi_!$ and appear in Saito's papers together with $\pi_* \Xi$. In fact, as was pointed out by Kashiwara (cf. [Sm7]), Saito's higher residue pairing is nothing but the formal microlocal Poincaré duality between $\pi_* \hat{\Xi}$ and $\pi_* \hat{\Xi}_!$.

Definition. (i) We denote by $\Lambda := \mathcal{O}_T((\partial_1^{-1}))$ the ring of formal Laurent series in ∂_1^{-1} with coefficients in $\mathcal{O}_T$, i.e., those of the form $\sum_{\nu \in Z} a_\nu \partial_1^\nu$ with $a_\nu \in \mathcal{O}_T$ and $a_\nu = 0$ for ν sufficiently large. We endow it with an increasing filtration $\{\Lambda^{(k)}; k \in Z\}$ defined by

$$\Lambda^{(k)} := \mathcal{O}_T[[\partial_1^{-1}]]\partial_1^k = \{\textstyle\sum_\nu a_\nu \partial_1^\nu;\ a_\nu = 0 \text{ for } k < \nu\}.$$

We denote by $\iota: \Lambda \to \Lambda$ the $\mathcal{O}_T$-algebra involution such that $\partial_1^\iota: = -\partial_1$. We regard Λ as a left $\pi_* \mathcal{D}_S$-module by letting $\theta \in \Theta_T \subset \pi_* \Theta_S$ act on the coefficients of Laurent series, letting $\partial_1 \in \pi_* \Theta_S$ act as multiplication by ∂_1 and letting $t_1 \in \pi_* \mathcal{O}_S$ act as the derivation $-\partial/\partial(\partial_1)$ (the microlocal Fourier transform!).

(ii) We define the ring of formal microdifferential operators on S by

$$\hat{\mathscr{E}}_S := \mathcal{D}_S \otimes_{\pi^{-1}(\mathcal{O}_T)[\partial_1]} \pi^{-1}(\Lambda).$$

Definition. $(\hat{K}^\bullet, \hat{\delta}) := (K^\bullet, \delta) \otimes_{q^{-1}(\mathcal{O}_T)} q^{-1}(\Lambda)$ is the complex

$$0 \longrightarrow \hat{K}^{-1} \xrightarrow{\hat{\delta}} \hat{K}^0 \xrightarrow{\hat{\delta}} \cdots \xrightarrow{\hat{\delta}} \hat{K}^n \longrightarrow 0$$

on X defined for each integer j by

$$\hat{K}^j := \Omega_{X/T}^{j+1}((\partial_1^{-1})) = \Omega_{X/T}^{j+1} \otimes_{q^{-1}(\mathcal{O}_T)} q^{-1}(\Lambda)$$

which consists of elements of the form $\sum_{\nu \in Z} \omega_\nu \partial_1^\nu$ with $\omega_\nu \in \Omega_{X/T}^{j+1}$ such that $\omega_\nu = 0$ for ν sufficiently large. We define an increasing filtration on $\hat{K}^j$ by

$$\hat{\mathscr{F}}^{(k)} \hat{K}^j := \{\textstyle\sum_\nu \omega_\nu \partial_1^\nu;\ \omega_\nu = 0 \text{ for } j+k-n < \nu\}$$

for $k \in Z$. The $q^{-1}(\Lambda)$-linear map $\hat{\delta}: \hat{K}^j \to \hat{K}^{j+1}$ sends $\omega \partial_1^\nu$ in $\hat{K}^j$ with $\omega \in \Omega_{X/T}^{j+1}$ to

$$\hat{\delta}(\omega \partial_1^\nu) := (d\omega)\partial_1^\nu - (dF_1 \wedge \omega)\partial_1^{\nu+1}.$$

A left $\varphi^{-1}(\mathcal{D}_S)$-module structure on each $\hat{K}^j$ is defined exactly as for K^j in Section 1, so that $\hat{K}^\bullet$ becomes a complex of left $\varphi^{-1}(\hat{\mathscr{E}}_S)$-modules and $\hat{\delta}$ is $\varphi^{-1}(\hat{\mathscr{E}}_S)$-linear satisfying $\hat{\delta}(\hat{\mathscr{F}}^{(k)} \hat{K}^j) \subset \hat{\mathscr{F}}^{(k)} \hat{K}^{j+1}$.

By direct calculation using Lemmas 1.3 and 1.4, we have the following, which is much easier to prove than Corollary 1.7:

Proposition 3.1. *We have* $\mathscr{H}^j(\hat{K}^\cdot, \hat{\delta})=0$ *for* $j \neq n$. $\mathscr{H}^n(\hat{K}^\cdot, \hat{\delta})$ *has support on* C *and, together with the filtration induced by* $\hat{\mathscr{F}}$, *coincides with the* $\mathscr{F}$*-adic completion of* $\mathscr{H}^n(K^\cdot, \delta)$, *i.e.,*

$$\mathscr{H}^n(\hat{K}^\cdot, \hat{\delta})=\text{proj lim}_{k\to\infty} \mathscr{H}^n(K^\cdot, \delta)/\mathscr{F}^{(-k)}\mathscr{H}^n(K^\cdot, \delta).$$

By this result, the second spectral sequences degenerate and we get the following, which is much simpler than the corresponding statements in Theorems 1.1 and 1.2:

Theorem 3.2. *Both* $\boldsymbol{R}^j\varphi_*(\hat{K}^\cdot, \hat{\delta})=\mathscr{H}^j(\varphi_*\hat{K}^\cdot, \varphi_*\hat{\delta})$ *and* $\boldsymbol{R}^j\varphi_!(\hat{K}^\cdot, \hat{\delta})=\mathscr{H}^{j-n}(R^n\varphi_!\hat{K}^\cdot, R^n\varphi_!\hat{\delta})$ *vanish for* $j\neq n$. *Moreover, we have canonical isomorphisms of filtered left* $\hat{\mathscr{E}}_S$*-modules*

$$\hat{\Xi}=\boldsymbol{R}^n\varphi_*(\hat{K}^\cdot, \hat{\delta})=\mathscr{H}^n(\varphi_*\hat{K}^\cdot, \varphi_*\hat{\delta})=\varphi_*\mathscr{H}^n(\hat{K}^\cdot, \hat{\delta})$$
$$\hat{\Xi}_!=\boldsymbol{R}^n\varphi_!(\hat{K}^\cdot, \hat{\delta})=\mathscr{H}^0(R^n\varphi_!\hat{K}^\cdot, R^n\varphi_!\hat{\delta})=\varphi_*\mathscr{H}^n(\hat{K}^\cdot, \hat{\delta}),$$

where the filtrations on the right hand terms are those induced by $\hat{\mathscr{F}}$ *and where*

$$\hat{\Xi}:=\text{proj lim}_{k\to\infty} \Xi/\Xi^{(-k)}, \qquad \hat{\Xi}_!:=\text{proj lim}_{k\to\infty}\Xi_!/\Xi_!^{(-k)},$$

with the filtrations $\{\hat{\Xi}^{(l)}: l\in \boldsymbol{Z}\}$ *and* $\{\hat{\Xi}_!^{(l)}; l\in \boldsymbol{Z}\}$ *induced by* $\{\Xi^{(l)}; l\in \boldsymbol{Z}\}$ *and* $\{\Xi_!^{(l)}; l\in \boldsymbol{Z}\}$. *Under these isomorphisms, the canonical* $\hat{\mathscr{E}}_S$*-homomorphism* $\hat{\kappa}\colon \hat{\Xi}_!\to\hat{\Xi}$ *induced by* $\kappa\colon \Xi_!\to\Xi$ *is a filtration preserving isomorphism.*

In the same way as Lemma 1.8, we obtain the following using Lemmas 1.3 and 1.4. The proof is easier this time, since $q\colon X\to T$ is a smooth Stein morphism with contractible fibers and of relative dimension $n+1$:

Lemma 3.3.

(i) $\mathscr{H}^j(q_*\Omega^\cdot_{X/T}, q_*(d))=\begin{cases}\mathcal{O}_T & \text{for } j=0\\ 0 & \text{for } j\neq 0.\end{cases}$

(ii) $\mathscr{H}^j(q_*\Omega^\cdot_{X/T}, q_*(dF_1\wedge))=\begin{cases}0 & \text{for } j\neq n+1\\ q_*\Omega^{n+1}_{X/S} & \text{for } j=n+1.\end{cases}$

(iii) $\mathscr{H}^j(R^{n+1}q_!\Omega^\cdot_{X/T}, R^{n+1}q_!(d))=\begin{cases}0 & \text{for } j\neq n+1\\ \mathcal{O}_T & \text{for } j=n+1.\end{cases}$

(iv) $\mathscr{H}^j(R^{n+1}q_!\Omega^\cdot_{X/T}, R^{n+1}q_!(dF_1\wedge))=\begin{cases}q_*\Omega^{n+1}_{X/S} & \text{for } j=0\\ 0 & \text{for } j\neq 0.\end{cases}$

By Proposition 3.1 and Lemma 3.3, we get the following by means of the spectral sequences exactly as in Theorems 1.1, 1.2 and 3.2 (see also Lemmas 2.4 and 2.5 as well as the paragraphs immediately before them):

Theorem 3.4. *Both* $\boldsymbol{R}^j q_*(\hat{K}^\bullet, \hat{\delta}) = \mathscr{H}^j(q_*\hat{K}^\bullet, q_*\hat{\delta})$ *and* $\boldsymbol{R}^j q_!(\hat{K}^\bullet, \hat{\delta}) = \mathscr{H}^{j-n-1}(R^{n+1}q_!\hat{K}^\bullet, R^{n+1}q_!\hat{\delta})$ *vanish for* $j \neq n$. *Moreover, we have canonical isomorphisms of filtered left* $\pi_*\hat{\mathscr{E}}_S$*-modules*

$$\pi_*\hat{\Xi} = \boldsymbol{R}^n q_*(\hat{K}^\bullet, \hat{\delta}) = \mathscr{H}^n(q_*\hat{K}^\bullet, q_*\hat{\delta}) = q_*\mathscr{H}^n(\hat{K}^\bullet, \hat{\delta})$$
$$\pi_*\hat{\Xi}_! = \boldsymbol{R}^n q_!(\hat{K}^\bullet, \hat{\delta}) = \mathscr{H}^{-1}(R^{n+1}q_!\hat{K}^\bullet, R^{n+1}q_!\hat{\delta}) = q_*\mathscr{H}^n(\hat{K}^\bullet, \hat{\delta}).$$

Under these isomorphisms, the $\pi_*\hat{\mathscr{E}}_S$*-homomorphism* $\pi_*(\hat{\kappa})\colon \pi_*\hat{\Xi}_! \to \pi_*\hat{\Xi}$ *is a filtration preserving isomorphism. Furthermore,*

$$\mathrm{gr}^k(\pi_*\hat{\Xi}_!) = \mathrm{gr}^k(\pi_*\hat{\Xi}) \cong q_*\Omega^{n+1}_{X/S}$$

are $\mathscr{O}_T$*-locally free of rank* μ *for each* k *and*

$$\mathrm{gr}(\pi_*\hat{\Xi}) = \mathrm{gr}(\pi_*\hat{\Xi}_!) = (q_*\Omega^{n+1}_{X/S}) \otimes_{\mathscr{O}_T} \mathrm{gr}(\Lambda)$$

so that $\pi_*\hat{\Xi}$ *and* $\pi_*\hat{\Xi}_!$ *are* Λ*-locally free of rank* μ, *while* $\pi_*\hat{\Xi}^{(k)}$ *and* $\pi_*\hat{\Xi}_!^{(k)}$ *for each* k *are locally free modules of rank* μ *over* $\Lambda^{(0)} = \mathscr{O}_T[[\partial_1^{-1}]]$.

Note that the associated graded rings and modules above are with respect to our filtrations, so that, for instance, $\mathrm{gr}(\Lambda) = \mathscr{O}_T[\bar{\partial}_1, \bar{\partial}_1^{-1}]$ is the Laurent polynomial ring in a variable $\bar{\partial}_1$. The last assertions above follow from the lemma of Krull-Azumaya-Nakayama.

Remark. Later, we need the following, which we can show using Lemma 3.3, (i) and (ii):

(i) $\pi_*\Xi = \mathscr{H}^n(q_*K^\bullet, q_*\delta)$.

(ii) The multiplication by ∂_1 is *bijective* on $\pi_*\Xi$, even though it is only surjective on Ξ by Theorem 1.1, (4), (ii).

(iii) $\pi_*\Xi^{(k)} = \partial_1^k \pi_*\Xi^{(0)}$ for all integers k.

The proof of (i) is exactly the same as that of Theorem 1.1, (i). We can show (ii), hence (iii), as in Lemma 1.5, (iii), thanks to Lemma 3.3, (i).

The following is essentially the higher residue pairing in Saito's papers. It was recognized to be the formal microlocal Poincaré duality by Kashiwara (cf. M. Saito [Sm7]).

Theorem 3.5 (The formal microlocal Poincaré duality). *For the locally free modules* $\pi_*\hat{\Xi}_!$ *and* $\pi_*\hat{\Xi}$ *of rank* μ *over the ring* $\Lambda = \mathscr{O}_T((\partial_1^{-1}))$, *we have a canonical nondegenerate pairing* $\rho\colon \pi_*\hat{\Xi}_! \times \pi_*\hat{\Xi} \to \Lambda$ *which is* Λ*-linear for the first factor and* Λ*-semilinear with respect to the involution* ι *of* Λ *for the second. It satisfies the following properties for* $\eta \in \pi_*\hat{\Xi}_!$ *and* $\xi \in \pi_*\hat{\Xi}$:

(i) *ρ is $\mathcal{O}_T$-bilinear and*

$$\rho(\eta, \xi)\partial_1 = \rho(\partial_1\eta, \xi) = \rho(\eta, -\partial_1\xi).$$

(ii) *For $\theta \in \Theta_T \subset \pi_*\hat{\mathcal{E}}_S$ we have*

$$\theta(\rho(\eta, \xi)) = \rho(\theta\eta, \xi) + \rho(\eta, \theta\xi),$$

where the left hand side denotes the action of θ on the coefficients in $\mathcal{O}_T$ of the Laurent series $\rho(\eta, \xi) \in \Lambda$.

(iii) *As for the action of $t_1 \in \pi_*\mathcal{O}_S \subset \pi_*\hat{\mathcal{E}}_S$, we have*

$$t_1(\rho(\eta, \xi)) = \rho(t_1\eta, \xi) - \rho(\eta, t_1\xi),$$

where the left hand side denotes the derivation $-\partial/\partial(\partial_1)$ acting on the Laurent series $\rho(\eta, \xi)$.

(iv) *For each pair k, l of integers, we have*

$$\rho(\pi_*\hat{\Xi}_!^{(n+1+k)}, \pi_*\hat{\Xi}^{(l)}) = \Lambda^{(k+l)}$$

and the induced pairing

$$\bar{\rho}_{n+1+k,l}\colon q_*\Omega^{n+1}_{X/S} \times q_*\Omega^{n+1}_{X/S} = \mathrm{gr}^{n+1+k}(\pi_*\hat{\Xi}_!) \times \mathrm{gr}^l(\pi_*\hat{\Xi}) \longrightarrow \mathrm{gr}^{k+l}(\Lambda) = \mathcal{O}_T$$

is a nondegenerate $\mathcal{O}_T$-bilinear pairing.

Remark. Saito calls $\bar{\rho}_{0,0}$ the residue pairing and shows it to be symmetric. Composing ρ with the filtration preserving $\pi_*\hat{\mathcal{E}}_S$-isomorphism $\pi_*(\hat{\kappa})^{-1}\colon \pi_*\hat{\Xi} \xrightarrow{\sim} \pi_*\hat{\Xi}_!$, we get a pairing $K\colon \pi_*\hat{\Xi} \times \pi_*\hat{\Xi} \to \Lambda$, which Saito calls the higher residue pairing and which satisfies $K(\eta, \xi) = K(\xi, \eta)^{\iota}$.

Proof. By Lemma 3.3, (iii), we have the trace homomorphism

$$\mathrm{tr}\colon R^{n+1}q_!\Omega^{n+1}_{X/T} \longrightarrow \mathcal{O}_T$$

which is surjective with kernel $dR^{n+1}q_!\Omega^n_{X/T}$. By Ramis-Ruget [RR], we have the relative Serre duality (cf. Theorem 2.6), i.e., the cup product induces a perfect topological $\mathcal{O}_T$-bilinear pairing

$$\rho\colon R^{n+1}q_!\Omega^{j+1}_{X/T} \times q_*\Omega^{n-j}_{X/T} \longrightarrow R^{n+1}q_!\Omega^{n+1}_{X/T} \xrightarrow{\mathrm{tr}} \mathcal{O}_T$$

for each j. Obviously, $R^{n+1}q_!(dF_1\wedge)$ and $q_*(dF_1\wedge)$ are mutually adjoint with respect to ρ, while $R^{n+1}q_!(d)$ and $-q_*(d)$ are mutually adjoint with respect to ρ, since the trace map has kernel equal to $dR^{n+1}q_!\Omega^n_{X/T}$.

We now extend this ρ uniquely to a pairing

$$\rho\colon R^{n+1}q_!\hat{K}^j \times q_*\hat{K}^{n-j-1} \longrightarrow \Lambda$$

so that it is Λ-linear for the first factor and Λ-*semilinear* with respect to the involution ι of Λ for the second factor. Thanks to this twisting, we easily see that $R^{n+1}q_!(\hat{\delta})$ and $-q_*(\hat{\delta})$ are mutually adjoint with respect to ρ. By Theorem 3.4, we have

$$\pi_*\hat{\Xi}_! = \ker [R^{n+1}q_!(\hat{\delta})\colon R^{n+1}q_!\hat{K}^{-1} \longrightarrow R^{n+1}q_!\hat{K}^0]$$
$$\pi_*\hat{\Xi} = \operatorname{coker} [q_*(\hat{\delta})\colon q_*\hat{K}^{n-1} \longrightarrow q_*\hat{K}^n].$$

Thus we have a pairing $\rho\colon \pi_*\hat{\Xi}_! \times \pi_*\hat{\Xi} \to \Lambda$ which is Λ-linear for the first factor and Λ-semilinear with respect to ι for the second. By the definition of our filtration, we clearly have

$$\rho(\pi_*\hat{\Xi}_!^{(n+1+k)}, \pi_*\hat{\Xi}^{(l)}) \subset \Lambda^{(k+l)}$$

for each pair k, l of integers.

The pairings $\bar{\rho}_{n+1+k,l}$ induced on the associated graded modules are easily seen to be nondegenerate bilinear forms on the $\mathcal{O}_T$-locally free module $q_*\Omega^{n+1}_{X/S}$ of rank μ. Hence, we have the equality in (iv) and ρ itself is necessarily nondegenerate.

(i) follows from the Λ-linearity for the first factor and Λ-semilinearity for the second.

(ii) and (iii) are immediate by direct calculation. q.e.d.

Remark. We can rephrase Theorem 3.5 as follows: By sending $\eta \in \pi_*\hat{\Xi}_!$ to the homomorphism which sends $\xi \in \pi_*\hat{\Xi}$ to $\rho(\eta, \xi) \in \Lambda$, we have a left $\pi_*\hat{\mathscr{E}}_S$-isomorphism

$$\pi_*\hat{\Xi}_! \xrightarrow{\sim} \mathscr{H}om_{\Lambda\text{-semi}}(\pi_*\hat{\Xi}, \Lambda)$$

which decreases the filtration by $n+1$, where the right hand side consists of Λ-semilinear homomorphisms u from $\pi_*\hat{\Xi}$ to Λ with respect to ι, where the left $\pi_*\hat{\mathscr{E}}_S$-module structure is defined for $\xi \in \pi_*\hat{\Xi}$ by

$$(\partial_1 u)(\xi) := u(-\partial_1\xi) = \partial_1 \cdot u(\xi)$$
$$(t_1 u)(\xi) := t_1(u(\xi)) + u(t_1\xi)$$
$$(\theta u)(\xi) := \theta(u(\xi)) - u(\theta\xi) \qquad \text{for } \theta \in \Theta_T$$

and where those u with $u(\pi_*\hat{\Xi}^{(l)}) \subset \Lambda^{(k+l)}$ for all $l \in \boldsymbol{Z}$ belong to the k-th filtration.

M. Saito [Sm7] identifies the right hand side as the left $\pi_*\hat{\mathscr{E}}_S$-module *adjoint* to $\pi_*\hat{\Xi}$ and relates Theorem 3.5 to Theorem 2.1.

§ 4. Universal unfoldings and primitive forms

For the holomorphic map $\varphi: X \to S$ of our concern, we again consider as in Section 2 the fiber product $Z := X \times_T S$ with the projections $p: Z \to S$ and $\Pi: Z \to X$. Thus φ gives rise to a section $\sigma: X \to Z$ for Π which is an isomorphism onto the nonsingular hypersurface $\sigma(X) = \{(x, s) \in Z; F(x, s) = 0\}$ with $F(x, s) := t_1 - F_1(x, t')$. Recall also that

$$C = \sigma^{-1}(\hat{C}) = \mathrm{Specan}\,(\mathcal{O}_X/(\partial F_1/\partial x_0, \cdots, \partial F_1/\partial x_n))$$
$$\hat{C} = \Pi^{-1}(C) = \mathrm{Specan}\,(\mathcal{O}_Z/(\partial F/\partial x_0, \cdots, \partial F/\partial x_n))$$
$$\sigma(C) = \hat{C} \cap \sigma(X) = \mathrm{Specan}\,(\mathcal{O}_Z/(F, \partial F/\partial x_0, \cdots, \partial F/\partial x_n)).$$

Remark. For $f(x) := F_1(x, 0)$, the fiber $X_O := \varphi^{-1}(O)$ over the origin is the hypersurface $\{x \in Z_O; f(x) = 0\}$ in $Z_O = p^{-1}(O) = \boldsymbol{C}^{n+1}$. The fiber of $p_*\mathcal{O}_{\hat{C}}$ at O is

$$(p_*\mathcal{O}_{\hat{C}})(O) = \mathcal{O}_{Z_O}/(\partial f/\partial x_0, \cdots, \partial f/\partial x_n),$$

while the fiber

$$(\varphi_*\mathcal{O}_C)(O) = \mathcal{O}_{Z_O}/(f, \partial f/\partial x_0, \cdots, \partial f/\partial x_n)$$

of $\varphi_*\mathcal{O}_C$ at O is nothing but

$$\mathcal{T}^1_{X_O} = \mathcal{E}xt^1_{\mathcal{O}_{X_O}}(\Omega^1_{X_O}, \mathcal{O}_{X_O})$$

and classifies the first order deformations of X_O without regard to the embedding.

Definition. $\varphi: X \to S$ is said to be a *universal unfolding* if the $\mathcal{O}_S$-linear map sending $\theta \in \Theta_S$ to the image $\overline{\theta F}$ in $p_*\mathcal{O}_{\hat{C}}$ of θF under the canonical projection $p_*\mathcal{O}_Z \to p_*\mathcal{O}_{\hat{C}}$ is an isomorphism $\Theta_S \xrightarrow{\sim} p_*\mathcal{O}_{\hat{C}}$.

Thus in this case, $m = \dim S$ coincides with the Milnor number μ.

Saito's theory describes the Gauss-Manin systems Ξ and $\Xi_!$ for $\varphi: X \to S$, which contain cohomological information on the fibers, in terms of interesting linear algebra on Θ_S. This feature is in marked contrast to the usual deformation theory, in which we describe the tangent spaces of the parameter space in terms of the cohomology groups of sheaves on the fibers.

As the first set of ingredients for the linear algebra on Θ_S, the universality of φ induces on Θ_S the structure of a commutative and associative $\mathcal{O}_S$-algebra as well as an $\mathcal{O}_S$-module endomorphism w of Θ_S as follows:

Proposition 4.1. *For a universal unfolding $\varphi\colon X\to S$, define an $\mathcal{O}_T$-submodule of $\pi_*\Theta_S$ by*

$$\mathcal{G}:=\{g\in\pi_*\Theta_S;\ [g,\partial_1]=0\}=\mathcal{O}_T\partial_1\oplus\Theta_T.$$

(1) *The $\mathcal{O}_T$-linear map which sends $g\in\mathcal{G}$ to the image $[gF]$ of gF under $\pi_*p_*\mathcal{O}_Z\to q_*\mathcal{O}_C$ is an isomorphism $\mathcal{G}\xrightarrow{\sim}q_*\mathcal{O}_C$. This induces on $\mathcal{G}$ a commutative and associative $\pi_*\mathcal{O}_S$-algebra structure with ∂_1 as the identity, where the multiplications $g*g'\in\mathcal{G}$ and $t_1*g\in\mathcal{G}$ for $g,g'\in\mathcal{G}$ are defined by*

$$[(g*g')F]:=[gF]\cdot[g'F]\qquad \textit{and}\qquad [(t_1*g)F]:=[F_1]\cdot[gF].$$

We denote by A the $\mathcal{O}_T$-linear endomorphism of $\mathcal{G}$ which sends $g\in\mathcal{G}$ to $A(g):=t_1*g\in\mathcal{G}$.

(2) *By the identification $\pi^*\mathcal{G}=\Theta_S=p_*\mathcal{O}_{\hat{C}}$, we have the structure of a commutative and associative $\mathcal{O}_S$-algebra on Θ_S with ∂_1 as the identity. Moreover, the $\mathcal{O}_S$-module endomorphism w of Θ_S defined by*

$$w(\theta):=t_1\theta-(\pi^*A)(\theta)\qquad \textit{for } \theta\in\Theta_S$$

coincides with the multiplication of $p_*(\bar{F})$ on $p_*\mathcal{O}_{\hat{C}}$, is injective and has $\mathcal{O}_S$-locally free image equal to the sheaf

$$\Theta_S(-\log D):=\{\theta\in\Theta_S;\ \theta\Delta\in\mathcal{O}_S\Delta\},$$

of germs of holomorphic vector fields on S with logarithmic zeros along D, where $\Delta=\det(p_*(\bar{F}))=\det(w)$ as in Lemma 2.4.

(3) *We have*

$$\Delta^{-1}w(\theta)\Delta=\theta(\operatorname{trace}(w))=\mu(\theta t_1)-\theta\,(\operatorname{trace}(A))$$

for $\theta\in\Theta_S$, with the Milnor number $\mu=\dim S$.

Proof. (1) follows from the definition of universality, since $p_*\mathcal{O}_{\hat{C}}=p_*\Pi^*\mathcal{O}_C=\pi^*q_*\mathcal{O}_C$.

(3) is a consequence of (2). Indeed, $w(\theta)$ is in $\Theta_S(-\log D)$ by (2), hence $w(\theta)\Delta$ belongs to $\Delta\mathcal{O}_S$. We get the desired equality by comparing the leading coefficients, since $\Delta=t_1^{\mu}-(\operatorname{trace} A)t_1^{\mu-1}+\cdots$.

Here is a sketch of the more involved proof of (2) due to Saito [Sk11, (2.3) and (4.3)]: In view of Lemma 2.4, it suffices to show that $\Theta_S(-\log D)$ coincides with the kernel of the $\mathcal{O}_S$-linear map sending $\theta\in\Theta_S$ to $[\theta F]\in\varphi_*\mathcal{O}_C$.

The scheme-theoretic image of C under φ is Specan $(\mathcal{O}_S/\mathcal{J})$, where

$$\mathcal{J}:=\ker[\mathcal{O}_S\to\varphi_*\mathcal{O}_C]=\{g\in\mathcal{O}_S;\ p^*(g)\in F\mathcal{O}_Z+\textstyle\sum_{0\le j\le n}(\partial F/\partial x_j)\mathcal{O}_S\}.$$

We first note that $\mathscr{J}$ is principal, i.e., $\mathscr{J}=h\mathcal{O}_S$ for some h, and has the same radical as $\Delta\mathcal{O}_S$, since C is an $(m-1)$-dimensional complete intersection in X and is finite over S. We thus have

$$\Theta_S(-\log D)=\{\theta \in \Theta_S;\ \theta h \in h\mathcal{O}_S\}.$$

Any $\theta \in \Theta_S$ can be written as $\theta=a\partial_1+\theta'$ with $a \in \mathcal{O}_S$ and $\theta' \in \pi^*\Theta_T$. Thus by the chain rule applied to $\varphi\colon X\to S$, we have

$$\varphi^*(\theta h)=(\theta F)|_X\cdot\varphi^*(\partial_1 h)+\hat{\theta}'(\varphi^*h),$$

where $(\theta F)|_X$ is the pull-back of $\theta F \in H^0(Z, \mathcal{O}_Z)$ by $\sigma\colon X\to Z$, while $\hat{\theta}'$ is the lifting of θ' to Θ_X so that $\hat{\theta}'t_j=\theta' t_j$ for $2\le j\le m$ and $\hat{\theta}'x_i=0$ for $0\le i\le n$.

$\varphi\colon X\to S$ is a universal unfolding by assumption. Hence, as Saito shows, $\mathscr{J}$ coincides with its radical and there exists a codimension one analytic subset $C'\subset C$ such that the germ of the map φ at each point in $C\setminus C'$ is analytically trivial. Thus $(\partial/\partial t_j)^\wedge(\varphi^*h)$ for $2\le j\le m$ vanishes on C, while the image of $\varphi^*(\partial_1 h)$ in $\mathcal{O}_C$ is not a zero divisor. By restricting the above identity to C, we get

$$[\varphi^*(\theta h)]=[\theta F]\cdot[\varphi^*(\partial_1 h)]$$

and conclude that $[\theta F]=0$ if and only if $[\varphi^*(\theta h)]=0$, i.e., $\theta h \in \mathscr{J}=h\mathcal{O}_S$.
q.e.d.

Saito introduced the notion of *primitive forms*, which enables us to describe $\pi_*\hat{\Xi}$ completely in terms of linear algebra on $\mathscr{G}=\{g \in \pi_*\Theta_S;\ [g, \partial_1]=0\}$. It was M. Saito who proved in [Sm7] the local existence of *good* primitive forms in general.

Theorem 4.2 (K. Saito and M. Saito). *If S is small enough, then there exists*

$$\zeta \in H^0(S, \Xi^{(-1)})=H^0(T, \pi_*\Xi^{(-1)})\subset H^0(T, \pi_*\hat{\Xi}^{(-1)}),$$

which satisfies the following properties:

(i) *The map sending $\theta \in \Theta_S$ to $\theta\zeta \in \Xi^{(0)}$ is an $\mathcal{O}_S$-isomorphism. In particular, $\Xi^{(0)}=\Theta_S\zeta$, hence $\Xi=\mathscr{D}_S\zeta$.*

(i′) *The image in $\mathrm{gr}^{-1}(\pi_*\hat{\Xi})\cong q_*\Omega^{n+1}_{X/S}$ of $\zeta \in \pi_*\hat{\Xi}^{(-1)}$ is a generator as an invertible $q_*\mathcal{O}_C$-module. Hence for each integer k, the map sending $g \in \mathscr{G}$ to the image in $\mathrm{gr}^k(\pi_*\hat{\Xi})$ of $g\partial_1^k\zeta \in \pi_*\hat{\Xi}^{(k)}$ is an $\mathcal{O}_T$-linear isomorphism*

$$\mathscr{G}\xrightarrow{\sim}\mathrm{gr}^k(\pi_*\hat{\Xi})$$

and the map sending $g\otimes\partial_1^k$ *to* $g\partial_1^k\zeta$ *is a filtration preserving* Λ*-linear isomorphism*

$$\mathcal{G}\otimes_{\mathcal{O}_T}\Lambda \xrightarrow{\sim} \pi_*\hat{\Xi}.$$

(ii) *For* g, g' *in* $\mathcal{G}\subset\pi_*\mathcal{D}_S\subset\pi_*\hat{\mathcal{E}}_S$ *we have*

$$gg'\zeta \in \mathcal{G}\zeta+\mathcal{G}\partial_1\zeta \qquad \text{and} \qquad t_1 g\zeta \in \mathcal{G}\partial_1^{-1}\zeta+\mathcal{G}\zeta.$$

(iii) *For the higher residue pairing*

$$K\colon \pi_*\hat{\Xi}\times\pi_*\hat{\Xi} \xrightarrow{\pi_*(\hat{\kappa})^{-1}\times\mathrm{id}} \pi_*\hat{\Xi}_!\times\pi_*\hat{\Xi} \xrightarrow{\rho} \Lambda$$

in the remark after Theorem 3.5, *we have*

$$K(g\zeta, g'\zeta) \in \mathcal{O}_T\partial_1^{-n-1} \qquad \text{for all } g, g' \in \mathcal{G}.$$

(iv) *For the Euler operator* $E:=w(\partial_1)=t_1\partial_1-A(\partial_1) \in \pi_*\Theta_S(-\log D)$, *we have* $E\zeta=r\zeta$ *for a rational number* r.

Remark. We do not reproduce the long proof here. A result of Malgrange [M2] first reduces it to a problem on the fiber $\pi^{-1}(O)$. Then a crucial role is played by the deep results due to Deligne, Varchenko, Scherk and Steenbrink in the theory of mixed Hodge structures.

K. Saito calls ζ satisfying (i′) through (iv) a *primitive form.* (i) is certainly stronger than (i′) and is a consequence of M. Saito's proof of the local existence. He constructs a section for $\hat{\Xi}\to\mathrm{gr}^0(\hat{\Xi})$ which is "good" with respect to various filtrations. Let us here call such a primitive form *good.*

According to both Saitos, however, we might need to impose additional conditions on good primitive forms for them to be of intrinsic interest in the general case. See also the remark immediately before Proposition 4.5 below.

By Proposition 4.1, we have a commutative and associative $\pi_*\mathcal{O}_S$-algebra multiplications $g*g'$ and $t_1*g=A(g)$ on $\mathcal{G}$. Hence clearly by (i), (ii), (iii) above, there exist $\nabla_g g' \in \mathcal{G}$, $N(g) \in \mathcal{G}$ and $J(g, g') \in \mathcal{O}_T$ for each pair g, g' in $\mathcal{G}$ such that

$$\begin{aligned} &gg'\zeta=(\nabla_g g')\zeta+(g*g')\partial_1\zeta, \\ &t_1 g\zeta=N(g)\partial_1^{-1}\zeta+A(g)\zeta \qquad \text{and} \\ &K(g\zeta, g'\zeta)=J(g, g')\partial_1^{-n-1}. \end{aligned}$$

The following is then immediate by straightforward computation:

Lemma 4.3. *A primitive form $\zeta \in H^0(S, \Xi^{(-1)})$ determines a biadditive map*

$$\nabla: \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G} \qquad \text{sending } (g, g') \text{ to } \nabla_g g',$$

an $\mathcal{O}_T$-linear endomorphism $N: \mathcal{G} \to \mathcal{G}$ and a nondegenerate symmetric $\mathcal{O}_T$-bilinear form $J: \mathcal{G} \times \mathcal{G} \to \mathcal{O}_T$ which together satisfy the following properties for all g, g', g'' in $\mathcal{G}$:

(1) *For $a \in \mathcal{O}_T$, we have $\nabla_{(ag)} g' = a(\nabla_g g')$ and $\nabla_g(ag') = (ga)g' + a(\nabla_g g')$, where $(ga) \in \mathcal{O}_T \subset \pi_* \mathcal{O}_S$ is the derivative of a with respect to $g \in \mathcal{G} \subset \pi_* \Theta_S$.*

(2) *$\nabla_{\partial_1} g = 0$ and $\nabla_g \partial_1 = 0$.*

(3) *$\nabla_{[g,g']} g'' = \nabla_g(\nabla_{g'} g'') - \nabla_{g'}(\nabla_g g'')$ and*

$$\nabla_g(g' * g'') - \nabla_{g'}(g * g'') + g * (\nabla_{g'} g'') - g' * (\nabla_g g'') = [g, g'] * g''.$$

This latter for $g'' = \partial_1$ is reduced to the torsion-freeness $\nabla_g g' - \nabla_{g'} g = [g, g']$, where $[g, g'] := gg' - g'g \in \mathcal{G}$ is the Lie bracket.

(4) *$\nabla_g(N(g')) = N(\nabla_g g')$ and*

$$\nabla_g(A(g')) - A(\nabla_g g') = (gt_1)g' + N(g * g') - g * (N(g') + g').$$

(5) *$J(g * g', g'') = J(g', g * g'')$ and*

$$g(J(g', g'')) = J(\nabla_g g', g'') + J(g', \nabla_g g'').$$

(6) *$J(A(g), g') = J(g, A(g'))$ and*

$$(n+1)J(g, g') = J(N(g), g') + J(g, N(g')).$$

(7) *$N(\partial_1) = r\partial_1$ holds for the rational number r in Theorem* 4.2, (iv). *Moreover, the Euler operator $E = w(\partial_1) = t_1\partial_1 - A(\partial_1)$ satisfies $\nabla_g E = -N(g) + (r+1)$ and $[E, g] = N(g) - (r+1)g - \nabla_{A(\partial_1)} g$.*

If we denote by L_{∂_1} the Lie derivative with respect to ∂_1, then the $\mathcal{O}_T$-submodule

$$\mathcal{F} := \{\omega \in \pi_* \Omega_S^1;\ L_{\partial_1}(\omega) = 0\} = \mathcal{O}_T dt_1 \oplus \Omega_T^1$$

of $\pi_* \Omega_S^1$ is dual to $\mathcal{G}$ with the nondegenerate $\mathcal{O}_T$-bilinear pairing, $\langle\ ,\ \rangle: \mathcal{F} \times \mathcal{G} \to \mathcal{O}_T$ induced by the canonical $\mathcal{O}_S$-bilinear pairing $\langle\ ,\ \rangle$: $\Omega_S^1 \times \Theta_S \to \mathcal{O}_S$. The $\mathcal{O}_T$-bilinear form J on $\mathcal{G}$ then gives rise to an $\mathcal{O}_T$-isomorphism (resp. $\mathcal{O}_S$-isomorphism)

$$J: \mathcal{G} \xrightarrow{\sim} \mathcal{F} \quad (\text{resp. } J: \Theta_S \xrightarrow{\sim} \Omega_S^1)$$

denoted by the same letter, so that $\langle J(g), g'\rangle = J(g, g')$ for $g, g' \in \mathscr{G}$ (resp. Θ_S). By Lemma 4.3, we have

$$\langle J(N(J^{-1}(\eta))), g\rangle = \langle \eta, (n+1)g - N(g)\rangle$$
$$\langle J(g' * J^{-1}(\eta)), g\rangle = \langle \eta, g' * g\rangle$$

for $g, g' \in \mathscr{G}$ (resp. Θ_S) and $\eta \in \mathscr{F}$ (resp. Ω_S^1) as well as

$$\langle J(A(J^{-1}(\eta))), g\rangle = \langle \eta, A(g)\rangle \qquad \text{for } g \in \mathscr{G} \text{ and } \eta \in \mathscr{F}$$
$$\langle J(w(J^{-1}(\eta))), \theta\rangle = \langle \eta, w(\theta)\rangle \qquad \text{for } \theta \in \Theta_S \text{ and } \eta \in \Omega_S^1.$$

By Lemma 4.3 we have an integrable connection $\nabla\colon \Theta_T \times \mathscr{G} \to \mathscr{G}$ on $\mathscr{G}$, i.e., a left $\mathscr{D}_T$-module structure. Thus in addition to $*$ and A on $\mathscr{G}$ and $*$ and w on Θ_S given by the universality as in Proposition 4.1, a choice of a primitive form provides the second set of ingredients for linear algebra on $\mathscr{G}$ and Θ_S as follows:

Proposition 4.4. *Let ∇, J and N correspond to a primitive form $\zeta \in H^0(S, \Xi^{(-1)})$ as in Lemma 4.3 and denote by the same letters the pullbacks to Θ_S by π.*

(i) *$\mathscr{G}^\nabla := \{\partial \in \mathscr{G};\ \nabla_g \partial = 0$ for all $g \in \mathscr{G}\}$ is a $\boldsymbol{C}_T$-submodule of $\mathscr{G}$ of rank equal to $\mu = \dim S$ such that*

$$\mathscr{G} = \mathscr{O}_T \otimes_{\boldsymbol{C}_T} \mathscr{G}^\nabla \qquad \text{and} \qquad \Theta_S = \pi^* \mathscr{G} = \mathscr{O}_S \otimes_{\boldsymbol{C}_S} \Theta_S^\nabla$$

with $\Theta_S^\nabla = \pi^{-1}(\mathscr{G}^\nabla)$. Moreover, $\mathscr{G}^\nabla$ contains ∂_1 and has the properties $[\partial, \partial'] = 0$ and

$$w(\partial')\partial - \partial w(\partial') = N(\partial * \partial') - \partial * \{N(\partial') + \partial'\}$$

for all $\partial, \partial' \in \mathscr{G}^\nabla$. N induces a $\boldsymbol{C}_T$-endomorphism of $\mathscr{G}^\nabla$ with $N(\partial_1) = r\partial_1$, while J induces a nondegenerate symmetric $\boldsymbol{C}_T$-bilinear form $J\colon \mathscr{G}^\nabla \times \mathscr{G}^\nabla \to \boldsymbol{C}_T$ such that

$$(n+1)J(\partial, \partial') = J(N(\partial), \partial') + J(\partial, N(\partial'))$$

for all $\partial, \partial' \in \mathscr{G}^\nabla$. In particular, $(n+1) - \alpha$ is an eigenvalue of N if so is α.

(ii) *We have the dual integrable connection $\nabla\colon \Theta_T \times \mathscr{F} \to \mathscr{F}$ and its pull-back $\nabla\colon \Theta_S \times \Omega_S^1 \to \Omega_S^1$ so that $\mathscr{F}^\nabla := \{\omega \in \mathscr{F};\ \nabla_g \omega = 0$ for all $g \in \Theta_T\}$ is a $\boldsymbol{C}_T$-module of rank μ satisfying*

$$\mathscr{F} = \mathscr{O}_T \otimes_{\boldsymbol{C}_T} \mathscr{F}^\nabla \qquad \text{and} \qquad \Omega_S^1 = \pi^* \mathscr{F} = \mathscr{O}_S \otimes_{\boldsymbol{C}_S} (\Omega_S^1)^\nabla$$

with $(\Omega_S^1)^\nabla = \pi^{-1}(\mathscr{F}^\nabla)$ killed by the exterior differentiation d_S on Ω_S^1. N induces a $\boldsymbol{C}_T$-endomorphism of $\mathscr{F}^\nabla$, while J induces a $\boldsymbol{C}_T$-isomorphism $J\colon \mathscr{G}^\nabla \xrightarrow{\sim} \mathscr{F}^\nabla$ and a $\boldsymbol{C}_S$-isomorphism $J\colon \Theta_S^\nabla \xrightarrow{\sim} (\Omega_S^1)^\nabla$.

Remark. We need the following simple observations later: If $\{\partial_1, \cdots, \partial_\mu\}$ and $\{\partial'_1, \cdots, \partial'_\mu\}$ are $\boldsymbol{C}_T$-bases of $\mathscr{G}^\nabla$ satisfying $J(\partial_i, \partial'_j)=\delta_{ij}$, then $\{J(\partial'_1), \cdots, J(\partial'_\mu)\}$ is a $\boldsymbol{C}_T$-basis of $\mathscr{F}^\nabla$ dual to the $\boldsymbol{C}_T$-basis $\{\partial_1, \cdots, \partial_\mu\}$ of $\mathscr{G}^\nabla$ so that we have

$$g=\sum_{1\leq j\leq\mu} J(g, \partial_j)\partial'_j \qquad \text{for } g \in \mathscr{G},$$
$$\eta=\sum_{1\leq j\leq\mu} \langle\eta, \partial_j\rangle J(\partial'_j) \qquad \text{for } \eta \in \mathscr{F}$$

as well as

$$d_S v=\sum_{1\leq j\leq\mu} (\partial_j v)J(\partial'_j) \qquad \text{and}$$
$$\nabla_\theta(d_S v)=\sum_{1\leq j\leq\mu} (\theta\partial_j v)J(\partial'_j)$$

for $v \in \mathcal{O}_S$ and $\theta \in \Theta_S$. Saito defines the sheaf of *flat coordinates* on S by

$$\begin{aligned}\mathscr{F}\mathscr{C}:&=\{u \in \mathcal{O}_S;\ \partial u \in \boldsymbol{C}_S \text{ for all } \partial \in \Theta_S^\nabla\}\\ &=\{u \in \mathcal{O}_S;\ \partial\partial' u=0 \text{ for all } \partial, \partial' \in \Theta_S^\nabla\}\\ &=\{u \in \mathcal{O}_S;\ (\theta\theta'-\nabla_\theta\theta')u=0 \text{ for all } \theta, \theta' \in \Theta_S\},\end{aligned}$$

which is a $\boldsymbol{C}_S$-submodule of $\mathcal{O}_S$ of rank $\mu+1$ containing $\boldsymbol{C}_S$. The exterior differentiation d_S gives rise to an exact sequence $0\to\boldsymbol{C}_S\to\mathscr{F}\mathscr{C}\to(\Omega_S^1)^\nabla\to 0$. Saito defines a *flat function* on S to be $\tau \in H^0(S, \mathscr{F}\mathscr{C})$, unique up to addition of constant functions, such that $J(\partial_1)=d_S\tau$. In particular, $d_S\tau$ is in $(\Omega_S^1)^\nabla$ and satisfies $d_S(w(\partial_1)\tau)=(2r-n)d_S\tau$.

Remark. If ζ is a *good* primitive form in the sense of the remark immediately after Theorem 4.2, then the $\boldsymbol{C}_T$-endomorphism N of $\mathscr{G}^\nabla$ is semisimple with rational numbers as eigenvalues. K. Saito expects them to be positive, hence also to be less than $n+1$ by Proposition 4.4. This is indeed the case for the universal unfoldings of weighted homogeneous polynomials with only isolated critical points.

M. Saito showed that the eigenvalues of N do not depend on the *good* primitive form chosen, but may depend on the primitive forms if they are not good.

Recall that the multiplication by ∂_1 on $\pi_*\Xi$ and $\pi_*\hat{\Xi}$ is bijective by the remark immediately after Theorem 3.4 and by the fact that $\pi_*\hat{\Xi}$ is a module over $\Lambda=\mathcal{O}_T((\partial_1^{-1}))$. Hence it is also bijective on $H^0(S, \Xi)=H^0(T, \pi_*\Xi)\subset H^0(T, \pi_*\hat{\Xi})$, although it is not on Ξ itself.

For a primitive form $\zeta \in H^0(S, \Xi^{(-1)})$ and $g, g' \in \mathscr{G}$, let

$$P(g, g'):=gg'-\nabla_g g'-(g*g')\partial_1 \in \pi_*\mathscr{D}_S$$
$$Q(g):=w(g)\partial_1-N(g)=t_1 g\partial_1-A(g)\partial_1-N(g) \in \pi_*\mathscr{D}_S$$

and *regard them also as elements of* $\mathcal{D}_S$. By Lemma 4.3, we easily see that $P(g, g')$ is $\mathcal{O}_T$-bilinear in g, g' and satisfies

$$P(g, g')\partial_1 = \partial_1 P(g, g'),$$

while $Q(g)$ is $\mathcal{O}_T$-linear in g and satisfies

$$(Q(g)+g)\partial_1 = \partial_1 Q(g).$$

For ∂ and ∂' in $\mathcal{G}^\nabla$, we have much simpler expressions

$$\begin{aligned} &P(\partial, \partial') = \partial\partial' - (\partial * \partial')\partial_1 \\ &Q(\partial) = w(\partial)\partial_1 - N(\partial) = t_1\partial\partial_1 - A(\partial)\partial_1 - N(\partial). \end{aligned}$$

The following is then immediate by Lemma 4.3 and Proposition 4.4:

Proposition 4.5. *For a fixed primitive form* $\zeta \in H^0(S, \mathcal{Z}^{(-1)})$ *let*

$$\zeta_{k-1} := \partial_1^k \zeta \in H^0(S, \mathcal{Z}) \qquad \textit{for } k \in \boldsymbol{Z}.$$

Then it satisfies the system of holomorphic linear differential equations

$$\begin{aligned} &P(g, g')\zeta_{k-1} = 0 \qquad \textit{for } g, g' \in \mathcal{G} \\ &\{Q(g) + (k+1)g\}\zeta_{k-1} = 0 \qquad \textit{for } g \in \mathcal{G} \\ &\{E - (r-k)\}\zeta_{k-1} = 0, \end{aligned}$$

or equivalently,

$$\begin{aligned} &\partial\partial'\zeta_{k-1} = (\partial * \partial')\partial_1\zeta_{k-1} \qquad \textit{for } \partial, \partial' \in \mathcal{G}^\nabla \\ &w(\partial)\partial_1\zeta_{k-1} = \{N(\partial) - (k+1)\partial\}\zeta_{k-1} \qquad \textit{for } \partial \in \mathcal{G}^\nabla \\ &E\zeta_{k-1} = (r-k)\zeta_{k-1} \qquad \textit{with } E := w(\partial_1). \end{aligned}$$

As a consequence, it also satisfies

$$\begin{aligned} &w(\partial)\partial'\zeta_{k-1} = \{N(\partial * \partial') - (k+1)(\partial * \partial')\}\zeta_{k-1} \\ &\partial w(\partial')\zeta_{k-1} = \{\partial * (N(\partial') - k\partial')\}\zeta_{k-1} \end{aligned}$$

for $\partial, \partial' \in \mathcal{G}^\nabla$ *by Proposition* 4.4.

The surprising algebraic description of $\kappa \colon \mathcal{Z}_1 \to \mathcal{Z}$ given below is due to Saito [Sk14]. He came across the formula in his explicit calculation in [Sk4] for simple (i.e., rational double when $n=2$) singularities.

Theorem 4.6. *Except when n is odd and $\varphi^{-1}(O)$ has only ordinary double points, the restriction $\kappa|_{S\setminus D}$ of the canonical $\mathcal{D}_S$-homomorphism* $\kappa \colon \mathcal{Z}_1$

$=\operatorname{ind}\lim_{k\to-\infty}\mathscr{H}om_{\mathcal{O}_S}(\Xi^{(k)}, \mathcal{O}_S)\to\Xi$ *is a nonzero constant multiple of the restriction* $\kappa'|_{S\backslash D}$ *of the left* $\mathscr{D}_S$*-homomorphism* $\kappa':=\operatorname{ind}\lim_{k\to-\infty}\kappa'_k$ *determined by a primitive form* $\zeta\in H^0(S, \Xi^{(-1)})$ *as follows*:

$$\kappa'_k\colon \mathscr{H}om_{\mathcal{O}_S}(\Xi^{(k)}, \mathcal{O}_S)\longrightarrow\Xi^{(n-k-1)}$$
$$\kappa'_k(u):=(-1)^k\sum_{1\le j\le\mu}u(\partial_j\zeta_{k-1})w(\partial'_j)\zeta_{n-k-1}$$

for $u\in\mathscr{H}om_{\mathcal{O}_S}(\Xi^{(k)}, \mathcal{O}_S)$, *where* $\{\partial_1, \cdots, \partial_\mu\}$ *and* $\{\partial'_1, \cdots, \partial'_\mu\}$ *are* C_T*-bases of* $\mathscr{G}^\nabla$ *such that* $J(\partial_i, \partial'_j)=\delta_{ij}$.

Proof. It is clear that the above expression for κ'_k does not depend on the choice of $\{\partial_i\}$ and $\{\partial'_j\}$.

Since w is self-adjoint with respect to J as we saw immediately before Proposition 4.4, we have

$$\kappa'_k(u)=(-1)^k\sum_{1\le j\le\mu}u(w(\partial_j)\zeta_{k-1})\partial'_j\zeta_{n-k-1}.$$

Moreover, $w(\partial'_j)\zeta_{n-k-1}=w(\partial'_j)\partial_1\zeta_{n-k-2}=(N(\partial'_j)-(n-k)\partial'_j)\zeta_{n-k-2}$ and $w(\partial_j)\zeta_k=w(\partial_j)\partial_1\zeta_{k-1}=(N(\partial_j)-(k+1)\partial_j)\zeta_{k-1}$. Also $(n+1)\,\mathrm{id}-N$ is the adjoint of N with respect to J. Hence we have

$$\kappa'_k(u)=(-1)^{k+1}\sum_{1\le j\le\mu}u(w(\partial_j)\zeta_k)\partial'_j\zeta_{n-k-2}=\kappa'_{k+1}(u)$$

for u in $\mathscr{H}om_{\mathcal{O}_S}(\Xi^{(k+1)}, \mathcal{O}_S)$. Thus $\{\kappa'_k\}$ gives rise to an $\mathcal{O}_S$-homomorphism κ' of the inductive limits.

We now show that κ' thus defined is left $\mathscr{D}_S$-linear, i.e., $\kappa'(\theta u)=\theta\kappa'(u)$ for all $\theta\in\Theta_S$ and $u\in\mathscr{H}om_{\mathcal{O}_S}(\Xi^{(k)}, \mathcal{O}_S)$. Indeed, we have $(\theta u)(\partial_j\zeta_{k-1})=\theta(u(\partial_j\zeta_{k-1}))-u(\theta\partial_j\zeta_{k-1})$ by definition. It thus suffices to show that

$$-\sum_{1\le j\le\mu}u(\theta\partial_j\zeta_{k-1})w(\partial'_j)\zeta_{n-k-1}=\sum_{1\le j\le\mu}u(\partial_j\zeta_{k-1})\theta w(\partial'_j)\zeta_{n-k-1},$$

which is an immediate consequence of $\theta\partial_j\zeta_{k-1}=(\theta*\partial_j)\zeta_k$, $\theta w(\partial'_j)\zeta_{n-k-2}=(\theta*w(\partial'_j))\zeta_{n-k-1}$ and $J(\theta*\partial, \partial')=J(\partial, \theta*\partial')$.

As Saito [Sk8] showed, a left $\mathscr{D}_{S\backslash D}$-homomorphism $\Xi_!|_{S\backslash D}\to\Xi|_{S\backslash D}$ is necessarily a constant multiple of $\kappa|_{S\backslash D}$ except when n is odd and $\varphi^{-1}(O)$ has only ordinary double points. q.e.d.

By the Poincaré duality (cf. Theorem 2.1 and Appendix), the canonical $\mathscr{D}_S$-homomorphism $\kappa\colon\Xi_!\to\Xi$ gives rise to a homomorphism of perverse C_S-modules $\kappa\colon \mathrm{Sol}_S(\Xi)=\mathrm{DR}_S(\Xi_!)\to\mathrm{DR}_S(\Xi)$. If we take the zero-th cohomology sheaves, we get a C_S-homomorphism

$$\kappa\colon\mathscr{H}om_{\mathscr{D}_S}(\Xi, \mathcal{O}_S)\longrightarrow\mathscr{H}om_{\mathscr{D}_S}(\mathcal{O}_S, \Xi).$$

We also have C_S-isomorphisms (cf. Appendix)

$$R^n\varphi_! C_X \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_S}(\Xi, \mathcal{O}_S) \qquad \text{and} \qquad R^n\varphi_* C_X \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_S}(\mathcal{O}_S, \Xi).$$

As a consequence of Theorem 4.6, we have the following in view of the remark immediately after Proposition 4.4:

Corollary 4.7. *Except when n is odd and $\varphi^{-1}(O)$ has only ordinary double points, the restriction $\kappa|_{S\setminus D}$ of the canonical C_S-homomorphism*

$$\kappa\colon R^n\varphi_! C_X = \mathscr{H}om_{\mathscr{D}_S}(\Xi, \mathcal{O}_S) \longrightarrow \mathscr{H}om_{\mathscr{D}_S}(\mathcal{O}_S, \Xi) = R^n\varphi_* C_X$$

coincides, up to a nonzero constant multiple, with the restriction $\kappa'|_{S\setminus D}$ of the C_S-homomorphism κ' determined in terms of a primitive form $\zeta \in H^0(S, \Xi^{(-1)})$ as follows: $\kappa'(u) \in \mathscr{H}om_{\mathscr{D}_S}(\mathcal{O}_S, \Xi)$ for $u \in \mathscr{H}om_{\mathscr{D}_S}(\Xi, \mathcal{O}_S)$ sends $1 \in \mathcal{O}_S$ to

$$\begin{aligned}\kappa'(u)(1) &= (-1)^k \textstyle\sum_{1\le j\le\mu} u(\partial_j\zeta_{k-1})w(\partial'_j)\zeta_{n-k-1}\\ &= (-1)^k \textstyle\sum_{1\le j\le\mu} \partial_j(u(\zeta_{k-1}))w(\partial'_j)\zeta_{n-k-1}\\ &= (-1)^k w(J^{-1}(d_S u(\zeta_{k-1})))\zeta_{n-k-1},\end{aligned}$$

which is an element of Ξ killed by Θ_S and is independent of k and the choice of $\{\partial_j\}$ and $\{\partial'_j\}$.

Remark. We have canonical homomorphisms

$$R^n\varphi_! Z_X \longrightarrow R^n\varphi_! C_X \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_S}(\Xi, \mathcal{O}_S).$$

Thus the evaluation at $\xi \in \Xi$ of the right hand side gives a Z_S-homomorphism $R^n\varphi_! Z_X \to \mathcal{O}_S$ sending $\gamma \in R^n\varphi_! Z_X$ to $(1/2\pi i)^n \int_\gamma \xi$ in Saito's notation, a motivation for which is as follows: On the one hand, a section γ of $R^n\varphi_! Z_X$ over $S\setminus D$ can be regarded as a family, parametrized by $s \in S\setminus D$, of n-dimensional homology classes $\gamma(s) \in H_n(X_s, Z)$ of the Milnor fiber $X_s = \varphi^{-1}(s)$. On the other hand, a section α of $\varphi_*\Omega^{n+1}_{X/T}$ gives rise to a section ξ of $\Xi^{(0)}$ as its image. Outside D, we can find $\omega \in \varphi_*\Omega^n_{X/T}$ such that $dF_1 \wedge \omega = \alpha$. Such $\omega =:$ Res $(\alpha/(t_1 - F_1))$ determines a closed holomorphic n-form $\omega(s)$ on each fiber $X_s = \varphi^{-1}(s)$ for $s \in S\setminus D$. Then $(1/2\pi i)^n \int_\gamma \xi$ is nothing but a holomorphic function on $S\setminus D$ whose value at $s \in S\setminus D$ is $(1/2\pi i)^n \int_{\gamma(s)} \omega(s)$, the period of $\omega(s)$ with respect to $\gamma(s) \in H_n(X_s, Z)$.

A primitive form $\zeta \in H^0(S, \Xi^{(-1)})$ determines a decreasing sequence of $\mathscr{D}_S$-submodules $\{\mathscr{D}_S\zeta_{k-1};\, k \in Z\}$ with the generator ζ_{k-1} of $\mathscr{D}_S\zeta_{k-1}$ satisfying the system of differential equations in Proposition 4.5. As in Proposition 4.8 below, we can show that $\mathscr{D}_S\zeta_{k-1} = \mathscr{D}_S\zeta_k$ if $k \neq r$ and if

$k+1$ is not an eigenvalue of the $\boldsymbol{C}_S$-endomorphism N of $\mathscr{G}^{\nabla}$. Thus if ζ is a good primitive form in the sense of the remark immediately after Theorem 4.2, we have

$$\Xi=\mathscr{D}_S\zeta=\mathscr{D}_S\zeta_{-1}\supset\mathscr{D}_S\zeta_0\supset\cdots\supset\mathscr{D}_S\zeta_{l-1}=\mathscr{D}_S\zeta_l=\cdots$$

for l large enough. However, these $\mathscr{D}_S$-submodules do not behave well with respect to the $\mathscr{O}_S$-rank outside D and to the $\boldsymbol{C}_S$-homomorphism

$$\iota_k\colon \mathscr{H}om_{\mathscr{D}_S}(\mathscr{D}_S\zeta_{k-1}, \mathscr{O}_S)\longrightarrow\mathscr{H}om_{\mathscr{D}_S}(\mathscr{O}_S, \mathscr{D}_S\zeta_{n-k-1})$$

such that $\iota_k(u)(1)=(-1)^k w(J^{-1}(d_S u(\zeta_{k-1})))\zeta_{n-k-1}$ for u in the left hand side as in Corollary 4.7.

Instead, Saito introduces new left $\mathscr{D}_S$-modules $\mathscr{M}^{(k)}$ below with *complex* parameters k such that there exists a surjective left $\mathscr{D}_S$-homomorphism $\mathscr{M}^{(k)}\to\mathscr{D}_S\zeta_{k-1}$ if k is an integer. These $\mathscr{D}_S$-modules $\mathscr{M}^{(k)}$ are reminiscent of those considered in Kashiwara [K1]:

Definition. For a fixed primitive form $\zeta\in H^0(S, \Xi^{(-1)})$, we define

$$\mathscr{M}^{(k)}:=\mathscr{D}_S m_k \qquad \text{for each } k\in\boldsymbol{C}$$

to be the left $\mathscr{D}_S$-module generated by a single element m_k whose annihilator is the left ideal in $\mathscr{D}_S$ generated by

$$P(g, g') \qquad \text{and} \qquad Q(g)+(k+1)g \qquad \text{for all } g, g'\in\mathscr{G}$$

in the notation defined immediately before Proposition 4.5. We denote by

$$\varepsilon_k\colon \mathscr{M}^{(k+1)}\longrightarrow\mathscr{M}^{(k)}, \qquad \varepsilon\colon \mathscr{M}^{(0)}\longrightarrow\Xi, \qquad \beta_k\colon \mathscr{M}^{(k)}\longrightarrow\mathscr{O}_S$$

the $\mathscr{D}_S$-homomorphisms defined by

$$\varepsilon_k(m_{k+1}):=\partial_1 m_k, \quad \varepsilon(m_0):=\zeta, \quad \beta_k(m_k):=1.$$

Obviously, β_k is surjective with kernel $\mathscr{D}_S\Theta_S m_k$. By Theorem 4.2, (i), ε is surjective if ζ is a good primitive form.

As in Proposition 4.5, we have much simpler alternative defining relations

$$\partial\partial' m_k=(\partial*\partial')\partial_1 m_k, \qquad w(\partial)\partial_1 m_k=\{N(\partial)-(k+1)\partial\}m_k$$

for all ∂, ∂' in $\mathscr{G}^{\nabla}$. Consequently, we have

$$\partial' w(\partial)m_k=\{\partial'*N(\partial)-k(\partial'*\partial)\}m_k, \qquad \partial\partial'\partial_1^{j-1}m_k=(\partial*\partial')\partial_1^j m_k \qquad \text{and}$$

$$w(\partial)\partial'\partial_1^{j-1}m_k=\{N(\partial'*\partial)-(k+j)(\partial'*\partial)\}\partial_1^{j-1}m_k$$

for $j\geq 1$ by Lemma 4.3 and Proposition 4.4. In particular, we have

$$w(\partial)\partial_1^j m_k=\{N(\partial)-(k+j)\partial\}\partial_1^{j-1}m_k \qquad \text{for } j\geq 1.$$

Proposition 4.8. *For each good primitive form ζ and for each $k\in \boldsymbol{C}$, we have the following:*

(i) *$\mathcal{M}^{(k)}$ is a holonomic left $\mathcal{D}_S$-module with regular singularities such that $\mathcal{M}^{(k)}|_{S\setminus D}$ is a locally free $\mathcal{O}_{S\setminus D}$-module of rank $\mu+1$.*

(ii) *For the rational number r appearing in Theorem* 4.2, (iv), *$(w(\partial_1)-r+k)m_k$ generates a $\mathcal{D}_S$-submodule which is $\mathcal{O}_S$-free of rank one. If $k\neq r$, this gives a $\mathcal{D}_S$-splitting for the surjective $\mathcal{D}_S$-homomorphism $\beta_k\colon \mathcal{M}^{(k)}\to\mathcal{O}_S$.*

(iii) *Let l be the rank of the kernel of the $\boldsymbol{C}_T$-endomorphism $N-(k+1)$* id *of $\mathcal{G}^\nabla$. Then* ker $[\varepsilon_k\colon \mathcal{M}^{(k+1)}\to\mathcal{M}^{(k)}]$ *and* coker $[\varepsilon_k\colon \mathcal{M}^{(k+1)}\to\mathcal{M}^{(k)}]$ *are $\mathcal{O}_S$-locally free of rank $l+1$ (resp. l), if $k+1\neq r$* (resp. $k+1=r$).

Proof. (ii) is clear by $N(\partial_1)=r\partial_1$ and the relations satisfied by m_k mentioned above.

To show (i), let us note $w(\Theta_S)=\Theta_S(-\log D)$ by Proposition 4.1. For each $\nu\geq 0$, denote by $\mathcal{D}_S^{(\nu)}$ the subsheaf of $\mathcal{D}_S$ consisting of differential operators of order not greater than ν and consider the filtration $\{\mathcal{D}_S^{(\nu)}m_k;\ \nu\geq 0\}$ of $\mathcal{M}^{(k)}$ by coherent $\mathcal{O}_S$-submodules. By the relations satisfied by m_k mentioned above, we have

$$\mathcal{D}_S^{(\nu)}m_k=\mathcal{O}_S m_k+\textstyle\sum_{1\leq j\leq \nu}\Theta_S\partial_1^{j-1}m_k \qquad \text{and}$$
$$w(\Theta_S)\mathcal{D}_S^{(\nu)}m_k\subset\mathcal{D}_S^{(\nu)}m_k \qquad \text{for } \nu\geq 1.$$

Hence $\mathcal{M}^{(k)}$ is a holonomic left $\mathcal{D}_S$-module with regular singularities as in the proof of Theorem 2.3. Since the determinant $\varDelta$ of w is the defining equation for D in S by Lemma 2.4, (ii), the restriction of w to $S\setminus D$ is surjective.

(iii) coker $[\varepsilon_k]$ is generated over $\mathcal{D}_S$ by the image $\bar{m}_k$ of m_k with the relations

$$\partial_1\bar{m}_k=0, \qquad \partial\partial'\bar{m}_k=0, \qquad \{N(\partial)-(k+1)\partial\}\bar{m}_k=0$$

for all ∂, ∂' in $\mathcal{G}^\nabla$. Note that the $\boldsymbol{C}_T$-endomorphism N of $\mathcal{G}^\nabla$ is semisimple as we saw in the remark immediately before Proposition 4.5. On the other hand, $(w(\partial_1)-r+k+1)m_{k+1}$ and $w(\partial)m_{k+1}$ with $N(\partial)=(k+1)\partial$ are obviously contained in ker $[\varepsilon_k]$. Using the filtration above, we easily see that they in fact generate ker $[\varepsilon_k]$ over $\mathcal{O}_S$. q.e.d.

Remark. Saito also considers the quotient $\mathscr{D}_S$-module

$$\tilde{\mathscr{M}}^{(k)}:=\mathscr{M}^{(k)}/\mathscr{O}_S(w(\partial_1)-r+k)m_k,$$

which, by (ii) above, is a direct factor of $\mathscr{M}^{(k)}$ if $k\neq r$.

Recall that the sheaf of flat coordinates on S was defined in the remark immediately after Proposition 4.4 as

$$\mathscr{F}\mathscr{C}:=\{v\in\mathscr{O}_S;\ \partial\partial' v=0 \text{ for all } \partial,\ \partial'\in\mathscr{G}^r\}.$$

By Proposition 4.8 and by the same formula as that in Corollary 4.7, we get the following:

Theorem 4.9. *Given a good primitive form ζ, define constructible $\boldsymbol{C}_S$-modules for each $k\in\boldsymbol{C}$ by*

$$M_k:=\mathscr{H}om_{\mathscr{D}_S}(\mathscr{M}^{(k)},\mathscr{O}_S) \qquad \textit{and} \qquad M'_k:=\mathscr{H}om_{\mathscr{D}_S}(\mathscr{O}_S,\mathscr{M}^{(k)}).$$

(1) *We have an injective (resp. surjective) $\boldsymbol{C}_S$-homomorphism $\beta_k^*: \boldsymbol{C}_S\to M_k$ (resp. $(\beta_k)_*: M'_k\to\boldsymbol{C}_S$). Moreover, the $\boldsymbol{C}_S$-homomorphism $\varepsilon^*: R^n\varphi_!\boldsymbol{C}_X=\mathscr{H}om_{\mathscr{D}_S}(\Xi,\mathscr{O}_S)\to M_0$ is injective.*

(2) *The restriction to $S\setminus D$ of M_k and M'_k are mutually dual locally free $\boldsymbol{C}_{S\setminus D}$-modules of rank $\mu+1$.*

(3) *The evaluation map $M_k\to\mathscr{O}_S$ at m_k respectively sends $\beta^*(\boldsymbol{C}_S)$ and* $\ker[\varepsilon_k^*: M_k\to M_{k+1}]$ *isomorphically onto the sheaf $\boldsymbol{C}_S$ of constant functions and*

$$\{v\in\mathscr{F}\mathscr{C}:\partial_1 v=0,\ N(\partial)v=(k+1)\partial v \textit{ for all } \partial\in\mathscr{G}^r\},$$

which is $\boldsymbol{C}_S$-locally free of rank $l+1$ (resp. l) if $k+1\neq r$ (resp. $k+1=r$) for l as in Proposition 4.8, (iii).

(4) *We have a canonical $\boldsymbol{C}_S$-homomorphism $j_k: M_k\to M'_{n-k}$ such that $j_k(u)$ for $u\in M_k$ sends $1\in\mathscr{O}_S$ to $j_k(u)(1):=w(J^{-1}(d_S u(m_k)))m_{n-k}\in\mathscr{M}^{(n-k)}$, which is killed by Θ_S. We have $(\varepsilon_{n-k-1})_*\circ j_k=-j_{k+1}\circ\varepsilon_k^*$ with obvious induced $\boldsymbol{C}_S$-homomorphisms $\varepsilon_k^*: M_k\to M_{k+1}$ and $(\varepsilon_{n-k})_*: M'_{n-k}\to M'_{n-k-1}$.*

(5) *The restriction of j_k to $S\setminus D$ induces a $\boldsymbol{C}_{S\setminus D}$-isomorphism*

$$\operatorname{coker}[\beta_k^*]|_{S\setminus D}\xrightarrow[\sim]{j_k}\ker[(\beta_{n-k})_*]|_{S\setminus D}.$$

For simplicity, let us adopt the following in view of (3) above:

Definition. For a fixed primitive form ζ, denote by

$$d_S M_k\subset\Omega_S^1 \qquad k\in\boldsymbol{C}$$

the image of the C_S-homomorphism $M_k \to \mathcal{O}_S \to \Omega_S^1$ with kernel $\beta_k^*(C_S)$ obtained as the composite of the evaluation map at m_k and the exterior differentiation d_S.

Proposition 4.10. *Let ∇ be the connection on Ω_S^1 as in Proposition* 4.4, (ii).

(1) *For $k \in C$, $\theta \in \Theta_S$ and $u \in M_k$ we have*

$$\nabla_{w(\theta)}(d_S u(m_k)) = J(\theta * \{([(n-k)\,\mathrm{id} - N] \circ J^{-1})(d_S u(m_k))\}).$$

(2) *For $u_1, \cdots, u_\mu \in M_k$ we have*

$$\begin{aligned}&\nabla_{w(\theta)}(d_S u_1(m_k) \wedge \cdots \wedge d_S u_\mu(m_k)) \\ &\quad = \left\{\left(\frac{n-1}{2} - k\right) w(\theta)(\log \Delta)\right\} d_S u_1(m_k) \wedge \cdots \wedge d_S u_\mu(m_k)\end{aligned}$$

with respect to the induced connection on $\Omega_S^\mu = \det \Omega_S^1$. In particular, if $d_S u_1(m_k) \wedge \cdots \wedge d_S u_\mu(m_k) = \det(\partial_i u_j(m_k)) J(\partial_1') \wedge \cdots \wedge J(\partial_\mu')$ does not vanish for C_S-bases $\{\partial_i\}$ and $\{\partial_j'\}$ of $\mathcal{G}^\nabla$ with $J(\partial_i, \partial_j') = \delta_{ij}$, then

$$d_S u_1(m_k) \wedge \cdots \wedge d_S u_\mu(m_k) = c \Delta^{(n-1)/2-k} J(\partial_1') \wedge \cdots \wedge J(\partial_\mu')$$

for a nonzero constant c.

Proof. (1) is clear by the remark immediately after Proposition 4.4, since for the $\mathcal{O}_S$-endomorphism B of Θ_S defined by $B(\theta') := \theta * \theta'$ for $\theta' \in \Theta_S$, we have

$$w(\theta)\partial_j u(m_k) = \{(N \circ B)(\partial_j) - (k+1) B(\partial_j)\} u(m_k).$$

(2) By (1) above, we have

$$\begin{aligned}&\nabla_{w(\theta)}(d_S u_1(m_k) \wedge \cdots \wedge d_S u_\mu(m_k)) \\ &\quad = \{\mathrm{trace}\,(B \circ [(n-k)\,\mathrm{id} - N])\} d_S u_1(m_k) \wedge \cdots \wedge d_S u_\mu(m_k).\end{aligned}$$

Thus it suffices to show that

$$\frac{2}{n+1}\,\mathrm{trace}\,(B \circ N) = \mathrm{trace}\,(B) = w(\theta)(\log \Delta).$$

We have the first equality, since $J \circ B \circ J^{-1}$ and $J \circ N \circ J^{-1}$ are adjoint to B and $(n+1)\,\mathrm{id} - N$, respectively, by Lemma 4.3. By Proposition 4.1, (3), we have $w(\theta)(\log \Delta) = \theta(\mathrm{trace}\,(w)) = \mu(\theta t_1) - \theta(\mathrm{trace}\,(A))$. The extreme right hand side is the trace of the $\mathcal{O}_S$-endomorphism of Θ_S sending $g \in \Theta_S$ to $(\theta t_1) - \nabla_\theta(A(g)) + A(\nabla_\theta(g))$, which is equal by Lemma 4.3, (4) to $-N(\theta * g) + \theta * (N(g) + g) = \{B \circ N - N \circ B + B\}(g)$. q.e.d.

The $\mathcal{D}_S$-homomorphism $\varepsilon_k: \mathcal{M}^{(k+1)} \to \mathcal{M}^{(k)}$ obviously induces $\nabla_{\partial_1}: d_S M_k \to d_S M_{k+1}$. By Theorem 4.9, (2), (5) we have a $C_{S\setminus D}$-isomorphism

$$j_k: d_S M_k|_{S\setminus D} \xrightarrow{\sim} \mathcal{H}om_{C_S}(d_S M_{n-k}, C_S)|_{S\setminus D}.$$

Moreover, if ζ is a good primitive form, the surjective $\mathcal{D}_S$-homomorphism $\varepsilon: \mathcal{M}^{(0)} \to \Xi$ induces a C_S-homomorphism

$$R^n \varphi_! C_X = \mathcal{H}om_{\mathcal{D}_S}(\Xi, \mathcal{O}_S) \longrightarrow M_0 \longrightarrow d_S M_0,$$

whose restriction to $S\setminus D$ is an isomorphism if $r \neq 0$. Thus by Corollary 4.7, Theorem 4.9 and Proposition 4.10, we get the following:

Corollary 4.11. *Let ζ be a good primitive form with $r \neq 0$. Then $d_S M_k|_{S\setminus D}$ and $d_S M_{n-k}|_{S\setminus D}$ for each $k \in C$ are locally free $C_{S\setminus D}$-modules of rank μ canonically dual to each other. We have a sequence of $C_{S\setminus D}$-homomorphisms*

$$\begin{aligned} R^n\varphi_! C_X|_{S\setminus D} \xrightarrow{\sim} d_S M_0|_{S\setminus D} \xrightarrow{\nabla_{\partial_1}} d_S M_1|_{S\setminus D} \longrightarrow \cdots \\ \longrightarrow d_S M_{n-1}|_{S\setminus D} \xrightarrow{\nabla_{\partial_1}} d_S M_n|_{S\setminus D} \xrightarrow{\sim} R^n\varphi_* C_X|_{S\setminus D}, \end{aligned}$$

the first and the last of which are isomorphisms. The composite coincides with $(-1)^n \kappa'|_{S\setminus D}$, where κ' is as in Corollary 4.7. Moreover, for $k \in Z$, we have

$$\det(d_S M_k|_{S\setminus D}) = \Delta^{(n-1)/2-k} \det((\Omega_S^1)^\Gamma|_{S\setminus D})$$

as $C_{S\setminus D}$-submodules of $\Omega^\mu_{S\setminus D}$. In particular, if n is even, then $d_S M_{n/2}|_{S\setminus D}$ is a canonically self dual locally free $C_{S\setminus D}$-module of rank μ such that

$$\det((\Omega_S^1)^\Gamma|_{S\setminus D}) = \Delta^{1/2} \det(d_S M_{n/2}|_{S\setminus D}).$$

Remark. It is quite likely that $r \neq 0$ for any good primitive form. As Saito poses, it might also be an interesting problem to study the constructible C_S-modules

$$\mathcal{E}xt^j_{\mathcal{D}_S}(\mathcal{M}^{(k)}, \mathcal{O}_S) \qquad \text{for } j > 0.$$

There is an attempt in a special case by Maisonobe-Miniconi [MM] by means of the Cauchy-Kowalevsky theorem.

From now on, we fix a good primitive form ζ with $r \neq 0$ and assume n to be even. Here is a brief sketch of the construction of Saito's period map.

By Theorem 4.9, $M_{n/2}|_{S\setminus D}$ is $\boldsymbol{C}_{S\setminus D}$-locally free of rank $\mu+1$. The evaluation at $m_{n/2}$ maps it injectively to $\mathcal{O}_{S\setminus D}$ sending the subsheaf $\beta^*_{n/2}(\boldsymbol{C}_{S\setminus D})$ isomorphically onto the sheaf $\boldsymbol{C}_{S\setminus D}$ of constant functions. There exists the smallest, possibly infinite, unramified covering λ: $Y\to S\setminus D$ such that $\lambda^*(M_{n/2}|_{S\setminus D})$ becomes $\boldsymbol{C}_Y$-free. Thus there exists a $(\mu+1)$-dimensional complex vector space V with a fixed nonzero $v_0 \in V$ such that $\lambda^*(M_{n/2}|_{S\setminus D})$ is the constant sheaf V_Y and that v_0 is mapped to the constant function 1 by the pulled-back evaluation map $V_Y\to\mathcal{O}_Y$. Thus we have a holomorphic map $Y\to V^*$ to the dual complex vector space V^* with the image contained in the μ-dimensional affine hyperplane

$$E:=\{l \in V^*;\ l(v_0)=1\}.$$

$Y\to E$ is necessarily a local isomorphism. Indeed, the image of the composite $V_Y\to\mathcal{O}_Y\xrightarrow{d_Y}\Omega^1_Y$ generates Ω^1_Y over $\mathcal{O}_Y$, since $d_S M_{n/2}|_{S\setminus D}$ generates $\Omega^1_{S\setminus D}$ over $\mathcal{O}_{S\setminus D}$. Note that $\Omega^1_E=\mathcal{O}_E\otimes_{\boldsymbol{C}}(V/\boldsymbol{C}v_0)$.

Saito then constructs a nonsingular complex manifold $\tilde{S}$ with a reduced effective divisor $\tilde{D}$ satisfying the following properties: (1) $Y=\tilde{S}\setminus\tilde{D}$, (2) λ: $Y\to S\setminus D$ extends to a flat holomorphic map α: $\tilde{S}\to S$ with the image equal to $\{s \in S;\ X_s=\varphi^{-1}(s)$ has at most simple singularities$\}$, (3) for any $s \in D$, there exists a neighborhood U in S of s such that the restriction $\tilde{U}\to U$ of α to any connected component $\tilde{U}$ of $\alpha^{-1}(U)$ is finite (i.e., proper with finite fibers) and (4) $Y\to E$ extends to a local isomorphism

$$P\colon \tilde{S}\longrightarrow E.$$

This P is *Saito's period map*.

By Proposition 4.10 and Corollary 4.11, the Jacobian determinant of the ramified covering map α: $\tilde{S}\to S$ is

$$\operatorname{Jac}(\alpha)=c\Delta^{1/2} \qquad \text{for a nonzero constant } c$$

with respect to flat coordinates of S.

By Theorem 4.9 and Corollary 4.11, the $\boldsymbol{C}_S$-homomorphism $j_{n/2}$: $M_{n/2}\to M'_{n/2}$ induces the self duality of $d_S M_{n/2}|_{S\setminus D}$. Hence we obtain a $\boldsymbol{C}$-linear map $i'_{n/2}$: $V\to V^*$ which induces an isomorphism

$$i_{n/2}\colon V/\boldsymbol{C}v_0\xrightarrow{\sim}(V/\boldsymbol{C}v_0)^*.$$

Note that $j_{n/2}$ corresponds to the symmetric $\boldsymbol{C}_S$-bilinear form $J_{n/2}$: $M_{n/2}\times M_{n/2}\to\boldsymbol{C}_S$ given by

$$J_{n/2}(u, u'):=\{w(J^{-1}(d_S u(m_{n/2})))\}u'(m_{n/2})$$

for $u, u' \in M_{n/2}$. This induces the symmetric $\boldsymbol{C}$-bilinear form $I'_{n/2}$ on V corresponding to $i'_{n/2}$ above, hence the nondegenerate symmetric $\boldsymbol{C}$-bilinear form $I_{n/2}$ on $V/\boldsymbol{C}v_0$ corresponding to $i_{n/2}$ above.

Consider the $\boldsymbol{C}_S$-submodule of $M_{n/2}$ defined by

$$\tilde{M}_{n/2} := \mathcal{H}om_{\mathcal{D}_S}(\tilde{\mathcal{M}}^{(n/2)}, \mathcal{O}_S) = \{u \in M_{n/2};\ w(\partial_1)u = (r - n/2)u\}$$

in the notation of the remark immediately after Proposition 4.8. If $r \neq n/2$, then by Proposition 4.8, (ii), $\tilde{M}_{n/2}|_{S\setminus D}$ is mapped isomorphically onto $d_S M_{n/2}|_{S\setminus D}$. Hence we have a codimension one $\boldsymbol{C}$-subspace $\tilde{V}$ of V which is mapped isomorphically onto $V/\boldsymbol{C}v_0$. In this case, the $\boldsymbol{C}$-bilinear form on $\tilde{V}$ induced by $I'_{n/2}$ can be identified with $I_{n/2}$ on $V/\boldsymbol{C}v_0$. $I_{n/2}$ induces a nondegenerate symmetric $\boldsymbol{C}$-bilinear form, hence a nondegenerate quadratic form Q on $(V/\boldsymbol{C}v_0)^* \cong \tilde{V}^*$. We can show that

$$P^*(Q) = \alpha^*(\tau)/(r - n/2) \qquad \text{if } r \neq n/2,$$

where $P^*(Q)$ is the pulled-back function to $\tilde{S}$ of Q and τ is a flat function on S as defined in the remark immediately after Proposition 4.4. For the proof, we use the fact that $w(J^{-1}(d_S\tau)) = w(\partial_1)$, which acts on $\tilde{M}_{n/2}$ as the multiplication by $(r - n/2)$.

Remark. Here is a comparison between the period map above and the conventional one found, for instance, in Looijenga [L1], Saito [Sk7] and Varchenko-Givental [VG]:

By Corollary 4.11, we have a $\boldsymbol{C}_{S\setminus D}$-homomorphism

$$(\nabla_{\partial_1})^{n/2}\colon R^n\varphi_! \boldsymbol{C}_X|_{S\setminus D} \longrightarrow d_S M_{n/2}|_{S\setminus D}.$$

This is an isomorphism if and only if the $\boldsymbol{C}_T$-endomorphism N of $\mathcal{G}^\nabla$ has no positive integral eigenvalues by Theorem 4.9, (3). If we use $R^n\varphi_!\boldsymbol{C}_X|_{S\setminus D}$ instead of $d_S M_{n/2}|_{S\setminus D}$ we get another period map $P'\colon \tilde{S}' \to E'$ from a ramified covering $\tilde{S}'$ of S to a μ-dimensional complex affine space E'. Since $(\nabla_{\partial_1})^{n/2}$ might have nontrivial kernel in general, P' has less chance of being injective.

Since this survey is already getting too long, we do not attempt to write out the interesting formulas in this section explicitly in the case of the universal unfolding of a weighted homogeneous polynomial $f(x)$ with simple or simple elliptic critical points. We refer the reader, for instance, to Looijenga [L1] ~ [L4], Saito [Sk4], [Sk15] ~ [Sk19], Saito-Yano-Sekiguchi [SYS], Ishiura-Noumi [IN], Noumi [N2], Varchenko [V1] ~ [V3] and Varchenko-Givental [VG].

Appendix. Algebro-analytic overview

In this appendix, we have a brief look at Saito's results in the previous sections from much more powerful perspective of algebraic analysis.

We have already given an introductory survey of algebraic analysis in [O1], to which we refer the reader for necessary notation and references. We point out, however, that Kashiwara [K2] and Kashiwara-Schapira [KS] have since appeared and contain the proofs of some of the basic results necessary here.

We do not go into the more recent theory of filtered regular holonomic $\mathscr{D}_S$-modules due to Brylinski [B2], [B3], M. Saito [Sm3], [Sm4] and others which combines algebraic analysis and the theory of Hodge structures.

Let us first recall the Riemann-Hilbert correspondence.

Let S be an m-dimensional complex manifold. On the one hand, $\boldsymbol{D}^b_{rh}(\mathscr{D}_S)$ denotes the derived category of bounded complexes of left $\mathscr{D}_S$-modules whose cohomology sheaves are holonomic with regular singularities. On the other hand, $\boldsymbol{D}^b_c(\boldsymbol{C}_S)$ denotes the derived category of bounded complexes of $\boldsymbol{C}_S$-modules whose cohomology sheaves are constructible $\boldsymbol{C}_S$-modules. Then the *de Rham functor* $\mathrm{DR}_S(M) := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_S}(\mathscr{O}_S, M)$ gives an equivalence of categories

$$\mathrm{DR}_S\colon \boldsymbol{D}^b_{rh}(\mathscr{D}_S) \xrightarrow{\sim} \boldsymbol{D}^b_c(\boldsymbol{C}_S),$$

which is due to Kashiwara-Kawai and Mebkhout and is called the *Riemann-Hilbert correspondence*. The subcategory $\mathscr{P}erv(\boldsymbol{C}_S)$ of *perverse sheaves*, by definition, corresponds via DR_S to the subcategory of $\boldsymbol{D}^b_{rh}(\mathscr{D}_S)$ of objects arising from single holonomic left $\mathscr{D}_S$-modules with regular singularities. For instance, we have

$$\mathrm{DR}_S(\mathscr{O}_S) = \boldsymbol{C}_S.$$

We have the *adjoint functor* A_S from $\boldsymbol{D}^b_{rh}(\mathscr{D}_S)$ to itself defined by

$$A_S(M) := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_S}(M, \mathscr{D}_S) \otimes_{\mathscr{O}_S} \omega_S^{-1}[3m] = M^*[2m],$$

where $M^* := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_S}(M, \mathscr{D}_S) \otimes_{\mathscr{O}_S} \omega_S^{-1}[m]$ with $m = \dim S$. In particular, we have

$$A_S(\mathscr{O}_S) = \mathscr{O}_S[2 \dim S] \qquad \text{and} \qquad \mathscr{O}_S^* = \mathscr{O}_S.$$

On the other hand, we have the *Verdier dual functor* V_S from $\boldsymbol{D}^b_c(\boldsymbol{C}_S)$ into itself defined by

$$V_S(F) := \boldsymbol{R}\mathscr{H}om_{\boldsymbol{C}_S}(F, \boldsymbol{C}_S[2m]),$$

which satisfies $V_S(\boldsymbol{C}_S)=\boldsymbol{C}_S[2m]$. The Riemann-Hilbert correspondence is known to be compatible with these dualities, i.e., the *Poincaré-Verdier duality* holds:

$$\mathrm{DR}_S \circ A_S = V_S \circ \mathrm{DR}_S.$$

Sometimes convenient is the anti-equivalence

$$\mathrm{Sol}_S\colon \boldsymbol{D}^b_{rh}(\mathscr{D}_S)^\circ \xrightarrow{\sim} \boldsymbol{D}^b_c(\boldsymbol{C}_S)$$

given by the contravariant *solution functor* $\mathrm{Sol}_S(M):=\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_S}(M,\mathscr{O}_S)=\mathrm{DR}_S(M^*)=\boldsymbol{R}\mathscr{H}om_{\boldsymbol{C}_S}(\mathrm{DR}_S(M),\boldsymbol{C}_S)$.

For a holomorphic map $\varphi\colon X\to S$ of complex manifolds, we have contravariant functors

$$\boldsymbol{R}\varphi_*,\ \boldsymbol{R}\varphi_!\colon \boldsymbol{D}^b_c(\boldsymbol{C}_X)\longrightarrow \boldsymbol{D}^b_c(\boldsymbol{C}_S)$$

taking the direct image and that with proper support, which satisfy the *Verdier duality*

$$\boldsymbol{R}\varphi_*\circ V_X = V_S\circ \boldsymbol{R}\varphi_! \qquad \text{and} \qquad \boldsymbol{R}\varphi_!\circ V_X = V_S\circ \boldsymbol{R}\varphi_*.$$

On the other hand, we have the notions of the integration along the fibers of φ and that with proper support: For a complex N of left $\mathscr{D}_X$-modules, we let

$$\int_\varphi N:=\boldsymbol{R}\varphi_*(\mathscr{D}_{S\leftarrow X}\otimes^{\boldsymbol{L}}_{\mathscr{D}_X}N)[-\dim X+\dim S]$$
$$\int_\varphi^{\mathrm{pr}} N:=\boldsymbol{R}\varphi_!(\mathscr{D}_{S\leftarrow X}\otimes^{\boldsymbol{L}}_{\mathscr{D}_X}N)[-\dim X+\dim S].$$

This definition is different from the usual one appearing in the theory of Kashiwara and Kawai which does not have the degree shift by $-\dim X+\dim S$. We adopt the present definition, since it behaves better under the de Rham functors.

One of the basic results in algebraic analysis is the following result of Kashiwara [K2, Theorem 8.1]: When $\varphi\colon X\to S$ is *proper*, the integration along the fibers induces the functor

$$\int_\varphi\colon \boldsymbol{D}^b_{rh}(\mathscr{D}_X)\longrightarrow \boldsymbol{D}^b_{rh}(\mathscr{D}_S),$$

which satisfies

$$\mathrm{DR}_S\circ\int_\varphi=\boldsymbol{R}\varphi_*\circ\mathrm{DR}_X \qquad \text{and} \qquad A_S\circ\int_\varphi=\int_\varphi\circ A_X,$$

the latter being the *Poincaré duality.*

When φ is not proper, $\int_\varphi$ and $\int_\varphi^{\mathrm{pr}}$ no longer give rise to functors from $\boldsymbol{D}_{rh}^b(\mathscr{D}_X)$ to $\boldsymbol{D}_{rh}^b(\mathscr{D}_S)$. Nevertheless, Kashiwara-Schapira [KS, Theorem 9.4.1] shows the following when φ and $N\in\boldsymbol{D}_{rh}^b(\mathscr{D}_X)$ satisfy certain conditions (e.g., φ is non-characteristic for N):

(1) $\int_\varphi N$ and $\int_\varphi^{\mathrm{pr}} N$ belong to $\boldsymbol{D}_{rh}^b(\mathscr{D}_S)$.

(2) $\mathrm{DR}_S\left(\int_\varphi N\right)=\boldsymbol{R}\varphi_*(\mathrm{DR}_X(N))$ and $\mathrm{DR}_S\left(\int_\varphi^{\mathrm{pr}} N\right)=\boldsymbol{R}\varphi_!(\mathrm{DR}_X(N))$.

(3) The Poincaré duality holds:

$$A_S\left(\int_\varphi N\right)=\int_\varphi^{\mathrm{pr}} A_X(N) \qquad \text{and} \qquad A_S\left(\int_\varphi^{\mathrm{pr}} N\right)=\int_\varphi A_X(N).$$

In particular, the above condition is satisfied in the case of our concern in this paper, i.e., $N=\mathcal{O}_X$ and the fibers of $\varphi\colon X\to S$ have at most isolated singularities. Hence we have the following:

(1′) The cohomology sheaves

$$\int_\varphi^j \mathcal{O}_X:=\mathscr{H}^j\left(\int_\varphi \mathcal{O}_X\right) \qquad \text{and} \qquad \int_\varphi^{\mathrm{pr},j} \mathcal{O}_X:=\mathscr{H}^j\left(\int_\varphi^{\mathrm{pr}} \mathcal{O}_X\right)$$

are holonomic left $\mathscr{D}_S$-modules with regular singularities.

(2′) $\mathrm{DR}_S\left(\int_\varphi \mathcal{O}_X\right)=\boldsymbol{R}\varphi_*(\boldsymbol{C}_X)$ and $\mathrm{DR}_S\left(\int_\varphi^{\mathrm{pr}} \mathcal{O}_X\right)=\boldsymbol{R}\varphi_!(\boldsymbol{C}_X)$.

(3′) The poincaré duality

$$A_S\left(\int_\varphi \mathcal{O}_X\right)=\int_\varphi^{\mathrm{pr}} A_X(\mathcal{O}_X) \qquad \text{and} \qquad A_S\left(\int_\varphi^{\mathrm{pr}} \mathcal{O}_X\right)=\int_\varphi A_X(\mathcal{O}_X)$$

holds. Since $A_X(\mathcal{O}_X)=\mathcal{O}_X[2\dim X]$, $A_S(M)=M^*[2\dim S]$ and $n=\dim X-\dim S$, we have

$$\left(\int_\varphi \mathcal{O}_X\right)^*=\int_\varphi^{\mathrm{pr}} \mathcal{O}_X[2n] \qquad \text{and} \qquad \left(\int_\varphi^{\mathrm{pr}} \mathcal{O}_X\right)^*=\int_\varphi \mathcal{O}_X[2n].$$

We now relate these results to Saito's results in the previous sections by showing

$$(*) \qquad \int_\varphi \mathcal{O}_X=\boldsymbol{R}\varphi_*(K^\cdot,\delta) \qquad \text{and} \qquad \int_\varphi^{\mathrm{pr}} \mathcal{O}_X=\boldsymbol{R}\varphi_!(K^\cdot,\delta).$$

Hence $\int_\varphi^j \mathcal{O}_X=\boldsymbol{R}^j\varphi_*(K^\cdot,\delta)$, $\int_\varphi^{\mathrm{pr},j} \mathcal{O}_X=\boldsymbol{R}^j\varphi_!(K^\cdot,\delta)$ and we in particular have the following:

$$\int_{\varphi}^{j} \mathcal{O}_X = \begin{cases} \mathcal{O}_S & j=0 \\ 0 & j\neq 0, n \\ \Xi & j=n \end{cases} \quad \text{and} \quad \int_{\varphi}^{\mathrm{pr},j} \mathcal{O}_X = \begin{cases} \Xi_! & j=n \\ 0 & j\neq n, 2n \\ \mathcal{O}_S & j=2n \end{cases}$$

where Ξ and $\Xi_!$ are holonomic left $\mathcal{D}_S$-modules with regular singularities (cf. Theorems 1.1, 1.2 and 2.3). We thus have canonical isomorphisms of $\boldsymbol{C}_S$-modules

$$\mathcal{H}om_{\mathcal{D}_S}(\mathcal{O}_S, \Xi_!) = R^n\varphi_! \boldsymbol{C}_X = \mathcal{H}om_{\mathcal{D}_S}(\Xi, \mathcal{O}_S),$$
$$\mathcal{H}om_{\mathcal{D}_S}(\mathcal{O}_S, \Xi) = R^n\varphi_* \boldsymbol{C}_X = \mathcal{H}om_{\mathcal{D}_S}(\Xi_!, \mathcal{O}_S).$$

As in Sections 2 and 4, consider the fiber product $Z:=X\times_T S$ with the projections $p: Z\to S$ and $\Pi: Z\to X$. Moreover, $\sigma: X\to Z$ is the closed embedding onto the nonsingular hypersurface $\sigma(X)=\{(x, s) \in Z; F(x, s)=0\}$. Since $\varphi=p\circ\sigma$ with σ proper, we have

$$\int_{\varphi} \mathcal{O}_X = \int_p \int_{\sigma} \mathcal{O}_X \quad \text{and} \quad \int_{\varphi}^{\mathrm{pr}} \mathcal{O}_X = \int_p^{\mathrm{pr}} \int_{\sigma} \mathcal{O}_X$$

by generality on the integration along the fibers. We also know that

$$M:=\int_{\sigma} \mathcal{O}_X = \mathcal{H}^1_{[\sigma(X)]}(\mathcal{O}_Z)[1] = (\mathcal{O}_Z[F^{-1}]/\mathcal{O}_Z)[1]$$

as left $\mathcal{D}_Z$-modules. Note that M is supported on $\sigma(X)$ but is not a module over $\sigma_*\mathcal{O}_X=\mathcal{O}_Z/F\mathcal{O}_Z$. By the quasi-isomorphism

$$\Omega^{\bullet}_{Z/S}\otimes_{\mathcal{O}_Z}\mathcal{D}_Z \longrightarrow \mathcal{D}_{S\leftarrow Z}[n+1]$$

we get

$$\int_{\varphi} \mathcal{O}_X = \int_p M := \boldsymbol{R}p_*(\mathcal{D}_{S\leftarrow Z}\otimes^{L}_{\mathcal{D}_Z} M)[-n-1] = \boldsymbol{R}p_*(\Omega^{\bullet}_{Z/S}\otimes_{\mathcal{O}_Z} M, \nabla)$$

and similarly

$$\int_{\varphi}^{\mathrm{pr}} \mathcal{O}_X = \int_p^{\mathrm{pr}} M = \boldsymbol{R}p_!(\Omega^{\bullet}_{Z/S}\otimes_{\mathcal{O}_Z} M, \nabla),$$

where $(\Omega^{\bullet}_{Z/S}\otimes_{\mathcal{O}_Z} M, \nabla)$ is the complex of left $p^{-1}(\mathcal{D}_S)$-modules arising from the integrable connection $\nabla: M\to\Omega^1_{Z/S}\otimes_{\mathcal{O}_Z} M$ defined by the left $\mathcal{D}_Z$-module structure of M and satisfying $\nabla(\omega m)=(d\omega)\otimes m+(-1)^j\omega\wedge\nabla m$ for $m\in M$, $\omega\in\Omega^j_{Z/S}$ and $d:=d_{Z/S}$.

To prove our claim $(*)$ above, it thus remains to show that the complex $(\Omega^{\bullet}_{Z/S}\otimes_{\mathcal{O}_Z} M, \nabla)$ of left $p^{-1}(\mathcal{D}_S)$-modules is isomorphic to the restriction $(\Pi^{-1}(K^{\bullet})|_{\sigma(X)}, \delta)$ to $\sigma(X)$ of the pull-back $(\Pi^{-1}(K^{\bullet}), \delta)$. Indeed, we have

$$\Omega^{\cdot}_{Z/S}=\Pi^*(\Omega^{\cdot}_{X/T})=\Pi^{-1}(\Omega^{\cdot}_{X/T})\otimes_{\Pi^{-1}(\mathcal{O}_X)}\mathcal{O}_Z.$$

Moreover, we have a $\Pi^{-1}(\mathcal{O}_X)$-isomorphism

$$\mathcal{O}_Z[F^{-1}]/\mathcal{O}_Z=\oplus_{k\geq 0}(\Pi^{-1}(\mathcal{O}_X)|_{\sigma(X)})\lambda_k,$$

where λ_k is the class of $(-1)^k k!/2\pi i F^{k+1}$ modulo $\mathcal{O}_Z$. Since $F=t_1-F_1(x, t')$, we see that

$$t_1\lambda_k=F_1\lambda_k-k\lambda_{k-1}, \qquad \nabla\lambda_k=-(dF_1)\lambda_{k+1} \qquad \text{and}$$
$$\theta\lambda_k=(\theta F)\lambda_{k+1} \qquad \text{for } \theta \in p^{-1}(\Theta_S).$$

Thus the obvious isomorphism from $\Pi^{-1}(\Omega^{j+1}_{X/T})|_{\sigma(X)}\partial_1^k$ to $\Pi^{-1}(\Omega^{j+1}_{X/T})|_{\sigma(X)}\lambda_k$ sending $\omega\partial_1^k$ to $\omega\lambda_k$ gives rise to a left $p^{-1}(\mathcal{D}_S)$-isomorphism from $\Pi^{-1}(K^{\cdot})|_{\sigma(X)}$ to

$$\begin{aligned}\Omega^{\cdot}_{Z/S}\otimes_{\mathcal{O}_Z}M&=\Pi^{-1}(\Omega^{\cdot}_{X/T})\otimes_{\Pi^{-1}(\mathcal{O}_X)}(\oplus_{k\geq 0}\Pi^{-1}(\mathcal{O}_X)|_{\sigma(X)}\lambda_k)[1]\\&=(\oplus_{k\geq 0}\Pi^{-1}(\Omega^{\cdot}_{X/T})|_{\sigma(X)}\lambda_k)[1],\end{aligned}$$

which is easily seen to be compatible with δ and ∇.

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