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Analytic Representations of SL_2 over a $\mathfrak{p}$-Adic Number Field, III

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§ 0. Introduction

0-1. In our former paper [12], we have constructed a p-adic analogue of the holomorphic discrete series of $\mathrm{SL}_2(\boldsymbol{R})$ which is related to the theory of p-adic Schottky groups of D. Mumford. We have also constructed a p-adic analogue of the non-unitary principal series in [11], and have studied the relation between our discrete series and our principal series. The main purpose of this paper is to study the irreducibilities and the equivalences of our principal series.

Let $\boldsymbol{Q}_p$ be the p-adic number field, let L be a finite extension of $\boldsymbol{Q}_p$, and let k be a field containing L. We assume (i) the p-adic valuation of $\boldsymbol{Q}_p$ can be extended to a valuation $|\ \ |$ of k, and (ii) k is maximally complete with respect to $|\ \ |$ (cf. § 1 for a definition). These conditions are satisfied if k is a finite extension of L. Let L^* and k^* be the multiplicative groups of L and k, respectively, and let χ: $L^* \to k^*$ be a homomorphism which can be expressed as $\chi(z) = \exp\{\alpha(\chi)\log(z)\}$ for some $\alpha(\chi) \in k$ if z is sufficiently close to 1. Hence χ is a locally analytic character of L^* with values in k^*.

Let G denote the group $\mathrm{SL}_2(L)$, and let P be the subgroup of G of all lower triangular matrices. We define a one-dimensional representation χ of P by

$$P \ni \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \longmapsto \chi(a) \in k^*,$$

and construct the induced representation Ind (P, G, χ) of G in the category of k-valued locally analytic functions (cf. § 1 and § 2 for the exact definition). Further we realize this representation of G on a space D_χ of k-valued locally analytic functions on L in a natural manner. Then it is not difficult to find all closed G-invariant subspaces of D_χ. For simplicity, we assume that $\alpha(\chi)$ is not a non-negative integer. Then our main result in this case can be stated as the following:

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Theorem. *Let the notation and assumptions be as above. Then each D_χ is a topologically irreducible G-module, and no two of the G-modules D_χ for all such χ's are equivalent.*

0-2. In Section 1, we define a natural topology on the space of the locally analytic functions on a given p-adic non-singular algebraic variety. In Section 2, we use the result of Section 1 and construct a locally analytic representation T_χ of G on a space D_χ of locally analytic functions on L, and state our main results (cf. Theorems 1 and 2). In Section 3, we define an action dT_χ of the Lie algebra $\mathfrak{g}=\{X \in M_2(L);\ \mathrm{tr}\, X=0\}$ of G on D_χ, and prove a lemma (Key Lemma) which plays a crucial role in the proof of the irreducibility of D_χ. In Section 4, we study the action of $\mathfrak{g}$ on the germ $D_{\chi,w}$ of functions of D_w at each point w of $\boldsymbol{P}^1(L)$, and obtain a local irreducibility assertion (cf. Proposition 2). In Section 5, we use differential operators L_w acting on $D_{\chi,w}$ and show that the family $\{D_{\chi,w};\ w \in \boldsymbol{P}^1(L)\}$ satisfies the assumption of the Key Lemma (cf. Proposition 3 for the exact statement). Then we use the Key Lemma to patch these local irreducibilities to the global irreducibility of D_χ.

If $\alpha(\chi)$ is a non-negative integer m, then D_χ has a clsoed G-invariant subspace $D_\chi^{\mathrm{loc},m}$ (cf. § 2). We prove in Section 5 the irreducibility of $D_\chi/D_\chi^{\mathrm{loc},m}$ also. In Section 6, we prove the Frobenius reciplocity law (cf. Proposition 4), and, by using it, construct closed G-invariant subspaces of $D_\chi^{\mathrm{loc},m}$ and intertwining operators between them (cf. Propositions 5 and 6). In Section 7, we prove that $D_\chi^{\mathrm{loc},m}$ has no other $(\mathfrak{g}, G)$-invariant subspaces. In Section 8, we study intertwining operators between these G-modules, and prove Theorem 2.

0-3. Remarks. (1) If $\alpha(\chi)$ is a non-negative integer, then we have $\mathrm{Hom}_G(D_\chi, D_\chi)=k$ id. though D_χ is not irreducible (cf. Proposition 5). Hence D_χ is not completely irreducible and Schur's lemma does not hold in our case. This phenomenon causes the main difficulty in studying the irreducibility of our module.

(2) Our Key Lemma is a generalization of an irreducibility criterion of induced modules of finite groups. T. Shintani used the criterion to study representations of p-adic groups on complex vector spaces (cf. Shintani [15]). So we modify Shintani's proof into the present form and overcome the difficulty in studying the irreducibility.

(3) Let $\mathfrak{o}$ be the integer ring of L, and let K be the maximal compact subgroup $\mathrm{SL}_2(\mathfrak{o})$ of G. Then the irreducibility of D_χ (or $D_\chi/D_\chi^{\mathrm{loc},m}$) holds not only as a G-module but also as a K-module.

(4) An essential part of Theorem 2 is due to W. Casselman. He proved the Frobenius reciplocity law for our module $\mathrm{Ind}\,(P, G, \chi)$ and

calculated $\mathrm{Hom}_G(D_\chi, D_\delta)$. The author would like to express his thanks to Casselman for this contribution.

§ 1. Spaces of locally analytic functions

1-1. Let k be a field with a non-trivial non-archimedean valuation $|\ |$. We assume that $(k, |\ |)$ is *maximally complete*. Namely, we assume that for any decreasing sequence $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ of balls in k, the intersection $\bigcap C_n$ is not empty. It is obivous that a maximally complete field is complete, and that a locally compact non-archimedean field is maximally complete. Let L be a locally compact subfield of k.

Let $\{r_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in the value group $|L^*|$ of the multiplicative group L^* of L satisfying $\lim_n r_n = 0$. For any positive integer n and for any $a=(a_1, \cdots, a_N) \in L^N$, let

$$B_{a,n} = \{z=(z_1, \cdots, z_N) \in L^N; |z_i - a_i| < r_n \ (i=1, \cdots, N)\}.$$

Since the valuation $|\ |$ is non-archimedean, $B_{a,n} \cap B_{b,n} \neq \phi$ iff $B_{a,n} = B_{b,n}$. We define a space $\mathscr{A}'(B_{a,n})$ by

$$\mathscr{A}'(B_{a,n}) = \{f(z) = \sum_M c_M (z-a)^M;\ c_M \in k,\ |c_M| r_n^{|M|} \text{ is bounded}\},$$

where $M=(m_1, \cdots, m_N)$ runs over the set of all N-tuples of non-negative integers, $|M| = m_1 + \cdots + m_N$, and $(z-a)^M = (z_1-a_1)^{m_1} \cdots (z_N-a_N)^{m_N}$. Then $\mathscr{A}'(B_{a,n})$ becomes a complete Banach space with the following norm:

$$\|f\|_{a,n} = \operatorname*{Sup}_M |c_M| r_n^{|M|}.$$

If $m>n$, then $r_m < r_n$. Hence $B_{a,n}$ can be expressed as a finite disjoint union of certain $B_{b,m}$'s: $B_{a,n} = \coprod B_{b,m}$ $(b \in B(m))$. Since $B_{b,n} = B_{a,n}$ for any $b \in B(m)$, $\mathscr{A}'(B_{b,n}) = \mathscr{A}'(B_{a,n})$. Let

$$\begin{array}{ccc} \rho_{b,n,m}: \mathscr{A}'(B_{b,n}) & \longrightarrow & \mathscr{A}'(B_{b,m}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sum_M c_M (z-b)^M & \longmapsto & \sum_M c_M (z-b)^M \end{array}$$

be the restriction map. Since $r_m < r_n$, $|c_M| r_m^{|M|} \to 0$ $(|M| \to \infty)$. Further the image of the unit ball

$$\{\sum c_M (z-b)^M;\ |c_M| r_n^{|M|} \leq 1\}$$

of $\mathscr{A}'(B_{b,n})$ with the induced topology from $\mathscr{A}'(B_{b,m})$ is homeomorphic to the direct product space

$$\{c=(c_M);\ c_M \in k,\ M=(m_1, \cdots, m_N),\ m_i \in \boldsymbol{Z},\ \geq 0,\ |c_M| r_n^{|M|} \leq 1\}$$

of a countable number of closed balls in k. Since k is maximally complete, it follows from Springer [16], 1.17 that this subset of $\mathscr{A}'(B_{b,m})$ is c-compact. Hence the restriction map

$$\rho_{n,m}^a = (\rho_{b,n,m}) \colon \mathscr{A}'(B_{a,n}) \longrightarrow \bigoplus_{b \in B(m)} \mathscr{A}'(B_{b,m})$$

is a c-compact map.

Let

$$\mathscr{A}(B_{a,n}) = \operatorname*{inj\,lim}_{m} \{ \bigoplus_{b \in B(m)} \mathscr{A}'(B_{b,m}) \}$$

be the injective limit of the Banach spaces $\bigoplus_b \mathscr{A}'(B_{b,m})$ with respect to the restriction maps. Then it follows from the result of [9], 3–1 that $\mathscr{A}(B_{a,n})$ is a Hausdorff complete reflexive bornologic locally convex k-vector space, and the strong dual of $\mathscr{A}(B_{a,n})$ is a Fréchet space.

Remark. Put

$$\mathscr{A}''(B_{a,n}) = \{ f(z) = \sum_M c_M (z-a)^M;\ c_M \in k,\ |c_M| r_n^{|M|} \to 0\ (|M| \to \infty) \}.$$

Then $\mathscr{A}''(B_{a,n})$ is a closed subspace of $\mathscr{A}'(B_{a,n})$, and the restriction map $\rho_{n,m}^a$ induces an injection $\mathscr{A}'(B_{a,n}) \to \bigoplus_{b \in B(m)} \mathscr{A}''(B_{b,m})$ for any $m > n$. Hence

$$\mathscr{A}(B_{a,n}) = \operatorname*{inj\,lim}_{m} \bigoplus_{b \in B(m)} \{ \mathscr{A}''(B_{b,m}) \}$$

holds. This expression will be used in Sections $2 \sim 7$.

1-2. Let V be an N-dimensional non-singular algebraic variety defined over L, let V_L be the set of all L-valued points of V, and let W be an open subset of V_L. Since L is locally compact, V_L and W are locally compact. Since V is non-singular, W has an open covering $\mathscr{U}$ such that each member U of $\mathscr{U}$ is contained in an affine L-subset of V_L which is L-isomorphic to a Zariski open L-subset of L^N. Since W is locally compact and paracompact, we may assume that each $U \in \mathscr{U}$ is an open compact subset, and that $\mathscr{U}$ is locally finite. Here, by taking a refinement if necessary, we may assume that $\mathscr{U}$ is a mutually disjoint covering. Since any open compact subset of L^N is a finite disjoint union of balls of the form $B_{a,n}$, by taking a refinement if necessary, we may assume that each $U \in \mathscr{U}$ is L-analytically isomorphic to a ball of the form $B_{a,n}$ ($a \in L^N$, $n \in \boldsymbol{Z}, \geq 0$). Therefore we have proved that W has an open disjoint covering $\mathscr{U}$ such that for each $U \in \mathscr{U}$, there is an L-analytic isomorphism $i_U: B_{a,n} \xrightarrow{\sim} U$ for some $a \in L^N$ and $n \in \boldsymbol{Z}, \geq 0$.

Let $\mathscr{A}(B_{a,n})$ be as in 1–1, and let

$$\mathscr{A}(U)=\{f: U \rightarrow k; f \circ i_U \in \mathscr{A}(B_{a,n})\}.$$

We choose the topology on $\mathscr{A}(U)$ which makes the map $\mathscr{A}(U) \ni f \mapsto f \circ i_U \in \mathscr{A}(B_{a,n})$ into a topological isomorphism. Since W is the disjoint union of the U's, we put

$$\mathscr{A}(W)=\prod_{U \in \mathscr{U}} \mathscr{A}(U).$$

It is easy to see that this definition of the locally convex k-vector space $\mathscr{A}(W)$ does not depend on a special choice of $\mathscr{U}$ and the i_U's. We call an element of $\mathscr{A}(W)$ a *locally analytic function on* W. Note that a k-valued function f: $W \rightarrow k$ is locally analytic in this sense iff for any $w \in W$, there exists an open neighbourhood U of w such that the restriction of f to U can be expressed as a convergent power series of local coordinates of W at w. Obviously, (i) $\mathscr{A}(W)$ is a complete Hausdorff locally convex space, and (ii) the addition and the multiplication are continuous in the topology of $\mathscr{A}(W)$. Further (iii), if G is an algebraic L-group, the right and the left translations of G_L induce automorphisms of $\mathscr{A}(G_L)$, and (iv), if $W \subset L^N$, then the partial differentiation $\partial/\partial z_i$ $(i=1, \cdots, N)$ induces a continuous map of $\mathscr{A}(W)$.

§ 2. The main results

2-1. *Construction of the representations.* Let k be a field with a non-trivial non-archimedean valuation $|\ \ |$, and let L be a locally compact subfield of k. Hereafter *we assume* that (1) $(k, |\ \ |)$ is maximally complete and (2) L is a finite extension of the p-adic number field $\boldsymbol{Q}_p$. Let $\mathfrak{o}$ be the integer ring of L, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$, let $\mathfrak{o}^*$ be the unit group of $\mathfrak{o}$, and let q be the cardinality of the residue field $\mathfrak{o}/\mathfrak{p}$ of L. We fix a prime element of L and denote it by π.

Let $G=\mathrm{SL}_2(L)$, $K=\mathrm{SL}_2(\mathfrak{o})$, and let P be the subgroup of G consisting of all lower triangular matrices. Let χ: $L^* \rightarrow k^*$ be a locally analytic homomorphism from the multiplicative group of L to the multiplicative group of k. Hence χ can be expressed as

$$\chi(z)=\exp\{\alpha(\chi) \log(z)\}$$

if $z \in L^*$ is sufficiently close to 1, where $\alpha(\chi)$ is a constant in k, and $\exp(w)$ and $\log(w)$ are the p-adic exponential function $\sum_n w^n/n!$ and the p-adic logarithmic function $\sum_n (-1)^{n+1}(z-1)^n/n$, respectively. Note that $\alpha(\chi)$ is the value of $(d/dz)\chi(z)$ at $z=1$. We extend χ to a representation of P by

$$\chi\colon P \ni \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \longmapsto \chi(a) \in k^*.$$

Let Ind (P, G, χ) be the space of locally analytic functions $F\colon G \to k$ such that

$$F(pg)=\chi(a)F(g) \quad \left(p=\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in P\right)$$

holds for any p. Since G is a non-singular algebraic variety defined over L, the space $\mathscr{A}(G)$ of all k-valued locally analytic functions on G has a natural locally convex topology (cf. § 1). Since Ind (P, G, χ) is a closed subspace of $\mathscr{A}(G)$, the topology of Ind (P, G, χ) is also Hausdorff and complete. For any element g_1 of G, we put

$$T(g_1)F(g)=F(gg_1) \quad (F \in \operatorname{Ind}(P, G, \chi)).$$

Then $T(g_1)$ is an automorphism of the k-vector space Ind (P, G, χ), and T defines a continuous representation of the p-adic group G on the p-adic vector space Ind (P, G, χ).

Let U be the unipotent radical of P. Then, for any elements g and g' of G, $Ug=Ug'$ holds iff the first rows of them coincide. Since $\chi(u)=1$ for any $u \in U$, $F(ug)=F(g)$ holds for any $u \in U$. Hence

$$F(g)=F(\alpha, \beta, \gamma, \delta) \quad \left(g=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)$$

depends only on (α, β). Therefore we write $F(g)=F(\alpha, \beta)$. Then the space Ind (P, G, χ) is identified with the space of all locally analytic functions

$$F\colon \{(\alpha, \beta) \in L^2;\ (\alpha, \beta)\neq(0, 0)\} \longrightarrow k$$

satisfying the condition

$$F(\mu\alpha, \mu\beta)=\chi(\mu)F(\alpha, \beta)$$

for any $\mu \in L^*$. Further the group G acts on this space by

$$T(g_1)F(\alpha, \beta)=F(a\alpha+c\beta, b\alpha+d\beta) \quad \left(g_1=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right),$$

and the topology on this space coincides with the induced topology from

$\mathscr{A}(\{(\alpha, \beta) \in L^2; (\alpha, \beta) \neq (0, 0)\})$.

For any element $F(\alpha, \beta)$ of Ind (P, G, χ), we define a function $f\colon L \to k$ by

$$f(z) = F(z, 1).$$

Then f is a locally analytic function on L, and the function

$$\chi(z)^{-1} f(z) = \chi(z)^{-1} F(z, 1) = F(1, z^{-1})$$

is expanded into a convergent power series of z^{-1} if z^{-1} is sufficiently small. Let D_χ be the space of locally analytic functions $f\colon L \to k$ such that $\chi(z)^{-1} f(z)$ can be expanded into a convergent power series of z^{-1} for $|z| \gg 0$. Then we see that the map

$$i\colon \mathrm{Ind}\,(P, G, \chi) \ni F(\alpha, \beta) \longmapsto f(z) = F(z, 1) \in D_\chi$$

is bijective. Further, if we define $T_\chi(g)$ $(g \in G)$ by $T_\chi(g) \circ i = i \circ T(g)$, then we have

$$T_\chi(g) f(z) = \chi(bz+d) f((az+c)/(bz+d)) \quad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\right).$$

We choose the topology on D_χ which makes the map i into a topological isomorphism.

Remark. Put

$$A(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad C(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$(a \in L^*, c \in L)$. Then I and the $C(c)$'s generate the whole group G, and they act on our space D_χ as

$$T_\chi(A(a)) f(z) = \chi(a)^{-1} f(a^2 z),$$
$$T_\chi(C(c)) f(z) = f(z+c),$$
$$T_\chi(I) f(z) = \chi(z) f(-1/z).$$

These formulas will be frequently used.

2-2. *The irreducibility.* We assume that $\alpha(\chi) = m$ is a non-negative integer. Then $\chi(z) = z^m \varepsilon(z)$ with a locally constant character ε. Let $D_\chi^{\mathrm{loc}, m}$ be the space of functions $f\colon L \to k$ such that (i) for any $z_0 \in L$, there exist a ball $V_{z_0} = \{z \in L; |z - z_0| \leq r\}$ $(r > 0)$ and a polynomial $p_{z_0}(z) \in k[z]$ with $\deg p_{z_0}(z) \leq m$ satisfying $f(z) = p_{z_0}(z)$ for any $z \in V_{z_0}$, and (ii) there exist a ball $V_\infty=\{z \in \boldsymbol{P}^1(L); |z|\geq r'\}$ $(r'>0)$ and a polynomial $p_\infty(z) \in k[z]$ with $\deg p_\infty \leq m$ satisfying $f(z)=\chi(z)p_\infty(z^{-1})$ for any $z \in L\cap V_\infty$. Since G is generated by I and the $C(c)$'s $(c \in L)$, we observe that $D_\chi^{\mathrm{loc},m}$ is a G-invariant k-subspace of D_χ. Since the derivation d/dz acts on D_χ continuously, the kernel $D_\chi^{\mathrm{loc},m}$ of the continuous map $(d/dz)^{m+1}$: $D_\chi \to D_\chi$ is a closed subspace of D_χ. Hence $D_\chi^{\mathrm{loc},m}$ is a closed G-invariant k-subspace of D_χ.

Now we assume that $\chi(z)=z^m$ $(m \in \boldsymbol{Z}, m\geq 0)$ holds for any $z \in L^*$ (resp. $z \in \mathfrak{o}^*$). Let P_m be the space of functions f: $L\to k$ such that there exists a polynomial $p(z) \in k[z]$ with $\deg p(z)\leq m$ satisfying $f(z)=p(z)$ for any $z \in L$. Then P_m is a finite dimensional G-invariant (resp. K-invariant) k-subspace of D_χ. Since D_χ is a Hausdorff space, the finite dimensional subspace P_m of D_χ is closed. Hence P_m is also a closed G-invariant (resp. K-invariant) subspace of D_χ. Obviously $D_\chi \supset D_\chi^{\mathrm{loc},m} \supset P_m \supset \{0\}$.

Let N: $L^* \to k^*$ be the locally constant character such that $N(z)=1$ holds for any $z \in \mathfrak{o}^*$ and such that $N(\pi^n)=q^n$ for any integer n. Then we have the following:

Theorem 1. (i) *If $\alpha(\chi)$ is not a non-negative integer, then D_χ is a topologically irreducible K-module.*

(ii) *If $\alpha(\chi)$ is a non-negative integer m, let $D_\chi^{\mathrm{loc},m}$ be as above. Then $D_\chi/D_\chi^{\mathrm{loc},m}$ is a topologically irreducible K-module. Let $\varepsilon(z)=z^{-m}\chi(z)$ and let $l=l(\chi)$ be the smallest positive integer such that $\varepsilon(z)=1$ holds for any $z \in \mathfrak{o}^*$ with $|z-1|\leq|\pi^l|$. Then*:

(ii-a) *If $\varepsilon(z)=1$ holds for any $z \in L^*$, then P_m is a topologically irreducible G-module, and any element of $D_\chi^{\mathrm{loc},m}\backslash P_m$ generates $D_\chi^{\mathrm{loc},m}$ as a topological G-module. In particular, $D_\chi^{\mathrm{loc}}/P_m$ is a topologically irreducible G-module.*

(ii-b) *If $\varepsilon(z)=N(z)^2$ holds for any $z \in L^*$, then $D_\chi^{\mathrm{loc},m}$ contains a topologically irreducible G-submodule Q_m such that $D_\chi^{\mathrm{loc},m}\supsetneqq Q_m \supsetneqq \{0\}$ and such that any element of $D_\chi^{\mathrm{loc},m}\backslash Q_m$ generates $D_\chi^{\mathrm{loc},m}$ as a topological G-module. In particular, $D_\chi^{\mathrm{loc},m}/Q_m$ is a topologically irreducible G-module.*

(ii-c) *We assume that $\varepsilon(z)^2=N(z)^2$ holds for any $z \in L^*$ and that $\varepsilon(z)\neq N(z)$ holds for some $z \in L^*$. If $\varepsilon(z)\neq 1$ holds for some $z \in \mathfrak{o}^*$, we also assume that $\sqrt{\varepsilon(-1)q^{l(\chi)}}$ is contained in k. Then $D_\chi^{\mathrm{loc},m}$ is a direct sum $D_{\chi,+}^{\mathrm{loc},m}\oplus D_{\chi,-}^{\mathrm{loc},m}$ of two non-equivalent topologically irreducible G-submodules $D_{\chi,+}^{\mathrm{loc},m}$ and $D_{\chi,-}^{\mathrm{loc},m}$.*

(ii-d) *If $\varepsilon(z)$ does not satisfy any of the conditions in* (ii-a), (ii-b) *and* (ii-c), *then $D_\chi^{\mathrm{loc},m}$ is a topologically irreducible G-module.*

2-3. *The equivalence.* We assume that $\alpha(\chi)$ is a non-negative integer m. Hence $\chi(z)=\varepsilon(z)z^m$ holds with a locally constant character ε: $L^*\to k^*$.

Put

$$\delta(z)=z^{-2m-2}\chi(z)=\varepsilon(z)z^{-m-2}.$$

For any element $f(z)$ of D_χ, put

$$g(z)=(d/dz)^{m+1}f(z).$$

It is obvious that $g(z)$ is a locally analytic function on L. Further, since we can write

$$f(z)=\chi(z)\sum_{n=0}^{\infty}c_nz^{-n}=\varepsilon(z)\sum_{n=0}^{\infty}c_nz^{m-n}$$

with a convergent power series $\sum c_nz^{-n}\in k\{z\}$ for $|z|\gg 0$, we have

$$\begin{aligned} g(z)&=\varepsilon(z)\sum_{n=0}^{\infty}(m-n)(m-n-1)\cdots(-n)c_nz^{-n-1}\\ &=(\varepsilon(z)z^{-m-2})\sum_{n=0}^{\infty}(-1)^{m+1}(n+1)(n+2)\cdots(n+m+1)c_{n+m+1}z^{-n}\end{aligned}$$

for $|z|\gg 0$. Hence $g(z)$ is an element of D_δ. Further we observe from this calculation that for any $g(z)\in D_\delta$, there exists a function $f(z)$ in D_χ satisfying $(d/dz)^{m+1}f(z)=g(z)$, Since the kernel of $D_\chi\ni f\mapsto g\in D_\delta$ is $D_\chi^{\mathrm{loc},m}$, the correspondence $f(z)\mapsto g(z)=(d/dz)^{m+1}f(z)$ induces a continuous bijection $S_\chi^*\colon D_\chi/D_\chi^{\mathrm{loc},m}\to D_\delta$. Here, if $f(z)$ is expanded into a convergent power series on a ball B, then $g(z)$ is also expanded into a convergent power series on B. Hence it follows from the open mapping theorem in the category of Banach k-vector spaces that $S=S_\chi^*$ is a topological isomorphism (cf. § 4 also).

Since $T_\chi(C(c))f(z)=f(z+c)$, we differentiate this formula $(m+1)$-times and obtain $S\circ T_\chi(C(c))f(z)=g(z+c)=S(f)(z+c)$. By induction on m, we obtain $(d/dz)^{m+1}[z^mf(-1/z)]=z^{-m-2}((d/dz)^{m+1}f)(-1/z)$. Since $T_\chi(I)f(z)=\chi(z)f(-1/z)=\varepsilon(z)z^mf(-1/z)$, we obtain

$$\begin{aligned} S\circ T_\chi(I)f(z)&=\varepsilon(z)(d/dz)^{m+1}(z^mf(-1/z))\\ &=\varepsilon(z)z^{-m-2}((d/dz)^{m+1}f)(-1/z)\\ &=\delta(z)S(f)(-1/z)=T_\delta(I)\circ S(f)(z).\end{aligned}$$

Hence $S_\chi^*\colon D_\chi/D_\chi^{\mathrm{loc},m}\to D_\delta$ is a G-isomorphism.

Now we have the following:

Theorem 2. *Let V_1 and V_2 be two different topologically irreducible G-modules constructed in Theorem 1. Hence V_1 and V_2 are one of the following topologically irreducible G-modules:* D_χ, $D_\chi^{\mathrm{loc},m}$, $D_\chi/D_\chi^{\mathrm{loc},m}$, P_m,

$D_\chi^{\mathrm{loc},m}/P_m$, Q_m, $D_\chi^{\mathrm{loc},m}/Q_m$, $D_{\chi,+}^{\mathrm{loc},m}$, $D_{\chi,-}^{\mathrm{loc},m}$ *(the corresponding χ's for V_1 and V_2 may be different). Then V_1 and V_2 are G-equivalent if and only if one of the following conditions is satisfied:*

(i) $\alpha(\chi)$ is a non-negative integer m, $\delta(z)=z^{-2m-2}\chi(z)$, $V_1=D_\chi/D_\chi^{\mathrm{loc},m}$ and $V_2=D_\delta$.

(ii) m is a non-negative integer, $\varepsilon(z)$ is a locally constant character such that $\varepsilon(z)\neq 1$, $N(z)^2$ and $\varepsilon(z)^2\neq N(z)^2$, $\chi(z)=z^m\varepsilon(z)$, $\delta(z)=z^m\varepsilon(z)^{-1}N(z)^2$, $V_1=D_\chi^{\mathrm{loc},m}$ and $V_2=D_\delta^{\mathrm{loc},m}$.

(iii) m is a non-negative integer, $\chi(z)=z^m$, $\delta(z)=z^mN(z)^2$, $V_1=P_m$ (resp. $V_1=D_\chi^{\mathrm{loc},m}/P_m$) and $V_2=D_\delta^{\mathrm{loc},m}/Q_m$ (resp, $V_2=Q_m$).

Remark. The construction of the subspaces Q_m and $D_{\chi,\pm}^{\mathrm{loc},m}$ and the construction of the intertwining operators

$$H_\chi\colon D_\chi^{\mathrm{loc},m}/P_m\longrightarrow Q_m,\qquad H_\delta\colon D_\delta^{\mathrm{loc},m}/Q_m\longrightarrow P_m$$

and the projection operators $H_{\chi,\pm}\colon D_\chi^{\mathrm{loc},m}\to D_{\chi,\pm}^{\mathrm{loc},m}$ are a bit complicated. We construct them explicitly in Section 6 by using the Frobenius reciplocity law.

§ 3. Action of the Lie algebra $\mathfrak{g}$ and the Key Lemma

3-1. *The Lie algebra* $\mathfrak{g}$. Let

$$\mathfrak{g}=\{X\in M_2(L);\ \mathrm{tr}(X)=0\}$$

be the Lie algebra of G. Then for any $X\in\mathfrak{g}$, the series

$$\exp(tX)=\sum_{n=0}^{\infty}(tX)^n/n!$$

converges in $M_2(L)$ to an element of G if t is sufficiently small. Further $\exp(tX)$ belongs to any given congruence subgroup of $K=\mathrm{SL}_2(\mathfrak{o})$ if t is sufficiently small.

Put

$$X_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad X_-=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad Y=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Then X_+, X_- and Y span the Lie algebra $\mathfrak{g}$ as an L-vector space, and

$$\exp(tX_+)=\begin{pmatrix}1&t\\0&1\end{pmatrix},\quad \exp(tX_-)=\begin{pmatrix}1&0\\t&1\end{pmatrix},\quad \exp(tY)=\begin{pmatrix}e^t&0\\0&e^{-t}\end{pmatrix}.$$

Let D_χ and T_χ be as in Section 2. For any element X of $\mathfrak{g}$, we define an operator $(dT_\chi)(X)$ on D_χ by

$$(dT_\chi)(X)f(z)=\lim_{t\to 0}\frac{1}{t}\{T_\chi(\exp(tX))f(z)-f(z)\}$$

$(f(z)\in D_\chi)$.

Since $\chi(xy)=\chi(x)\chi(y)$ for any $x, y\in L$,

$$\chi'(x)=\chi'(1)\chi(x)/x=\alpha(\chi)\chi(x)/x.$$

Hence we obtain

$$\begin{aligned}(dT_\chi)(X_+)f(z)&=\lim t^{-1}(\chi(tz+1)f(z/(tz+1))-f(z))\\&=\alpha(\chi)zf(z)-z^2f'(z),\\(dT_\chi)(X_-)f(z)&=\lim t^{-1}(f(z+t)-f(z))\\&=f'(z),\\(dT_\chi)(Y)f(z)&=\lim t^{-1}(\chi(e^{-t})f(e^{2t}z)-f(z))\\&=-\alpha(\chi)f(z)+2zf'(z),\end{aligned}$$

where $f'(z)=(d/dz)f(z)$. Since X_+, X_- and Y span $\mathfrak{g}$, $(dT_\chi)(X)$ is a well-defined continuous k-linear endomorphism on D_χ for any $X\in\mathfrak{g}$. Hence dT_χ defines a continuous representation of the Lie algebra $\mathfrak{g}$ on D_χ. It is obvious that any closed K-invariant subspace of D_χ is $\mathfrak{g}$-invariant.

3-2. *The Key Lemma.* Let V_i $(i=1, \cdots, n)$ be a finite number of Banach spaces over k. We assume: (i) $V_i\neq\{0\}$; (ii) Each V_i is a topologically irreducible $\mathfrak{g}$-module; (iii) If $i\neq j$, then there exists no triple (W, f, g) such that (a) W is a Banach space over k on which $\mathfrak{g}$ acts as continuous endomorphisms, (b) f: $V_i\to W$ and g: $V_j\to W$ are injective continuous k-linear $\mathfrak{g}$-homomorphisms, and (c) the image $\mathrm{im}(f)$ of f and the image $\mathrm{im}(g)$ of g are dense in W. Note that the condition (iii) implies that V_i and V_j are not $\mathfrak{g}$-equivalent. Further, if V_i and V_j are finite dimensional Banach spaces over k, then the condition (iii) is equivalent to the non-equivalence of V_i and V_j.

Key Lemma. *Let V_i $(i=1, \cdots, n)$ be as above, and let $V=\oplus V_i$ be the direct sum of these Banach $\mathfrak{g}$-modules. Let U be a closed $\mathfrak{g}$-invariant k-subspace of V. Then there exists a subset I of $\{1, \cdots, n\}$ such that U is the direct sum of the V_i's $(i\in I)$: $U=\bigoplus_{i\in I}V_i$.*

Proof. We are going to prove the lemma by induction on n. It is obvious that the lemma holds for $n=1$. Hence we assume that the lemma holds for any smaller integer.

Let $Q: V \to V/U = V^*$ be the natural map. Since U is a closed subspace of the Banach space V, V^* has a natural structure of a Banach space (cf. e.g. van Rooij [14]). Further, V^* is a continuous $\mathfrak{g}$-module and Q is a continuous $\mathfrak{g}$-homomorphism. Put $V_i^* = Q(V_i)$.

Since the kernel $\ker(Q \mid V_i)$ of the restriction of Q to V_i is a closed $\mathfrak{g}$-invariant subspace, $\ker(Q \mid V_i)$ is either V_i or $\{0\}$. If $\ker(Q \mid V_i) = V_i$, then $V_i \subset U$, and hence

$$U = V_i \oplus ((\bigoplus_{j \neq i} V_j) \cap U).$$

Since $(\bigoplus_{j \neq i} V_j) \cap U$ is a closed $\mathfrak{g}$-invariant subspace of $\bigoplus_{j \neq i} V_j$, it follows from the assumption on n that $(\bigoplus_{j \neq i} V_j) \cap U$ is a direct sum of a finite number of the V_j's $(j \neq i)$. Hence U is a direct sum of a finite number of the V_j's $(j = 1, \cdots, n)$. Since the lemma holds in this case, we may assume $\ker(Q \mid V_i) = \{0\}$ for each i. Then $Q \mid V_i$; $V_i \to V^*$ is an injective continuous k-linear $\mathfrak{g}$-homomorphism.

Let $V' = \bigoplus_{i \neq n} V_i$. Since the kernel $\ker(Q \mid V')$ of the restriction $Q \mid V'$ of Q to V' is a closed $\mathfrak{g}$-invariant subspace of V', it follows from the assumption on n that $\ker(Q \mid V')$ is a direct sum of a finite number of the V_i's $(i \neq n)$. Hence $\ker(Q \mid V') \neq \{0\}$ iff it contains some V_i $(i \neq n)$. Since this contradicts our assumption, we may assume that $Q \mid V'$: $\bigoplus_{i \neq n} V_i \to V^*$ is an injective $\mathfrak{g}$-homomorphism. Note that the image of $Q \mid V'$ is $V'^* = V_1^* + \cdots + V_{n-1}^*$.

Let V_n^{*c} be the closure of V_n^* in V^*. Then $V_n^{*c} \cap V'^*$ is a closed $\mathfrak{g}$-invariant subspace of V'^*. Since $Q \mid V'$: $V' \to V'^*$ is a continuous map, $(Q \mid V')^{-1} (V_n^{*c} \cap V'^*)$ is a closed $\mathfrak{g}$-invariant subspace of $V' = \bigoplus_{i \neq n} V_i$. By our assumption on n, this subspace of V' is a direct sum of a finite number of the V_i's $(i \neq n)$.

If this space is not the total space V', then there is an index $j \leq n-1$ such that $(Q \mid V')^{-1} (V_n^{*c} \cap V'^*)$ is a subspace of $\bigoplus_{i \neq j, n} V_i$. It follows that

$$V_n^{*c} \cap (V_1^* + \cdots + V_{n-1}^*) \subset V_1^* + \cdots + V_{j-1}^* + V_{j+1}^* + \cdots + V_{n-1}^*.$$

Let $u = u_1 + \cdots + u_n$ $(u_i \in V_i)$ be any element of U. Then

$$Q(u_1 + \cdots + u_{n-1}) + Q(u_n) = Q(u) = 0.$$

Hence $Q(u_1 + \cdots + u_{n-1}) = Q(-u_n)$ is an element of

$$V_n^* \cap V'^* = V_n^* \cap (V_1^* + \cdots + V_{n-1}^*).$$

By our assumption on j, there exists $v_1, \cdots, v_{n-1}$ such that $v_i \in V_i$, $v_j = 0$ and $Q(v_1 + \cdots + v_{n-1}) = Q(u_1 + \cdots + u_{n-1})$. Since $Q \mid V'$ is injective,

$v_1+\cdots+v_{n-1}=u_1+\cdots+u_{n-1}$. Hence $u=u_1+\cdots+u_{n-1}+u_n=v_1+\cdots+v_{n-1}+u_n$ is an element of $\bigoplus_{i\neq j} V_i$. It follows that U is a closed $\mathfrak{g}$-invariant subspace of $\bigoplus_{i\neq j} V_i$. By our assumption on n, the lemma holds in this case. Hence we may assume $(Q\,|\,V')^{-1}(V_n^{*c}\cap V'^*)=V'$.

Now we have $V_n^{*c}\cap V'^*=Q(V')=V'^*$. Hence V_n^* is dense in V'^*. Since $V^*=V'^*+V_n^*$, V_n^* is dence in V^*. We repeat similar arguments for each i, and observe that the lemma does not hold only in the case such that $n\geq 2$ and each V_i^* is dense in V^*. Put $W=V^*$, $f=Q\,|\,V_n: V_n\to V^*$ and $g=Q\,|\,V_1: V_1\to V^*$. Then the triple (W, f, g) contradicts the assumption (iii). It follows that no such case occurs. Therefore the Key Lemma is proved.

§ 4. Proof of Theorem 1, I (the local study)

4-1. *The space* $V_{\infty,n}\oplus\bigoplus_w V_{w,n}$. Let n be a positive integer, let

$$K_n=\left\{g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathfrak{o});\ |a-1|,\ |b|,\ |c|,\ |d-1|\leq|p^n|\right\}$$

and, for any element w of L satisfying $|w|<|p^{-n}|$, let $r_{w,n}=|p^n|$ if $|w|\leq 1$ and $r_{w,n}=|p^n w^2|$ if $1\leq|w|<|p^{-n}|$. Then $|p^n|\leq r_{w,n}<|p^{-n}|$. Put

$$B_{w,n}=\{z\in L;\ |z-w|\leq r_{w,n}\}$$
$$B_{\infty,n}=\{z\in P^1(L);\ |z|\geq|p^{-n}|\}.$$

Since the valuation $|\ \ |$ is non-archimedean, $B_{w,n}\cap B_{v,n}\neq\phi$ iff $B_{w,n}=B_{v,n}$. Hence the one-dimensional projective space $P^1(L)=L\cup\{\infty\}$ can be expressed as a disjoint union $B_{\infty,n}\coprod\coprod_{w\in W} B_{w,n}$ with a certain subset $W=W_n$ of $\{w\in L;\ |w|<|p^{-n}|\}$ containing 0. Since I and the $C(c)$ $(c\in\mathfrak{o})$ generate the group K, it follows from the definition of $r_{w,n}$ that this decomposition $B_{\infty,n}\coprod\coprod B_{w,n}$ is preserved by any $g\in K$. Further, if $g\in K_n$, then it is easy to see that $g(B_{\infty,n})=B_{\infty,n}$ and $g(B_{w,n})=B_{w,n}$ hold.

Let $V_{w,n}=V_{w,n,\chi}$ be the space of formal power series

$$f(z)=\sum_{0\leq m<\infty} c_m(z-w)^m\in k[[z-w]]$$

such that $|c_m|r_{w,n}^m\to 0$ $(m\to\infty)$. Since k is complete, $V_{w,n}$ becomes a Banach space over k with $\|f\|=\|f\|_{w,n}=\mathrm{Max}\,|c_m|r_{w,n}^m$. Obviously any $f(z)\in V_{w,n}$ gives a (locally analytic) function on $B_{w,n}$. We extend $f(z)$ to a function $\tilde{f}(z)$ on L by putting $\tilde{f}(z)=0$ for any $z\in L\backslash B_{w,n}$. Then this extended function $\tilde{f}(z)$ belongs to D_χ, and we can regard $V_{w,n}$ as a subspace of D_χ by this injection $V_{w,n}\ni f(z)\mapsto\tilde{f}(z)\in D_\chi$ (cf. Remark in 1–1). Similarly, let $V_{\infty,n}=V_{\infty,n,\chi}$ be the space of series

$$f(z)=\chi(z)\sum_{0\leq m<\infty} c_m z^{-m} \quad (c_m \in k)$$

such that $|c_m p^m|\to 0$ $(m\to\infty)$. Then $V_{\infty,n}$ becomes a Banach space over k with $\|f\|=\|f\|_{\infty,n}=\mathrm{Max}|c_m p^m|$, and there exists a similar injection $V_{\infty,n}\to D_\chi$. Further the direct sum $V_n=V_{n,\chi}=V_{\infty,n,\chi}\oplus\bigoplus_{w\in W} V_{w,n,\chi}$ is regarded as a subspace of D_χ, the inclusion maps $V_n\hookrightarrow D_\chi$ induce injective maps $V_{n,\chi}\to V_{n+1,\chi}$, and the injective limit inj lim $V_{n,\chi}$ of the $V_{n,\chi}$'s coincides with the space D_χ (cf. § 1 and 2–1).

Proposition 1. *If $\chi(z)$ is analytic for $|z-1|\leq|p^n|$, then $V_{\infty,n}$ and the $V_{w,n}$'s are preserved by the endomorphism*

$$T_\chi(g):\ D_\chi \ni f(z)\longmapsto \chi(bz+d)f((az+c)/(bz+d)) \in D_\chi$$

of D_χ for any $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K_n$ satisfying $|b|\leq|p^{2n}|$.

Since this proposition can be proved in a straight way, we omit the proof. We note that this proposition shows that the Lie algebra $\mathfrak{g}$ acts continuously on $V_{\infty,n}$ and the $V_{w,n}$'s.

4-2. *The local irreducibility.* Let X_-, X_+ and Y be as in 2–2. Then they act on $V_{\infty,n}$ and the $V_{w,n}$'s by

$$(dT_\chi)(X_-)f(z)=f'(z),\quad (dT_\chi)(X_+)f(z)=\alpha(\chi)zf(z)-z^2f'(z),$$
$$(dT_\chi)(Y)f(z)=-\alpha(\chi)f(z)+2zf'(z).$$

Hence they act on $(z-w)^m \in V_{w,n}$ and $\chi(z)z^{-m} \in V_{\infty,n}$ as

$$\begin{aligned}
&(dT_\chi)(X_-)(z-w)^m=m(z-w)^{m-1},\\
&(dT_\chi)(X_+)(z-w)^m=\alpha(\chi)z(z-w)^m-z^2m(z-w)^{m-1}\\
&\qquad=(\alpha(\chi)-m)(z-w)^{m+1}+(\alpha(\chi)-2m)w(z-w)^m-mw^2(z-w)^{m-1},\\
&(dT_\chi)(Y)(z-w)^m=-\alpha(\chi)(z-w)^m+2zm(z-w)^{m-1}\\
&\qquad=(-\alpha(\chi)+2m)(z-w)^m+2mw(z-w)^{m-1},\\
&(dT_\chi)(X_-)\chi(z)z^{-m}=(\alpha(\chi)-m)\chi(z)z^{-m-1},\\
&(dT_\chi)(X_+)\chi(z)z^{-m}=\alpha(\chi)\chi(z)z^{-m+1}-z^2(\alpha(\chi)-m)\chi(z)z^{-m-1}=m\chi(z)z^{-m+1},\\
&(dT_\chi)(Y)\chi(z)z^{-m}=-\alpha(\chi)\chi(z)z^{-m}+2z(\alpha(\chi)-m)\chi(z)z^{-m-1}\\
&\qquad=(\alpha(\chi)-2m)\chi(z)z^{-m}.
\end{aligned}$$

Now we have the following:

Proposition 2. (1) *If $\alpha(\chi)$ is not a non-negative integer, then $V_{w,n}$ and $V_{\infty,n}$ are topologically irreducible $\mathfrak{g}$-modules.*

(2) *If $\alpha(\chi)$ is a non-negative integer, then the subspace*

$$P_{w,n}=\{f(z)=\sum_{0\leq m\leq\alpha(\chi)} c_m(z-w)^m;\ c_m\in k\}$$

of $V_{w,n}$ and the subspace

$$P_{\infty,n}=\{f(z)=\chi(z)\sum_{0\leq m\leq\alpha(\chi)} c_m z^{-m};\ c_m\in k\}$$

of $V_{\infty,n}$ are $\mathfrak{g}$-invariant. Further $P_{w,n}$ and $P_{\infty,n}$ are irreducible $\mathfrak{g}$-modules, and any element of $V_{w,n}$ (resp. $V_{\infty,n}$) which does not belong to the space $P_{w,n}$ (resp. $P_{\infty,n}$) spans the space $V_{w,n}$ (resp. $V_{\infty,n}$) topologically as a $\mathfrak{g}$-module. In particular, $V_{w,n}/P_{w,n}$ and $V_{\infty,n}/P_{\infty,n}$ are topologically irreducible $\mathfrak{g}$-modules.

Proof. If $\alpha(\chi)$ is a non-negative integer, then

$$(dT_\chi)(X_+)(z-w)^{\alpha(\chi)}=-\alpha(\chi)w(z-w)^{\alpha(\chi)}-\alpha(\chi)w^2(z-w)^{\alpha(\chi)-1}$$

is an element of $P_{w,n}$, and

$$(dT_\chi)(X_-)\chi(z)z^{-\alpha(\chi)}=0.$$

Since $\mathfrak{g}$ is spanned by X_+, X_- and Y, it follows that $P_{w,n}$ and $P_{\infty,n}$ are $\mathfrak{g}$-invariant subspaces.

In general, put

$$L_w=[(dT_\chi)(Y)-2w(dT_\chi)(X_-)+\alpha(\chi)\,\mathrm{id.}]/2.$$

Then we have

$$L_w(z-w)^m=m(z-w)^m$$

for any m. Let $f(z)=\sum c_m(z-w)^m$ be an element of $V_{w,n}$, and let M and N be two different non-negative integers. Then

$$(N-M)^{-1}(L_w-M\,\mathrm{id.})f(z)=\sum_{m=0}^{\infty}\frac{m-M}{N-M}c_m(z-w)^m.$$

Hence, repeating this process for $0\leq M\leq H$ $(N\leq H)$, $M\neq N$, we obtain

$$\begin{aligned}&(\prod_M(N-M)^{-1}(L_w-M\,\mathrm{id.}))f(z)\\ &\quad=c_N(z-w)^N\\ &\qquad+\sum_{m>H}\frac{m(m-1)\cdots(m-N+1)(m-N-1)\cdots(m-H)}{N(N-1)\cdots(N-N+1)(N-N-1)\cdots(N-H)}c_m(z-w)^m.\end{aligned}$$

Since the coefficient

$$(-1)^{H-N}\frac{m!}{N!(m-N)!}\cdot\frac{(m-N-1)!}{(H-N)!(m-H-1)!}$$

of $c_m(z-w)^m$ is an integer, the second term of the right hand side of this equation belongs to $V_{w,n}$. Further the norm of this term satisfies

$$\leq \operatorname*{Max}_{m>H}|c_m|r^m_{w,n}\longrightarrow 0 \quad (H\rightarrow+\infty).$$

Hence

$$\prod_{N\neq M\leq H}(N-M)^{-1}(L_w-M\,\mathrm{id.})\sum_{0\leq m<\infty}c_m(z-w)^m\longrightarrow c_N(z-w)^N$$

in $V_{w,n}$ for $H\rightarrow+\infty$. Hence the minimal closed $\mathfrak{g}$-invariant subspace U_f of $V_{w,n}$ containing $f(z)=\sum_m c_m(z-w)^m$ contains $c_N(z-w)^N$ for any N.

We assume that $f(z)=\sum c_m(z-w)^m$ is a non-zero element of $V_{w,n}$. Then there is a non-zero coefficient c_N. Hence the minimal closed $\mathfrak{g}$-invariant subspace U_f contains $(z-w)^N$. If $\alpha(\chi)$ is a non-negative integer, then we assume $f(z)\notin P_{w,n}$. In this case, there is a non-zero coefficient c_N with $N>\alpha(\chi)$. Hence U_f contains $(z-w)^N$ with $N>\alpha(\chi)$.

Since

$$\begin{aligned}&(dT_\chi)(X_-)(z-w)^m=m(z-w)^{m-1},\\&[(dT_\chi)(X_+)-(\alpha(\chi)-2m)w\,\mathrm{id.}+w^2(dT_\chi)(X_-)](z-w)^m\\&\qquad=(\alpha(\chi)-m)(z-w)^{m+1},\end{aligned}$$

it follows that U_f contains all polynomials of $z-w$. Since they are dense in $V_{w,n}$, U_f coincides with $V_{w,n}$. Hence $V_{w,n}$ (or $V_{w,n}/P_{w,n}$) is a topologically irreducible $\mathfrak{g}$-module. The irreducibility of $P_{w,n}$ can be proved similarly.

Let

$$L_\infty=-[(dT_\chi)(Y)-\alpha(\chi)\,\mathrm{id.}]/2.$$

Then we have

$$L_\infty\chi(z)z^{-m}=\chi(z)mz^{-m}.$$

Further

$$(dT_\chi)(X_+)\chi(z)z^{-m}=\chi(z)mz^{-m+1}\text{ and }(dT_\chi)(X_-)\chi(z)z^{-m}=(\alpha(\chi)-m)\chi(z)z^{-m-1}.$$

Hence the irreducibility of $V_{\infty,n}$ (or $V_{\infty,n}/P_{\infty,n}$ or $P_{\infty,n}$) can be proved similarly.

§ 5. Proof of Theorem 1, II (the irreducibility of D_χ or $D_\chi/D_\chi^{\mathrm{loc},m}$)

5-1. *Proof of the assumption* (iii) *of the Key Lemma.* Let n be a positive integer, and let $V_{n,\chi}=V_{\infty,n,\chi}\oplus\bigoplus_w V_{w,n,\chi}$ ($w \in W_n$) be as in 4–1. Then $V_{\infty,n}=V_{\infty,n,\chi}$ and the $V_{w,n}=V_{w,n,\chi}$'s are non-zero Banach spaces over k. If $\alpha(\chi)$ is not a non-negative integer, then, by Proposition 2, each of them is a topologically irreducible $\mathfrak{g}$-module. If $\alpha(\chi)$ is a non-negative integer, then $V_{\infty,n}/P_{\infty,n}$ and the $V_{w,n}/P_{w,n}$'s are non-zero Banach spaces over k, and topologically irreducible $\mathfrak{g}$-modules.

Proposition 3. *If $\alpha(\chi)$ is not a non-negative integer, then any two of $V_{\infty,n}$ and the $V_{w,n}$'s ($w \in W_n$) satisfy the condition* (iii) *of* 3–2. *If $\alpha(\chi)$ is a non-negative integer, then any two of $V_{\infty,n}/P_{\infty,n}$ and the $V_{w,n}/P_{w,n}$'s ($w \in W_n$) satisfy the condition* (iii) *of* 3–2.

Proof. First we reduce the proof of the proposition to the case where k is an algebraically closed field.

Let $(k', |\ \ |')$ be any extension of $(k, |\ \ |)$ such that $(k', |\ \ |')$ is maximally complete. For any Banach space U over k, let U' be the complete tensor product $k'\hat{\otimes}_k U$ (cf. van Rooij [14], Chap. 4). Then $V'_{w,n}=k'\hat{\otimes}_k V_{w,n}$ is the space of all elements of $k'[[z-w]]$ which converge for $|z-w|\le r_{w,n}$, and $V'_{\infty,n}=k'\hat{\otimes}_k V_{\infty,n}$ is the space consisting of all $\chi(z)h(z^{-1})$ such that $h(z)$ is an element of $k'[[z]]$ which converges for $|z|\le|p^n|$. Hence each $V'_{v,n}$ ($v=\infty$ or $v \in W_n$) is made from $V_{v,n}$ simply by replacing k with k'.

If $\alpha(\chi)$ is not a non-negative integer, then $V'_{v,n}$ is a non-zero topologically irreducible $\mathfrak{g}$-module. Let h: $V_{v,n}\to W$ be a continuous k-linear $\mathfrak{g}$-homomorphism. Then h can be extended to a continuous k'-linear $\mathfrak{g}$-homomorphism h': $V'_{v,n}\to W'$. Since the complete tensor products give an exact functor (cf. van Rooij [14], Chap. 4), h is injective (resp. has a dense image) iff h' is injective (resp. has a dense image). Hence, to prove the proposition, we may replace k by any maximally complete extension k' of k. Therefore we may assume that k is algebraically closed because there exists a pair $(k', |\ \ |')$ satisfying this condition (cf. ibid.).

The reduction of the proof in the case where $\alpha(\chi)$ is a non-negative integer is similar.

Let $\alpha(\chi)$ be arbitrary, let w be an element of W_n, and let

$$L_w=[(dT_\chi)(Y)-2w(dT_\chi)(X_-)+\alpha(\chi)\,\mathrm{id.}]/2$$

be as in 4–2. Then

$$L_w f(z)=(z-w)(d/dz)f(z)$$

holds for any $f(z) \in V_{w,n}$. Let s be a positive integer ≥ 2, and put

$$L_{w,s}=(L_w-(p^s-1)\,\mathrm{id.})(L_w-(p^s-2)\,\mathrm{id.})\cdots(L_w-\mathrm{id.})L_w.$$

Then for any element $f(z)=\sum_m c_m(z-w)^m$ of $V_{w,n}$, we have

$$L_{w,s}f(z)=\sum_m (m-p^s+1)(m-p^s+2)\cdots(m-1)mc_m(z-w)^m.$$

Since $m!/\{(m-p^s)!\,p^s!\}$ is an integer, $|(m-p^s+1)(m-p^s+2)\cdots(m-1)m| \leq |p^s!|$. Hence

$$\|L_{w,s}f(z)\| \leq |p^s!|\,\|f(z)\|.$$

Since k is algebraically closed, the valuation of k is dense. Hence there is an element ρ of k which satisfies $|p^s!|<|\rho|<|p^s!\,p^{-1}|$. Then the operator norm $\|\rho^{-1}L_{w,s}\|$ of $\rho^{-1}L_{w,s}$ is smaller than 1. Hence

$$(\rho^{-1}L_{w,s})^l \longrightarrow 0 \quad (l \in \boldsymbol{Z},\ l \to +\infty)$$

strongly on $V_{w,n}$.

Let $F(T)$ be the polynomial in $k[T]$ defined by

$$F(T)=(T-p^s+1)(T-p^s+2)\cdots(T-1)T-\rho,$$

and let $t \in k$ be a solution of $F(T)=0$. Since $|\rho|<|p^s!\,p^{-1}|<1$,

$$|(t-p^s+1)\cdots(t-1)t|=|\rho|<1.$$

Hence there is an integer i such that $0 \leq i \leq p^s-1$ and $|t-i|<1$. We assume $|t-i| \leq |t-j|$ for any j with $0 \leq j < p^s$. Then

$$|t-j|=|(t-i)+(i-j)| \geq |t-i|.$$

Hence $|t-j| \geq |i-j|$ for any j with $0 \leq j < p^s$. Let $e=t-i$. Then

$$\begin{aligned} |p^s!\,p^{-1}|>|\rho| &= |(t-p^s+1)\cdots(t-1)t| \\ &\geq |(i-p^s+1)\cdot \cdots \cdot(-1)\cdot e\cdot 1\cdot 2\cdot \cdots \cdot i| \\ &= |(p^s-i-1)!\,i!|\,|e| \geq |(p^s-1)!|\,|e|. \end{aligned}$$

Therefore $|e|<|p^{s-1}|$. If $|e| \leq |p^s|$, then $|e|=|t-i|<|i-j|$ for any $j \neq i$. Hence $|t-j|=|i-j|$ and $|\rho|=|(t-p^s+1)\cdots(t-1)t|=|e|\,|(p^s-1)!| \leq |p^s!|$. Since this is a contradiction, we obtain $|p^s|<|e|<|p^{s-1}|$. In particular, e is not an integer.

Let v be an element of W_n such that $B_{v,n} \neq B_{w,n}$. Then $r_{v,n}<|v-w|$. Since $|e-m| \leq \mathrm{Max}\,(|e|, |m|)$, $|(e-m)/m| \leq |e/m|$ (resp. 1) if $|m|<|e|$ (resp.

$|m|\geq|e|$). Since $|p^s|<|e|<|p^{s-1}|$, $|m|<|e|$ holds iff m is divisible by p^s. Hence

$$\left|\frac{(e-1)\cdots(e-m)}{m!}\right|=|(ep^{-s})^{[m/p^s]}([m/p^s]!)^{-1}|$$
$$\leq|ep^{-s-1/(p-1)}|^{[m/p^s]}<|p^{-2p-s}|^m,$$

where $[m/p^s]$ is the largest integer m^* satisfying $m^*\leq[m/p^s]$. Since $p^{-s}\to 0$ $(s\to+\infty)$, $|p^{2p^{-s}}|\to 1$ $(s\to+\infty)$. We choose a sufficiently large integer s so that $r_{v,n}/|v-w|<|p^{2p^{-s}}|<1$ holds. Then

$$\left(1+\frac{z-v}{v-w}\right)^e=\sum_{0\leq m<\infty}\frac{e(e-1)\cdots(e-m+1)}{m!}\left(\frac{z-v}{v-w}\right)^m$$

converges for $|z-v|\leq r_{v,n}$. Hence

$$h(z)=(z-w)^i\left(1+\frac{z-v}{v-w}\right)^e$$

is a non-zero element of $V_{v,n}$. Since

$$\frac{d}{dz}h(z)=i(z-w)^{i-1}\left(1+\frac{z-v}{v-w}\right)^e+(z-w)^i\frac{e}{v-w}\left(1+\frac{z-v}{v-w}\right)^{e-1}$$
$$=(i+e)(z-w)^{i-1}\left(1+\frac{z-v}{v-w}\right)^e,$$

$L_wh(z)=(z-w)(d/dz)h(z)=(i+e)h(z)=th(z)$. Hence

$$\rho^{-1}L_{w,s}h(z)=\rho^{-1}(L_w-p^s+1)\cdots(L_w-1)L_wh(z)$$
$$=\rho^{-1}(t-p^s+1)\cdots(t-1)th(z)=h(z).$$

Now we assume that $\alpha(\chi)$ is not a non-negative integer. Suppose that there is a triple (W,f,g) such that W is a Banach space over k on which $\mathfrak{g}$ acts continuously, and f: $V_{w,n}\to W$ and g: $V_{v,n}\to W$ are injective continuous $\mathfrak{g}$-homomorphisms with dense images. Since $(\rho^{-1}L_{w,s})^l\to 0$ $(l\to+\infty)$ strongly on $V_{w,n}$, it follows from the continuity of f that $(\rho^{-1}L_{w,s})^l\to 0$ $(l\to+\infty)$ strongly on the image of f. Since the image of f is dense in W, $(\rho^{-1}L_{w,s})^l\to 0$ $(l\to+\infty)$ strongly on W. On the other hand, $h(z)$ is a non-zero element of $V_{v,n}$ satisfying $\rho^{-1}L_{w,s}h(z)=h(z)$. Since g is an injective $\mathfrak{g}$-homomorphism, $0\neq g(h(z))\in W$ satisfies $\rho^{-1}L_{w,s}g(h(z))=g(h(z))$. Since this contradicts the assumption $(\rho^{-1}L_{w,s})^l\to 0$ $(l\to+\infty)$ on W, there exists no triple (W,f,g) such that f: $V_{w,n}\to W$ and g: $V_{v,n}\to W$ $(w, v\in W_n, B_{w,n}\neq B_{v,n})$ satisfy the conditions (a), (b), (c) of 3–2. Hence the proposition holds in this case.

We can prove that the condition (iii) of 3–2 for $V_{w,n}/P_{w,n}$ and $V_{v,n}/P_{v,n}$ ($w, v \in W_n$, $B_{w,n} \neq B_{v,n}$) holds in the case when $\alpha(\chi)$ is a non-negative integer in the same way, because $h(z)$ does not belong to $P_{v,n}$.

Now we assume that $\alpha(\chi)$ is not a non-negative integer, and that $f: V_{w,n} \to W$ ($w \in W_n$) and $g: V_{\infty,n} \to W$ satisfy the conditions (a), (b), (c) of 3–2. Let c be an element of $\mathfrak{o}$ satisfying $|p^n|<|w+c|$, and let T be the endomorphism

$$T_\chi(I_c): D_\chi \longrightarrow D_\chi \quad \left(I_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in K\right)$$

of D_χ. Then T induces local homomorphisms

$$T_w: V_{-1/(w+c),n} \longrightarrow V_{w,n} \quad \text{and} \quad T_\infty: V_{0,n} \longrightarrow V_{\infty,n},$$

and they satisfy the condition

$$T_* \circ (dT_\chi)(X) = (dT_\chi)(I_c X I_c^{-1}) \circ T_*,$$

for any $X \in \mathfrak{g}$. Hence, if we twist the action of $\mathfrak{g}$ on W by I_c, then $f \circ T_w: V_{-1/(w+c),n} \to W$ and $g \circ T_\infty: V_{0,n} \to W$ satisfy the conditions (a), (b), (c) of 3–2. Since this contradicts to what we have proved, the condition (iii) of 3–2 is satisfied also in this case.

We can prove the condition (iii) for $V_{w,n}/P_{w,n}$ and $V_{\infty,n}/P_{\infty,n}$ in the same way. Therefore the proof of Proposition 3 is completed.

5-2. *Proof of the irreducibility of $D_\chi/D_\chi^{\mathrm{loc},m}$.* We assume in 5–2 that $\alpha(\chi)$ is a non-negative integer m. Let f be an element of D_χ which does not belong to $D_\chi^{\mathrm{loc},m}$, and let U_f be the minimal closed K-invariant subspace of D_χ containing f. We are going to show $U_f = D_\chi$.

Since K acts transitively on $P^1(L)$, replacing f by $T_\chi(g)f$ ($g \in K$) if necessary, we may assume that the Taylor expansion of f at $z=0$ is not a polynomial of degree $\leq m$. We choose a positive integer n such that (i) $\chi(z)f(z^{-1})$ is analytic for $|z| \geq |p^n|$ (i.e. $\chi(z)f(z^{-1})$ is expanded into a convergent power series of z^{-1} for $|z| \geq |p^n|$), and (ii) for any $w \in L$ satisfying $|p^n w|<1$, $f(z)$ is analytic for $|z-w| \leq r_{w,n}$. Then $f(z)$ is an element of V_n. Hence we can write $f(z) = f_\infty(z) + \sum_w f_w(z)$ ($f_\infty(z) \in V_{\infty,n}$, $f_w(z) \in V_{w,n}$, $w \in W_n$). By our assumption, $f_0(z) \in V_{0,n}$ does not belong to $P_{0,n}$. Since $(V_{\infty,n}/P_{\infty,n}) \oplus \bigoplus_w (V_{w,n}/P_{w,n})$ satisfies the assumption of the Key Lemma, the minimal closed $\mathfrak{g}$-invariant subspace $U_{f,n}$ of V_n containing f has an element $h(z)$ of the form

$$h(z) = h_\infty(z) + \sum_w h_w(z) \in V_{\infty,n} \oplus \bigoplus_w V_{w,n},$$

where $h_\infty(z) \in P_{\infty,n}$, $h_w(z) \in P_{w,n}$ if $B_{w,n} \neq B_{0,n}$, $h_0(z) - z^{m+1} \in P_{0,n}$. Then

$$[(dT_\chi)(X_-)]^{m+1}h_\infty(z)=[(dT_\chi)(X_-)]^{m+1}h_w(z)=0\ (0\neq w\in W_n)\quad\text{and}$$
$$[(dT_\chi)(X_-)]^{m+1}h_0(z)=(m+1)!\in P_{0,n}.$$

By the assertion (ii) of Proposition 2, $P_{0,n}$ is an irreducible $\mathfrak{g}$-module. Hence $U_{f,n}$ contains $P_{0,n}$. Since $T_\chi(C(-w))P_{0,n}=P_{w,n}$ ($w\in W_n$, $|w|\leq 1$), $U_{f,n}$ contains the direct sum $P_n^{(1)}$ of the $P_{w,n}$'s with $w\in W_n$, $|w|\leq 1$. Similarly $U_{f,n}$ contains the direct sum $P_n^{(1)*}$ of the $P_{w,n}$'s with $w\in W_n$, $|w|<1$. Since n can be arbitrarily small, U_f contains the injective limit of the $P_n^{(1)*}$'s. Since $T_\chi(I)$ (inj lim $P_n^{(1)*}$) contains the direct sum $P_n^{(2)}$ of $P_{\infty,n}$ and the $P_{w,n}$'s with $w\in W_n$, $|w|>1$, U_f contains the direct sum of $P_{\infty,n}$ and the $P_{w,n}$'s ($w\in W_n$). Hence, by substracting an element of this space if necessary, we may assume that $h_0(z)=z^{m+1}$ and $h_\infty(z)=h_w(z)=0$ for any non-zero $w\in W_n$.

By the assertion (ii) of Proposition 2, $h_0(z)$ spans $V_{0,n}$ topologically as a $\mathfrak{g}$-module. Since $T_\chi(C(-w))P_{0,n}=V_{w,n}$ ($w\in W_n$, $|w|\leq 1$), U_f contains the direct sum $V_n^{(1)}$ of the $V_{w,n}$'s with $w\in W_n$, $|w|\leq 1$. Since n can be arbitrarily small, U_f contains the injective limit of the $V_n^{(1)}$'s. Similarly, U_f contains the injective limit of the direct sum $V_n^{(1)*}$ of the $V_{w,n}$'s with $w\in W_n$, $|w|<1$. Then $T_\chi(I)$ (inj lim $V_n^{(1)*}$) is the injective limit of the direct sum $V_n^{(2)}$ of $V_{\infty,n}$ and the $V_{w,n}$'s with $w\in W_n$, $|w|>1$. Hence U_f contains (inj lim $V_n^{(1)}$)$\oplus$(inj lim $V_n^{(2)}$)$=D_\chi$. Therefore we have proved $U_f=D_\chi$.

5-3. *Proof of the irreducibility of* D_χ. We assume in 5–3 that $\alpha(\chi)$ is not a non-negative integer. Let f be a non-zero element of D_χ, and let U_f be the minimal closed K-invariant subspace of D_χ containing f. We must show $U_f=D_\chi$. We can prove this fact in the same way as in 5–2. Simply replace $D_\chi^{\mathrm{loc},m}$, $P_{\infty,n}$ and the $P_{w,n}$'s ($w\in W_n$) by the zero space $\{0\}$, and do the same arguments. Since the argument about $P_{\infty,n}\oplus\bigoplus_w P_{w,n}$ is trivial in this case, the proof is far easier than in 5–2.

§ 6. The Frobenius reciplocity law

6-1. *The Frobenius reciplocity law.* For any topological group N and for any linear topological k-vector spaces V_1 and V_2 on which the group N acts continuously, let $\mathrm{Hom}_N(V_1, V_2)$ denote the k-module of all continuous N-linear homomorphisms from V_1 to V_2.

Let $G=SL_2(L)$, let V be a linear topological k-vector space, and let $T: G\to \mathrm{Aut}_k(V)$ be a continuous representation. We assume that for any fixed v, $T(g)v$ is a locally analytic function of g with values in V. Let χ: $L^*\to k^*$ be a locally analytic character, and let P, Ind (P, G, χ) and D_χ be as in Section 2. Note that for each $f(z)\in D_\chi\simeq\mathrm{Ind}\,(P, G, \chi)$, $T_\chi(g)f(z)$ is a locally analytic function of g with values in D_χ.

Let H be a continuous G-homomorphism from V to Ind (P, G, χ). Then for any $v \in V$, $H(v)$ is a locally analytic function on G with values in k and satisfies

$$H(v)(pg)=\chi(a)H(v)(g) \quad \left(g \in G, p=\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in P\right).$$

Since H is a G-homomorphism, we have

$$H(g_1v)(g)=T(g_1)H(v)(g)=H(v)(gg_1)$$

for any $g, g_1 \in G$ and $v \in V$. Since $p=\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}$ acts on k by the multiplication of $\chi(a)$,

$$H^*=H(\ )(1)\colon V \ni v \longmapsto H(v)(1) \in k$$

is a continuous P-homomorphism.

Conversely, if H^* is a P-homomorphism from V to k, then we define

$$H(v)(g)=H^*(gv) \quad (v \in V, g \in G).$$

Then $H(v)\colon G \rightarrow k$ is a locally analytic function, and satisfies

$$H(v)(pg)=H^*(pgv)=\chi(a)H^*(gv)=\chi(a)H(v)(g)$$

for any $p \in P$. Hence $H(v)$ is an element of Ind (P, G, χ). Since $H(\ )(1) = H^*$ as k-valued functions on V, we have the following Frobenius reciplocity law:

Proposition 4 (Casselman). *Let the notation and assumptions be as before. Then the correspondence $H \mapsto H^*=H(\ )(1)$ induces a k-linear bijection*

$$\mathrm{Hom}_G(V, \mathrm{Ind}\,(P, G, \chi)) \xrightarrow{\sim} \mathrm{Hom}_P(V, k),$$

where $p=\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \in P$ acts on k by the multiplication of $\chi(a)$.

6-2. $\mathrm{Hom}_G(D_\chi, D_\delta)$. Let χ and δ be k-valued locally analytic characters of L^*. Then, by Proposition 4, H is a continuous G-homomorphism from D_χ to Ind (P, G, δ) iff $H^*=H(\ )(1)$ is a continuous P-homomorphism from D_χ to k, where P acts on k by δ. Since P is generated by the $A(a)$'s and the $C(c)$'s ($a \in L^*$, $c \in L$), $H \in \mathrm{Hom}_G(D_\chi$, Ind $(P, G, \delta))$ iff $H^*\colon D_\chi \rightarrow k$ is a continuous k-linear operator and satisfies

$$H^*(f(z+c))=H^*(f(z)) \qquad (c \in L),$$
$$H^*(f(a^2z))=\chi(a)\delta(a)H^*(f(z)) \qquad (a \in L^*)$$

for any $f(z) \in D_\chi$. Therefore, taking the limits for $c\to 0$ and for $a\to 1$, we obtain

$$H^*((dT_\chi)(X_-)f(z))=0 \quad \text{and} \quad H^*((dT_\chi)(Y)f(z))=\alpha(\delta)H^*(f(z)).$$

For any $n \in \boldsymbol{Z}$, $n\geq 0$, $r \in |L^*|$ and $c \in L$, put

$$f_{\chi,\infty,n,r}(z)=\begin{cases}\chi(z)z^{-n} & |z|\geq r\\ 0 & |z|<r\end{cases} \quad \text{and} \quad f_{c,n,r}(z)=\begin{cases}0 & |z-c|>r\\ (z-c)^n & |z-c|\leq r.\end{cases}$$

Since $(dT_\chi)(X_-)f_{c,n+1,r}(z)=(n+1)f_{c,n,r}(z)$, $H^*(f_{c,n,r}(z))=0$ for any c, n and r. Since

$$(dT_\chi)(X_-)f_{\chi,\infty,n,r}(z)=(\alpha(\chi)-n)f_{\chi,\infty,n+1,r}(z),$$

$H^*(f_{\chi,\infty,n,r}(z))\neq 0$ only if $\alpha(\chi)=n-1$ or $n=0$. Since $f_{\chi,\infty,n,r'}(z)-f_{\chi,\infty,n,r}(z)$ $(r'>r)$ vanishes for $|z|\geq r'$, this element of D_χ can be expressed as an infinite linear combination of the $f_{c,n^*,r^*}(z)$'s. It follows that

$$H^*(f_{\chi,\infty,n,r'}(z))=H^*(f_{\chi,\infty,n,r}(z)),$$

and hence $H^*(f_{\chi,\infty,n,r}(z))$ does not depend on r. Since $f_{\chi,\infty,n,r}(a^2z)=\chi(a)^2a^{-2n}f_{\chi,\infty,n,|a^{-2}|r}(z)$, we have

$$H^*(f_{\chi,\infty,n,r}(a^2z))=\chi(a)^2a^{-2n}H^*(f_{\chi,\infty,n,r}(z)).$$

Therefore $H^*(f_{\chi,\infty,n,r}(z))\neq 0$ only if $\chi(a)^2a^{-2n}=\chi(a)\delta(a)$ holds for any $a \in L^*$. We note that $H^*\neq 0$ iff $H^*(f_{\chi,\infty,n,r}(z))\neq 0$ for some n, because the $f_{\chi,\infty,n,r}(z)$'s and the $f_{c,n,r}(z)$'s span a dense subspace of D_χ.

We assume that $\alpha(\chi)$ is not a non-negative integer. If $H^*\neq 0$, then $H^*(f_{\chi,\infty,0,r}(z))\neq 0$. Hence we have $\chi(a)=\delta(a)$ for any $a \in L^*$. In this case, $\mathrm{Hom}_G(D_\chi, D_\delta)$ contains the identity mapping id. and $\mathrm{Hom}_G(D_\chi, D_\delta)\simeq\mathrm{Hom}_P(D_\chi, k)$ is one-dimensional. Hence $\mathrm{Hom}_G(D_\chi, D_\delta)=k$ id. and $\mathrm{Hom}_G(D_\chi, D_\delta)=0$ if $\chi(a)\neq\delta(a)$ for some a.

Next we assume that $\alpha(\chi)$ is a non-negative integer m. If $H^*\neq 0$, then $H^*(f_{\chi,\infty,0,r}(z))\neq 0$ or $H^*(f_{\chi,\infty,m+1,r}(z))\neq 0$. If $H^*(f_{\chi,\infty,0,r}(z))\neq 0$, then we have $\chi(a)=\delta(a)$ for any $a \in L^*$. If $H^*(f_{\chi,\infty,m+1,r}(z))\neq 0$, then $\chi(a)^2a^{-2m-2}=\chi(a)\delta(a)$. Hence $\chi(a)=a^{2m+2}\delta(a)$ holds for any $a \in L^*$. Since $m\geq 0$, $a^{2m+2}\neq 1$ for some a. Hence $\mathrm{Hom}_G(D_\chi, D_\delta)\simeq\mathrm{Hom}_P(D_\chi, k)$ is a one dimensional k-vector space in either case. In particular, $\mathrm{Hom}_G(D_\chi, D_\chi)=k$ id.

We assume that $H^*(f_{\chi,\infty,m+1,r}(z))=\gamma\neq 0$. Put $S_\chi f(z)=(d/dz)^{m+1}f(z)$ for any $f(z)\in D_\chi$. Then, by our identification $\mathrm{Ind}\,(P, G, \delta)\simeq D_\delta$, $f(z)\in D_\chi$ corresponds to

$$H(f(z))\begin{pmatrix} w & 1 \\ 0 & w^{-1}\end{pmatrix}=H^*\left(T_\chi\begin{pmatrix} w & 1 \\ 0 & w^{-1}\end{pmatrix}f(z)\right)=H^*\left(\chi(z+w^{-1})f\left(\frac{wz}{z+w^{-1}}\right)\right)$$
$$=H^*\left(\chi(z)f\left(\frac{w(z-w^{-1})}{z}\right)\right)=H^*(\chi(z)f(w-z^{-1}))\in D_\delta$$

as a function of w. If $f(z)=\sum_{n=0}^{\infty}c_n(z-w)^n$ for $|z-w|\leq r$, then

$$\begin{aligned}H^*(\chi(z)f(w-z^{-1}))&=H^*\left(\sum_{n=0}^{\infty}c_n(-1)^n f_{\chi,\infty,n,r}(z)\right)\\&=(-1)^{m+1}\gamma c_{m+1}=(-1)^{m+1}((m+1)!)^{-1}\gamma(d/dw)^{m+1}f(w)\\&=(-1)^{m+1}((m+1)!)^{-1}\gamma S_\chi(f(w)).\end{aligned}$$

Therefore $\mathrm{Hom}_G(D_\chi, D_\delta)=kS_\chi$ if $\alpha(\chi)$ is a non-negative integer m and $\delta(z)=z^{-2m-2}\chi(z)$. Summarizing, we obtain the following:

Proposition 5 (Casselman). (i) *If $\alpha(\chi)$ is a non-negative integer m, put $\delta(z)=z^{-2m-2}\chi(z)$ for any $z\in L^*$. Then*

$$S_\chi\colon D_\chi\ni f(z)\longmapsto(d/dz)^{m+1}f(z)\in D_\delta$$

is a continuous G-homomorphism and we have $\mathrm{Hom}_G(D_\chi, D_\delta)=kS_\chi$.

(ii) *In general, let χ, $\delta\colon L^*\to k^*$ be locally analytic characters. Then* $\mathrm{Hom}_G(D_\chi, D_\chi)=k\ \mathrm{id.}$, *and* $\mathrm{Hom}_G(D_\chi, D_\delta)=0$ *if $\chi(z)\neq\delta(z)$ and if $\chi(z)$ and $\delta(z)$ do not satisfy the condition of* (i).

6-3. $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\delta)$. We assume in 6–3 that $\alpha(\chi)$ is a non-negative integer m. Hence $\chi(z)=\varepsilon(z)z^m$ holds with a locally constant character $\varepsilon\colon L^*\to k^*$. Let $D_\chi^{\mathrm{loc},m}$ be as in Section 2. Then $f_{\chi,\infty,n,r}(z)$ and $f_{c,n,r}(z)$ ($n\in\boldsymbol{Z}$, $0\leq n\leq m$, $r\in|L^*|$, $c\in L$) are elements of $D_\chi^{\mathrm{loc},m}$, and any element of $D_\chi^{\mathrm{loc},m}$ can be expressed as a finite k-linear combination of them. Let H be a non-zero element of $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, \mathrm{Ind}\,(P, G, \delta))$. Then $H^*=H(\ )(1)$ is a continuous mapping of $D_\chi^{\mathrm{loc},m}$ to k and satisfies

$$\begin{aligned}&H^*(f(z+c))=H^*(f(z)) &&(c\in L),\\&H^*(f(a^2z))=\chi(a)\delta(a)H^*(f(z)) &&(a\in L^*)\end{aligned}$$

for any $f(z)\in D_\chi^{\mathrm{loc},m}$. Hence, as in 6–2, we have $H^*(f_{c,n,r}(z))=0$ $(0\leq n<m)$ and $H^*(f_{\chi,\infty,n,r}(z))=0$ $(0<n\leq m)$. Put

$$H^*(f_{0,m,1}(z))=\alpha \quad\text{and}\quad H^*(f_{\chi,\infty,0,1}(z))=\beta.$$

Since $H^*(f(z+c))=H^*(f(z))$, we have $H^*(f_{c,m,|\pi^n|}(z))=H^*(f_{0,m,|\pi^n|}(z))$. Since

$$f_{0,m,1}(z)=\sum_{c\in\mathfrak{o}/\mathfrak{p}^n} f_{c,m,|\pi^n|}(z),$$

and since H^* is linear, we obtain $H^*(f_{0,m,|\pi^n|}(z))=q^{-n}\alpha$. Hence we have

$$H^*(f_{c,m,|\pi^n|}(z))=q^{-n}\alpha.$$

It follows that

$$\begin{aligned}\chi(a)\delta(a)H^*(f_{0,m,1}(z))&=H^*(f_{0,m,1}(a^2z))\\&=a^{2m}H^*(f_{0,m,|a^{-2}|}(z))=a^{2m}N(a)^2H^*(f_{0,m,1}(z)).\end{aligned}$$

Therefore, if $\alpha\neq 0$, then $\chi(z)\delta(z)=z^{2m}N(z)^2$ holds for any $z\in L^*$.

Let $l=l(\chi)$ be the smallest positive integer l such that $\varepsilon(z)=1$ holds for $|z-1|\leq|\pi^l|$. Then we have

$$f_{\chi,\infty,0,|\pi^n|}(z)=f_{\chi,\infty,0,1}(z)-\sum_{c\in(\mathfrak{p}^{1-n}\backslash\mathfrak{p})/\mathfrak{p}^l}\varepsilon(c)\sum_{i=0}^{m}\binom{m}{i}c^{m-i}f_{c,i,|\pi^l|}(z)$$

for any $n\in \boldsymbol{Z}$, $n\geq 0$, and hence

$$H^*(f_{\chi,\infty,0,|\pi^n|}(z))=\beta-\sum_{c\in(\mathfrak{p}^{1-n}\backslash\mathfrak{p})/\mathfrak{p}^l}\varepsilon(c)q^{-l}\alpha.$$

If ε is not trivial on $\mathfrak{o}^*$, then $\sum_{c\in\mathfrak{o}^*/\mathfrak{p}^l}\varepsilon(c)=0$ and hence $\sum_{c\in(\mathfrak{p}^{1-n}\backslash\mathfrak{p})/\mathfrak{p}^l}\varepsilon(c)=0$. If ε is trivial on $\mathfrak{o}^*$, then we obtain $l=1$ and $\sum_{c\in(\mathfrak{o}\backslash\mathfrak{p})/\mathfrak{p}}\varepsilon(c)=q-1$. Hence

$$\sum_{c\in(\mathfrak{p}^{1-n}\backslash\mathfrak{p})/\mathfrak{p}^l}\varepsilon(c)=(q-1)\varepsilon(1)+(q^2-q)\varepsilon(\pi)^{-1}+\cdots+(q^n-q^{n-1})\varepsilon(\pi)^{-n+1}.$$

If $\varepsilon(\pi)=q$, then this sum is equal to $n(q-1)$. If $\varepsilon(\pi)\neq q$, then this sum is equal to $(q-1)\{1-q\varepsilon(\pi)^{-1}\}^{-1}\{1-(q\varepsilon(\pi)^{-1})^n\}$. Therefore $H^*(f_{\chi,\infty,0,|\pi^n|}(z))$ is equal to the following:

(i) β if $\varepsilon(z)$ is not trivial on $\mathfrak{o}^*$;

(ii) $\beta-n(q-1)q^{-1}\alpha$ if $\varepsilon(z)=N(z)$ holds for any $z\in L^*$;

(iii) $\beta-\{1-(q\varepsilon(\pi)^{-1})^n\}\{1-q\varepsilon(\pi)^{-1}\}^{-1}(q-1)q^{-1}\alpha$ if ε is trivial on $\mathfrak{o}^*$ and $\varepsilon(\pi)\neq q$.

It is easy to see that this formula holds also for $n\in\boldsymbol{Z}$, $n<0$, and that $H^*(f_{\chi,\infty,0,|\pi^n|}(z+c))=H^*(f_{\chi,\infty,0,|\pi^n|}(z))$ holds for any $c\in L$. Since

$$\chi(a)^2H^*(f_{\chi,\infty,0,|a^{-2}|}(z))=H^*(f_{\chi,\infty,0,1}(a^2z))=\chi(a)\delta(a)H^*(f_{\chi,\infty,0,1}(z))$$

for any $a\in L^*$, the following condition must be satisfied:

(i) $\chi(z)=\delta(z)$ if $\beta\neq 0$, and if $\alpha=0$ or ε is not trivial on $\mathfrak{o}^*$;

(ii) $\chi(\pi)^n\{\beta+2n(q-1)q^{-1}\alpha\}=\delta(\pi)^n\beta$ if $\alpha\neq 0$ and $\varepsilon(z)=N(z)$;

(iii) $\chi(\pi)^n[\beta-\{1-(q\varepsilon(\pi)^{-1})^{2n}\}\{1-q\varepsilon(\pi)^{-1}\}^{-1}(q-1)q^{-1}\alpha]=\delta(\pi)^n\beta$ if $\alpha\neq 0$, $\varepsilon(\pi)\neq q$ and ε is trivial on $\mathfrak{o}^*$.

The second case is obviously impossible. In the third case, $\varepsilon(z)=z^{-m}\chi(z)=z^mN(z)^2\delta(z)^{-1}$ must satisfy

$$\{1-(q\varepsilon(\pi)^{-1})^{2n}\}\{\beta-(1-q\varepsilon(\pi)^{-1})^{-1}(q-1)q^{-1}\alpha\}=0.$$

Hence either $\varepsilon(\pi)=-q$ or $\beta=(1-q\varepsilon(\pi)^{-1})^{-1}(q-1)q^{-1}\alpha$. Note that

$$H^*(f_{\chi,\infty,0,|\pi^n|}(z))=(1-q\varepsilon(\pi)^{-1})^{-1}(q-1)q^{-1}\alpha(q\varepsilon(\pi)^{-1})^n$$

holds if $\beta=(1-q\varepsilon(\pi)^{-1})^{-1}(q-1)q^{-1}\alpha$. We observe that $\alpha(\delta)=\alpha(\chi)=m$ holds in general, and that $\mathrm{Hom}_P(D_\chi^{\mathrm{loc},m},k)\simeq\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)$ is two dimensional if $\chi(z)=\delta(z)$, $\varepsilon(z)^2=N(z)^2$ and $\varepsilon(z)\neq N(z)$, is one dimensional if $\chi(z)=\delta(z)$ and $\varepsilon(z)^2\neq N(z)^2$, or if $\chi(z)=\delta(z)=z^mN(z)$, or if $\chi(z)\delta(z)=z^{2m}N(z)^2$ and $\varepsilon(z)^2\neq N(z)^2$, and is zero dimensional otherwise.

If $\chi(z)=\delta(z)$ and $\varepsilon(z)^2\neq N(z)^2$, or if $\chi(z)=\delta(z)=z^mN(z)$, then $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)$ contains the identity mapping id. Hence $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)=k\,\mathrm{id}$. If $\chi(z)\delta(z)=z^{2m}N(z)^2$ and $\varepsilon(z)^2\neq N(z)^2$, we denote by H_χ^* the element of $\mathrm{Hom}_P(D_\chi^{\mathrm{loc},m},k)$ which satisfies $H_\chi^*(f_{0,m,1}(z))=1$, and denote by H_χ the element of $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)\simeq\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},\mathrm{Ind}\,(P,G,\delta))\simeq\mathrm{Hom}_P(D_\chi^{\mathrm{loc},m},k)$ corresponding to H_χ^* by this isomorphism. Then we have $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)=kH_\chi$.

In general, for any $f(z)\in D_\chi$, we assume that $H\in\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)$ corresponds to $H^*\in\mathrm{Hom}_P(D_\chi^{\mathrm{loc},m},k)\simeq\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},\mathrm{Ind}\,(P,G,\delta))\simeq\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta)$. Then

$$H(f(z))(w)=H^*\left(T_\chi\begin{pmatrix}w & 1\\ 0 & w^{-1}\end{pmatrix}f(z)\right)=H^*(\chi(z)f(w-z^{-1}))$$

holds for any $f(z)\in D_\chi^{\mathrm{loc},m}$. Since $[(dT_\chi)(X_-)]^{m+1}f(z)=0$,

$$\begin{aligned}(d/dw)^{m+1}H(f(z))(w)&=[(dT_\delta)(X_-)]^{m+1}H(f(z))(w)\\&=H([(dT_\chi)(X_-)]^{m+1}f(z))(w)=0.\end{aligned}$$

Hence $\mathrm{im}\,(H)\subset D_\delta^{\mathrm{loc},m}$, and $H\in\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\delta^{\mathrm{loc},m})$.

We assume that $\varepsilon(z)=1$ holds for any $z\in L^*$ with $|z-1|<1$. If $|w-c|\leq|\pi^n|$, then $|(w-z^{-1})-c|\leq|\pi^n|$ holds iff $|z|\geq|\pi^{-n}|$. Since the coefficient of z^m in $z^m(w-z^{-1}-c)^i$ is $(w-c)^i$ $(0\leq i\leq m)$, we obtain $H^*(\chi(z)f_{c,i,|\pi^n|}(w-z^{-1}))=H^*(f_{\chi,\infty,0,|\pi^n|}(z))(w-c)^i$. If $|w-c|>|\pi^n|$, then $|(w-z^{-1})-c|\leq|\pi^n|$ holds iff $|z-(w-c)^{-1}|\leq|\pi^n(w-c)^{-2}|$. Since $|1-(w-c)z|\leq|\pi^n(w-c)^{-1}|<1$, $\varepsilon(z)=\varepsilon(w-c)^{-1}$. Hence

$$H^*(\chi(z)f_{c,i,|\pi^n|}(w-z^{-1}))=\varepsilon(w-c)^{-1}(w-c)^iH^*(f_{(w-c)^{-1},m,|\pi^n(w-c)^{-2}|}(z))$$
$$=\varepsilon(w-c)^{-1}N(w-c)^2q^{-n}\alpha(w-c)^i.$$

Similarly, if $|\pi^{-n}|\leq|w|$, then $|w-z^{-1}|\geq|\pi^{-n}|$ holds iff $|z-w^{-1}|\geq|\pi^{-n}w^{-2}|$. Since

$$\chi(z)(w-z^{-1})^{-i}\chi(w-z^{-1})=\chi(w)w^{-i}\chi(z-w^{-1})\{(z-w^{-1})+w^{-1}\}^i(z-w^{-1})^{-i},$$
$$H^*(\chi(z)f_{\chi,\infty,i,|\pi^n|}(w-z^{-1}))=\chi(w)^{-1}w^{-i}H^*(f_{\chi,\infty,0,|\pi^nw^2|}(z))$$
$$=\chi(w)^{-1}w^{-i}H^*(f_{\chi,\infty,0,|\pi^n|}(w^2z))=H^*(f_{\chi,\infty,0,|\pi^n|}(z))\delta(w)w^{-i}.$$

If $|\pi^{-n}|>|w|$, then $|w-z^{-1}|\geq|\pi^{-n}|$ holds iff $|z|\leq|\pi^n|$, Since $|zw|<1$, $\varepsilon(zw-1)=\varepsilon(-1)$. Since $\chi(z)(w-z^{-1})^{-i}\chi(w-z^{-1})=\varepsilon(wz-1)z^m(w-z^{-1})^{m-i}$, we obtain

$$H^*(\chi(z)f_{\chi,\infty,i,|\pi^n|}(w-z^{-1}))=\varepsilon(-1)w^{m-i}H^*(f_{0,m,|\pi^n|}(z))=\varepsilon(-1)q^{-n}\alpha w^{m-i}.$$

If $\chi(z)=z^m$ and $\delta(z)=z^mN(z)^2$, then

$$H_\chi(f_{c,i,|\pi^n|}(z))(w)=\begin{cases}q^{-n}\delta(w-c)(w-c)^{-m+i} & |w-c|>|\pi^n|\\ -q^{n-1}(w-c)^i & |w-c|\leq|\pi^n|,\end{cases}$$
$$H_\chi(f_{\chi,\infty,i,|\pi^n|}(z))(w)=\begin{cases}-q^{n-1}\delta(w)w^{-i} & |w|\geq|\pi^{-n}|\\ q^{-n}w^{m-i} & |w|<|\pi^{-n}|,\end{cases}$$
$$H_\delta(f_{c,i,|\pi^n|}(z))(w)=q^{-n}(w-c)^i,$$
$$H_\delta(f_{\delta,\infty,i,|\pi^n|}(z))(w)=q^{-n}w^{m-i}.$$

We observe that $H_\delta\circ H_\chi=H_\chi\circ H_\delta=0$. Let P_m and Q_m be the kernels of H_χ and H_δ, respectively. Then H_χ and H_δ induce continuous G-homomorphisms $D_\chi^{\mathrm{loc},m}/P_m\to Q_m$ and $D_\delta^{\mathrm{loc},m}/Q_m\to P_m$, respectively.

If $p\neq 2$, then $\varepsilon(z)^2=N(z)^2$ implies $\varepsilon(z)=1$ for $|z-1|<1$. If $\varepsilon(z)^2=N(z)^2$ for any $z\in L^*$, $\varepsilon(z)=1$ for any $z\in 1+\mathfrak{p}$, and $\varepsilon(z)\neq 1$ for some $z\in\mathfrak{o}^*$, then for any α, $\beta\in k$, the conditions

$$H^*(f_{c,m,|\pi^n|}(z))=q^{-n}\alpha \quad\text{and}\quad H^*(f_{\chi,\infty,|\pi^n|}(z))=\beta$$

define an element H of $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\delta^{\mathrm{loc},m})$. In this case, we have

$$H(f_{c,i,|\pi^n|}(z))(w)=\begin{cases}q^{-n}\alpha\chi(w-c)(w-c)^{-m+i} & |w-c|>|\pi^n|\\ \beta(w-c)^i & |w-c|\leq|\pi^n|,\end{cases}$$
$$H(f_{\chi,\infty,i,|\pi^n|}(z))(w)=\begin{cases}\beta\chi(w)w^{-i} & |w|\geq|\pi^{-n}|\\ \varepsilon(-1)q^{-n}\alpha w^{m-i} & |w|<|\pi^{-n}|.\end{cases}$$

Hence we have

$$H(f_{0,i,|\pi^n|}(z))(w)=\beta f_{0,i,|\pi^n|}(w)+q^{-n}\alpha f_{\chi,\infty,m-i,|\pi^{-n+1}|}(w),$$
$$H(f_{\chi,\infty,i,|\pi^n|}(z))(w)=\beta f_{\chi,\infty,i,|\pi^n|}(w)+\varepsilon(-1)q^{-n}\alpha f_{0,m-i,|\pi^{-n+1}|}(w).$$

If $\varepsilon(z)^2=N(z)^2$ holds for any $z\in L^*$ and if $\varepsilon(z)$ is not trivial on $1+\mathfrak{p}$, then $p=2$ and $l=l(\chi)>1$. In this case, these formulas are modified as

$$H(f_{0,i,|\pi^n|}(z))(w)=\beta f_{0,i,|\pi^n|}(w)+q^{-n}\alpha f_{\chi,\infty,m-i,|\pi^{l-n}|}(w),$$
$$H(f_{\chi,\infty,i,|\pi^n|}(z))(w)=\beta f_{\chi,\infty,i,|\pi^n|}(w)+\varepsilon(-1)q^{-n}\alpha f_{0,m-i,|\pi^{l-n}|}(w).$$

In particular,

$$H^2(f_{0,i,|\pi^n|}(z))(w)=(\beta^2+\varepsilon(-1)q^{-l}\alpha^2)f_{0,i,|\pi^n|}(w)+2q^{-n}\alpha\beta f_{\chi,\infty,m-i,|\pi^{l-n}|}(w),$$
$$H^2(f_{\chi,\infty,i,|\pi^n|}(z))(w)=(\beta^2+\varepsilon(-1)q^{-l}\alpha^2)f_{\chi,\infty,i,|\pi^n|}(w)$$
$$+2\varepsilon(-1)q^{-n}\alpha\beta f_{0,m-i,|\pi^{l-n}|}(w).$$

If $\sqrt{\varepsilon(-1)q^l}$ is not contained in k, then the operator H corresponding to $(\alpha,\beta)=(q^l,0)$ is denoted by I_χ. Then we have

$$I_\chi^2=\varepsilon(-1)q^l\text{ id. and }\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m},D_\chi)=k\text{ id.}\oplus kI_\chi.$$

In particular, the endomorphism ring $\mathrm{End}_G(D_\chi^{\mathrm{loc},m})$ is isomorphic to the field $k(\sqrt{\varepsilon(-1)q^l})$.

If $\sqrt{\varepsilon(-1)q^l}$ is contained in k, then the operator H corresponding to

$$(\alpha,\beta)=\left(\pm\frac{1}{2}\sqrt{\varepsilon(-1)q^l},\ \frac{1}{2}\right)$$

is denoted by $H_{\chi,\pm}$. Then we have

$$H_{\chi,+}^2=H_{\chi,+}\neq 0,\quad H_{\chi,-}^2=H_{\chi,-}\neq 0,\quad H_{\chi,+}+H_{\chi,-}=\text{id}.$$

Therefore $H_{\chi,+}$ and $H_{\chi,-}$ are projection operators. Put

$$D_{\chi,+}^{\mathrm{loc},m}=\mathrm{im}\,(H_{\chi,+})\quad\text{and}\quad D_{\chi,-}^{\mathrm{loc},m}=\mathrm{im}\,(H_{\chi,-}).$$

Then these spaces $D_{\chi,\pm}^{\mathrm{loc},m}$ are closed G-invariant subspaces of $D_\chi^{\mathrm{loc},m}$, and

$$D_\chi^{\mathrm{loc},m}=D_{\chi,+}^{\mathrm{loc},m}\oplus D_{\chi,-}^{\mathrm{loc},m}.$$

If $\varepsilon(z)=1$ for any $z\in\mathfrak{o}^*$ and $\varepsilon(\pi)=-q$, then we have

$$H(f_{c,i,|\pi^n|}(z))(w)=\begin{cases}q^{-n}\alpha\chi(w-c)(w-c)^{-m+i} & |w-c|>|\pi^n|\\ [\beta-2^{-1}(1-q^{-1})(1-(-1)^n)\alpha](w-c)^i & |w-c|\leq|\pi^n|,\end{cases}$$

$$H(f_{\chi,\infty,i,|\pi^n|}(z))(w)=\begin{cases}[\beta-2^{-1}(1-q^{-1})(1-(-1)^n)\alpha]\chi(w)w^{-i} & |w|\geq|\pi^{-n}|\\ q^{-n}\alpha w^{m-i} & |w|<|\pi^{-n}|.\end{cases}$$

We denote by $H_{\chi,+}$ and $H_{\chi,-}$ the operators corresponding to

$$(\alpha,\beta)=\left(\frac{q}{q+1},\ \frac{q}{q+1}\right)\quad\text{and}\quad\left(\frac{-q}{q+1},\ \frac{1}{q+1}\right),$$

respectively. Then we have $H^2_{\chi,+}=H_{\chi,+}\neq 0$, $H^2_{\chi,-}=H_{\chi,-}\neq 0$, $H_{\chi,+}+H_{\chi,-}=\mathrm{id}$. Hence $D^{\mathrm{loc},m}_{\chi,+}=\mathrm{im}\,(H_{\chi,+})$ and $D^{\mathrm{loc},m}_{\chi,-}=\mathrm{im}\,(H_{\chi,-})$ are closed G-invariant subspaces of $D^{\mathrm{loc},m}_{\chi}$, and $D^{\mathrm{loc},m}_{\chi}=D^{\mathrm{loc},m}_{\chi,+}\oplus D^{\mathrm{loc},m}_{\chi,-}$.

Summarizing, we obtain the following:

Proposition 6. *Let m be a non-negative integer, and let $D^{\mathrm{loc},m}_{\chi}$ be as in* 2–2. *Let $N(z)$ be as in* 2–2. *Then*:

(i) *If $\chi(z)=z^m$ and $\delta(z)=z^mN(z)^2$, then there are non-zero continuous G-homomorphisms $H_\chi\colon D^{\mathrm{loc},m}_{\chi}\to D^{\mathrm{loc},m}_{\delta}$ and $H_\delta\colon D^{\mathrm{loc},m}_{\delta}\to D^{\mathrm{loc},m}_{\chi}$ such that $H_\delta\circ H_\chi=H_\chi\circ H_\delta=0$. Hence $P_m=\ker(H_\chi)$ and $Q_m=\ker(H_\delta)$ are non-trivial closed G-invariant subspaces. Further we have* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi},D_\delta)=kH_\chi$ *and* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\delta},D_\chi)=kH_\delta$.

(ii) *We assume $\chi(z)=z^m\varepsilon(z)$, $\varepsilon(z)^2=N(z)^2$, $\varepsilon(z)\neq N(z)$. Let $l=l(\chi)$ be the smallest positive integer l such that $\varepsilon(z)=1$ holds for $|z-1|\leq|\pi^l|$. If $\varepsilon(z)$ is not trivial on $\mathfrak{o}^*$, and if $\sqrt{\varepsilon(-1)q^l}$ is not contained in k, then there is a continuous G-endomorphism I_χ of $D^{\mathrm{loc},m}_{\chi}$ such that $I^2_\chi=\varepsilon(-1)q^l$* id. *and* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi},D_\chi)=k\,\mathrm{id.}\oplus kI_\chi$. *If $\varepsilon(z)$ is trivial on $\mathfrak{o}^*$, or if $\sqrt{\varepsilon(-1)q^l}$ is contained in k, then there are continuous G-endomorphisms $H_{\chi,+}$ and $H_{\chi,-}$ of $D^{\mathrm{loc},m}_{\chi}$ such that $H^2_{\chi,+}=H_{\chi,+}\neq 0$, $H^2_{\chi,-}=H_{\chi,-}\neq 0$, $H_{\chi,+}+H_{\chi,-}=$*id. *Put $D^{\mathrm{loc},m}_{\chi,\pm}=\mathrm{im}\,(H_{\chi,\pm})$. Then these spaces are closed G-invariant subspaces of $D^{\mathrm{loc},m}_{\chi}$ and $D^{\mathrm{loc},m}_{\chi}=D^{\mathrm{loc},m}_{\chi,+}\oplus D^{\mathrm{loc},m}_{\chi,-}$ holds. Further we have* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi},D_\chi)=kH_{\chi,+}\oplus kH_{\chi,-}$. *In particular, $D^{\mathrm{loc},m}_{\chi,+}$ and $D^{\mathrm{loc},m}_{\chi,-}$ are not G-equivalent.*

(iii) *If $\chi(z)=z^m\varepsilon(z)$, $\delta(z)=z^m\varepsilon(z)^{-1}N(z)^2$, and $\varepsilon(z)^2\neq N(z)^2$, then there is a non-zero continuous G-homomorphism $H_\chi\colon D^{\mathrm{loc},m}_{\chi}\to D^{\mathrm{loc},m}_{\delta}$ and we have* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi},D_\delta)=kH_\chi$.

(iv) *If $\chi(z)=\delta(z)=z^m\varepsilon(z)$ and if $\varepsilon(z)$ does not satisfy the conditions $\varepsilon(z)^2=N(z)^2$ and $\varepsilon(z)\neq N(z)$, then we have* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi},D_\delta)=k$ id.

(v) *If $\alpha(\chi)=m$, and if χ and δ do not satisfy any one of* (i)–(iv), *then we have* $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi},D_\delta)=0$.

§ 7. Proof of Theorem 1, III (the irreducibility of $D^{\mathrm{loc},m}_{\chi}$)

In Section 7, we assume that $\alpha(\chi)$ is a non-negative integer m. Hence $\varepsilon(z)=z^{-m}\chi(z)$ is a locally constant character. Let $l=l(\chi)$ be the smallest positive integer such that $\varepsilon(z)=1$ holds for any $z \in L^*$ with $|z-1| \leq |\pi^l|$. For any element f of $D_\chi^{\mathrm{loc},m}$, let U_f be the minimal $(\mathfrak{g}, G)$-invariant subspace of $D_\chi^{\mathrm{loc},m}$ containing f. Then U_f is contained in any closed G-invariant subspace of $D_\chi^{\mathrm{loc},m}$ containing f. We study in Section 7 this $(\mathfrak{g}, G)$-module U_f, and prove the irreducibility of (ii) of Theorem 1 not as topological G-modules but as algebraic $(\mathfrak{g}, G)$-modules. Let $f_{\chi,\infty,n,r}(z)$ and $f_{c,n,r}(z)$ be as in 6–2.

7-1. *A standard generator of $D_\chi^{\mathrm{loc},m}$.*

Lemma 2. *Let $f(z)=f_{0,0,1}(z) \in D_\chi^{\mathrm{loc},m}$ be the characteristic function of $\mathfrak{o}$, then $f(z)$ generates $D_\chi^{\mathrm{loc},m}$ as an algebraic $(\mathfrak{g}, G)$-module.*

Proof. For any positive integer n, $\chi(p)^{-n}T_\chi(A(p^{-n}))f(z)=f_{0,0,|p^{2n}|}(z)$ is contained in U_f. Since $|1-z^{-1}| \leq |p^{2n}|$ iff $|z-1| \leq |p^{2n}|$, $f_{0,0,|p^{2n}|}(1-z^{-1})$ $=1$ (resp. 0) if $|z-1| \leq |p^{2n}|$ (resp. otherwise). We choose a large integer n such that $\varepsilon(z)=1$ holds for $|z-1| \leq |p^{2n}|$. Then

$$\begin{aligned} &((m-i)!)^{-1}[(dT_\chi)(X_-)]^{m-i} \circ T_\chi(I) \circ T_\chi(C(1))f_{0,0,|p^{2n}|}(z) \\ &\quad =((m-i)!)^{-1}(d/dz)^{m-i}z^m\varepsilon(z)f_{1,0,|p^{2n}|}(z) \end{aligned}$$

$(0 \leq i \leq m)$ is z^i (resp. 0) if $|z-1| \leq |p^{2n}|$ (resp. otherwise). Since these elements span $P_{1,2n}$ (cf. Proposition 2), U_f contains $P_{1,2n}$. Since the translations $C(c)$: $z \to z+c$ $(c \in L)$ act homogeneously on L, U_f contains the direct sum $\bigoplus_w P_{w,2n}$ $(w \in W_n)$. Since $T_\chi(I)P_{0,2n}=P_{\infty,2n}$, U_f contains $P_{\infty,2n} \oplus \bigoplus_w P_{w,2n}$. Since n can be arbitrarily small, U_f contains the injective limit space inj lim $(P_{\infty,2n} \oplus \bigoplus_w P_{w,2n})=D_\chi^{\mathrm{loc},m}$.

Corollary. *Let $f(z)$ be one of the $f_{c,i,|\pi^n|}(z)$'s and the $f_{\chi,\infty,i,|\pi^n|}(z)$'s $(i \in \boldsymbol{Z}$, $0 \leq i \leq m$, $n \in \boldsymbol{Z}$, $c \in L)$. Then $f(z)$ generates $D_\chi^{\mathrm{loc},m}$ as an algebraic $(\mathfrak{g}, G)$-module.*

Proof. Let N be a positive integer satisfying $2N \geq -n$. Then

$$\begin{aligned} &\sum_{c^* \in \mathfrak{o}/\mathfrak{p}^{2N+n}} T_\chi(C(c^*)) \circ T_\chi(A(\pi^{-N})) \circ T_\chi(C(c)) \circ [(dT_\chi)(X_-)]^i f_{c,i,|\pi^n|}(z) \\ &\quad = i!\,\chi(\pi)^N f_{0,0,1}(z). \end{aligned}$$

It follows from Lemma 2 that $f_{c,i,|\pi^n|}(z)$ generates $D_\chi^{\mathrm{loc},m}$ as a $(\mathfrak{g}, G)$-module. Since

$$T_\chi(I)f_{\chi,\infty,i,|\pi^n|}(z)=\chi(-1)(-1)^i f_{0,i,|\pi^n|}(z),$$

$f_{\chi,\infty,i,|\pi^n|}(z)$ generates $D_\chi^{\mathrm{loc},m}$ as a $(\mathfrak{g}, G)$-module.

7-2. *Irreducibility of* $D_\chi^{\mathrm{loc},m}$. We assume in 7–2 that the locally constant character $\varepsilon(z)$ is neither 1 nor $N(z)^2$, and that $\varepsilon(z)^2=N(z)^2$ implies $\varepsilon(z)=N(z)$ as functions on L^*. Let $f(z)$ be a non-zero element of $D_\chi^{\mathrm{loc},m}$. We want to show $D_\chi^{\mathrm{loc},m}\subset U_f$.

Since the translations $C(c)\colon z\mapsto z+c$ ($c\in L$) act homogeneously on L, we may assume $f(0)\neq 0$. We write $f(z)=\sum_{0\leq i\leq m}f_i(z)z^i$ with locally constant functions $f_i(z)\in D_\varepsilon^{\mathrm{loc},0}$. Since $f(0)$ is not zero, $f_0(0)$ is not zero. Since

$$[(dT_\chi)(X_-)]^m\circ T_\chi(I)f(z)=m!\,\varepsilon(z)f_0(-1/z)$$

is contained in U_f, to prove $U_f=D_\chi^{\mathrm{loc},m}$, we may assume that $f(z)$ is a locally constant function, and $f(z)=\gamma\varepsilon(z)$ for $|z|\geq|p^{-2n}|$ ($\gamma\in k^*$, $n\in\boldsymbol{Z}$, $n\geq 0$). Since $p^{-mn}\varepsilon(p)^nT_\chi(A(p^{-n}))f(z)=\varepsilon(p)^{2n}f(p^{-2n}z)=\gamma\varepsilon(z)$ for $|z|\geq 1$, we may assume that $f(z)=\gamma\varepsilon(z)$ for $|z|\geq 1$. We note that for any locally constant function $h(z)$ in $D_\chi^{\mathrm{loc},m}$,

$$T_\varepsilon(I)h(z)=(m!)^{-1}[(dT_\chi)(X_-)]^m\circ T_\chi(I)h(z)$$

is contained in U_h.

Put $l=l(\chi)$, let π be a prime element of $\mathfrak{p}$, and let n be a positive integer such that $n>l$ and $f(x)=f(y)$ holds for $|x|,|y|<1$ and $|x-y|\leq|\pi^n|$. Let C be a representative of $\mathfrak{p}$ modulo $\mathfrak{p}^n$. Put

$$h(z)=\sum_{c\in C}T_\chi(C(c))f(z).$$

Then $h(z)=q^{n-1}\gamma\varepsilon(z)$ for $|z|\geq|\pi^{1-l}|$, $h(z)=0$ for $|\pi^{1-l}|>|z|\geq 1$, and $h(z)=\sum_{c\in C}f(c)$ for $|z|<1$. Put $\gamma_1=q^{n-1}\gamma$ and $\gamma_2=\sum_{c\in C}f(c)$.

If $\gamma_2=0$, then $h(z)=\gamma_1f_{\chi,\infty,m,1}(z)$ is contained in U_f. It follows from the corollary to Lemma 2 that $U_f=D_\chi^{\mathrm{loc},m}$ holds in this case.

If $\gamma_2\neq 0$, then the function

$$h^*(z)=\sum_{c\in\mathfrak{p}^{-1}/\mathfrak{p}}T_\chi(C(c))h(z)$$

$=q^2\gamma_1\varepsilon(z)$ for $|z|\geq|\pi^{-1-l}|$, $=0$ for $|\pi^{-1-l}|>|z|>|\pi^{-1}|$, $=\gamma_2$ for $|z|\leq|\pi^{-1}|$ if $\varepsilon(z)$ is not trivial on $\mathfrak{o}^*$, $=\gamma_2+(q-1)\gamma_1+(q^2-q)\gamma_1\varepsilon(\pi)^{-1}$ for $|z|\leq|\pi^{-1}|$ if $\varepsilon(z)$ is trivial on $\mathfrak{o}^*$.

If $\varepsilon(z)$ is not trivial on $\mathfrak{o}^*$, then

$$h(z)-q^{-2}\pi^{-m}\varepsilon(\pi)T_\chi(A(\pi^{-1}))h^*(z)=h(z)-q^{-2}\varepsilon(\pi)^2h^*(\pi^{-2}z)\in U_f$$

is equal to $(1-q^{-2}\varepsilon(\pi)^2)\gamma_2f_{0,0,|\pi|}(z)$. Since $\varepsilon(z)^2\neq N(z)^2$, we can choose a suitable prime element π so that $\varepsilon(\pi)^2\neq N(\pi)^2=q^2$. Hence $f_{0,0,|\pi|}(z)$ is contained in U_f. It follows from the corollary to Lemma 2 that $U_f=D_\chi^{\mathrm{loc},m}$ holds in this case.

If $\varepsilon(z)$ is trivial on $\mathfrak{o}^*$, then $l=1$ and $h(z)-q^{-2}\pi^{-m}\varepsilon(\pi)T_\chi(A(\pi^{-1}))h^*(z) =[\gamma_2-q^{-2}\varepsilon(\pi)^2\{\gamma_2+(q-1)\gamma_1+(q^2-q)\varepsilon(\pi)^{-1}\gamma_1\}]f_{0,0,|\pi|}(z)$. Since

$$\begin{aligned}\gamma_2-q^{-2}\varepsilon(\pi)^2\{\gamma_2+(q-1)\gamma_1+(q^2-q)\varepsilon(\pi)^{-1}\gamma_1\}\\ =q^{-2}(\varepsilon(\pi)+q)\{(q-\varepsilon(\pi))\gamma_2-(q-1)\varepsilon(\pi)\gamma_1\},\end{aligned}$$

and since $\varepsilon(\pi)\neq -q$, U_f contains $f_{0,0,|\pi|}(z)$ if $(q-\varepsilon(\pi))\gamma_2\neq(q-1)\varepsilon(\pi)\gamma_1$. In this case, U_f contains $f_{0,0,1}(z)$ and hence $U_f=D_\chi^{\mathrm{loc},m}$.

If $(q-\varepsilon(\pi))\gamma_2=(q-1)\varepsilon(\pi)\gamma_1$, put

$$h^{**}(z)=\sum_{c\in\mathfrak{o}/\mathfrak{p}} T_\chi(C(c))h(z).$$

Then $h^{**}(z)=q\gamma_1\varepsilon(z)$ for $|z|>1$, $=(q-1)\gamma_1+\gamma_2$ for $|z|\leq 1$. Then $\{(q-1)\gamma_1+\gamma_2\}h(z)-\gamma_2h^{**}(z)=(q-1)\gamma_1(\gamma_1-\gamma_2)\varepsilon(z)$ for $|z|>1$, $=\{(q-1)\gamma_1+\gamma_2\}(\gamma_1-\gamma_2)$ for $|z|=1$, and $=0$ for $|z|<1$. Hence

$$\sum_{c\in\mathfrak{o}/\mathfrak{p}} T_\chi(C(c))\circ T_\varepsilon(I)[\{(q-1)\gamma_1+\gamma_2\}h(z)-\gamma_2h^{**}(z)]=0 \qquad \text{for } |z|>1,$$

and $=(q-1)(q\gamma_1+\gamma_2)(\gamma_1-\gamma_2)$ for $|z|\leq 1$. Since $(q-\varepsilon(\pi))\gamma_2=(q-1)\varepsilon(\pi)\gamma_1$, $\gamma_1=\gamma_2$ implies $\varepsilon(\pi)=1$. Since $\varepsilon(z)\neq 1$, this is a contradiction. If $\gamma_2=-q\gamma_1$, then $\varepsilon(\pi)=q^2$. Since $\varepsilon(z)\neq N(z)^2$, this is a contradiction. Hence $(q-1)(q\gamma_1+\gamma_2)(\gamma_1-\gamma_2)\neq 0$. Therefore U_f contains $f_{0,0,1}(z)$ and hence $U_f=D_\chi^{\mathrm{loc},m}$. Therefore we have proved that $U_f=D_\chi^{\mathrm{loc},m}$ holds in any case.

7-3. *Irreducibility of Q_m and $D_\delta^{\mathrm{loc},m}/Q_m$.* Let $\delta(z)=z^mN(z)^2$ and let $\eta(z)=N(z)^2$. Let $H_\delta\colon D_\delta^{\mathrm{loc},m}\to P_m$, $Q_m=\ker(H_\delta)$ etc. be as in Proposition 6. Let $f(z)$ be a non-zero element of $D_\delta^{\mathrm{loc},m}$. We want to study U_f. We repeat the argument in 7–2 and may assume that $f(z)$ is a locally constant function, and that $f(z)=\gamma\eta(z)$ ($\gamma\in k^*$) holds for $|z|\geq 1$. Let n be a positive integer such that $f(x)=f(y)$ holds for $|x|,|y|<1$ and $|x-y|\leq|\pi^n|$. Put $\gamma_1=q^{n-1}\gamma$ and $\gamma_2=\sum_{c\in\mathfrak{p}/\mathfrak{p}^n}f(c)$. Then

$$h(z)=\sum_{c\in\mathfrak{p}/\mathfrak{p}^n} T_\delta(C(c))f(z)$$

$=\gamma_1$ for $|z|\geq 1$, and $=\gamma_2$ for $|z|<1$. We repeat the argument in 7–2 and observe that $-(q+1)(q-1)(\gamma_2+q\gamma_1)f_{0,0,|\pi|}(z)$ is contained in U_f.

If $f(z)$ is not contained in Q_m, $H_\delta^*(f(z))=\gamma+q^{-n}\gamma_2=q^{-n}(q\gamma_1+\gamma_2)\neq 0$. Hence U_f contains $f_{0,0,|\pi|}(z)$. It follows from the corollary to Lemma 2 that $U_f=D_\delta^{\mathrm{loc},m}$.

If $f(z)$ is contained in Q_m, then $\gamma_2=-q\gamma_1$. Since $\gamma_1\neq 0$, dividing by γ_1 if necessary, we may assume $\gamma_1=1$. We note that $h(z)$ is contained in the image of H_χ ($\chi(z)=z^m$) because $h(z)=qH_\chi(f_{0,0,|\pi|}(z))$. For any integer n,

$$h_{2n}(z)=q(q+1)^{-1}\chi(\pi)^{-n}T_\chi(A(\pi^{-n}))\{h(z)+q^{-1}T_\varepsilon(I)h(z)\}$$

$=0$ for $|z|>|\pi^{2n}|$, $=1$ for $|z|=|\pi^{2n}|$, $=-(q-1)$ for $|z|<|\pi^{2n}|$. Similarly

$$h_{2n-1}(z)=-q^3(q+1)^{-1}\chi(\pi)^{-n}T_\chi(A(\pi^{-n}))$$
$$\circ\{q^{-1}T_\varepsilon(I)h(z)+q^{-2}\sum_{c\in\mathfrak{p}^{-1}/\mathfrak{p}}T_\chi(C(c))h(z)\}$$

$=0$ for $|z|>|\pi^{2n-1}|$, $=1$ for $|z|=|\pi^{2n-1}|$, $=-(q-1)$ for $|z|<|\pi^{2n-1}|$. Taking a suitable linear combination of $h(z)$ and the $h_i(z)$'s $(-n<i<n)$, we see that there is a function $h_{0,n}(z)$ in U_f such that $h_{0,n}(z)=\varepsilon(z)$ for $|z|\geq|\pi^{-n}|$, $=0$ for $|\pi^{-n}|>|z|>|\pi^n|$, and $=-1$ for $|z|\leq|\pi^n|$. Then for any $c\in L$ with $|c|\leq|\pi^{-n}|$, $h_{c,n}(z)=T_\chi(C(-c))h_{0,n}(z)=\varepsilon(z)$ for $|z|\geq|\pi^{-n}|$, $=0$ for $|\pi^{-n}|>|z-c|>|\pi^n|$, and $=-1$ for $|z-c|\leq|\pi^n|$.

Let $q(z)$ be a locally constant function in Q_m. Let n be a large integer such that (i) $q(z)=\gamma_3\varepsilon(z)$ $(\gamma_3\in k)$ for $|z|\geq|\pi^{-n}|$, and (ii) $q(x)=q(y)$ for $|x|$, $|y|<|\pi^{-n}|$ and $|x-y|\leq|\pi^n|$. Then we can write

$$q(z)=\gamma_3 f_{\chi,\infty,m,|\pi^n|}(z)+\sum_{c\in\mathfrak{p}^{1-n}/\mathfrak{p}^n}\gamma_c f_{c,0,|\pi^n|}(z)$$

with $\gamma_c\in k$. Since $q(z)$ is contained in Q_m, $\gamma_3+\sum\gamma_c=0$. Since $h_{c,n}(z)=f_{\chi,\infty,m,|\pi^n|}(z)-f_{c,0,|\pi^n|}(z)$ is contained in U_f, we see that $q(z)=-\sum_c\gamma_c h_{c,n}(z)$ is contained in U_f. Since

$$[(dT_\chi)(X_-)]^{m-i}\circ T_\chi(I)\circ T_\varepsilon(I)q(z)=(m-i)!\,z^i q(z),$$

and since any element of Q_m can be written as $\sum_{0\leq i\leq m}z^i q_i(z)$ with locally constant functions $q_i(z)$ in Q_m, U_f contains Q_m. Note that we have also proved $\mathrm{im}(H_\chi)=Q_m$ $(\chi(z)=z^m)$.

7-4. *Irreducibility of P_m and $D_\chi^{\mathrm{loc},m}$.* Let $\chi(z)=z^m$, $H_\chi\colon D_\chi^{\mathrm{loc},m}\to Q_m$, $P_m=\ker(H_\chi)$ be as in Proposition 6. Since $H_\chi\circ H_\delta=0$, $\mathrm{im}(H_\delta)$ is contained in $P_m=\ker(H_\chi)$. Hence P_m contains the space P_m^* of all polynomial functions $f\colon L\to k$ of degree $\leq m$. Let $B_{0,n}$ and $P_{0,n}$ be as in 4–2. Then the restriction map

$$P_m^*\ni f(z)\longmapsto(f\,|\,B_{0,n})(z)\in P_{0,n}$$

is a k-linear $\mathfrak{g}$-isomorphism. It follows from Proposition 2 that P_m^* is an algebraically irreducible $\mathfrak{g}$-module.

Let $f(z)$ be an element of $D_\chi^{\mathrm{loc},m}$ which is not contained in P_m^*. We claim $U_f=D_\chi^{\mathrm{loc},m}$. Then we also have $P_m=P_m^*$ because $D_\chi^{\mathrm{loc},m}\supsetneqq P_m\supset P_m^*$.

We repeat the argument in 7–2 and may assume that $f(z)$ is a non-constant locally constant function. Let n be the largest integer n such that

$f(x) \neq f(y)$ holds for some $x, y \in L$ with $|x-y|=|\pi^n|$. Since $T_\chi(C(-x))f(z)$ is also contained in U_f, we may assume $x=0$. Then, replacing $f(z)$ by

$$T_\varepsilon(I) \circ (\sum_{c \in \mathfrak{p}^{1-n}/\mathfrak{p}^N} T_\chi(C(c))) \circ T_\varepsilon(I) f(z)$$

for a sufficiently large integer N if necessary, we may assume that $f(z)$ is constant for $|z|>|\pi^n|$. Let C be a representaive of $\mathfrak{p}^n \backslash \mathfrak{p}^{n+1}$ modulo $\mathfrak{p}^{n+1}$. Then we have constants γ_0, γ_c $(c \in C)$, γ_∞ such that $f(z)=\gamma_0$ for $|z|<|\pi^n|$, $f(z)=\gamma_c$ for $|z-c|<|\pi^n|$ and $f(z)=\gamma_\infty$ for $|z|>|\pi^n|$. By our assumptions, $\gamma_c \neq \gamma_0$ at least for one c. Let $\gamma=q^{-1}(\gamma_0+\sum_c \gamma_c)$.

If $\gamma_\infty \neq \gamma$, then put

$$h(z)=q^{-1} \sum_{c \in \mathfrak{p}^n/\mathfrak{p}^{n+1}} T_\chi(C(c)) f(z).$$

Then $h(z)=\gamma_\infty$ for $|z|>|\pi^n|$, and $=\gamma$ for $|z| \leq |\pi^n|$. Since

$$\sum_{c \in \mathfrak{p}^{n-2}/\mathfrak{p}^n} T_\chi(C(c))\{h(z)-\pi^{-m} T_\chi(A(\pi^{-1}))h(z)\}=(1-q^{-2})(\gamma_\infty-\gamma) f_{0,0,|\pi^{n-2}|}(z),$$

U_f contains $f_{0,0,|\pi^{n-2}|}(z)$. It follows from the corollary to Lemma 2 that U_f contains $D_\chi^{\mathrm{loc},m}$.

If $\gamma_\infty=\gamma$, replacing $f(z)$ by $T_\chi(C(c))f(z)$ $(c \in C)$ if necessary, we may assume $\gamma_0 \neq \gamma$. Then $T_\varepsilon(I)f(z)=\gamma_0$ for $|z|>|\pi^{-n}|$, $=\gamma_c$ for $|z-(-c^{-1})| \leq |\pi^{-n}|$, and $=\gamma_\infty$ for $|z|<|\pi^{-n}|$. By our assumption,

$$\gamma_0-q^{-1}(\gamma_\infty+\sum_c \gamma_c)=q^{-1}(q+1)(\gamma_0-\gamma_\infty)$$

is not zero. Hence we repeat a similar argument as above and obtain $U_f \supset D_\chi^{\mathrm{loc},m}$. Therefore the claim is proved. Note that we have also proved that H_χ and H_δ induce bijective G-homomorphisms $D_\chi^{\mathrm{loc},m}/P_m \to Q_m$ and $D_\delta^{\mathrm{loc},m}/Q_m \to P_m$, respectively. It follows from the open mapping theorem that these maps are topological isomorphisms.

7-5. *Irreducibility of* $D_{\chi,\pm}^{\mathrm{loc},m}$. We assume in 7–5 that $\varepsilon(z)^2=N(z)^2$ and $\varepsilon(z) \neq N(z)$ as functions on L^*. If $\varepsilon(z)$ is not trivial on $\mathfrak{o}^*$, then put $l=l(\chi)$.

First we assume that $\sqrt{\varepsilon(-1)q^l}$ is not contained in k. Let $f(z)$ be a non-zero element of $D_\chi^{\mathrm{loc},m}$. We want to show $U_f=D_\chi^{\mathrm{loc},m}$. We repeat the argument in 7–2 and may assume that $f(z)=\gamma_1\varepsilon(z)$ $(\gamma_1 \in k^*)$ for $|z| \geq |\pi^{1-l}|$, $=0$ for $|\pi^{1-l}|>|z| \geq 1$, and $=\gamma_2$ $(\gamma_2 \in k)$ for $|z|<1$.

If l is an even integer, put $2i=l-2$. Then

$$\gamma_2 f(z)-\gamma_1 \pi^{im} \varepsilon(\pi)^{-i} T_\chi(A(\pi^i)) \circ T_\varepsilon(I) f(z)=(\gamma_2^2-q^{-2i}\varepsilon(-1)\gamma_1^2) f_{0,0,|\pi|}(z).$$

Since $\sqrt{\varepsilon(-1)q^l} \notin k$, $\gamma_2^2-q^{2-l}\varepsilon(-1)\gamma_1^2\neq 0$. Hence U_f contains $f_{0,0,|\pi|}(z)$, and hence $U_f=D_\chi^{\mathrm{loc},m}$.

If l is an odd integer, put $2i=l-1$. Then

$$\sum_{c\in\mathfrak{p}^{1-l}/\mathfrak{p}} T_\chi(C(c))\{\gamma_2 f(z)-\gamma_1\pi^{im}\varepsilon(\pi)^{-i}T_\chi(A(\pi^i))\circ T_\varepsilon(I)f(z)\}$$
$$=\{-q^{2-l}\varepsilon(-1)\gamma_1^2+\gamma_2^2\}f_{0,0,|\pi^{1-l}|}(z).$$

Since $\sqrt{\varepsilon(-1)q^l}\notin k$, $-q^{2-l}\varepsilon(-1)\gamma_1^2+\gamma_2^2\neq 0$. Hence U_f contains $f_{0,0,|\pi^{1-l}|}(z)$, and hence $U_f=D_\chi^{\mathrm{loc},m}$.

Now we assume that $\varepsilon(z)$ is not trivial on $\mathfrak{o}^*$, and that $l=l(\chi)$ satisfies $\sqrt{\varepsilon(-1)q^l}\in k$. Then, by Proposition 6, we have continuous G-endomorphisms $H_{\chi,\pm}\colon D_\chi^{\mathrm{loc},m}\to D_\chi^{\mathrm{loc},m}$ such that $H_{\chi,\pm}^2=H_{\chi,\pm}\neq 0$ and $H_{\chi,+}+H_{\chi,-}=\mathrm{id}$. Let $f(z)$ be any non-zero element of $D_{\chi,\pm}^{\mathrm{loc},m}=\mathrm{im}(H_{\chi,\pm})$. We want to show $U_f=D_{\chi,\pm}^{\mathrm{loc},m}$.

By repeating the argument in 7–2, we may assume that $f(z)=\gamma_1\varepsilon(z)$ ($\gamma_1\in k^*$) for $|z|\geq|\pi^{1-l}|$, $=0$ for $|\pi^{1-l}|\geq|z|\geq 1$, and $=\gamma_2$ ($\gamma_2\in k$) for $|z|<1$. Since $f(z)$ is contained in $D_{\chi,\pm}^{\mathrm{loc},m}$, $\gamma_1=\pm\sqrt{\varepsilon(-1)q^l}q^{-1}\gamma_2$. Hence we can write $f(z)=\gamma_2 H_{\chi,\pm}(f_{0,0,|\pi|}(z))$. Since $f_{0,0,|\pi|}(z)$ generates $D_\chi^{\mathrm{loc},m}$ as an algebraic $(\mathfrak{g}, G)$-module, $f(z)$ generates $H_{\chi,\pm}(D_\chi^{\mathrm{loc},m})=D_{\chi,\pm}^{\mathrm{loc},m}$ as an algebraic $(\mathfrak{g}, G)$-module. Hence $U_f=D_{\chi,\pm}^{\mathrm{loc},m}$.

If $\varepsilon(z)=1$ for $|z|=1$ and $\varepsilon(\pi)=-q$, then we can prove in the same way that $D_{\chi,\pm}^{\mathrm{loc},m}$ is an algebraically irreducible $(\mathfrak{g}, G)$-module.

§ 8. Proof of Theorem 2

In Section 8, we study $\mathrm{Hom}_G(U, V)$ in the case where U is a closed G-invariant subspace of D_χ and V is a quotient space of D_δ by a G-invariant subspace, and prove Theorem 2. Note that we have already determined $\mathrm{Hom}_G(D_\chi, D_\delta)$ and $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\delta)$.

8-1. $\mathrm{Hom}_G(P_m, D_\theta)$ *and* $\mathrm{Hom}_G(Q_m, D_\theta)$. Let $\chi(z)=z^m$, $\delta(z)=z^m N(z)^2$, $H_\chi\colon D_\chi^{\mathrm{loc},m}\to Q_m$, $H_\delta\colon D_\delta^{\mathrm{loc},m}\to P_m$ etc. be as in Proposition 6. Let $\theta\colon L^*\to k^*$ be another locally analytic character. Then we have the following exact sequences:

$$0\longrightarrow\mathrm{Hom}_G(Q_m, D_\theta)\longrightarrow\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\theta)\longrightarrow\mathrm{Hom}_G(P_m, D_\theta);$$

$$0\longrightarrow\mathrm{Hom}_G(P_m, D_\theta)\longrightarrow\mathrm{Hom}_G(D_\delta^{\mathrm{loc},m}, D_\theta)\longrightarrow\mathrm{Hom}_G(Q_m, D_\theta).$$

If $\mathrm{Hom}_G(P_m, D_\theta)\neq 0$, then $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\theta)\neq 0$. It follows from Proposition 6 that either (i) $\theta(z)=\chi(z)$ and $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\theta)=k\,\mathrm{id}.$, or (ii) $\theta(z)=\delta(z)$ and $\mathrm{Hom}_G(D_\chi^{\mathrm{loc},m}, D_\theta)=kH_\chi$. Therefore, if $\mathrm{Hom}_G(P_m, D_\theta)\neq 0$, then $\theta(z)=\chi(z)$ and $\mathrm{Hom}_G(P_m, D_\theta)=k$ id. If $\mathrm{Hom}_G(Q_m, D_\theta)\neq 0$, then we repeat a similar argument and obtain that $\theta(z)=\delta(z)$ and $\mathrm{Hom}_G(Q_m, D_\theta)$ $=k$ id.

8-2. $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi,\pm}, D_\delta)$. Let $\chi(z)=z^m\varepsilon(z)$, $\varepsilon(z)^2=N(z)^2$, $\varepsilon(z)\neq N(z)$, $D^{\mathrm{loc},m}_\chi=D^{\mathrm{loc},m}_{\chi,+}\oplus D^{\mathrm{loc},m}_{\chi,-}$ etc. be as in Proposition 6. Let $\delta\colon L^*\to k^*$ be another locally analytic character. Then we have the following split exact sequence:

$$0\longrightarrow \mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi,-}, D_\delta)\longrightarrow \mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi}, D_\delta)\longrightarrow \mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi,+}, D_\delta)\longrightarrow 0.$$

By Proposition 6, $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi}, D_\delta)\neq 0$ iff $\delta(z)=\chi(z)$. Further

$$\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi}, D_\chi)=kH_{\chi,+}\oplus kH_{\chi,-}$$

holds. Therefore $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi,\pm}, D_\delta)\neq 0$ iff $\delta(z)=\chi(z)$, and in this case, $\mathrm{Hom}_G(D^{\mathrm{loc},m}_{\chi,\pm}, D_\chi)=k$ id.

8-3. *Generalization.* Let U be a closed G-invariant subspace of D_χ. We have already determined $\mathrm{Hom}_G(U, V)$ in the case of $V=D_\delta$. If $V=D_\delta/D^{\mathrm{loc},n}_\delta$ ($\alpha(\delta)=n\in \boldsymbol{Z}$, $n\geq 0$), then we have a G-isomorphism

$$S^*_\delta\colon D_\delta/D^{\mathrm{loc},n}_\delta \xrightarrow{\sim} D_{\delta^*}\quad (\delta^*(z)=z^{-2n-2}\delta(z)).$$

Hence $\mathrm{Hom}_G(U, D_\delta/D^{\mathrm{loc},n}_\delta)\simeq \mathrm{Hom}_G(U, D_{\delta^*})$. Since $\alpha(\delta^*)=-n-2\leq -2$, $\mathrm{Hom}_G(U, D_{\delta^*})\neq 0$ iff $\chi(z)=\delta^*(z)$, $U=D_\chi$, and $\mathrm{Hom}_G(U, D_{\delta^*})=k$ id. Therefore $\mathrm{Hom}_G(U, D_\delta/D^{\mathrm{loc},n}_\delta)\neq 0$ holds iff $\chi(z)=z^{-2n-2}\delta(z)$, $U=D_\chi$, and $\mathrm{Hom}_G(U, D_\delta/D^{\mathrm{loc},n}_\delta)=k(S^*_\delta)^{-1}$ holds in this case.

If $V=D_\delta/P_n$ ($\delta(z)=z^n$, $n\in\boldsymbol{Z}$, $n\geq 0$), then put $\delta^*(z)=z^{-n-2}$. Then we have a G-isomorphism $S^*_\delta\colon D_\delta/D^{\mathrm{loc},n}_\delta\simeq D_{\delta^*}$. Hence we have the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_G(U, D^{\mathrm{loc},n}_\delta/P_n)\longrightarrow \mathrm{Hom}_G(U, D_\delta/P_n)\longrightarrow \mathrm{Hom}_G(U, D_{\delta^*}).$$

If $\mathrm{Hom}_G(U, D_\delta/P_n)\neq 0$, then either $\mathrm{Hom}_G(U, D_{\delta^*})\neq 0$ or $\mathrm{Hom}_G(U, D^{\mathrm{loc},n}_\delta/P_n)$ $\neq 0$. In the first case, we have $\chi(z)=\delta^*(z)$ and $\mathrm{Hom}_G(U, D_{\delta^*})=k$ id. Since $U\hookrightarrow D_{\delta^*}\xrightarrow{S^{*-1}_\delta} D_\delta/D^{\mathrm{loc},n}_\delta$ is injective, it does not come from $\mathrm{Hom}_G(U, D_\delta/P_n)$. Hence $\mathrm{Hom}_G(U, D_\delta/P_n)=\mathrm{Hom}_G(U, D^{\mathrm{loc},n}_\delta/P_n)\neq 0$. Since

$$\mathrm{Hom}_G(U, D^{\mathrm{loc},n}_\delta/P_n)\simeq \mathrm{Hom}_G(U, Q_m)\hookrightarrow \mathrm{Hom}_G(U, D_\theta)\quad (\theta(z)=z^nN(z)^2),$$

it follows that either (i) $\chi(z)=\theta(z)$ and $\mathrm{Hom}_G(U, D_\theta)=k$ id. or (ii) $\chi(z)=$ $\delta(z)$, $U\subset D^{\mathrm{loc},n}_\delta$ and $\mathrm{Hom}_G(U, D_\theta)=kH_\chi$. In the first case, $\chi(z)=z^nN(z)^2$, $U=Q_n$ and $\mathrm{Hom}_G(Q_n, D_\delta/P_n)=kH^{-1}_\delta$. In the second case, $\chi(z)=\delta(z)$,

$U=D_\delta^{\mathrm{loc},n}$ and $\mathrm{Hom}_G(D_\delta^{\mathrm{loc},n}, D_\delta/P_n)=kR_\delta$ with the natural map $R_\delta\colon D_\delta^{\mathrm{loc},n} \to D_\delta^{\mathrm{loc},n}/P_n$.

Similarly, if $V=D_\delta/Q_n$ $(\delta(z)=z^nN(z)^2)$, then either (i) $\chi(z)=z^n$, $U=P_n$ and $\mathrm{Hom}_G(P_n, D_\delta/Q_n)=kH_\delta^{-1}$, or (ii) $\chi(z)=\delta(z)$, $U=D_\delta^{\mathrm{loc},n}$ and $\mathrm{Hom}_G(D_\delta^{\mathrm{loc},n}, D_\delta/Q_n)=kR_\delta$ with the natural map $R_\delta\colon D_\delta^{\mathrm{loc},n}\to D_\delta^{\mathrm{loc},n}/Q_n$.

If $V=D_\delta/D_{\delta,\pm}^{\mathrm{loc},n}$ $(\delta(z)=z^n\eta(z),\ \eta(z)^2=N(z)^2,\ \eta(z)\neq N(z))$, then we have the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_G(U, D_{\delta,\mp}^{\mathrm{loc},n})\to \mathrm{Hom}_G(U, D_\delta/D_{\delta,\pm}^{\mathrm{loc},n})\longrightarrow \mathrm{Hom}_G(U, D_\delta/D_\delta^{\mathrm{loc},n}).$$

If $\mathrm{Hom}\ (U, D_\delta/D_{\delta,\pm}^{\mathrm{loc},n})\neq 0$, then either $\mathrm{Hom}_G(U, D_\delta/D_\delta^{\mathrm{loc},n})\neq 0$ or $\mathrm{Hom}_G(U, D_{\delta,\mp}^{\mathrm{loc},n})\neq 0$. In the first case, we have either (i) $\chi(z)=\delta(z)$, $U=D_\delta$, $\mathrm{Hom}_G(D_\delta, D_\delta/D_\delta^{\mathrm{loc},n})=kR_\delta^*$ with the natural map $R_\delta^*\colon D_\delta\to D_\delta/D_\delta^{\mathrm{loc},n}$, or (ii) $\chi(z)=z^{-2n-2}\delta(z)$, $U=D_\chi$ and $\mathrm{Hom}_G(D_\chi, D_\delta/D_\delta^{\mathrm{loc},n})=kS_\delta^{*-1}$. In the case (i), we have $\mathrm{Hom}_G(D_\delta, D_\delta)=k$ id. and hence $\mathrm{Hom}_G(D_\delta, D_{\delta,\mp}^{\mathrm{loc},n})=0$. Hence

$$\mathrm{Hom}_G(D_\delta, D_\delta/D_{\delta,\pm}^{\mathrm{loc},n})=kR_{\delta,\pm}^*$$

with the natural map $R_{\delta,\pm}^*\colon D_\delta\to D_\delta/D_{\delta,\pm}^{\mathrm{loc},n}$. In the case (ii), we also have $\mathrm{Hom}_G(D_\delta, D_{\delta,\mp}^{\mathrm{loc},n})=0$. Since S_δ^{*-1} is injective, it does not come from $\mathrm{Hom}_G(D_\delta, D_\delta/D_{\delta,\pm}^{\mathrm{loc},n})$. Hence this case does not occur. If $\mathrm{Hom}_G(U, D_{\delta,\mp}^{\mathrm{loc},n})\neq 0$, we have $\mathrm{Hom}_G(U, D_\delta^{\mathrm{loc},n})\neq 0$. Further we may also assume that $\mathrm{Hom}_G(U, D_\delta/D_\delta^{\mathrm{loc},n})=0$ holds because we have already studied the other case. It follows from Proposition 6 that $\chi(z)=\delta(z)$, $U=D_{\delta,\pm}^{\mathrm{loc},n}$ and $\mathrm{Hom}_G(D_{\delta,\mp}^{\mathrm{loc},n}, D_{\delta,\mp}^{\mathrm{loc},n})=k$ id. Since $D_{\delta,\mp}^{\mathrm{loc},n}=D_\delta^{\mathrm{loc},n}/D_{\delta,\pm}^{\mathrm{loc},n}\subset D_\delta/D_{\delta,\pm}^{\mathrm{loc},n}$, we have

$$\mathrm{Hom}_G(D_{\delta,\mp}^{\mathrm{loc},n}, D_\delta/D_{\delta,\pm}^{\mathrm{loc},n})=k\ \mathrm{id}.$$

8-4. *Proof of Theorem* 2. Let U and V be two different topologically irredicuble G-modules constructed in Theorem 1. Hence U and V are one of the D_χ, $D_\chi^{\mathrm{loc},m}$, $D_\chi/D_\chi^{\mathrm{loc},m}$, P_m, $D_\chi^{\mathrm{loc},m}/P_m$, Q_m, $D_\chi^{\mathrm{loc},m}/Q_m$, $D_{\chi,+}^{\mathrm{loc},m}$, $D_{\chi,-}^{\mathrm{loc},m}$ (the corresponding χ's for U and V may be different). Since $S_\chi^*\colon D_\chi/D_\chi^{\mathrm{loc},m}\to D_\delta$ $(\delta(z)=z^{-2n-2}\chi(z))$ is a topological G-isomorphism, we may omit $D_\chi/D_\chi^{\mathrm{loc},m}$ from the above list of candidates. Since $H_\chi\colon D_\chi^{\mathrm{loc},m}/P_m\to Q_m$ and $H_\delta\colon D_\delta^{\mathrm{loc},m}/Q_m\to P_m$ are topological G-isomorphisms, we may also omit $D_\chi^{\mathrm{loc},m}/P_m$ and $D_\chi^{\mathrm{loc},m}/Q_m$ from the above list. Then we may assume that U is a closed G-invariant subspace of D_χ and V is a closed G-invariant subspace of D_δ for certain χ and δ. Then $\mathrm{Hom}_G(U, V)\hookrightarrow \mathrm{Hom}_G(U, D_\delta)$. Since we have already determined $\mathrm{Hom}_G(U, D_\delta)$, it is easy to check that Theorem 2 holds.

Added in proof. After this paper was submitted, the author and W. Schikhof have succeeded to generalize the results of [9], § 3 to non maximally complete fields. As a consequence, the results of this paper hold without assuming that k is maximally complete. A detailed proof will be published in a following paper.

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