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Coverings of Algebraic Varieties

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Introduction

In this article we survey and prove some of the results about (unramified) coverings of algebraic varieties. Recently Madhav Nori has asked the following question.

Conjecture A. *Let S be a projective, non-singular surface over the field of complex numbers $\boldsymbol{C}$. Suppose D is an effective divisor on S with $D^2>0$. Let N be the normal subgroup of $\pi_1(S)$ generated by the images of the fundamental groups of the non-singular models of all the irreducible components of D. Then the index $[\pi_1(S):N]$ is finite.*

If the conjecture is true, then any surface (smooth, projective) possessing a (possibly singular) rational curve of positive self-intersection would have a finite fundamental group! In [7] Nori verifies the conjecture in a special case. Surprisingly, this conjecture is related to the following old question:

Conjecture B. *Let X be a smooth, projective variety over $\boldsymbol{C}$. Then the universal covering space of X is holomorphically convex.*

Recall that a complex manifold M is said to be holomorphically convex if given a sequence of distinct points $x_1, x_2, \cdots$ in M without a limit point in M, there exists a holomorphic function f on M such that the set $\{f(x_n)\}_{n=1,2,\ldots}$ is unbounded.

A compact, complex manifold is vacuously holomorphically convex. We will prove the following results in this paper.

(1) (*See* § 1, *Proposition* 1). *Suppose every covering space $\tilde{S}\to S$ is holomorphically convex. Then Conjecture* A *is valid for S, if D is an irreducible curve. If the universal covering space is holomorphically convex, then Conjecture* A *is true, if D is a rational curve* (*possibly singular*).

(2) (*See* § 1, *Theorem*). *Let $\pi: S\to\Delta$ be an elliptic surface. If*

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$\chi(S, \mathcal{O}_s)=\sum_j(-1)^j \dim_{\boldsymbol{C}} H^j(S, \mathcal{O}_s)>0$, *then any covering space of* S *is holomorphically convex. If* $\chi(S, \mathcal{O}_s)=0$, *then the universal covering space of* S *is holomorphically convex.*

First we discuss related results and then turn to the sketch of the proofs.

(i) If C is a rational curve on S with $C^2>0$, then we shall show that $H^1(S, \mathcal{O})=0$ and hence $\chi(S, \mathcal{O})>0$. In particular, if S is an elliptic surface then Nori's conjecture is valid for S in the case C is a rational curve on S with $C^2>0$. Hence $\pi_1(S)$ is finite.

(ii) There exists an elliptic surface $S \simeq C_1 \times C_2$, where C_i are elliptic curves, such that S has a covering space which is not holomorphically convex. See [3] for details.

(iii) Using the Enriques classification of surfaces it follows that Conjecture B is verified for all algebraic surfaces which are not of general type.

(iv) The universal covering $\tilde{S}$ of a projective, non-singular surface S may not be embeddable as an analytic subset (open or closed) in any $\boldsymbol{C}^n$. For, if $C \subset S$ is a rational curve, then there exists a lift $\bar{C} \to \tilde{S}$ where $\bar{C}$ is the non-singular model of C. Since $\bar{C} \simeq \boldsymbol{P}^1$, the map $\bar{C} \to \tilde{S}$ must be constant (if $\tilde{S} \subset \boldsymbol{C}^n$).

Conjecture B has been verified in some cases. In [9] Siegel proved that if $\tilde{S}$ is a bounded open subset of $\boldsymbol{C}^n$, then $\tilde{S}$ is a Stein manifold. Recently Mok and Wong have generalized this to coverings of quasi-projective varieties. See [6].

Using results of Kajiwara-Sakai, James Carlson and Reese Harvey have verified Conjecture B for a compact Moishezon manifold whose universal covering space is a domain spread over an open subset of a Stein manifold. See [2].

In [8] Shabat has proved that if $\pi: S \to \Delta$ is a holomorphic map with Δ a compact Riemann surface of genus ≥ 2 and all fibres of π are compact Riemann surfaces of genus ≥ 2, then the universal covering space of S is holomorphically convex.

A result of Griffiths asserts that any smooth projective variety has a Zariski open subset whose universal covering space is a bounded domain in $\boldsymbol{C}^n$ (and holomorphically convex). The results of Griffiths and Shabat are easy consequences of Bers' simultaneous uniformization theorem. See [1].

§ 1.

We sketch the proofs of some of the results.

Proposition 1. *Suppose every regular covering space* $\tilde{S}$ *of* S *is holomorphically convex. If* $C \subset S$ *is an irreducible curve with* $C^2>0$, *then the* *normal subgroup generated by the image of* $\pi_1(\overline{C})$ *in* $\pi_1(S)$ *has finite index.* (*Here* $\overline{C}$ *is the non-singular model of* C).

Proof. Let $H=$ image $\{\pi_1(\overline{C})\to\pi_1(S)\}$ and consider the covering φ: $\tilde{S}\to S$ such that $\varphi_*\pi_1(\tilde{S})$ is the normal subgroup generated by H. By assumption, $\tilde{S}$ is holomorphically convex.

By generalization of the Lefschetz hyperplane section theorem proved in [7], it follows that $\pi_1(C)\to\pi_1(S)$ is surjective (since $C^2>0$). Hence $\varphi^{-1}(C)$ is a connected curve on $\tilde{S}$. By construction, there exists a lift $\overline{C}\to\tilde{S}$. Let $\varphi^{-1}(C)=\bigcup_{i=1}^r C_i$, where C_i are irreducible components of $\varphi^{-1}(C)$. Corresponding to any C_i, there exists a lift $\overline{C}\to\tilde{S}$ with image C_i, because φ is a regular covering. Thus each C_i is compact. Choose points $x_i\in C_i$. If the set $\{x_i\}$ is infinite, it has no limit point in $\tilde{S}$. There exists a holomorphic function f on $\tilde{S}$ which is unbounded on $\{x_1, x_2, \cdots\}$. But $f|C_i$ is constant for every i and $\bigcup_{i=1}^r C_i$ is connected, a contradiction.

The main result in this paper is the following.

Theorem. *Let* $\pi\colon S\to\Delta$ *be an elliptic surface* (*which is projective*). *If* $\chi(S,\mathcal{O})>0$, *then any covering space of* S *is holomorphically convex. If* $\chi(S,\mathcal{O})=0$, *then the universal covering space of* S *is holomorphically convex.*

It is easy to see that, for proving the above result, we can assume that no fibre of π contains an exceptional curve of the first kind. The proof depends on the following:

Proposition 2. *Let* $\pi\colon S\to\Delta$ *be a minimal elliptic fibration with* Δ *a compact Riemann surface of genus* g. *For a general fibre* F *of* π, *let* I *denote the image of* $\pi_1(F)$ *in* $\pi_1(S)$. *Then we have an exact sequence*

$$(1)\longrightarrow I\longrightarrow\pi_1(S)\longrightarrow\Gamma\longrightarrow(1);$$

where $\Gamma=\langle x_i, y_i, \alpha_j;\ 1\le i\le g,\ 1\le j\le r\mid \prod_{i=1}^g[x_i, y_i]\prod_{j=1}^r\alpha_j=1,\ \alpha_j^{m_j}=1\rangle$. *Here* r *is the number of singular fibres of* π, *with multiplicities* $m_1,\cdots,m_r$ (≥1). *If* π *has at least one singular fibre not of the type* mI_0, *then* I *is a cyclic group of odd order. If further* $g=0$, *then* $I=\{1\}$.

We will refer the reader to the paper [3] for the proof of this proposition. The proof involves detailed description of the singular fibres and the monodromy around singular fibres described in the fundamental papers of Kodaira [4], [5]. To deduce the Theorem from Proposition 2, we use:

Lemma 1. *Suppose* $\Delta\simeq\mathbf{P}^1$ *and* π *has at most two singular fibres, both of type* ${}_mI_0$. *Then* S *is birationally a ruled surface and hence any convering of* S *is holomorphically convex.*

Proof. The fact that $|nK_s|=\phi$ for all $n\geq 1$ follows easily from the canonical bundle formula for S. For a ruled surface $X\to C$, any covering $\tilde{X}$ of X arises from a covering of C and base change. Finally, one knows that any Riemann surface is holomorphically convex, from which it follows trivially that $\tilde{X}$ is holomorphically convex.

Proof of Theorem. It is easy to see that for a minimal elliptic fibration $\pi\colon S\to\Delta$, $\chi(S,\mathcal{O})>0$ if and only if π has at least one singular fibre not of type ${}_mI_0$. Suppose first $g=0$. If $\chi(S,\mathcal{O})>0$ and π has at most two multiple fibres, then we can show that $\pi_1(S)$ is finite. For a proof, see [3]. In this case the universal covering space $\tilde{S}$ of S is compact and we are done. If $\chi(S,\mathcal{O})=0$ and π has at most two singular fibers (necessarily of type ${}_mI_0$), then by Lemma 1 we are done. So we can assume that either $g>0$ or $g=0$ and π has more than two multiple fibres. In the exact sequence

$$(1)\longrightarrow I\longrightarrow\pi_1(S)\xrightarrow{\phi}\Gamma\longrightarrow(1),$$

the significance of the group Γ is as follows.

There exists a ramified (possibly infinite) covering $\psi\colon\tilde{\Delta}\to\Delta$ such that Γ acts as a properly discontinuous group of analytic automorphisms of $\tilde{\Delta}$ and ψ is the quotient map $\tilde{\Delta}\to\tilde{\Delta}/\Gamma\simeq\Delta$. Further, letting $a_i\in\Delta$ be the point for which the singular fibre $\pi^*(a_i)$ has multiplicity m_i, for every point $p\in\tilde{\Delta}$ with $\psi(P)=a_i$, the local ramification index of ψ at P is m_i.

Suppose first that $\chi(S,\mathcal{O})>0$. Then I is finite. For any subgroup H of $\pi_1(S)$, let $\Gamma_1=\phi(H)$ and $H_1=\phi^{-1}(\Gamma_1)=I\cdot H$. Then H is a subgroup of finite index in H_1. Letting $\tilde{S}$ be the universal covering of S, $\tilde{S}/H\to\tilde{S}/H_1$ is a finite, unramified map. To show that $\tilde{S}/H$ is holomorphically convex, it suffices to show that S/H_1 is holomorphically convex. But we obtain $\tilde{S}/H_1$ by pulling back the elliptic fibration via base change $\Delta_1\to\Delta$ where $\Delta_1=\Delta/\Gamma_1$. The fibration $\tilde{S}=\Delta_1\times_\Delta S$, after normalization, gives an unramified covering of S which is nothing but $\tilde{S}/H_1$.

Since Δ_1 is holomorphically convex and the fibres of $\tilde{S}/H_1\to\Delta_1$ are compact, $\tilde{S}/H_1$ is also holomorphically convex.

Now consider the case $\chi(S,\mathcal{O})=0$. In this case, we consider $\tilde{\Delta}\to\Delta$ and the pulled-back elliptic fibration $\lambda\colon\tilde{S}'\to\tilde{\Delta}$. Then $\tilde{S}'$ is an unramified covering of S, λ has no singular fibres, all the fibres of λ are complex-analytically the same and $\tilde{\Delta}$ is contractible. Then by a theorem of Grauert, $\tilde{S}'$ is biholomorphic with $\tilde{\Delta}\times E$, E being isomorphic to a fibre of λ. Clearly, $\tilde{S}$ is biholomorphic with $\tilde{\Delta}\times\boldsymbol{C}$ as the universal covering of E is $\boldsymbol{C}$. Then $\tilde{S}$ is trivially holomorphically convex.

To complete the arguments, we prove:

Lemma 2. *Let S be a smooth, projective surface over $\mathbf{C}$. If $C \subset S$ is a rational curve with $C^2 > 0$, then $H'(S, \mathcal{O}) = \{0\}$.*

Proof (by M. P. Murthy). Consider the Albanese morphism $f: S \to \mathrm{Alb}(S)$. Then, on the one hand $f(C)$ generates $\mathrm{Alb}(S)$ as $C^2 > 0$, on the other hand $f(C)$ is a point since C is a rational curve. Hence $\mathrm{Alb}(S) = \{0\}$ and $H^1(S, \mathcal{O}) = \{0\}$.

Note: *Added in proof.* In the situation of Proposition 2, I is actually trivial if $\chi(\mathrm{S}, \mathcal{O}_S) > 0$. For a proof, see D. A. Cox and S. Zucker's paper, "Intersection Numbers of Sections of Elliptic Surfaces", Invent. Math., Vol. 53, Fasc. 1, 1979.

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