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CARTAN SUBALGEBRAS OF TENSOR PRODUCTS OF FREE QUANTUM GROUP FACTORS WITH ARBITRARY FACTORS

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Let $\mathbb{G}$ be a free (unitary or orthogonal) quantum group. We prove that for any nonamenable subfactor $N \subset L^\infty(\mathbb{G})$ which is an image of a faithful normal conditional expectation, and for any σ-finite factor B, the tensor product $N \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras. This generalizes our previous work that provides the same result when B is finite. In the proof, we establish Ozawa–Popa and Popa–Vaes's weakly compact action on the continuous core of $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} B$ as the one *relative to* B, by using an operator-valued weight to B and the central weak amenability of $\widehat{\mathbb{G}}$.

1. Introduction

Let M be a von Neumann algebra. A *Cartan subalgebra* $A \subset M$ is an abelian von Neumann subalgebra which is an image of a faithful normal conditional expectation such that (i) A is maximal abelian and (ii) the normalizer $\mathcal{N}_M(A)$ generates M as a von Neumann algebra [Feldman and Moore 1977]. Here $\mathcal{N}_M(A)$ is given by $\{u \in \mathcal{U}(M) \mid uAu^* = A\}$.

The group measure space construction of Murray and von Neumann gives a typical example of a Cartan subalgebra. Indeed, the canonical subalgebra $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma$ is Cartan whenever the given action $\Gamma \curvearrowright (X,\mu)$ is free. More generally, one can associate any (not necessarily free) group action with a Cartan subalgebra by its orbit equivalence relation. Conversely when M has separable predual, any Cartan subalgebra $A \subset M$ is realized by an orbit equivalence relation (with a cocycle), and hence by a group action. Thus the notion of Cartan subalgebras is closely related to group actions. In particular if M has no Cartan subalgebras, then it cannot be constructed by any group actions. It was an open problem to find such a von Neumann algebra.

The first result in this direction was given by Connes [1975]. He constructed a II_1 factor which is not isomorphic to its opposite algebra, so it is particularly not isomorphic to any group action (without cocycle) von Neumann algebra. Voiculescu [1996] then provided a complete solution to this problem, by proving free group factors $L\mathbb{F}_n$ $(n \geq 2)$ have no Cartan subalgebras. He used his celebrated *free entropy* technique, and it was later developed to give other examples [Shlyakhtenko 2000; Jung 2007].

After these pioneering works, Ozawa and Popa [2010] introduced a completely new framework to study this subject. Among other things, they proved that free group factors are *strongly solid*, that is, for any diffuse amenable subalgebra $A \subset L\mathbb{F}_n$, the von Neumann algebra generated by the normalizer $\mathcal{N}_{L\mathbb{F}_n}(A)$

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remains amenable. Since $L\mathbb{F}_n$ itself is nonamenable, this immediately yields that $L\mathbb{F}_n$ has no Cartan subalgebras. Note that strong solidity is stable under taking subalgebras and hence any nonamenable subfactor of $L\mathbb{F}_n$ also has no Cartan subalgebras.

The proof of Ozawa and Popa consist of two independent steps. First, by using weak amenability of $\mathbb{F}_n$, they observed that the normalizer group acts *weakly compactly* on a given amenable subalgebra. Second, combining this weakly compact action with *Popa's deformation and intertwining techniques* [Popa 2006a; 2006b], they constructed a state which is central with respect to the normalizer group. Thus they obtained that the normalizer group generates an amenable von Neumann algebra. Since these techniques are applied to any finite crossed product $B \rtimes \mathbb{F}_n$ with the W*CMAP (weak* completely metric approximation property, see Section 2D), they also proved that for any finite factor B with the W*CMAP, the tensor product $L\mathbb{F}_n \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras.

To remove the W*CMAP assumption on $B \rtimes \mathbb{F}_n$, Popa and Vaes [2014a] introduced a notion of *relative weakly compact action*. This is an appropriate "relativization" of the first step above in the view of the relative tensor product $L^2(B \rtimes \mathbb{F}_n) \otimes_B L^2(B \rtimes \mathbb{F}_n)$. In particular this *only* requires the weak amenability of $\mathbb{F}_n$. Thus by modifying the proof in the second step above, they obtained, among other things, the tensor product $L\mathbb{F}_n \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras for any finite factor B.

The aim of the present paper is to develop these techniques to study type III von Neumann algebras. More specifically we replace the free group factor $L\mathbb{F}_n$ with the *free quantum group factor*, which is a type III factor in most cases. We have already studied this [Isono 2015a; 2015b] when B is finite. In the general case however, namely, when B is a type III factor, we could not provide a satisfactory answer to this problem, and this will be discussed in this article.

We note that the first solution to the Cartan subalgebra problem for type III factors in our framework was obtained by Houdayer and Ricard [2011]. They followed the proof of [Ozawa and Popa 2010] by exploiting techniques in [Chifan and Houdayer 2010], that is, the use of Popa's deformation and intertwining techniques together with the *continuous core decomposition*. While Houdayer and Ricard followed the idea of [Ozawa and Popa 2010], our approach in [Isono 2015a; 2015b] was based on [Popa and Vaes 2014b]. In particular, in the second step above, we made use of *Ozawa's condition* (*AO*) [2004] (or *biexactness*, see Section 2C) at the level of the continuous core. In this article, we stand again on the use of biexactness, and we will further develop techniques of [Isono 2015b]. See [Boutonnet et al. 2014] for other examples of type III factors with no Cartan subalgebras, and [Chifan and Sinclair 2013; Chifan et al. 2013] for other works on Cartan subalgebras of biexact group von Neumann algebras.

The following theorem is the main observation of this article. This should be regarded as a generalization of [Isono 2015b, Theorem B], and this allows us to obtain a satisfactory answer to the Cartan problem in the type III setting. See Section 2 for items in this theorem.

Theorem A. *Let $\mathbb{G}$ be a compact quantum group with the Haar state h, and B a type* III_1 *factor with a faithful normal state φ_B. Put $M := L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} B$ and $\varphi := h \otimes \varphi_B$. Let $C_{\varphi_B}(B)$ and $C_\varphi(M)$ be continuous cores of B and M with respect to φ_B and φ, and regard $C_{\varphi_B}(B)$ as a subset of $C_\varphi(M)$. Let* Tr *be a semifinite trace on $C_\varphi(M)$ with $\mathrm{Tr}|_{C_{\varphi_B}(B)}$ semifinite, and $p \in C_\varphi(M)$ a projection with $\mathrm{Tr}(p) < \infty$.*

Assume that $\widehat{\mathbb{G}}$ is biexact and centrally weakly amenable with Cowling–Haagerup constant 1. *Then for any amenable von Neumann subalgebra $A \subset pC_\varphi(M)p$, we have either one of the following conditions*:

(i) *We have $A \preceq_{C_\varphi(M)} C_{\varphi_B}(B)$.*

(ii) *The von Neumann algebra $\mathcal{N}_{pC_\varphi(M)p}(A)''$ is amenable relative to $C_{\varphi_B}(B)$.*

As a consequence of the main theorem, we obtain the following corollary. This is the desired one since our main example, free quantum groups, satisfies the assumptions in this corollary. See [Isono 2015b, Theorem C] for other examples of quantum groups satisfying these assumptions. Below we say that an inclusion of von Neumann algebras $A \subset M$ is *with expectation* if there is a faithful normal conditional expectation.

Corollary B. *Let $\mathbb{G}$ be a compact quantum group as in Theorem A. Then for any nonamenable subfactor $N \subset L^\infty(\mathbb{G})$ with expectation and any σ-finite factor B, the tensor product $N \mathbin{\overline{\otimes}} B$ has no Cartan subalgebras.*

For the proof of Theorem A, we will establish a weakly compact action on the continuous core of $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} B$ as the one *relative to B*. The central weak amenability of $\widehat{\mathbb{G}}$ is used to find approximation maps on the continuous core which are relative to $B \rtimes \mathbb{R}$. Then combined with the amenability of $\mathbb{R}$, we construct appropriate approximation maps on the core relative to B. In this process, since B is not with expectation in the core, we use operator-valued weights instead. This is our strategy for the first step.

For the second step, although we go along a very similar line to [Isono 2015b], we need a rather different (and general) approach to the proof. We note that this is why we assume only biexactness of $\widehat{\mathbb{G}}$, and do not need the notion of *condition* $(AOC)^+$ as in [Isono 2015a; 2015b].

This paper is organized as follows. In Section 2, we recall fundamental facts for our paper, such as Tomita–Takesaki theory, free quantum groups, biexactness, weak amenability, and Popa's intertwining techniques.

In Section 3, we study a generalization of the relative weakly compact action on the continuous core by constructing appropriate approximation maps on the core. The main tools for this construction are: operator-valued weights; central weak amenability; and weak containment, together with the amenability of $\mathbb{R}$. This is the most technical part of this paper.

In Section 4, we prove the main theorem. We follow the proof of [Popa and Vaes 2014b; Isono 2015b], using the weakly compact action given in Section 3.

2. Preliminaries

2A. *Tomita–Takesaki theory and operator-valued weights.* We first recall some notions in Tomita–Takesaki theory. We refer the reader to [Takesaki 1979] for this theory, and to [Haagerup 1979a; 1979b] and [Takesaki 1979, Chapter IX, §4] for operator-valued weights.

Let M be a von Neumann algebra and φ a faithful normal semifinite weight on M. Put $\mathfrak{n}_\varphi := \{x \in M \mid \varphi(x^*x) < \infty\}$ and denote by $\Lambda_\varphi : \mathfrak{n}_\varphi \to L^2(M, \varphi)$ the canonical embedding. We denote the *modular operator, modular conjugation*, and *modular action* for $M \subset \mathbb{B}(L^2(M, \varphi))$ by Δ_φ, J_φ and σ^φ respectively. The Hilbert space $L^2(M, \varphi)$ with J_φ and with its *positive cone* $\mathcal{P}_\varphi$ is called the *standard representation* *for M* [Takesaki 1979, Chapter IX, §1] and does not depend on the choice of φ. Any state on M is represented by a vector state, from which the vector is uniquely chosen from $\mathcal{P}_\varphi$. Any element $\alpha \in \mathrm{Aut}(M)$ is written as $\alpha = \mathrm{Ad}\, u$ by a unique $u \in \mathbb{B}(L^2(M, \varphi))$ which preserves the standard representation structure. The crossed product $M \rtimes_{\sigma^\varphi} \mathbb{R}$ by the modular action is called the *continuous core* [loc. cit., Chapter XII, §1] and is written as $C_\varphi(M)$, which is equipped with the dual weight $\hat{\varphi}$ and the canonical trace $\mathrm{Tr}_\varphi := \hat{\varphi}(h_\varphi^{-1} \cdot)$, where h_φ is a self-adjoint positive closed operator affiliated with $L\mathbb{R}$. For any other faithful normal semifinite weight ψ, there is a family of unitaries $([D\varphi, D\psi]_t)_{t\in\mathbb{R}}$ in M called the *Connes cocycle* [loc. cit., Definition VIII.3.4]. This gives a cocycle conjugate for modular actions of φ and ψ, and hence there is a $*$-isomorphism

$$\Pi_{\psi,\varphi} : C_\varphi(M) \to C_\psi(M), \qquad \Pi_{\psi,\varphi}(x) = x \quad (x \in M), \qquad \Pi_{\psi,\varphi}(\lambda_t^\varphi) = [D\psi, D\varphi]_t^* \lambda_t^\psi \quad (t \in \mathbb{R}).$$

It holds that $\Pi_{\psi,\varphi} \circ \Pi_{\varphi,\omega} = \Pi_{\psi,\omega}$ for any other ω on M, and $\Pi_{\psi\circ E_M,\varphi\circ E_M}|_{C_\varphi(M)} = \Pi_{\psi,\varphi}$ for any $M \subset N$ with expectation E_M. It preserves traces $\mathrm{Tr}_\psi \circ \Pi_{\psi,\varphi} = \mathrm{Tr}_\varphi$ [loc. cit., Theorem XII.6.10(iv)]. So the pair $(C_\varphi(M), \mathrm{Tr}_\varphi)$ does not depend on the choice of φ, and we call Tr_φ the canonical trace. A von Neumann algebra is said to be a *type III_1 factor* if its continuous core is a II_∞ factor.

Let $B \subset M$ be any inclusion of von Neumann algebras. We denote by $\widehat{M}^+$ the *extended positive cone* of M. For any *operator-valued weight* $T : \widehat{M}^+ \to \widehat{B}^+$, we use the notation

$$\mathfrak{n}_T := \{x \in M \mid \|T(x^*x)\|_\infty < +\infty\},$$
$$\mathfrak{m}_T := (\mathfrak{n}_T)^*\mathfrak{n}_T = \Big\{\textstyle\sum_{i=1}^n x_i^* y_i \;\Big|\; n \geq 1,\ x_i, y_i \in \mathfrak{n}_T \text{ for all } 1 \leq i \leq n\Big\}.$$

Then T has a unique extension $T : \mathfrak{m}_T \to B$ as a B-bimodule linear map. In this paper, all the operator-valued weights that we consider are assumed to be *faithful, normal* and *semifinite*. Note that since the operator-valued weight is nothing but a weight when $B = \mathbb{C}$, we may also extend a faithful normal semifinite weight φ on $\mathfrak{m}_\varphi$.

For any inclusion $B \subset M$ of von Neumann algebras with faithful normal weights φ_B and φ_M on B and M respectively, the modular actions of them satisfy $\sigma^{\varphi_M}|_B = \sigma^{\varphi_B}$ if and only if there is an operator-valued weight E_B from M to B which satisfies $\varphi_B \circ E_B = \varphi_M$, and E_B is determined uniquely by this equality [loc. cit., Theorem IX.4.18]. We call E_B the *operator-valued weight from* (M, φ_M) *to* (B, φ_B). In this case, the cores satisfy the inclusion $C_{\varphi_B}(B) \subset C_{\varphi_M}(M)$ since $\sigma^{\varphi_M}|_B = \sigma^{\varphi_B}$. When $\varphi_M|_B = \varphi_B$, E_B is a faithful normal conditional expectation [loc. cit., Theorem IX.4.2].

Let M be a von Neumann algebra and φ a faithful normal semifinite weight on M. Put $L^2(M) := L^2(M, \varphi)$ and let α be an action of $\mathbb{R}$ on M. In this article, as a representation of $M \rtimes_\alpha \mathbb{R}$, we use that for any $\xi \in L^2(\mathbb{R}) \otimes L^2(M) \simeq L^2(\mathbb{R}, M)$ and $s, t \in \mathbb{R}$,

$$M \ni x \mapsto \pi_\alpha(x), \qquad (\pi_\alpha(x)\xi)(s) := \alpha_{-s}(x)\xi(s),$$
$$L\mathbb{R} \ni \lambda_t \mapsto 1_M \otimes \lambda_t, \quad ((1 \otimes \lambda_t)\xi)(s) := \xi(s-t).$$

Let $C_c(\mathbb{R}, M)$ be the set of all $*$-strongly continuous functions from $\mathbb{R}$ to M with compact supports. Then there is an embedding

$$\hat{\pi}_\alpha : C_c(\mathbb{R}, M) \ni f \mapsto \int_{\mathbb{R}} (1 \otimes \lambda_t)\pi_\alpha(f(t))\, dt \in M \rtimes_\alpha \mathbb{R},$$

where the integral here should be understood as the map $T \in \mathbb{B}(L^2(\mathbb{R}) \otimes L^2(M))$ given by

$$\langle T\xi, \eta\rangle = \int_{\mathbb{R}} \langle (1 \otimes \lambda_t)\pi_\alpha(f(t))\xi, \eta\rangle \, dt$$

for all $\xi, \eta \in L^2(\mathbb{R}) \otimes L^2(M)$. We note that by

$$(f * g)(t) := \int_{\mathbb{R}} \alpha_s(f(t+s))g(-s)\, ds \quad \text{and} \quad f^\sharp(t) := \alpha_t^{-1}(f(-t)^*) \quad \text{for } f, g \in C_c(\mathbb{R}, M) \text{ and } t \in \mathbb{R},$$

$C_c(\mathbb{R}, M)$ is a $*$-algebra, so that $\hat{\pi}_\alpha$ is a $*$-homomorphism. For $f \in C_c(\mathbb{R}, M)$ and $x \in M$, we define $(f \cdot x)(t) := f(t)x$ for $t \in G$. Let $C_c(\mathbb{R}, M)\mathfrak{n}_\varphi \subset C_c(\mathbb{R}, M)$ be the set of linear spans of $f \cdot x$ for $f \in C_c(\mathbb{R}, M)$ and $x \in \mathfrak{n}_\varphi$. With this notation, the dual weight satisfies

$$\hat{\varphi}(\hat{\pi}_\alpha(g)^*\hat{\pi}_\alpha(f)) = \varphi((g^\sharp * f)(0)) = \int_{\mathbb{R}} \varphi(g(t)^* f(t))\, dt \quad \text{for any } f, g \in C_c(\mathbb{R}, M)\mathfrak{n}_\varphi$$

[Takesaki 1979, Theorem X.1.17]. The modular objects of $\hat{\varphi}$ are given by

$$\sigma_t^{\hat{\varphi}}|_M = \sigma_t^{\varphi} \quad \text{and} \quad \sigma_t^{\hat{\varphi}}(\lambda_s) = \lambda_s [D(\varphi \circ \alpha_s), D\varphi]_t \quad \text{for } s, t \in \mathbb{R},$$
$$(J_{\hat{\varphi}}\xi)(t) = u^*(t) J_\varphi \xi(-t) \quad \text{for } t \in \mathbb{R} \text{ and } \xi \in L^2(\mathbb{R}, L^2(M)),$$

where $u(t)$ is the unitary such that $\alpha_t = \operatorname{Ad} u(t)$, which preserves the standard structure of $L^2(M, \varphi)$. In particular $\sigma^{\hat{\varphi}}$ globally preserves M and so there is a canonical operator-valued weight E_M from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to (M, φ). By the equality $\varphi \circ E_M = \hat{\varphi}$, it holds that for any $f, g \in C_c(\mathbb{R}, M)$,

$$E_M(\hat{\pi}_\alpha(g)^*\hat{\pi}_\alpha(f)) = (g^\sharp * f)(0) = \int_{\mathbb{R}} g(t)^* f(t)\, dt.$$

Here we prove a few lemmas.

Lemma 2.1. *Let (N, φ_N) and (B, φ_B) be von Neumann algebras with faithful normal semifinite weights with $\varphi_N(1) = 1$. Let α^B be an action of $\mathbb{R}$ on B, and put $M := N \mathbin{\overline{\otimes}} B$, $\varphi := \varphi_N \otimes \varphi_B$, $\alpha := \sigma^{\varphi_N} \otimes \alpha^B$. Let E_M, E_B, $E_{B\rtimes\mathbb{R}}$ be the canonical operator-valued weights from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to (M, φ), from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to (B, φ_B), and from $(M \rtimes_\alpha \mathbb{R}, \hat{\varphi})$ to $(B \rtimes_{\alpha^B} \mathbb{R}, \hat{\varphi}_B)$ respectively. Then we have $E_{B\rtimes\mathbb{R}} \circ E_M = E_B$.*

Proof. Let P_N be the one-dimensional projection from $L^2(N, \varphi_N)$ onto $\mathbb{C}\Lambda_{\varphi_N}(1_N)$ and observe that the compression map by $P_N \otimes 1_B \otimes 1_{L^2(\mathbb{R})}$ on $N \mathbin{\overline{\otimes}} B \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$ gives a normal conditional expectation $E : M \rtimes_\alpha \mathbb{R} \to B \rtimes_{\alpha^B} \mathbb{R}$ satisfying $E((x \otimes b)\lambda_t) = \varphi_N(x) b \lambda_t$ for $x \in N$, $b \in B$, and $t \in \mathbb{R}$. It is faithful on $M \rtimes_\alpha \mathbb{R}$ since it is faithful on $N \mathbin{\overline{\otimes}} B \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$. A simple computation shows that $E = E_{B\rtimes\mathbb{R}}$ and $E_{B\rtimes\mathbb{R}}((x \otimes b)\lambda_t) = \varphi_N(x) b \lambda_t$ for $x \in N$, $b \in B$, and $t \in \mathbb{R}$. In particular $E_{B\rtimes\mathbb{R}}|_M$ is the canonical conditional expectation E_B^M from (M, φ) to (B, φ_B). Then by definition, $\varphi_B \circ E_B^M \circ E_M = \varphi \circ E_M = \hat{\varphi}$, and hence $E_B^M \circ E_M = E_B$. Since $E_B^M \circ E_M = E_{B\rtimes\mathbb{R}} \circ E_M$, we obtain the conclusion. □

We next recall the following well-known fact. We include a proof for the reader's convenience.

Lemma 2.2. *Let M be a type III_1 factor and N a von Neumann algebra. Then the center of the continuous core of $M \mathbin{\overline{\otimes}} N$ coincides with the center of N.*

Proof. Since M is a type III_1 factor, there is a faithful normal semifinite weight φ_M on M such that $(M_{\varphi_M})' \cap M = \mathbb{C}$ [Takesaki 1979, Theorem XII.1.7], where M_{φ_M} is the fixed point algebra of the modular action of φ_M. Let φ_N be a faithful normal semifinite weight on N and put $\varphi := \varphi_M \otimes \varphi_N$. Observe that the center of $C_\varphi(M \mathbin{\overline{\otimes}} N)$ is contained in

$$(M_{\varphi_M} \otimes \mathbb{C}1_{L^2(N)\otimes L^2(\mathbb{R})})' \cap M \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R})) = \mathbb{C}1_{L^2(M,\varphi_M)} \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R})).$$

On the other hand, since $\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N))$ commutes with $L\mathbb{R}$, it is contained in $(M \mathbin{\overline{\otimes}} N)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R}$; see, e.g., [Houdayer and Ricard 2011, Proposition 2.4]. Hence

$$\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N)) \subset \mathbb{C} \mathbin{\overline{\otimes}} N \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R})) \cap (M \mathbin{\overline{\otimes}} N)_\varphi \mathbin{\overline{\otimes}} L\mathbb{R} = \mathbb{C} \mathbin{\overline{\otimes}} N_{\varphi_N} \mathbin{\overline{\otimes}} L\mathbb{R}.$$

Finally since $\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N))$ commutes with M, and N_{φ_N} commutes with M and $L\mathbb{R}$, (up to exchanging positions of M and N) we have

$$\mathcal{Z}(C_\varphi(N \mathbin{\overline{\otimes}} M)) \subset M' \cap N_{\varphi_N} \mathbin{\overline{\otimes}} \mathbb{C} \mathbin{\overline{\otimes}} L\mathbb{R} = N_{\varphi_N} \mathbin{\overline{\otimes}} (M' \cap \mathbb{C} \mathbin{\overline{\otimes}} L\mathbb{R}) = N_{\varphi_N} \mathbin{\overline{\otimes}} \mathbb{C}1,$$

where we used $M' \cap \mathbb{C} \mathbin{\overline{\otimes}} L\mathbb{R} \subset \mathcal{Z}(C_{\varphi_M}(M)) = \mathbb{C}$. Since $N' \cap N_{\varphi_N} = \mathcal{Z}(N)$, we conclude that $\mathcal{Z}(C_\varphi(M \mathbin{\overline{\otimes}} N)) = \mathcal{Z}(N)$. Since all continuous cores are isomorphic with each other, preserving the position of $M \mathbin{\overline{\otimes}} N$, for any other faithful normal semifinite weight ψ, we obtain $\mathcal{Z}(C_\psi(M \mathbin{\overline{\otimes}} N)) = \mathcal{Z}(N)$. □

2B. ***Relative tensor products, basic constructions and weak containments.*** Let M and N be von Neumann algebras and H a Hilbert space. Throughout this paper, we denote *opposite* objects with a circle superscript (e.g., $N^\circ := N^{\mathrm{op}}$, $x^\circ := x^{\mathrm{op}} \in N^\circ$, $(xy)^\circ = y^\circ x^\circ$ for $x, y \in N$). We say that H is a *left M-module* (resp. a *right N-module*) if there is a normal unital injective $*$-homomorphism $\pi_H : M \to \mathbb{B}(H)$ (resp. $\theta_H : N^\circ \to \mathbb{B}(H)$). We say H is an *M-N-bimodule* if H is a left M-module and a right N-module with commuting ranges. The *standard bimodule* of M is a standard representation $L^2(M)$ as an M-bimodule, where the right action is given by $M^\circ \ni x^\circ \mapsto Jx^*J \in M' \subset \mathbb{B}(L^2(M))$.

Let N be a von Neumann algebra, φ a faithful normal semifinite weight, and $H = H_N$ a right N-module with the right action θ. A vector $\xi \in H$ is said to be *left φ-bounded* if there is a constant $C > 0$ such that $\|\theta(x^\circ)\xi\| \le C\|J_\varphi\Lambda_\varphi(x^*)\|$ for all $x \in \mathfrak{n}_\varphi^*$. We denote by $D(H,\varphi)$ all left φ-bounded vectors in H. It is known that the subspace $D(H,\varphi) \subset H$ is always dense [Takesaki 1979, Lemma IX.3.3(iii)]. For $\xi \in D(H,\varphi)$, define a bounded operator

$$L_\xi : L^2(N,\varphi) \to H;\ \ L_\xi J_\varphi \Lambda_\varphi(a^*) = \theta(a^\circ)\xi.$$

It is easy to verify that

$$\theta(x^\circ)L_\xi = L_\xi J_\varphi x^* J_\varphi \quad (x \in N),$$
$$L_\xi L_\eta^* \in \theta(N^\circ)' \quad \text{and} \quad L_\eta^* L_\xi \in (J_\varphi N J_\varphi)' = N \quad (\xi, \eta \in D(H,\varphi)),$$
$$x L_\xi y = L_{x\theta(\sigma_{i/2}^\varphi(y)^\circ)\xi} \quad (x \in \theta(N^\circ)',\ y \in N_a),$$

where $N_a \subset N$ is the subalgebra consisting of all *analytic* elements with respect to (σ_t^φ) (see [Takesaki 1979, Lemma IX.3.3(v)] for the third statement). For a left N-module $K = {}_N K$, the *relative tensor product* $H \otimes_N K$ is defined as the Hilbert space obtained by separation and compression of $D(H,\varphi) \otimes_{\mathrm{alg}} K$ with an inner product $\langle \xi_1 \otimes_N \eta_1, \xi_2 \otimes_N \eta_2\rangle := \langle L_{\xi_2}^* L_{\xi_1}\eta_1, \eta_2\rangle_K$. When $H = {}_M H_N$ is an M-N-bimodule and $K = {}_N K_A$ is an N-A-bimodule for von Neumann algebras M and A, the Hilbert space $H \otimes_N K$ is an M-A-bimodule given by $\pi(x)\theta(a^\circ)(\xi \otimes_N \eta) := (\pi_H(x)\xi) \otimes_B (\theta_K(a^\circ)\eta)$ for $x \in M$, $a \in A$, $\xi \in D(H, \varphi)$ and $\eta \in K$.

Since all standard representations $L^2(M)$ of M are isomorphic as M-bimodules, when we consider $H = K = L^2(M)$ and $N \subset M$, the Hilbert space $L^2(M) \otimes_N L^2(M)$ is determined canonically, and does not depend on the choice of a faithful normal semifinite weight φ on M with $L^2(M) = L^2(M, \varphi)$.

Let $B \subset M$ be an inclusion of von Neumann algebras and φ a faithful normal semifinite weight on M. The *basic construction* of the inclusion $B \subset M$ is defined by

$$\langle M, B\rangle := (J_\varphi B J_\varphi)' \cap \mathbb{B}(L^2(M, \varphi)).$$

Since all standard representations are canonically isomorphic, the basic construction does not depend on the choice of φ. Assume that the inclusion $B \subset M$ is with an operator-valued weight E_B. Fix a faithful normal semifinite weight φ_B on B and put $\varphi := \varphi_B \circ E_B$. Here we observe that any $x \in \mathfrak{n}_{E_B} \cap \mathfrak{n}_\varphi$ is left φ-bounded and $L_{\Lambda_\varphi(x)}\Lambda_{\varphi_B}(a) = \Lambda_\varphi(xa)$ for $a \in \mathfrak{n}_{\varphi_B}$. Indeed, for any analytic $a \in \mathfrak{n}_{\varphi_B} \cap \mathfrak{n}_{\varphi_B}^*$, we have $J_{\varphi_B}\Lambda_{\varphi_B}(a^*) = \Delta_{\varphi_B}^{1/2}\Lambda_{\varphi_B}(a) = \Lambda_{\varphi_B}(\sigma_{-i/2}^{\varphi_B}(a))$, see, e.g., the equation just before [Takesaki 1979, Lemma VIII.2.4], and hence by Lemma V.III.3.18(ii) of the same work,

$$L_{\Lambda_\varphi(x)}\Lambda_{\varphi_B}(\sigma_{-i/2}^{\varphi_B}(a)) = L_{\Lambda_\varphi(x)} J_{\varphi_B}\Lambda_{\varphi_B}(a^*) = J_\varphi a^* J_\varphi \Lambda_\varphi(x) = \Lambda_\varphi(x\sigma_{-i/2}^{\varphi}(a)).$$

Since $\sigma_{-i/2}^{\varphi_B}(a) = \sigma_{-i/2}^{\varphi}(a)$ (because $\sigma_t^\varphi|_B = \sigma_t^{\varphi_B}$ for $t \in \mathbb{R}$, and the analytic extension is unique if exists), this means that $L_{\Lambda_\varphi(x)}\Lambda_{\varphi_B}(b) = \Lambda_\varphi(xb)$ for any analytic $b \in \mathfrak{n}_{\varphi_B} \cap \mathfrak{n}_{\varphi_B}^*$. At the same time, we can define a bounded operator $L_x : \Lambda_{\varphi_B}(a) \mapsto \Lambda_\varphi(xa)$ for $a \in \mathfrak{n}_{\varphi_B}$ (use $x \in \mathfrak{n}_{E_B}$). So the map $L_{\Lambda_\varphi(x)}$ has a bounded extension on $L^2(B, \varphi_B)$ and coincides with L_x, as desired. Now it is easy to verify that

$$L_{\Lambda_\varphi(y)}^* L_{\Lambda_\varphi(x)} = E_B(y^*x) \in (J_\varphi B J_\varphi)' = B \subset \mathbb{B}(L^2(B, \varphi_B)) \quad (x, y \in \mathfrak{n}_{E_B} \cap \mathfrak{n}_\varphi).$$

We will use this formula for calculations in the proposition below and in Section 3.

Here we observe that a relative tensor product has a useful identification. We will use this proposition in Sections 3 and 4.

Proposition 2.3. *Let N and B be von Neumann algebras, and α^N and α^B actions of $\mathbb{R}$ on N and B respectively. Put $M := N \mathbin{\overline{\otimes}} B$ and $\alpha := \alpha^N \otimes \alpha^B$, and define $H := L^2(M \rtimes_\alpha \mathbb{R}) \otimes_B L^2(M \rtimes_\alpha \mathbb{R})$ as an $M \rtimes_\alpha \mathbb{R}$-bimodule with left and right actions π_H and θ_H.*

Then there is a

$U : H \to L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})$ such that, putting $\tilde{\pi}_H := \operatorname{Ad} U \circ \pi_H$ and $\tilde{\theta}_H := \operatorname{Ad} U \circ \theta_H$,

- $\tilde{\pi}_H(M \rtimes_\alpha \mathbb{R}) \subset \mathbb{B}(L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B)) \otimes \mathbb{C}1_N \otimes \mathbb{C}1_{L^2(\mathbb{R})}$,
$$\tilde{\pi}_H(\lambda_t) = \lambda_t \otimes 1_N \otimes 1_B \quad \textit{and} \quad \tilde{\pi}_H(x) = \pi_\alpha(x) \quad (t \in \mathbb{R},\ x \in N \mathbin{\overline{\otimes}} B = M);$$
- $\tilde{\theta}_H((M \rtimes_\alpha \mathbb{R})^\circ) \subset \mathbb{C}1_{L^2(\mathbb{R})} \otimes \mathbb{C}1_N \otimes \mathbb{B}(L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R}))$
$$\tilde{\theta}_H(\lambda_t^\circ) = 1_B \otimes 1_N \otimes \rho_t \quad \textit{and} \quad \tilde{\theta}_H(y^\circ) = \theta_\alpha(y^\circ) \quad (t \in \mathbb{R}, y \in B \mathbin{\overline{\otimes}} N \simeq M),$$
$$\textit{where } (\theta_\alpha(y^\circ)\xi)(s) := \alpha_s(y)^\circ \xi(s) \quad \textit{for } \xi \in L^2(\mathbb{R}, L^2(B) \otimes L^2(N)) \textit{ and } s \in \mathbb{R}.$$

Proof. We fix a faithful normal semifinite weight φ_B on B and put $\varphi := \varphi_N \otimes \varphi_B$. Denote by $\hat{\varphi}$ the dual weight of φ and then the standard representation of $M \rtimes_\alpha \mathbb{R}$ is given by

$$L^2(M \rtimes_\alpha \mathbb{R}, \hat{\varphi}) = L^2(N, \varphi_N) \otimes L^2(B, \varphi_B) \otimes L^2(\mathbb{R}) \simeq L^2(\mathbb{R}, L^2(N, \varphi_N) \otimes L^2(B, \varphi_B)).$$

For simplicity we put $L^2(N) := L^2(N, \varphi_N)$ and $L^2(B) := L^2(B, \varphi_B)$. Let E_B be the canonical operator-valued weight from $\widetilde{M}$ to B given by $\hat{\varphi} = \varphi_B \circ E_B$. Then for $E_B^M := \varphi_N \otimes \mathrm{id}_B$ on M and for the canonical operator-valued weight E_M from $(M \rtimes \mathbb{R}, \hat{\varphi})$ to (M, φ), we have $\hat{\varphi} = \varphi \circ E_M = \varphi_B \circ E_B^M \circ E_M$, and hence $E_B = E_B^M \circ E_M$ by the uniqueness condition. Observe then for any $f, g \in C_c(\mathbb{R}, M)$,

$$E_B(\hat{\pi}_\alpha(g)^* \hat{\pi}_\alpha(f)) = \int_{\mathbb{R}} E_B^M(g(t)^* f(t))\, dt.$$

Define a well-defined linear map

$$V : \Lambda_\varphi(\mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B}) \otimes_{\mathrm{alg}} J_\varphi \Lambda_\varphi(\mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N}) \to L^2(N) \otimes L^2(B) \otimes L^2(N)$$

by $V(\Lambda_\varphi(x \otimes a) \otimes J_\varphi \Lambda_\varphi(b \otimes y)) := \Lambda_{\varphi_N}(x) \otimes a J_{\varphi_B} \Lambda_{\varphi_B}(b) \otimes J_{\varphi_N} \Lambda_{\varphi_N}(y)$. We then define a linear map

$$U : L^2(\mathbb{R}, L^2(N) \otimes L^2(B)) \otimes_B L^2(\mathbb{R}, L^2(B) \otimes L^2(N)) \to L^2(\mathbb{R} \times \mathbb{R}, L^2(N) \otimes L^2(B) \otimes L^2(N))$$

by $(U(f \otimes_B J_{\hat{\varphi}} g))(t, s) := V(\Lambda_\varphi(f(t)) \otimes J_\varphi \Lambda_\varphi(g(-s)))$ for $f \in C_c(\mathbb{R}, N \otimes_{\mathrm{alg}} B)(\mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B})$ and $g \in C_c(\mathbb{R}, B \otimes_{\mathrm{alg}} N)(\mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N})$. (Note that we are identifying $\Lambda_{\hat{\varphi}}(\hat{\pi}_\alpha(f))$ and $\Lambda_{\hat{\varphi}}(\hat{\pi}_\alpha(g))$ as f and g.) We have to show that it is a well-defined unitary map. For $f_i \in C_c(\mathbb{R}, N \otimes_{\mathrm{alg}} B)(\mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B})$ and $g_i \in C_c(\mathbb{R}, B \otimes_{\mathrm{alg}} N)(\mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N})$, straightforward but rather complicated computations yield, on the one hand,

$$\Big\| \sum_i f_i \otimes_B J_{\hat{\varphi}} g_i \Big\|_2^2 = \sum_{i,j} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle F_{j,i} J_\varphi \Lambda_\varphi(g_i(-s)), J_\varphi \Lambda_\varphi(g_j(-s)) \rangle\, ds\, dt,$$

where $F_{j,i} := E_B^M(f_j(t)^* f_i(t))$, and on the other hand,

$$\Big\| U \sum_i (f_i \otimes_B J_{\hat{\varphi}} g_i) \Big\|_2^2 = \sum_{i,j} \int_{\mathbb{R} \times \mathbb{R}} \big\langle V\big(\Lambda_\varphi(f_i(t)) \otimes J_\varphi \Lambda_\varphi(g_i(-s))\big), V\big(\Lambda_\varphi(f_j(t)) \otimes J_\varphi \Lambda_\varphi(g_i(-s))\big) \big\rangle\, dt\, ds.$$

Hence if we show

$$\langle V(\Lambda_\varphi(x) \otimes J_\varphi \Lambda_\varphi(a)), V(\Lambda_\varphi(y) \otimes J_\varphi \Lambda_\varphi(b)) \rangle = \langle E_B^M(y^* x) J_\varphi \Lambda_\varphi(a), J_\varphi \Lambda_\varphi(b) \rangle$$

for any $x, y \in \mathfrak{n}_{\varphi_N} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_B}$ and $a, b \in \mathfrak{n}_{\varphi_B} \otimes_{\mathrm{alg}} \mathfrak{n}_{\varphi_N}$, then U is a well-defined unitary map. However this equation follows easily if we use elementary elements.

Finally $L^2(\mathbb{R} \times \mathbb{R}, L^2(N) \otimes L^2(B) \otimes L^2(N))$ is canonically isomorphic to $L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})$, where the first (resp. the second) variable in $\mathbb{R} \times \mathbb{R}$ corresponds to $L\mathbb{R}$ of the left one (resp. the right one) in the Hilbert space. It is then easy to see that $\tilde{\pi}_H$ and $\tilde{\theta}_H$ satisfy the desired condition. □

Let M and N be von Neumann algebras, and let H and K be M-N-bimodules. We denote by π_H and θ_H (resp. π_K and θ_K) left and right actions on H (resp. K). We say that K *is weakly contained in* H, denoted by $K \prec H$, if for any $\varepsilon > 0$, finite subsets $\mathcal{E} \subset M$ and $\mathcal{F} \subset N$, and any vector $\xi \in K$, there are vectors $(\eta_i)_{i=1}^n \subset H$ such that

$$\left|\sum_{i=1}^n \langle \pi_H(x)\theta_H(y^\circ)\eta_i, \eta_i\rangle_H - \langle \pi_K(x)\theta_K(y^\circ)\xi, \xi\rangle_K\right| < \varepsilon \quad (x \in \mathcal{E},\ y \in \mathcal{F}).$$

This is equivalent to saying that the algebraic $*$-homomorphism given by $\pi_H(x)\theta_H(y^\circ) \mapsto \pi_K(x)\theta_K(y^\circ)$ for $x \in M$ and $y \in N$ is bounded on $*\text{-alg}\{\pi_H(M), \theta_H(N^\circ)\}$. We denote by $\nu_{K,H}$ the associated $*$-homomorphism for $K \prec H$.

Let M and N be σ*-finite* von Neumann algebras and let X be a self-dual M-N-correspondence (i.e., a Hilbert N-module with a normal left M-action, see [Paschke 1973, Section 3] for self-duality and normality). Then the *interior tensor product*, see, e.g., [Lance 1995, Section 4], $H(X) := X \otimes_N L^2(N)$ is an M-N-bimodule. Conversely if H is an M-N-bimodule, then one can define a self-dual M-N-correspondence (i.e., a W*-Hilbert N-module with a left M-action)

$$X(H) := \{T : L^2(N) \to H \mid \text{bounded, } N^\circ\text{-module linear map}\}.$$

They in fact give a one-to-one correspondence between M-N-bimodules and self-dual M-N-correspondences, up to unitary equivalence; see [Baillet et al. 1988, Theorem 2.2] and [Rieffel 1974, Proposition 6.10]. By [Anantharaman-Delaroche 1990, §1.12, Proposition], $K \prec H$ if and only if $X(K) \prec X(H)$ in the following sense: for any σ-weak neighborhood $\mathcal{V}$ of $0 \in N$, finite subsets $\mathcal{E} \subset M$ and $\mathcal{F} \subset N$, and any $\xi \in X(K)$, there are vectors $(\eta_i)_{i=1}^n \subset X(H)$ such that

$$\sum_{i=1}^n \langle \eta_i, x\eta_i y\rangle_{X(H)} - \langle \xi, x\xi y\rangle_{X(K)} \in \mathcal{V} \quad (x \in \mathcal{E},\ y \in \mathcal{F}).$$

Suppose that $M = N$, $L^2(M) = K$, and $M = X(K)$. Then if $L^2(M) \prec H$, putting $\xi := 1_M$, for any finite subset $\mathcal{E} \subset M$ and for any σ-weak neighborhood $\mathcal{V}$ of $0 \in N$, there are vectors $(\eta_i)_{i=1}^n \subset X(H)$ such that

$$\sum_{i=1}^n \langle \eta_i, x\eta_i\rangle_{X(H)} - x \in \mathcal{V} \quad (x \in \mathcal{E}).$$

So putting $\psi_{(\mathcal{E},\mathcal{V})}(x) := \sum_{i=1}^n \langle \eta_i, x\eta_i\rangle_{X(H)}$ for $x \in M$, we find a net $(\psi_i)_i$ such that each ψ_i is given by a sum of compression maps by vectors in $X(H)$ and such that it converges to id_M in the point σ-weak topology. In this case, up to replacing η_i, we may assume that each ψ_i is a contraction [Anantharaman-Delaroche and Havet 1990, Lemma 2.2]. Then it is known that the existence of such a net is equivalent to $L^2(M) \prec H$ as follows, although we do not need this equivalence. See Proposition 2.4 of the same work for a more general statement.

Proposition 2.4. *Let M be a σ-finite von Neumann algebra and H an M-bimodule. Then $L^2(M) \prec H$ as M-bimodules if and only if there is a net $(\psi_i)_i$ of normal contractive completely positive (c.c.p.) maps on M, which converges to id_M point σ-weakly, such that each ψ_i is a finite sum of $\langle \eta, \cdot\, \eta\rangle_{X(H)}$ for some $\eta \in X(H)$.*

We recall the following well-known fact. This will be used in Section 3.

Lemma 2.5. *Let $B \subset M$ be an inclusion of σ-finite von Neumann algebras with an operator-valued weight E_B. Then the vector space $\mathfrak{n}_{E_B}$ is a pre-Hilbert B-module with the inner product $\langle x, y\rangle := E_B(x^*y)$ for $x, y \in \mathfrak{n}_{E_B}$, and its self-dual completion $\bar{\mathfrak{n}}_{E_B}$ is an M-B-correspondence.*

Let X be the self-dual completion of the interior tensor product $\bar{\mathfrak{n}}_{E_B} \otimes_B M$. Then as an M-M-correspondence, X is the unique one corresponding to the M-bimodule $L^2(M) \otimes_B L^2(M)$, using the one-to-one correspondence above.

Proof. It is easy to see that the B-valued inner product on $\mathfrak{n}_{E_B}$ in the statement is well-defined, so that $\mathfrak{n}_{E_B}$ is a pre-Hilbert B-module with a left M-action. Since the left M-action is faithful on $\mathfrak{n}_{E_B}$, so is on the self-dual completion; see, e.g., [Paschke 1973, Corollary 3.7]. This left M-action is normal, since the functional $M \ni x \mapsto \omega(\langle \xi, x\eta\rangle)$ is normal for all $\omega \in M_*$ and $\xi, \eta \in \mathfrak{n}_{E_B}$, and hence for all $\xi, \eta \in \bar{\mathfrak{n}}_{E_B}$ by [Paschke 1976, Lemma 2.3]. Thus $\bar{\mathfrak{n}}_{E_B}$ is an M-B-correspondence.

Let X be as in the statement. Then as in the first paragraph, it is easy to see that it is really an M-M-correspondence (i.e., the left M-action is well-defined, injective, and normal). Let us fix faithful normal states φ_B and φ on B and M respectively. Then the interior tensor product $X \otimes_M L^2(M, \varphi)$ is canonically identified as $L^2(M, \varphi_B \circ E_M) \otimes_B L^2(M, \varphi)$, so that X is identified as $X(L^2(M) \otimes_B L^2(M))$. □

2C. *Free quantum groups and biexactness.* For compact quantum groups, we refer the reader to [Woronowicz 1998; Maes and Van Daele 1998].

Let $\mathbb{G}$ be a compact quantum group. In this paper, we use the following notation, which will only be used in Section 4. We denote the Haar state by h, the set of equivalence classes of all irreducible unitary corepresentations by $\mathrm{Irred}(\mathbb{G})$, and right and left regular representations by ρ and λ respectively. We regard $C_{\mathrm{red}}(\mathbb{G}) := \rho(C(\mathbb{G}))$ as our main object and we frequently omit ρ when we see the dense Hopf $*$-algebra. The GNS representation of h is written as $L^2(\mathbb{G})$ and it has a decomposition $L^2(\mathbb{G}) = \sum_{x\in\mathrm{Irred}(\mathbb{G})} \oplus (H_x \otimes H_{\bar{x}})$. Along the decomposition, the modular operator of h is of the form $\Delta_h^{it} = \sum_{x\in\mathrm{Irred}(\mathbb{G})} \oplus (Q_x^{it} \otimes Q_{\bar{x}}^{-it})$ for some positive matrices Q_x.

Let F be a matrix in $\mathrm{GL}(n, \mathbb{C})$. The *free unitary quantum group* (resp. *free orthogonal quantum group*) for F [Wang 1995; Van Daele and Wang 1996] is the C*-algebra $C(A_u(F))$ (resp. $C(A_o(F))$) defined as the universal unital C*-algebra generated by all the entries of a unitary n by n matrix $u = (u_{i,j})_{i,j}$ satisfying that $F(u_{i,j}^*)_{i,j} F^{-1}$ is a unitary (resp. $F(u_{i,j}^*)_{i,j} F^{-1} = u$). We simply say that $\mathbb{G}$ is a *free quantum group* if $\mathbb{G}$ is a free unitary or orthogonal quantum group.

Here we recall the notion of biexactness introduced in [Isono 2015b, Definition 3.1], based on the group case [Brown and Ozawa 2008, Lemma 15.1.2].

Definition 2.6. Let $\mathbb{G}$ be a compact quantum group. We say that the dual $\widehat{\mathbb{G}}$ is *biexact* if it satisfies following conditions:

(i) $\widehat{\mathbb{G}}$ is exact (i.e., $C_{\mathrm{red}}(\mathbb{G})$ is exact).

(ii) There exists a unital completely positive (u.c.p.) map $\Theta : C_{\mathrm{red}}(\mathbb{G}) \otimes_{\min} C_{\mathrm{red}}(\mathbb{G})^\circ \to \mathbb{B}(L^2(\mathbb{G}))$ such that

$$\Theta(a \otimes b^\circ) - ab^\circ \in \mathbb{K}(L^2(\mathbb{G})) \quad \text{for any } a, b \in C_{\mathrm{red}}(\mathbb{G}).$$

Biexactness of free quantum groups was proved in [Vergnioux 2005; Vaes and Vergnioux 2007; Vaes and Vander Vennet 2010]. See [Isono 2015b, Theorem C] for other examples of biexact quantum groups.

Theorem 2.7. *Let* $\mathbb{G}$ *be a free quantum group* (more generally, a compact quantum group in [Isono 2015b, Theorem C]). *Then the dual* $\widehat{\mathbb{G}}$ *is biexact.*

2D. ***Central weak amenability and the W*CMAP.*** Let $\mathbb{G}$ be a compact quantum group. Denote the dense Hopf $*$-algebra by $\mathscr{C}(\mathbb{G})$. To any element $a \in \ell^\infty(\widehat{\mathbb{G}})$ we can associate a linear map m_a on $\mathscr{C}(\mathbb{G})$, given by $(m_a \otimes \iota)(u^x) = (1 \otimes ap_x)u^x$ for any $x \in \mathrm{Irred}(\mathbb{G})$, where $p_x \in c_0(\widehat{\mathbb{G}})$ is the canonical projection onto the x-component. We say $\widehat{\mathbb{G}}$ is *weakly amenable* (*with Cowling–Haagerup constant 1*) if there exists a net $(a_i)_i$ of elements of $\ell^\infty(\widehat{\mathbb{G}})$ such that:

- Each a_i has finite support; namely, $a_i p_x = 0$ except for finitely many $x \in \mathrm{Irred}(\mathbb{G})$.
- $(a_i)_i$ converges to 1 pointwise; namely, $a_i p_x$ converges to p_x in $\mathbb{B}(H_x)$ for any $x \in \mathrm{Irred}(\mathbb{G})$.
- Each m_{a_i} is extended on $L^\infty(\mathbb{G})$ as a completely contractive (say c.c.) map.

Note that, since a_i is finitely supported, each m_{a_i} is actually a map from $L^\infty(\mathbb{G})$ to $\mathscr{C}(\mathbb{G})$. We say $\widehat{\mathbb{G}}$ is *centrally weakly amenable* if each $a_i p_x$ above is taken as a scalar matrix for all i and $x \in \mathrm{Irred}(\mathbb{G})$. In this case, the associated multiplier m_{a_i} commutes with the modular action of the Haar state. This commutativity is important to us since such multipliers can be extended naturally on the continuous core with respect to the Haar state. Indeed, the maps $m_{a_i} \otimes \mathrm{id}_{L^2(\mathbb{R})}$ on $L^\infty(\mathbb{G}) \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$ restrict to approximation maps on the core. With this phenomenon in mind, we introduce the following terminology.

Definition 2.8. Let M be a von Neumann algebra and φ a fixed faithful normal state on M. We say that M has the *weak* completely metric approximation property with respect to* φ (or φ*-W*CMAP*, in short) if there exists a net $(\psi_i)_i$ of normal c.c. maps on M such that:

- Each ψ_i commutes with σ^φ; that is, $\psi_i \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \psi_i$ for all i and $t \in \mathbb{R}$.
- Each ψ_i is a finite sum of $\varphi(b^* \cdot a)z$ for some $a, b, z \in M$.
- ψ_i converges to id_M in the point σ-weak topology.

It is easy to see that the central weak amenability of $\widehat{\mathbb{G}}$ implies the W*CMAP with respect to the Haar state.

Weak amenability of the free quantum group was first obtained in [Freslon 2013], using the Haagerup property [Brannan 2012]. This is for the Kac type and hence is equivalent to the central weak amenability. The general case was solved later in [De Commer et al. 2014] and its proof in fact shows the central weak amenability as follows.

Theorem 2.9. *Let* $\mathbb{G}$ *be a free quantum group* (more generally a quantum group in [Isono 2015b, Theorem C]). *Then the dual* $\widehat{\mathbb{G}}$ *is centrally weakly amenable.*

In particular there is a net $(\psi_i)_i$ *of normal c.c. maps on* $L^\infty(\mathbb{G})$, *possessing the W*CMAP with respect to the Haar state, such that* $\psi_i(L^\infty(\mathbb{G})) \subset \mathscr{C}(\mathbb{G})$ *for all* i.

2E. *Popa's intertwining techniques.* Popa [2006a; 2006b] introduced a powerful tool called *intertwining techniques*. This is one of the main ingredients in the recent development of the von Neumann algebra theory. Here we introduce the one defined and studied in [Houdayer and Isono 2017, Definition 4.1 and Theorem 4.3] which treats general von Neumann algebras.

Definition 2.10. Let M be any σ-finite von Neumann algebra, 1_A and 1_B any nonzero projections in M, $A \subset 1_A M 1_A$ and $B \subset 1_B M 1_B$ any von Neumann subalgebras with expectation. We say that A *embeds with expectation into B inside M* and write $A \preceq_M B$ if there exist projections $e \in A$ and $f \in B$, a nonzero partial isometry $v \in eMf$ and a unital normal $*$-homomorphism $\theta : eAe \to fBf$ such that the inclusion $\theta(eAe) \subset fBf$ is with expectation and $av = v\theta(a)$ for all $a \in eAe$.

Theorem 2.11. *Keep the same notation as in Definition 2.10 and assume that A is finite. Then the following conditions are equivalent*:

1. *We have $A \preceq_M B$.*
2. *There exists no net $(w_i)_{i\in I}$ of unitaries in $\mathcal{U}(A)$ such that $E_B(b^* w_i a) \to 0$ in the σ-$*$-strong topology for all $a, b \in 1_A M 1_B$, where E_B is a fixed faithful normal conditional expectation from $1_B M 1_B$ onto B.*

For the proof of Corollary B, we prove a lemma. In the proof below, we make use of the *ultraproduct* von Neumann algebras [Ocneanu 1985]. We will actually use a more general one used in [Houdayer and Isono 2017], which treats a general directed set instead of $\mathbb{N}$. Recall from Section 2 of that paper that for any σ-finite von Neumann algebra M and any free ultrafilter $\mathcal{U}$ on a directed set I, we may define the *ultraproduct von Neumann algebra* $M^{\mathcal{U}}$, using $\ell^\infty(I) \mathbin{\overline{\otimes}} M$. In the proof below, we only need the following elementary properties: with the standard notation $(x_i)_{\mathcal{U}} \in M^{\mathcal{U}}$ for $(x_i)_{i\in I}$:

- $M \subset M^{\mathcal{U}}$ is with expectation by $E_{\mathcal{U}}((x_i)_{\mathcal{U}}) := \lim_{i\to\mathcal{U}} x_i$.
- For any σ-finite von Neumann algebras $A \subset M$ with expectation E_A, $A^{\mathcal{U}} \subset M^{\mathcal{U}}$ is with expectation defined by $E_{A^{\mathcal{U}}}((x_i)_{\mathcal{U}}) := (E_A(x_i))_{\mathcal{U}}$.
- If the subalgebra A is finite, then any norm bounded net $(a_i)_{i\in I}$ determines an element $(a_i)_{\mathcal{U}}$ in $M^{\mathcal{U}}$.

Lemma 2.12. *Let (B, φ_B) and (N, φ_N) be von Neumann algebras with faithful normal states. Put $M := B \mathbin{\overline{\otimes}} N$, $\varphi := \varphi_B \otimes \varphi_N$, $E_B = \mathrm{id}_B \otimes \varphi_N$ and $E_N = \varphi_B \otimes \mathrm{id}_N$. Let $p \in M$ be a projection and $A \subset pMp$ a von Neumann subalgebra with expectation. Fix $a := (a_i)_{i\in I} \in \ell^\infty(I) \mathbin{\overline{\otimes}} A$ and a free ultrafilter $\mathcal{U}$ on I such that $(a_i)_{\mathcal{U}} \in A^{\mathcal{U}}$. Then $E_{B^{\mathcal{U}}}(y^* a x) = 0$ for all $x, y \in M$ if and only if $E_N \circ E_{\mathcal{U}}(c^* a b)$ for all $b, c \in B^{\mathcal{U}}$.*

In particular, if A is finite, then $A \preceq_M B$ if and only if $A \preceq_{B \mathbin{\overline{\otimes}} N_0} B$ for any $N_0 \subset N$ with expectation E_{N_0} such that $\varphi_N \circ E_{N_0} = \varphi_N$, $p \in B \mathbin{\overline{\otimes}} N_0$ and $A \subset p(B \mathbin{\overline{\otimes}} N_0)p$.

Proof. Observe first that $E_{B^{\mathcal{U}}}(y^* a x) = 0$ for all $x, y \in M$ if and only if $E_{B^{\mathcal{U}}}((1 \otimes y^*) a (1 \otimes x)) = 0$ for all $x, y \in N$, which is equivalent to

$$\left\langle E_{B^{\mathcal{U}}}((1 \otimes y^*) a (1 \otimes x)) \Lambda_{\varphi_B^{\mathcal{U}}}(b), \Lambda_{\varphi_B^{\mathcal{U}}}(c) \right\rangle_{\varphi_B^{\mathcal{U}}} = 0$$

for all $x, y \in N$ and $b, c \in B^{\mathcal{U}}$. Writing $b = (b_i)_{\mathcal{U}}$ and $c = (c_i)_{\mathcal{U}}$, we calculate that

$$\begin{aligned}
\big\langle E_{B^{\mathcal{U}}}((1\otimes y^*)a(1\otimes x))\Lambda_{\varphi_B^{\mathcal{U}}}(b), \Lambda_{\varphi_B^{\mathcal{U}}}(c)\big\rangle_{\varphi_B^{\mathcal{U}}} &= \lim_{i\to\mathcal{U}}\big\langle E_B((1\otimes y^*)a_i(1\otimes x))\Lambda_{\varphi_B}(b_i), \Lambda_{\varphi_B}(c_i)\big\rangle_{\varphi_B} \\
&= \lim_{i\to\mathcal{U}} \varphi_B\big(c_i^* E_B((1\otimes y^*)a_i(1\otimes x))b_i\big) \\
&= \lim_{i\to\mathcal{U}} \varphi_B\circ E_B((c_i^*\otimes y^*)a_i(b_i\otimes x)) \\
&= \lim_{i\to\mathcal{U}} \varphi_N\circ E_N((c_i^*\otimes y^*)a_i(b_i\otimes x)) \\
&= \lim_{i\to\mathcal{U}} \varphi_N\big(y^* E_N((c_i^*\otimes 1)a_i(b_i\otimes 1))x\big) \\
&= \varphi_N\big(y^* E_N(\lim_{i\to\mathcal{U}}((c_i^*\otimes 1)a_i(b_i\otimes 1)))x\big) \\
&= \varphi_N\big(y^* E_N\circ E_{\mathcal{U}}((c^*\otimes 1)a(b\otimes 1))x\big).
\end{aligned}$$

Then since functionals of the form $\varphi_N(y^*\cdot x)$ for $x, y \in N$ are norm dense in N_*, the final term above is zero for all $x, y \in N$ if and only if $E_N\circ E_{\mathcal{U}}((c^*\otimes 1)a(b\otimes 1)) = 0$. Thus we proved that $E_{B^{\mathcal{U}}}(y^*ax) = 0$ for all $x, y \in M$ if and only if $E_N\circ E_{\mathcal{U}}((c^*\otimes 1)a(b\otimes 1)) = 0$ for all $b, c \in B^{\mathcal{U}}$.

For the second half of the statement, suppose that A is finite and $A \not\preceq_{B\bar{\otimes}N_0} B$. We will show $A \not\preceq_M B$. Since A is finite, there is a net $(u_i)_{i\in I} \subset \mathcal{U}(A)$ for a directed set I such that $E_B(y^*u_ix) \to 0$ strongly as $i\to\infty$ for all $x, y \in B\mathbin{\bar{\otimes}} N_0$. Fix any cofinal ultrafilter $\mathcal{U}$ on I. Since A is finite, $u := (u_i)_{\mathcal{U}} \in A^{\mathcal{U}}$ and hence $E_{B^{\mathcal{U}}}(y^*ux) = 0$ for all $x, y \in B\mathbin{\bar{\otimes}} N_0$. By the first half of the statement, this is equivalent to $E_{N_0}\circ E_{\mathcal{U}}(c^*ub) = 0$ for all $b, c \in B^{\mathcal{U}}$. Then since $E_{\mathcal{U}}(c^*ub)$ is contained in $B\mathbin{\bar{\otimes}} N_0$ and since $E_N|_{B\bar{\otimes}N_0} = (\varphi_B\otimes \mathrm{id}_N)|_{B\bar{\otimes}N_0} = E_{N_0}$, we have $E_N\circ E_{\mathcal{U}}(c^*ub) = 0$ for all $b, c \in B^{\mathcal{U}}$, which is in turn equivalent to $E_{B^{\mathcal{U}}}(y^*ux) = 0$ for $x, y \in M$ by the first half of the statement. Since this holds for arbitrary $\mathcal{U}$ on I, we conclude that $E_B(y^*u_ix) \to 0$ $*$-strongly as $i\to\infty$ for all $x, y \in M$. Thus we proved that $A \not\preceq_{B\bar{\otimes}N_0} B$ implies $A \not\preceq_M B$. □

3. Weakly compact actions

In this section, we define and study weakly compact actions on continuous cores. The main observation is Theorem 3.10, and the key item for the proof is Lemma 3.3.

3A. ***Relative amenability and approximation maps.*** In this subsection, we recall relative amenability for general von Neumann algebras introduced in [Isono 2017], which generalizes [Ozawa and Popa 2010; Popa and Vaes 2014a].

Definition 3.1. Let $B \subset M$ be von Neumann algebras, $p \in M$ a projection and $A \subset pMp$ a von Neumann subalgebra with expectation E_A. We say that the pair (A, E_A) *is injective relative to* B *in* M, and write $(A, E_A) \lessdot_M B$, if there exists a conditional expectation from $p\langle M, B\rangle p$ onto A which restricts to E_A on pMp.

Using amenability of $\mathbb{R}$ and the notion of relative amenability, we prove a lemma for approximation maps on the continuous core. For this we fix the following notation.

Let (M, φ) be a von Neumann algebra with a faithful normal semifinite weight, and $\widetilde{M} := M \rtimes \mathbb{R}$ the continuous core of M with the modular action σ^φ. We denote by $\hat{\varphi}$ the dual weight of φ, and by E_M the canonical operator-valued weight from $\widetilde{M}$ to M given by $\hat{\varphi} = \varphi \circ E_M$. We denote by $M \rtimes_{\text{alg}} G$ all the linear spans of $x\lambda_t$ for $x \in M$ and $t \in G$, which is a $*$-strongly dense subalgebra in $\widetilde{M}$.

Lemma 3.2. *In this setting, we have*

$$ {}_{\widetilde{M}}L^2(\widetilde{M})_{\widetilde{M}} \prec {}_{\widetilde{M}}L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M})_{\widetilde{M}}. $$

Proof. Recall first that

$$ M \rtimes \mathbb{R} = (M^\circ \otimes 1)' \cap \{\Delta_\varphi^{it} \otimes \rho_t \mid t \in \mathbb{R}\}', \quad \langle M \rtimes \mathbb{R}, M\rangle = (M^\circ \otimes 1)', $$

where ρ is the right regular representation. Since $\mathbb{R}$ is amenable, there are positive functionals $(f_n)_n \subset L^1(\mathbb{R})$ with $\|f_n\|_1 = 1$ satisfying $\lambda_g f_n - f_n \to 0$ weakly for all $g \in \mathbb{R}$. For each n, define a positive map

$$ F_n : \mathbb{B}(L^2(M) \otimes L^2(\mathbb{R})) \to \mathbb{B}(L^2(M) \otimes L^2(\mathbb{R})) $$

by

$$ F_n(T) := \int_{\mathbb{R}} (\Delta_\varphi^{it} \otimes \rho_t)\, T\, (\Delta_\varphi^{it} \otimes \rho_t)^* f_n(t) \cdot dt. $$

Since $\|F_n\| = 1$, we can take a cluster point of $(F_n)_n$, which we write as F. Then it satisfies

$$ (\Delta_\varphi^{it} \otimes \rho_t) F(T) (\Delta_\varphi^{it} \otimes \rho_t)^* = F(T) $$

for all $t \in \mathbb{R}$ and hence F is a conditional expectation onto $\{\Delta_\varphi^{it} \otimes \rho_t \mid t \in \mathbb{R}\}'$. It is easy to see that $F(T) \in (M^\circ \otimes 1)'$ for any $T \in (M^\circ \otimes 1)'$. Hence F restricts to a conditional expectation from $\langle M \rtimes \mathbb{R}, M\rangle$ onto $M \rtimes \mathbb{R}$. We obtain $(M \rtimes \mathbb{R}, \text{id}) \lessdot_{M \rtimes \mathbb{R}} M$. Finally since $M \rtimes \mathbb{R}$ is semifinite, using [Isono 2017, Theorem A.5], we get the conclusion. □

Lemma 3.3. *In this setting, there is a net $(\omega_j)_j$ of c.c.p. maps on $\widetilde{M}$ such that $\omega_j \to \text{id}_{\widetilde{M}}$ point σ-weakly and each ω_j is a finite sum of $\lambda_q^* E_M(z^* \cdot y)\lambda_p$ for some $y, z \in \mathfrak{n}_{E_M}$ and $p, q \in \mathbb{R}$.*

Proof. By Lemma 3.2 and Proposition 2.4, there is a net $(\omega_j)_j$ of c.c.p. maps on $\widetilde{M}$ such that $\omega_j \to \text{id}_{\widetilde{M}}$ point σ-weakly and each ω_j is a finite sum of $\langle \eta, \cdot\, \eta\rangle_{X(L^2(\widetilde{M})\otimes_M L^2(\widetilde{M}))}$ for some $\eta \in X(L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M}))$. We first replace each η in ω_j with some "algebraic" element in $X(L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M}))$.

By Lemma 2.5, the self dual completion X of $\bar{\mathfrak{n}}_{E_M} \otimes_{\text{alg}} \widetilde{M}$ is identified as the one corresponding to $L^2(\widetilde{M}) \otimes_M L^2(\widetilde{M})$. We denote by X_0 the image of $\bar{\mathfrak{n}}_{E_M} \otimes_{\text{alg}} \widetilde{M}$ in X. By [Paschke 1976, Lemma 2.3], $X_0 \subset X$ is dense in the *s-topology*; that is, for any $\eta \in X$ there is a net $(\eta_i)_i \subset X_0$ such that $\langle \eta - \eta_i, \eta - \eta_i\rangle_X \to 0$ in the σ-weak topology in $\widetilde{M}$. In our case, since $\mathfrak{n}_{E_B} \subset \bar{\mathfrak{n}}_{E_B}$ is dense in the s-topology and since $M \rtimes_{\text{alg}} G \subset \widetilde{M}$ is $*$-strongly dense, the image of $\mathfrak{n}_{E_M} \otimes_{\text{alg}} (M \rtimes_{\text{alg}} G)$ in X is dense in the s-topology. Hence we may replace each vector $\eta \in X$, appearing in ω_j above, with the one represented by elements in $\mathfrak{n}_{E_M} \otimes_{\text{alg}} (M \rtimes_{\text{alg}} G)$.

Thus, we may assume that each ω_j is a finite sum of $\lambda_q^* E_M(z^* \cdot y)\lambda_p$ for some $y, z \in \mathfrak{n}_{E_M}$ and $p, q \in \mathbb{R}$. However the completely bounded (c.b.) norms of the resulting net $(\omega_j)_j$ are no longer uniformly bounded.

So we have to again replace $(\omega_j)_j$ with c.c.p. maps. For this, we assume that, up to convex combinations, the convergence $\omega_j \to \mathrm{id}_{\widetilde{M}}$ is in the point strong topology.

Recall from (the first half of) the proof of [Anantharaman-Delaroche 1990, Lemma 2.2] that if we put $\varphi_i(x) := c_j \omega_j(x) c_j$ for $x \in \widetilde{M}$, where $c_j := 2(1+\omega_j(1))^{-1}$, then the net $(\varphi_i)_i$ satisfies that each φ_i is c.c.p. and that $\varphi_i \to \mathrm{id}_{\widetilde{M}}$ in the point strong topology. We will replace c_j with elements in $M \rtimes_{\mathrm{alg}} G$. For this, fix j and observe that, since $1+\omega_j(1)$ is in $M \rtimes_{\mathrm{alg}} G$, each c_j is actually contained in $\mathrm{C}^*\{M \rtimes_{\mathrm{alg}} G\}$, which is the norm closure of $M \rtimes_{\mathrm{alg}} G$. So there is a sequence $(a_n)_n$ in $M \rtimes_{\mathrm{alg}} G$ such that $\|a_n\|_\infty \le \|c_j^{1/2}\|_\infty$ and $\|a_n - c_j^{1/2}\|_\infty \to 0$. Put $b_n := a_n^* a_n \in M \rtimes_{\mathrm{alg}} G$ and observe that it satisfies $\|b_n\|_\infty \le \|c_j\|_\infty$ and $\|b_n - c_j\|_\infty \to 0$. It then holds that for any $x \in \widetilde{M}$,

$$\|c_j \omega_j(x) c_j - b_n \omega_j(x) b_n\|_\infty \le 2\|c_j\|_\infty \, \|\omega_j\|_{\mathrm{cb}} \, \|x\|_\infty \, \|c_j - b_n\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Now fix any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset (\widetilde{M})_1$ such that $1 \in \mathcal{F}$, and choose b_n such that

$$\|c_j \omega_j(x) c_j - b_n \omega_j(x) b_n\|_\infty < \varepsilon$$

for all $x \in \mathcal{F}$. Then since $1 \in \mathcal{F}$, we have

$$\|b_n \omega_j(\,\cdot\,) b_n\|_{\mathrm{cb}} = \|b_n \omega_j(1) b_n\|_\infty < \|c_j \omega_j(1) c_j\|_\infty + \varepsilon \le 1 + \varepsilon.$$

So $(1+\varepsilon)^{-1} b_n \omega_j(\,\cdot\,) b_n$ is a c.c.p. map which is still close to $c_j \omega_j(\,\cdot\,) c_j$ on $\mathcal{F}$. Thus we proved that for any j there is a net of c.c.p. maps converging to $c_j \omega_j(\,\cdot\,) c_j$ in the point *norm* topology such that each map is a finite sum of $\lambda_q^* E_M(z^* \cdot y) \lambda_p$ for some $y, z \in \mathfrak{n}_{E_M}$ and $p, q \in G$. Using this observation, since $c_j \omega_j(\,\cdot\,) c_j \to \mathrm{id}_{\widetilde{M}}$ as $j \to \infty$ in the point strong topology, it is easy to construct a desired net. □

3B. *Definition of weakly compact actions.* We introduce the following notion, which is an appropriate generalization of [Ozawa and Popa 2010, Definition 3.1] in our setting; see also [Popa and Vaes 2014a, Theorem 5.1]. Indeed, in the definition below, if we take $\mathcal{M} = M \mathbin{\overline{\otimes}} M^\circ$, this coincides with the original definition of weakly compact actions.

Definition 3.4. Let M be a semifinite von Neumann algebra with trace Tr, and let $\mathcal{M}$ be a von Neumann algebra which contains M and M° as von Neumann subalgebras, which we denote by $\pi(M)$ and $\theta(M^\circ)$, such that $[\pi(M), \theta(M^\circ)] = 0$.

Let $p \in M$ be a projection with $\mathrm{Tr}(p) = 1$, $A \subset pMp$ be a von Neumann subalgebra, and $\mathcal{G} \le \mathcal{N}_{pMp}(A)$ a subgroup. We say that the adjoint action of $\mathcal{G}$ on A is *weakly compact for* $(M, \mathrm{Tr}, \pi, \theta, \mathcal{M})$ if there is a net $(\xi_i)_i$ of unit vectors in the positive cone of $L^2(\mathcal{M})$ such that

(i) $\langle \pi(x)\xi_i, \xi_i\rangle_{L^2(\mathcal{M})} \to \mathrm{Tr}(pxp)$ for any $x \in M$;

(ii) $\|\pi(a)\theta(\bar{a})\xi_i - \xi_i\|_{L^2(\mathcal{M})} \to 0$ for any $a \in \mathcal{U}(A)$;

(iii) $\|\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\xi_i - \xi_i\|_{L^2(\mathcal{M})} \to 0$ for any $u \in \mathcal{G}$.

Here $\bar{a}$ means $(a^\circ)^*$ and $\mathcal{J}_{\mathcal{M}}$ is the modular conjugation for $L^2(\mathcal{M})$.

Remark 3.5. In this definition, since $\mathcal{J}_{\mathcal{M}}\xi_i = \xi_i$ for all i, condition (ii) for $a \in \mathcal{U}(A)$ implies condition (iii) for $a \in \mathcal{U}(A)$. Hence up to replacing $\mathcal{G}$ with the group generated by $\mathcal{U}(A)$ and $\mathcal{G}$, we may always assume that $\mathcal{G}$ contains $\mathcal{U}(A)$.

Below we record a characterization for weakly compact actions.

Proposition 3.6. *Keep the notation in Definition 3.4. The following conditions are equivalent*:

(1) *The group $\mathcal{G}$ acts on A as a weakly compact action for* $(M, \mathrm{Tr}, \pi, \theta, \mathcal{M})$.

(2) *There exists a net $(\omega_i)_i$ of normal states on $\mathcal{M}$ such that*

  (i) $\omega_i(\pi(x)) \to \mathrm{Tr}(pxp)$ *for any $x \in pMp$;*

  (ii) $\omega_i(\pi(a)\theta(\bar{a})) \to 1$ *for any $a \in \mathcal{U}(A)$;*

  (iii) $\|\omega_i \circ \mathrm{Ad}(\pi(u)\theta(\bar{u})) - \omega_i\| \to 0$ *for any $u \in \mathcal{G}$.*

(3) *There is a $\mathcal{G}$-central state ω on $\mathcal{M}$ such that for any $x \in M$ and $a \in \mathcal{U}(A)$*

$$\omega(x) = \mathrm{Tr}(pxp) \quad \textit{and} \quad \omega(\pi(a)\theta(\bar{a})) = 1.$$

(4) *There is a state Ω on $\mathbb{B}(L^2(\mathcal{M}))$ such that for any $x \in M$, $a \in \mathcal{U}(A)$ and $u \in \mathcal{G}$,*

$$\Omega(x) = \mathrm{Tr}(pxp), \quad \Omega(\pi(a)\theta(\bar{a})) = 1, \quad \textit{and} \quad \Omega((\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}) = 1.$$

Proof. This theorem follows from well-known arguments; see, e.g., the proof of [Ozawa and Popa 2010, Theorem 2.1]. So we give a sketch of proofs.

If (1) holds, then put $\Omega := \mathrm{Lim}_i \langle \cdot\, \xi_i, \xi_i\rangle_{L^2(\mathcal{M})}$ and obtain (4). If (4) holds, then the restriction of Ω on $\mathcal{M}$ gives (3). If (3) holds, then we can approximate ω by a net of normal states $(\omega_i)_i \subset \mathcal{M}_*$ weakly. Then by the Hahn–Banach separation theorem, up to convex combinations, we may assume that the convergence is in the norm and obtain (2). Finally if (2) holds, then for each i one can find a unique $\xi_i \in L^2(\mathcal{M})$ which is in the positive cone such that $\omega_i = \langle \cdot\, \xi_i, \xi_i\rangle_{L^2(\mathcal{M})}$. By the Powers–Størmer inequality [Takesaki 1979, Theorem IX.1.2(iv)], we obtain

$$\|\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\pi(u)\theta(\bar{u})\mathcal{J}_{\mathcal{M}}\xi_i - \xi_i\|^2 \le \|\omega_i \circ \mathrm{Ad}(\pi(u^*)\theta(u^\circ)) - \omega_i\| \to 0$$

for any $u \in \mathcal{G}$ and hence (1) holds. □

3C. *W*CMAP with respect to a state produces approximation maps on continuous cores.* We construct a family of approximation maps on continuous cores by assuming the W*CMAP with respect to a state.

For this, we fix the following setting. Let N and B be von Neumann algebras and φ_N and φ_B faithful normal states on N and B respectively. Put

$$M := N \mathbin{\overline{\otimes}} B, \quad \varphi := \varphi_N \otimes \varphi_B, \quad E_N := \mathrm{id}_N \otimes \varphi_B, \quad E_B := \varphi_N \otimes \mathrm{id}_B,$$

and we regard $\widetilde{B} := B \rtimes_{\sigma^{\varphi_B}} \mathbb{R}$ and $\widetilde{N} := N \rtimes_{\sigma^{\varphi_N}} \mathbb{R}$ as subalgebras of $\widetilde{M} := M \rtimes_{\sigma^{\varphi}} \mathbb{R}$. We denote by E_M the canonical operator-valued weight from $\widetilde{M}$ to M given by $\hat{\varphi} = \varphi \circ E_M$, where $\hat{\varphi}$ is the dual weight on $\widetilde{M}$. We also denote by E_B the canonical operator-valued weight from $\widetilde{M}$ to B given by $\hat{\varphi} = \varphi_B \circ E_B$.

Lemma 3.7. *Let $\omega : \widetilde{M} \to \widetilde{M}$ and $\psi : N \to N$ be c.b. maps given by*

$$\omega := \lambda_q^* E_M(z^* \cdot y)\lambda_p \quad \text{and} \quad \psi := \sum_{i=1}^{n} \varphi_N(z_i^* \cdot y_i)c_i$$

for some $p, q \in \mathbb{R}$, $y, z \in \mathfrak{n}_{E_M}$ and $c_i, y_i, z_i \in N$. Suppose $\psi \circ \sigma_t^{\varphi_N} = \sigma_t^{\varphi_N} \circ \psi$ for all $t \in \mathbb{R}$, so that the map $\tilde{\psi} := \psi \otimes \mathrm{id}_B \otimes \mathrm{id}_{L^2(\mathbb{R})}$ on $M \mathbin{\overline{\otimes}} \mathbb{B}(L^2(\mathbb{R}))$ induces the map $\widetilde{M} \to \widetilde{M}$ given by $\tilde{\psi}(x\lambda_t) = (\psi \otimes \mathrm{id}_B)(x)\lambda_t$ for $x \in M$ and $t \in \mathbb{R}$. Then the composition $\tilde{\psi} \circ \omega$ is given by

$$\tilde{\psi} \circ \omega(x) = \sum_{i=1}^{n} \lambda_q^* E_B\big(\sigma_q^{\varphi_N}(z_i^*)z^* x y \sigma_p^{\varphi_N}(y_i)\big)\lambda_p c_i, \quad x \in \widetilde{M}.$$

Proof. Recall from the proof of Lemma 2.1 that the canonical conditional expectation from $(\widetilde{M}, \hat{\varphi})$ to $(\widetilde{B}, \hat{\varphi}_B)$ is given by $E_{B \rtimes \mathbb{R}}((x \otimes b)\lambda_t) = \varphi_N(x)b\lambda_t$ for $x \in N$, $b \in B$ and $t \in \mathbb{R}$. For $x \in \widetilde{M}$, we calculate that

$$\begin{aligned}
\tilde{\psi} \circ \omega(x) &= \tilde{\psi}(\lambda_q^* E_M(z^* x y)\lambda_p) \\
&= \sum_{i=1}^{n} (\varphi_N(z_i^* \cdot y_i) \otimes \mathrm{id}_B \otimes \mathrm{id}_{L^2(\mathbb{R})})(\lambda_q^* E_M(z^* x y)\lambda_p)c_i \\
&= \sum_{i=1}^{n} E_{B \rtimes \mathbb{R}}(z_i^* \lambda_q^* E_M(z^* x y)\lambda_p y_i)c_i \\
&= \sum_{i=1}^{n} \lambda_q^* E_{B \rtimes \mathbb{R}} \circ E_M(\sigma_q^{\varphi_N}(z_i^*)z^* x y \sigma_p^{\varphi_N}(y_i))\lambda_p c_i.
\end{aligned}$$

Since $E_{B \rtimes \mathbb{R}} \circ E_M = E_B$ by Lemma 2.1, we obtain the conclusion. □

Lemma 3.8. *Suppose that N has the φ_N-W*CMAP. Then there exists a net $(\varphi_\lambda)_\lambda$ of c.c. maps on $\widetilde{M}$ such that $\varphi_\lambda \to \mathrm{id}_{\widetilde{M}}$ point σ-weakly and such that each φ_λ is a finite sum of $d^* E_B(z^* \cdot y)c$ for some $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$.*

Proof. Fix a net $(\psi_i)_i$ of normal c.c. maps on N as in Definition 2.8 and put $(\tilde{\psi}_i)_i$ as in the statement of the previous lemma. Let $(\omega_j)_j$ be a net of c.c.p. maps on $\widetilde{M}$ given by Lemma 3.3. Then by Lemma 3.7 the composition $\tilde{\psi}_i \circ \omega_j$ is a finite sum of $d^* E_B(z^* \cdot y)c$ for some $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$. Since $\lim_i(\lim_j \tilde{\psi}_i \circ \omega_j) = \mathrm{id}_{\widetilde{M}}$ in the point σ-weak topology, it is easy to show that for any finite subset $\mathcal{F} \subset \widetilde{M}$ and any σ-weak neighborhood $\mathcal{V}$ of 0, there are i and j such that $\tilde{\psi}_i \circ \omega_j(x) - x \in \mathcal{V}$ for all $x \in \mathcal{F}$. So putting this $\tilde{\psi}_i \circ \omega_j$ as $\varphi_{(\mathcal{F},\mathcal{V})}$, one can construct a desired net $(\varphi_\lambda)_\lambda := (\varphi_{(\mathcal{F},\mathcal{V})})_{(\mathcal{F},\mathcal{V})}$. □

3D. *Relative weakly compact actions on continuous cores.* We keep the notation from the previous subsection, such as $M = N \mathbin{\overline{\otimes}} B$ and $\varphi = \varphi_N \otimes \varphi_B$. Let Tr be an arbitrary semifinite trace on $\widetilde{M}$, $p \in \widetilde{M}$ a projection with $\mathrm{Tr}(p) = 1$, and $A \subset p\widetilde{M}p$ a von Neumann subalgebra with expectation E_A. In this subsection, we prove that under some assumptions on A and M, the normalizer of A in pMp acts on A as a weakly compact action with an appropriate representation.

Since our proof is a generalization of the one of [Popa and Vaes 2014a, Theorem 5.1], we make use of the following notation, which is similar to notation used in that theorem:

$$\begin{aligned}
H &:= L^2(\widetilde{M}, \hat{\varphi}) \otimes_B L^2(\widetilde{M}, \mathrm{Tr}), \text{ with left, right actions } \pi_H, \theta_H, \\
\mathcal{M}_H &:= \mathrm{W}^*\{\pi_H(\widetilde{M}), \theta_H(\widetilde{M}^\circ)\} \subset \mathbb{B}(H), \\
\mathcal{H} &:= (\theta_H(p)H) \otimes_A pL^2(\widetilde{M}, \mathrm{Tr}), \\
\pi_{\mathcal{H}} &: \widetilde{M} \ni x \mapsto (x \otimes_B p^\circ) \otimes_A p \in \mathbb{B}(\mathcal{H}), \\
\theta_{\mathcal{H}} &: \widetilde{M}^\circ \ni y^\circ \mapsto (1 \otimes_B p^\circ) \otimes_A y^\circ \in \mathbb{B}(\mathcal{H}), \\
\mathcal{M} &:= \mathrm{W}^*\{\pi_{\mathcal{H}}(\widetilde{M}), \theta_{\mathcal{H}}(\widetilde{M}^\circ)\} \subset \mathbb{B}(\mathcal{H}).
\end{aligned}$$

As we observed in Proposition 3.6, we actually use the weakly compact action with the standard representation of $\mathcal{M}$. So we first observe that $\mathcal{M}$ admits a useful identification as a crossed product, and so its standard representation is taken as a simple form.

Lemma 3.9. *Let $X \subset \mathcal{M}$ be the von Neumann subalgebra generated by $\pi_{\mathcal{H}}(B)$ and $\theta_{\mathcal{H}}(\widetilde{M}^\circ)$, and let $X \subset \mathbb{B}(L^2(X))$ be a standard representation, so that B and $\widetilde{M}^\circ$ acts on $L^2(X)$. Then $\mathcal{M}$ is isomorphic to the crossed product von Neumann algebra $\mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ by the diagonal action $\sigma^{\varphi_N} \otimes \alpha^X$, where α^X is given by $\alpha_t^X(\pi_{\mathcal{H}}(b)\theta_{\mathcal{H}}(y^\circ)) = \pi_{\mathcal{H}}(\sigma_t^{\varphi_B}(b))\theta_{\mathcal{H}}(y^\circ)$ for $t \in \mathbb{R}$, $b \in B$, and $y \in \widetilde{M}$.*

In particular the standard representation of $\mathcal{M}$ is given by $L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(X)$ with the following representation: for any $\xi \in L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(X) = L^2(\mathbb{R}, L^2(N) \otimes L^2(X))$ and $s \in \mathbb{R}$,

$$\begin{aligned}
&L\mathbb{R} \ni \lambda_t \mapsto \lambda_t \otimes 1_N \otimes 1_X, && ((\lambda_t \otimes 1_N \otimes 1_X)\xi)(s) := \xi(s-t), \\
&N \ni x \mapsto \pi_{\sigma^{\varphi_N}}(x) \otimes 1_X, && ((\pi_{\sigma^{\varphi_N}}(x) \otimes 1_X)\xi)(s) := (\sigma_{-s}^{\varphi_N}(x) \otimes 1_X)\xi(s), \\
&B \ni b \mapsto \pi_{\sigma^{\varphi_B}}(b)_{13}, && ((\pi_{\sigma^{\varphi_B}}(b)_{13})\xi)(s) := (1_N \otimes \sigma_{-s}^{\varphi_B}(b))\xi(s), \\
&\widetilde{M}^\circ \ni y^\circ \mapsto 1_{L^2(\mathbb{R})} \otimes 1_N \otimes y^\circ, && ((1_\mathbb{R} \otimes 1_N \otimes y^\circ)\xi)(s) := (1_N \otimes y^\circ)\xi(s).
\end{aligned}$$

Proof. By Proposition 2.3, H is isomorphic to $L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})$. Since the right $\widetilde{M}$-action acts only on the right three Hilbert spaces, the Hilbert space $\mathcal{H} = H \otimes_A pL^2(\widetilde{M}, \mathrm{Tr})$ is identified as $L^2(\mathbb{R}) \otimes L^2(N) \otimes K$, where

$$K := \theta_H(p^\circ)(L^2(B) \otimes L^2(N) \otimes L^2(\mathbb{R})) \otimes_A pL^2(\widetilde{M}, \mathrm{Tr}).$$

Note that $\widetilde{M}^\circ$ acts on K by $\theta_{\mathcal{H}}$, and B acts on $L^2(\mathbb{R}) \otimes K$ by $\pi_{\mathcal{H}}$, so that X acts on $L^2(\mathbb{R}) \otimes K$. More precisely we have $X \subset L^\infty(\mathbb{R}) \mathbin{\overline{\otimes}} \mathbb{C}1_N \mathbin{\overline{\otimes}} \mathbb{B}(K)$.

Let W be a unitary on $L^2(\mathbb{R}) \otimes L^2(N)$ given by $(W\xi)(t) := \Delta_{\varphi_N}^{it}\xi(t)$ for $t \in \mathbb{R}$ and $\xi \in L^2(\mathbb{R}) \otimes L^2(N) = L^2(\mathbb{R}, L^2(N))$. It satisfies that for any $f \in L^\infty(\mathbb{R})$, $t \in \mathbb{R}$, and $x \in N$,

$$W\pi_{\sigma^{\varphi_N}}(x)W^* = 1_{L^2(\mathbb{R})} \otimes x, \quad W(\lambda_t \otimes 1_N)W^* = \lambda_t \otimes \Delta_{\varphi_N}^{it}, \quad \text{and} \quad W(f \otimes 1_N)W^* = f \otimes 1_N.$$

Let next V be a unitary on $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ defined similarly to W exchanging $\Delta_{\varphi_N}^{it}$ with λ_t, so that it satisfies for $t \in \mathbb{R}$ and $f \in L^\infty(\mathbb{R})$,

$$V(1 \otimes \lambda_t)V^* = \lambda_t \otimes \lambda_t \quad \text{and} \quad V(1 \otimes f)V^* = 1 \otimes f.$$

Define then a unitary on $L^2(\mathbb{R}) \otimes \mathcal{H}$ by $U := (V \otimes 1_N \otimes 1_K)(1_{L^2(\mathbb{R})} \otimes W \otimes 1_K)$. One can show that $\operatorname{Ad} U = \mathrm{id}$ on $\mathbb{C}1_{L^2(\mathbb{R})} \otimes X \subset \mathbb{C}1_{L^2(\mathbb{R})} \mathbin{\overline{\otimes}} L^\infty(\mathbb{R}) \mathbin{\overline{\otimes}} \mathbb{C}1_N \mathbin{\overline{\otimes}} \mathbb{B}(K)$, and

$$\operatorname{Ad} U(1_{L^2(\mathbb{R})} \otimes \lambda_t \otimes 1_N \otimes 1_K) = (\lambda_t \otimes \lambda_t \otimes \Delta_{\varphi_N}^{it} \otimes 1_K) \qquad \text{for } t \in \mathbb{R},$$
$$\operatorname{Ad} U(1_{L^2(\mathbb{R})} \otimes \pi_{\sigma^{\varphi_N}}(x) \otimes 1_K) = (1_{L^2(\mathbb{R})} \otimes 1_{L^2(\mathbb{R})} \otimes x \otimes 1_K) \quad \text{for } x \in N.$$

Then $\operatorname{Ad} U(\mathcal{M})$ is identified as the crossed product von Neumann algebra $\mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ given by the $\mathbb{R}$-action $\sigma^{\varphi_N} \otimes \alpha^X$, where α^X is given by $\operatorname{Ad}(\lambda_t \otimes 1_N \otimes 1_K)$ using $X \subset L^\infty(\mathbb{R}) \otimes \mathbb{C}1_N \otimes \mathbb{B}(K)$, which is exactly the action given in the statement. Finally one can choose the standard representation of $\mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ as in the statement and we can end the proof. □

Now we prove the main observation of this section. This is a generalization of [Ozawa and Popa 2010, Theorem 3.5] and [Popa and Vaes 2014a, Theorem 5.1]. Since we already obtained approximation maps for $\widetilde{M}$ in Lemma 3.8, which are "relative to B", almost the same arguments as the above-cited theorems work. However, since our approximation maps are not defined directly on $\mathcal{M}_H$, we need a stronger assumption on the subalgebra A; namely, we need *amenability*, instead of relative amenability. See Step 1 in the proof below and observe that we really need amenability for a subalgebra $Q \subset pMp$.

Theorem 3.10. *Keep the setting above and suppose the following conditions*:

- *The algebra B is a type* III_1 *factor.*
- *The algebra A is amenable.*
- *The algebra N has the φ_N-W*CMAP.*

Then $\mathcal{N}_{p\widetilde{M}p}(A)$ acts on A as a weakly compact action for $(\widetilde{M}, \mathrm{Tr}, \pi_{\mathcal{H}}, \theta_{\mathcal{H}}, \mathcal{M})$.

Proof. The proof consists of several steps. For any von Neumann subalgebra $Q \subset p\widetilde{M}p$, we denote by $\mathcal{C}_{H,Q}$ (resp. $\mathcal{M}_{H,Q}$) the C*-algebra (resp. the von Neumann algebra) generated by $\pi_H(p\widetilde{M}p)\theta_H(Q^\circ)$.

Step 1. Using the φ_N-W*CMAP of N, we construct a net of normal functionals on $\mathcal{M}_H$ which are contractive on $\mathcal{M}_{H,Q}$ for any amenable Q.

In this step, we show that there is a net $(\mu_i)_i$ of normal functional on $\mathcal{M}_H$ such that

- $\mu_i(\pi_H(a)\theta_H(b^\circ)) = \mathrm{Tr}(p\varphi_i(a)pbp)$ for all $a, b \in \widetilde{M}$,
- we have $\|\mu_i|_{\mathcal{M}_{H,Q}}\| \le 1$ for any amenable von Neumann subalgebra $Q \subset p\widetilde{M}p$.

By Lemma 3.8, there exists a net $(\varphi_i)_i$ of c.c. maps on $\widetilde{M}$ such that $\varphi_i \to \mathrm{id}_{\widetilde{M}}$ point σ-weakly and that each φ_i is a finite sum of $d^*E_B(z^* \cdot y)c$ for $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$. Observe that for any functional $d^*E_B(z^* \cdot y)c$ for some $c, d \in \widetilde{M}$ and $y, z \in \mathfrak{n}_{E_B}$, one can define an associated normal functional on $\mathcal{M}_H$ by

$$\mathcal{M}_H \ni T \mapsto \langle T(\Lambda_{\hat{\varphi}}(y) \otimes_B \Lambda_{\mathrm{Tr}}(cp)), \Lambda_{\hat{\varphi}}(z) \otimes_B \Lambda_{\mathrm{Tr}}(dp) \rangle_H.$$

In this way, since φ_i is a finite sum of such maps, one can associate each φ_i with a normal functional on $\mathcal{M}_H$, which we denote by μ_i. Then by the formula $L^*_{\Lambda_{\hat{\varphi}}(z)} a L_{\Lambda_{\hat{\varphi}}(y)} = E_B(z^*ay)$ for $x, y \in \mathfrak{n}_{E_B} \cap \mathfrak{n}_\varphi$ and $a \in \widetilde{M}$, it is easy to verify that $\mu_i(\pi_H(a)\theta_H(b^\circ)) = \mathrm{Tr}(p\varphi_i(a)pbp)$ for $a, b \in \widetilde{M}$. We need to show that $\|\mu_i|_{\mathcal{M}_{H,Q}}\| \le 1$ for any amenable $Q \subset p\widetilde{M}p$. For this, since μ_i is normal, we have only to show that $\|\mu_i|_{\mathcal{C}_{H,Q}}\| \le 1$.

By Lemma 3.11 below, since B is a type III_1 factor, the $*$-algebra generated by $\pi_H(\widetilde{M})$ and $\theta_H(\widetilde{M}^\circ)$ is isomorphic to $\widetilde{M} \otimes_{\mathrm{alg}} \widetilde{M}^\circ$. So for any amenable $Q \subset p\widetilde{M}p$, the C*-algebra generated by $\pi_H(\widetilde{M})\theta_H(Q^\circ)$ is isomorphic to $\widetilde{M} \otimes_{\min} Q^\circ$. Hence one can define c.c. maps $\varphi_i \otimes \mathrm{id}_{Q^\circ}$ on $\mathcal{C}_{H,Q}$. Since Q is amenable, one has

$$ {}_{\widetilde{M}} L^2(\widetilde{M}p)_Q \prec {}_{\widetilde{M}}(\theta_H(p^\circ)H)_Q. $$

Finally if we denote by ν the associated $*$-homomorphism with this weak containment, then the functional $T \mapsto \langle \nu \circ (\varphi_i \otimes \mathrm{id}_{Q^\circ})(T)\Lambda_{\mathrm{Tr}}(p), \Lambda_{\mathrm{Tr}}(p)\rangle_{\mathrm{Tr}}$ coincides with μ_i on $\mathcal{C}_{H,Q}$, and hence we obtain $\|\mu_i|_{\mathcal{C}_{H,Q}}\| \le 1$. Thus we obtained a desired net $(\mu_i)_i$.

Step 2. Using the amenability of A, the absolute values of normal functionals $(\mu_i)_i$ constructed in Step 1 satisfy desired properties on $\mathcal{M}_{H,A}$.

Before this step, recall from the first part of the proof of [Ozawa and Popa 2010, Theorem 3.5] that for any C*-algebra C, any state ω on C and any partial isometry $u \in C$ with $p := uu^*$ and $q := u^*u$, one has

$$ \max\left\{ \|\omega(\,\cdot\, u^*) - \omega(\,\cdot\, q)\|^2, \|\omega(u \cdot u^*) - \omega(q \cdot q)\|^2 \right\} \le 4(\omega(p) + \omega(q) - \omega(u) - \omega(u^*)). $$

Let $(\mu_i)_i$ be a net constructed in Step 1. For notational simplicity, for any amenable von Neumann subalgebra $Q \subset p\widetilde{M}p$ we denote by μ_i^Q the restriction of μ_i on $\mathcal{M}_{H,Q}$.

Claim. *For any amenable Q, one has*

$$ \|\mu_i^Q\| \to 1 \quad \text{and} \quad \|\mu_i^Q - |\mu_i^Q|\| \to 0, $$

where $|\mu_i^Q|$ is the absolute value of μ_i^Q.

Proof of Claim. By Step 1, we know $\|\mu_i^Q\| \le 1$ and hence $\|\mu_i^Q\| \to 1$, since $\mu_i(\pi_H(p)\theta_H(p^\circ)) \to 1$. Let $\mu_i^Q = |\mu_i^Q|(\,\cdot\, u_i)$ be the polar decomposition with a partial isometry $u_i \in \mathcal{M}_{H,Q}$. For $p_i := u_i u_i^*$ and $q_i := u_i^* u_i$, it holds that

$$ |\mu_i^Q| = \mu_i^Q(\,\cdot\, u_i^*), \quad |\mu_i^Q| = |\mu_i^Q|(q_i \cdot q_i), \quad \text{and} \quad \mu_i^Q = \mu_i^Q(\,\cdot\, p_i) = \mu_i^Q(q_i \,\cdot\,). $$

The final equation says that $\mu_i^Q(p_i) = \mu_i^Q(1_Q) \to 1$. Then by the inequality at the beginning of this step, we have

$$ \begin{aligned} \|\mu_i^Q - |\mu_i^Q|\|^2 &= \||\mu_i^Q|(\,\cdot\, u_i^*) - |\mu_i^Q|(\,\cdot\, q_i)\|^2 \\ &\le 4\big(|\mu_i^Q|(p_i) + |\mu_i^Q|(q_i) - |\mu_i^Q|(u_i) - |\mu_i^Q|(u_i^*)\big) \\ &\le 4\big(\|\mu_i^Q\| + \|\mu_i^Q\| - 2\,\mathrm{Re}(\mu_i^Q(p_i))\big) \to 0. \end{aligned} $$

□

Put $\omega_i := |\mu_i^A| / \|\mu_i^A\|$. In this step, we show that $(\omega_i)_i$ satisfies the following conditions:

(1) $\omega_i(\pi_H(x)\theta_H(p^\circ)) \to \mathrm{Tr}(pxp)$ for all $x \in p\widetilde{M}p$.

(2) $\omega_i(\pi_H(a)\theta_H(\bar{a})) \to 1$ for all $a \in \mathcal{U}(A)$.

(3) $\|\omega_i \circ \mathrm{Ad}(\pi_H(u)\theta_H(\bar{u})) - \omega_i\|_{\mathcal{M}_{H,A}^*} \to 0$ for all $u \in \mathcal{N}_{p\widetilde{M}p}(A)$.

Since $\|\mu_i^A\| \to 1$ and $\|\mu_i^A - |\mu_i^A|\| \to 0$, to verify these three conditions, we have only to show that $(\mu_i)_i$ satisfies the same conditions. Then by construction, it is easy to verify (i) and (ii). So we will check only the final condition.

Fix $u \in \mathcal{N}_{p\widetilde{M}p}(A)$ and recall that the von Neumann algebra A^u generated by A and u is amenable [Ozawa and Popa 2010, Lemma 3.4]. Hence by Step 1, $\||\mu_i^{A^u}| - \mu_i^{A^u}\|_{\mathcal{M}^*_{H,A^u}} \to 0$. Combined with the inequality at the beginning of this step, putting $U := \pi_H(u)\theta_H(\bar{u})$, we have

$$\begin{aligned}\lim_i \|\mu_i^A \circ \operatorname{Ad} U - \mu_i^A\|^2_{\mathcal{M}^*_{H,A}} &\le \lim_i \|\mu_i^{A^u} \circ \operatorname{Ad} U - \mu_i^{A^u}\|^2_{\mathcal{M}^*_{H,A^u}} \\ &= \lim_i \||\mu_i^{A^u}| \circ \operatorname{Ad} U - |\mu_i^{A^u}|\|^2_{\mathcal{M}^*_{H,A^u}} \\ &\le \lim_i 4(2 - 2\operatorname{Re}(|\mu_i^{A^u}|(U))) \\ &= \lim_i 4(2 - 2\operatorname{Re}(\mu_i^{A^u}(U))) = 0.\end{aligned}$$

Thus we proved that the net $(\omega_i)_i$ of normal states on $\mathcal{M}_H$ satisfies conditions (i), (ii) and (iii) above.

Step 3. Using a normal u.c.p. map from $\mathcal{M}$ to $\mathcal{M}_{H,A}$, we obtain desired functionals on $\mathcal{M}$.

In this step, we first construct a normal u.c.p. map $\mathcal{E} : \mathcal{M} \to \mathcal{M}_{H,A}$ satisfying

$$\mathcal{E}(\pi_{\mathcal{H}}(a)\theta_{\mathcal{H}}(b^\circ)) = \pi_H(pap)\theta_H(E_A(pbp)^\circ) \quad \text{for any } a, b \in \widetilde{M},$$

where E_A is the unique Tr-preserving conditional expectation from $p\widetilde{M}p$ onto A.

For this, observe first that for any right A-module K with the right action θ_K, there is an isometry $V_K : K \to K \otimes_A pL^2(\widetilde{M}, \mathrm{Tr})$ given by $V\xi = \xi \otimes_A \Lambda_{\mathrm{Tr}}(p)$ for any left Tr-bounded vector $\xi \in K$. Indeed, using the fact $\Lambda_{\mathrm{Tr}}(p) = J_{\mathrm{Tr}}\Lambda_{\mathrm{Tr}}(p)$, one has

$$\|V\xi\| = \|\xi \otimes_A \Lambda_{\mathrm{Tr}}(p)\| = \|L_\xi \Lambda_{\mathrm{Tr}}(p)\|_{2,\mathrm{Tr}} = \|L_\xi \Lambda_{\mathrm{Tr}}(p)\|_{2,\mathrm{Tr}} = \|\theta_K(p^\circ)\xi\|_K = \|\xi\|_K.$$

Hence, since $\pi_H(p)\theta_H(p^\circ)H$ is a right A-module, one can define an isometry

$$V : \pi_H(p)\theta_H(p^\circ)H \to \pi_{\mathcal{H}}(p)\theta_{\mathcal{H}}(p^\circ)\mathcal{H} \subset \mathcal{H}, \quad V\xi := \xi \otimes_A \Lambda_{\mathrm{Tr}}(p).$$

It is then easy to verify that

$$V^*\pi_{\mathcal{H}}(a)\theta_{\mathcal{H}}(b^\circ)V = \pi_H(pap)\theta_H(E_A(pbp)^\circ) \quad \text{for any } a, b \in \widetilde{M}.$$

Thus we obtain a normal u.c.p. map $\mathcal{E} : \mathcal{M} \to \mathcal{M}_{H,A}$ by $\mathcal{E}(T) := V^*TV$.

Let now $(\omega_i)_i$ be the net of normal states on $\mathcal{M}_{H,A}$ constructed in Step 2. By conditions (i) and (ii) on $(\omega_i)_i$, it is easy to see that normal states $\gamma_i := \omega_i \circ \mathcal{E}$ on $\mathcal{M}$ satisfy

(i)′ $\gamma_i(\pi_{\mathcal{H}}(x)) \to \tau(pxp)$ for all $x \in \widetilde{M}$;

(ii)′ $\gamma_i(\pi_{\mathcal{H}}(a)\theta_{\mathcal{H}}(\bar{a})) \to 1$ for all $a \in \mathcal{U}(A)$.

Finally since E_A satisfies $E_A \circ \operatorname{Ad} u = \operatorname{Ad} u \circ E_A$ for any $u \in \mathcal{N}_{p\widetilde{M}p}(A)$, one has

$$\gamma_i \circ \operatorname{Ad}(\pi_{\mathcal{H}}(u)\theta_{\mathcal{H}}(\bar{u})) = \omega_i \circ \operatorname{Ad}(\pi_{\mathcal{H}}(u)\theta_{\mathcal{H}}(\bar{u})) \circ \mathcal{E}$$

on $\pi_{\mathcal{H}}(\widetilde{M})\theta_{\mathcal{H}}(\widetilde{M})$, and hence on $\mathcal{M}$ by normality. So condition (iii) on $(\omega_i)_i$ shows

(iii)′ $\|\gamma_i \circ \mathrm{Ad}(\pi_{\mathcal{H}}(u)\theta_{\mathcal{H}}(\bar{u})) - \gamma_i\| \to 0$ for all $u \in \mathcal{N}_{p\widetilde{M}p}(A)$.

Thus the net $(\gamma_i)_i$ on $\mathcal{M}$ satisfies conditions (i)′, (ii)′ and (iii)′. By Proposition 3.6(2), we conclude that $\mathcal{N}_{p\widetilde{M}p}(A)$ acts on A weakly compactly for $(\widetilde{M}, \mathrm{Tr}, \pi_{\mathcal{H}}, \theta_{\mathcal{H}}, \mathcal{M})$. □

We prove a lemma used in the proof above.

Lemma 3.11. *Assume that B is a type* III_1 *factor. Then the $*$-algebra generated by $\pi_H(\widetilde{M})$ and $\theta_H(\widetilde{M}^\circ)$ is isomorphic to $\widetilde{M} \otimes_{\mathrm{alg}} \widetilde{M}^\circ$.*

Proof. Let $\nu : \widetilde{M} \otimes_{\mathrm{alg}} \widetilde{M}^\circ \to *\text{-alg}\{\pi_H(\widetilde{M}), \theta_H(\widetilde{M}^\circ)\}$ be a $*$-homomorphism given by $\nu(x \otimes y^\circ) = \pi_H(x)\theta_H(y^\circ)$ for $x, y \in \widetilde{M}$. We will show that ν is injective.

Assume that $\nu\big(\sum_{i=1}^n x_i \otimes y_i^\circ\big) = \sum_{i=1}^n \pi_H(x_i)\theta_H(y_i^\circ) = 0$ for some $x_i, y_i \in \widetilde{M}$. We may assume $y_i \neq 0$ for all i. Put

$$X := \begin{bmatrix} \pi_H(x_1) & \pi_H(x_2) & \cdots & \pi_H(x_n) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad Y := \begin{bmatrix} \theta_H(y_1^\circ) & 0 & \cdots & 0 \\ \theta_H(y_2^\circ) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \theta_H(y_n^\circ) & 0 & \cdots & 0 \end{bmatrix}$$

and observe $XY = 0$. We regard them as elements in $\mathbb{B}(H) \otimes \mathbb{M}_n$. Let p be the left support projection of Y which is contained in $\theta_H(\widetilde{M}^\circ) \otimes \mathbb{M}_n$ and satisfies $Xp = 0$. Since $Xupu^* = 0$ for any unitary $u \in \mathbb{B}(H) \otimes \mathbb{M}_n$ which commutes with X, and since $\theta_H(\widetilde{M}^\circ) \otimes \mathbb{C}^n$ commutes with X (where $\mathbb{C}^n \subset \mathbb{M}_n$ is the diagonal embedding), we have $Xz = 0$ for $z := \sup\{upu^* \mid u \in \mathcal{U}(\theta_H(\widetilde{M}^\circ) \otimes \mathbb{C}^n)\}$. Observe that z is contained in

$$(\theta_H(\widetilde{M}^\circ) \otimes \mathbb{M}_n) \cap (\theta_H(\widetilde{M}^\circ) \otimes \mathbb{C}^n)' = \theta_H(\mathcal{Z}(\widetilde{M})^\circ) \otimes \mathbb{C}^n$$

and hence we can write $z = (z_i)_{i=1}^n$ for some $z_i \in \theta_H(\mathcal{Z}(\widetilde{M})^\circ)$. Then the condition $Xz = 0$ is equivalent to $\pi_H(x_i)z_i = 0$ for all i. Observe also that $z_i \neq 0$ for all i. Indeed, since $z \geq p$ and $pY = Y$, we have $zY = Y$ and hence $z_i\theta_H(y_i^\circ) = \theta_H(y_i^\circ)$. This implies $z_i \neq 0$ since we assume $y_i \neq 0$ for all i.

Now we claim that $\pi_H(x_i)z_i = 0$ is equivalent to $x_i = 0$ or $z_i = 0$. Once we prove the claim, since $z_i \neq 0$, we have $x_i = 0$ and so $\sum_{i=1}^n x_i \otimes y_i^\circ = 0$, which gives the injectivity of ν.

By Lemma 2.2, the center of $\widetilde{M}$ coincides with $\mathcal{Z}(N)$. Then by Proposition 2.3, we identify $H = L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B, \psi_B) \otimes L^2(N) \otimes L^2(\mathbb{R})$ on which we have

$$\pi_H(\widetilde{M}) \subset \mathbb{B}(L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(B, \psi_B)) \otimes \mathbb{C}1_{L^2(N)\otimes L^2(\mathbb{R})},$$
$$\theta_H(\widetilde{M}^\circ) \subset \mathbb{C}1_{L^2(\mathbb{R})\otimes L^2(N)} \otimes \mathbb{B}(L^2(B, \psi_B) \otimes L^2(N) \otimes L^2(\mathbb{R})).$$

In particular $\theta_H(\mathcal{Z}(\widetilde{M})^\circ) = \theta_H(\mathcal{Z}(N)) \subset \mathbb{C}1_{L^2(\mathbb{R})\otimes L^2(N)\otimes L^2(B,\psi_B)} \otimes \mathbb{B}(L^2(N) \otimes L^2(\mathbb{R}))$, and hence the C*-algebra generated by $\pi_H(\widetilde{M})$ and $\theta_H(\mathcal{Z}(\widetilde{M})^\circ)$ is isomorphic to $\widetilde{M} \otimes_{\min} \mathcal{Z}(\widetilde{M})^\circ$. Thus since $z_i \in \theta_H(\mathcal{Z}(\widetilde{M})^\circ)$, the condition $\pi_H(x_i)z_i = 0$ is equivalent to $x_i = 0$ or $z_i = 0$. □

4. Proof of Theorem A

To prove Theorem A we follow the proof of [Isono 2015b, Theorem B], which originally comes from the one of [Popa and Vaes 2014b, Theorem 1.4].

4A. ***Some general lemmas.*** Let $\mathbb{G}$ be a compact quantum group with the Haar state h and put $N_0 := C_{\mathrm{red}}(\mathbb{G}) \subset L^\infty(G) =: N$ and $\varphi_N := h$. Let (X, φ_X) be a von Neumann algebra with a faithful normal semifinite weight. Let α^X be an action of $\mathbb{R}$ on X and put $\alpha := \sigma^{\varphi_N} \otimes \alpha^X$ and $\mathcal{M} := (N \mathbin{\bar{\otimes}} X) \rtimes_\alpha \mathbb{R}$.

In this setting, we prove two general lemmas. We use the following general fact for quantum groups.

- For any $x \in \mathrm{Irred}(\mathbb{G})$, there is an orthonormal basis $\{u^x_{i,j}\}_{i,j} \subset C_{\mathrm{red}}(\mathbb{G})$ of H_x with $\lambda^x_{i,j} > 0$ such that $\sigma^h_t(u^x_{i,j}) = \lambda^x_{i,j} u^x_{i,j}$ for all $t \in \mathbb{R}$.

Recall that all the linear spans of such a basis, which is usually called a dense Hopf $*$-algebra, make a norm-dense $*$-subalgebra of $C_{\mathrm{red}}(\mathbb{G})$. We note that each matrix $(u^x_{i,j})_{i,j}$ may not be a unitary, since we assume $\{u^x_{i,j}\}_{i,j}$ is orthonormal (i.e., they are normalized).

Convention. Throughout this section, we fix such a basis $\{u^x_{i,j}\}^x_{i,j}$. For notation simplicity, we identify any subset $\mathcal{E} \subset \mathrm{Irred}(\mathbb{G})$ (possibly $\mathcal{E} = \mathrm{Irred}(\mathbb{G})$) with the set $\{u^x_{i,j} \mid x \in \mathcal{E}, i, j\}$.

Note that this identification will not cause any confusion, since in proofs of this section we only use the property that $\mathcal{E} \subset \mathrm{Irred}(\mathbb{G})$ is a finite set.

Here we record an elementary lemma.

Lemma 4.1. *For any $a \in N_0$, the element $\pi_{\sigma^{\varphi_N}}(a) \in N \rtimes_{\sigma^{\varphi_N}} \mathbb{R} \subset \mathbb{B}(L^2(N) \otimes L^2(\mathbb{R}))$ is contained in $N_0 \otimes_{\min} C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the set of all norm continuous bounded functions on $\mathbb{R}$.*

Proof. We may assume that a is an eigenvector; namely, $\sigma^{\varphi_N}_t(a) = \lambda^{it} a$ for some $\lambda > 0$. Then since $(\pi_{\sigma^{\varphi_N}}(a)\xi)(t) = \sigma^{\varphi_N}_{-t}(a)\xi(t) = \lambda^{-it} a\xi(t)$ for $t \in \mathbb{R}$, one has $\pi_{\sigma^{\varphi_N}}(a) = a \otimes f$, where $f \in C_b(\mathbb{R})$ is given by $f(t) := \lambda^{-it}$. Hence we get the conclusion. □

We fix a faithful normal semifinite weight φ_X on X and put $\psi := \varphi_N \otimes \varphi_X$ with its dual weight $\hat{\psi}$. Recall that the compression map $P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})}$, where P_N is the one-dimensional projection from $L^2(N)$ onto $\mathbb{C}\Lambda_{\varphi_N}(1_N)$, is a conditional expectation $E_{X\rtimes\mathbb{R}} : \mathcal{M} \to X \rtimes \mathbb{R}$, which satisfies $\hat{\psi} = \hat{\varphi}_X \circ E_{X\rtimes\mathbb{R}}$ (this was shown in the first half of the proof of Lemma 2.1). For any $a \in \mathcal{M}$ and $f \in C_c(\mathbb{R}, \mathcal{M})\mathfrak{n}_\psi$, we denote by af an element in $C_c(\mathbb{R}, \mathcal{M})\mathfrak{n}_\psi$ given by $t \mapsto \alpha_{-t}(a) f(t)$. Observe that $\Lambda_{\hat{\psi}}(\hat{\pi}_\alpha(af)) = \pi_\alpha(a)\Lambda_{\hat{\psi}}(\hat{\pi}_\alpha(f))$. A simple computation shows that for any $a, b \in N$ and $f, g \in C_c(\mathbb{R}, X)\mathfrak{n}_{\varphi_X}$,

$$\langle af, bg\rangle_{\hat{\psi}} = \langle a, b\rangle_{\varphi_N} \langle f, g\rangle_{\hat{\varphi}_X}.$$

Observe that all the linear spans of uf for $u \in \mathrm{Irred}(\mathbb{G})$ and $f \in C_c(\mathbb{R}, X)\mathfrak{n}_{\varphi_X}$ are dense in $L^2(N) \otimes L^2(X) \otimes L^2(\mathbb{R})$. So if $\{f_\lambda\}_\lambda \subset C_c(\mathbb{R}, X)\mathfrak{n}_{\varphi_X}$ is an orthonormal basis in $L^2(X) \otimes L^2(\mathbb{R})$, then the set $\{uf_\lambda\}_{u,\lambda}$ is an orthonormal basis of $L^2(N) \otimes L^2(X) \otimes L^2(\mathbb{R})$. Along this basis, any $a \in \mathfrak{n}_{\hat{\psi}}$ can be decomposed in $L^2(N) \otimes L^2(X) \otimes L^2(\mathbb{R})$ as, for some $\alpha_{u,\lambda} \in \mathbb{C}$,

$$\Lambda_{\hat{\psi}}(a) = \sum_{u,\lambda} \alpha_{u,\lambda} uf_\lambda = \sum_{u,\lambda} \alpha_{u,\lambda} \pi_{\varphi_N}(u) \Lambda_{\hat{\psi}}(\hat{\pi}_\alpha(f_\lambda)) = \sum_u \pi_{\sigma^{\varphi_N}}(u) a_u,$$

where $a_u = \sum_\lambda \alpha_{u,\lambda} f_\lambda \in L^2(\mathbb{R}, X)$. If we apply $(P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\pi_{\sigma^{\varphi_N}}(v^*)$ for some $v \in \mathrm{Irred}(\mathbb{G})$ to this decomposition, then on the one hand

$$(P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\pi_{\sigma^{\varphi_N}}(v^*)\,\Lambda_{\hat{\psi}}(a) = (P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\,\Lambda_{\hat{\psi}}(v^* a) = \Lambda_{\hat{\psi}}(E_{X\rtimes\mathbb{R}}(v^* a))$$

and on the other hand

$$(P_N \otimes 1_X \otimes 1_{L^2(\mathbb{R})})\pi_{\sigma^{\varphi_N}}(v^*)\sum_u \pi_{\sigma^{\varphi_N}}(u)a_u = \sum_u \varphi_N(v^* u)a_u = \varphi_N(v^* v)a_v = a_v.$$

Hence we have $a_v = \Lambda_{\hat{\psi}}(E_{X\rtimes\mathbb{R}}(v^* a))$ for all $v \in \mathrm{Irred}(\mathbb{G})$. Thus we observe that any element $a \in \mathfrak{n}_{\hat{\psi}}$ has the *Fourier expansion* in the sense that

$$\Lambda_{\hat{\psi}}(a) = \sum_u \pi_{\sigma^{\varphi_N}}(u)a_u = \sum_u \Lambda_{\hat{\psi}}(uE_{X\rtimes\mathbb{R}}(u^* a)), \quad \text{where } a_u = \Lambda_{\hat{\psi}}(E_{X\rtimes\mathbb{R}}(u^* a)).$$

Using this property, we can prove the following lemma. We omit the proof, since it is straightforward.

Lemma 4.2. *Let $\mathcal{M}_0 \subset \mathcal{M}$ be the C^*-subalgebra generated by N_0 and $X \rtimes \mathbb{R}$. Then one has*

$$\begin{aligned}\mathcal{M}_0 &= \overline{\mathrm{span}}^{\mathrm{norm}}\{ax \mid a \in N_0,\ x \in X \rtimes \mathbb{R}\}\\ &= \overline{\mathrm{span}}^{\mathrm{norm}}\{xa \mid a \in N_0,\ x \in X \rtimes \mathbb{R}\}.\end{aligned}$$

4B. *Proof of Theorem A*. Let $\mathbb{G}$ be a compact quantum group with the Haar state h and put $N_0 := C_{\mathrm{red}}(\mathbb{G}) \subset L^\infty(\mathbb{G}) =: N$ and $\varphi_N := h$. Let (B, φ_B) be a von Neumann algebra with a faithful normal state. We keep the notation from Sections 3C and 3D, such as $M, \varphi, \widetilde{B}, \widetilde{M}, \mathrm{Tr}, p, A, \mathcal{H}, \pi_{\mathcal{H}}, \theta_{\mathcal{H}}, \mathcal{M}$, except for the Hilbert space H (which is used just below in a different manner). Assume that $\mathrm{Tr}|_{\widetilde{B}}$ is semifinite. Recall that by Lemma 3.9, $\mathcal{M} = \mathbb{R} \ltimes (N \mathbin{\overline{\otimes}} X)$ with the standard representation $L^2(\mathcal{M}) = L^2(\mathbb{R}) \otimes L^2(N) \otimes L^2(X)$. Set $\pi := \pi_{\mathcal{H}}$ and $\theta := \theta_{\mathcal{H}}$ for simplicity, and we sometimes omit π and θ by regarding $\widetilde{M}, \widetilde{M}^\circ$ as subsets of $\mathcal{M}$. Using Proposition 2.3, we put

$$\begin{aligned}H &:= L^2(\mathcal{M}) \otimes_X L^2(\mathcal{M}) = L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N) \otimes L^2(X) \otimes L^2_r(N) \otimes L^2_r(\mathbb{R}),\\ K &:= L^2(\mathcal{M}) \otimes_{(N \overline{\otimes} X)} L^2(\mathcal{M}) = L^2_\ell(\mathbb{R}) \otimes L^2(N) \otimes L^2(X) \otimes L^2_r(\mathbb{R}),\end{aligned}$$

and we denote by π_H, ρ_H, π_K and ρ_K corresponding left and right actions of $\mathcal{M}$. Here we are using symbols ℓ and r for $L^2(\mathbb{R})$ and $L^2(N)$, so that π_H and π_K act on $L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N) \otimes L^2(X)$ and $L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N) \otimes L^2(X)$ respectively, and θ_H and θ_K act on $L^2(X) \otimes L^2_r(N) \otimes L^2_r(\mathbb{R})$ and $L^2(N) \otimes L^2(X) \otimes L^2_r(\mathbb{R})$ respectively. We denote by $\nu_{K,H}$ the corresponding $*$-homomorphism as $\mathcal{M}$-bimodules, which is *not* bounded in general.

In this setting, we prove two lemmas. The first one uses biexactness of quantum groups, which corresponds to [Isono 2015a, Lemma 4.1.3], while the second one uses Popa's intertwining techniques, which corresponds to [Isono 2015a, Lemma 4.1.2; 2015b, Lemma 4.4]. See also [Popa and Vaes 2014b, Sections 3.2 and 3.5] for the origins of them.

Lemma 4.3. *Assume that $\widehat{\mathbb{G}}$ is biexact with a u.c.p. map Θ as in the definition of biexactness. Let $\mathcal{M}_0$ be the C^*-algebra generated by N_0 and $\mathbb{R} \ltimes X$. Then Θ can be extended to a u.c.p. map*

$$\widetilde{\Theta} : \mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0)\} \to \mathbb{B}(K)$$

which satisfies, using the flip $\Sigma_{12} : K \simeq L^2(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})$,

$$\Sigma_{12}\big(\widetilde{\Theta}(\pi_H(xa)\theta_H(b^\circ y^\circ)) - \pi_K(xa)\theta_K(b^\circ y^\circ)\big)\Sigma_{12} \in \mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))$$

for any $a, b \in N_0$ *and* $x, y \in \mathbb{R} \ltimes X$.

Proof. By applying flip maps, we identify

$$H = L^2_\ell(N) \otimes L^2_r(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}),$$
$$K = L^2(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}).$$

We define a u.c.p. map $\widetilde{\Theta}$ by

$$\widetilde{\Theta} := \Theta \otimes \mathrm{id}_{L^2_\ell(\mathbb{R})} \otimes \mathrm{id}_{L^2(X)} \otimes \mathrm{id}_{L^2_r(\mathbb{R})} : N_0 \otimes_{\min} N_0^\circ \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})) \to \mathbb{B}(K).$$

Observe that by Lemma 4.1, $\pi_H(\mathcal{M}_0)$ and $\rho_H(\mathcal{M}_0)$ are contained in

$$N_0 \otimes_{\min} N_0^\circ \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})).$$

Recall that for $a, b \in N$, $\pi_H(a)$ and $\theta_H(b^\circ)$ are given by $\pi_{\sigma^{\varphi_N}}(a)$ on $L^2_\ell(\mathbb{R}) \otimes L^2_\ell(N)$ and $\theta_{\sigma^{\varphi_N}}(b^\circ)$ on $L^2_r(N) \otimes L^2_r(\mathbb{R})$. So if a and b are eigenvectors, they are of the form $\pi_H(a) = f \otimes a$ and $\theta_H(b^\circ) = b^\circ \otimes g$ for some $f, g \in C_b(\mathbb{R})$ by Lemma 4.1. It then holds that for any $x, y \in \mathbb{R} \ltimes X$,

$$\begin{aligned}
&\widetilde{\Theta}(\pi_H(xa)\theta_H(b^\circ y^\circ)) - \pi_K(xa)\theta_K(b^\circ y^\circ)\\
&\quad = \widetilde{\Theta}(\pi_H(x)\pi_H(a)\theta_H(b^\circ)\theta_H(y^\circ)) - \pi_K(x)\pi_K(a)\theta_K(b^\circ)\theta_K(y^\circ)\\
&\quad = \widetilde{\Theta}\big(\pi_H(x)(a \otimes b^\circ \otimes f \otimes 1_{L^2(X)} \otimes g)\theta_H(y^\circ)\big) - \pi_K(x)(ab^\circ \otimes f \otimes 1_{L^2(X)} \otimes g)\theta_K(y^\circ)\\
&\quad = \pi_K(x)\big((\Theta(a \otimes b^\circ) - ab^\circ) \otimes f \otimes 1_{L^2(X)} \otimes g\big)\theta_K(y^\circ).
\end{aligned}$$

Since $\Theta(a \otimes b^\circ) - ab^\circ \in \mathbb{K}(L^2(N))$ and $\pi_K(x), \theta_K(y^\circ) \in \mathbb{C}1_N \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))$, the last term above is contained in $\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})))$. Then by Lemma 4.2, we obtain the conclusion. □

Lemma 4.4. *Let* Ω *be a state on* $\mathbb{B}(K)$ *satisfying for any* $x \in \widetilde{M}$ *and* $a \in \mathcal{U}(A)$,

$$\Omega(\pi_K(\pi(x))) = \mathrm{Tr}(pxp) \quad \textit{and} \quad \Omega(\pi_K(\pi(a)\theta(\bar{a}))) = 1.$$

If $A \not\preceq_{\widetilde{M}} \widetilde{B}$, *then using the flip* $\Sigma_{12} : K \simeq L^2(N) \otimes L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})$, *it holds that*

$$\Omega \circ \mathrm{Ad}(\Sigma_{12})\big(\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))\big) = 0.$$

Proof. Since Ω is a state, by the Cauchy–Schwarz inequality, we have only to show that

$$\Omega \circ \mathrm{Ad}(\Sigma_{12})(\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{C}1_{L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R})}) = 0.$$

In this setting we can follow the proof of [Isono 2015b, Lemma 4.4]. Indeed suppose by contradiction that there exist $\delta > 0$ and a finite subset $\mathcal{F} \subset \mathrm{Irred}(\mathbb{G})$ such that

$$\Omega(1_{L^2_\ell(\mathbb{R})} \otimes P_{\mathcal{F}} \otimes 1_{L^2(X) \otimes L^2_r(\mathbb{R})}) > \delta,$$

where $P_{\mathcal{F}}$ is the orthogonal projection onto $\sum_{x\in\mathcal{F}} H_x \otimes H_{\bar{x}}$. Then the argument in [loc. cit., Lemma 4.4] works by replacing $\|\cdot\|$ with Ω. Hence we omit the proof. □

Now we are in position to prove the main theorem. We actually prove the following more general theorem. Theorem A then follows immediately with Theorem 3.10.

Theorem 4.5. *Let $A \subset p\widetilde{M}p$ be a von Neumann subalgebra and $\mathcal{G} \leq \mathcal{N}_{p\widetilde{M}p}(A)$ a subgroup. Assume the following three conditions*:

(A) *The group $\mathcal{G}$ acts on A by conjugation as a weakly compact action for $(\widetilde{M}, \pi, \theta, \mathcal{M})$.*

(B) *The quantum group $\widehat{\mathbb{G}}$ is biexact and centrally weakly amenable.*

(C) *We have $A \not\preceq_{\widetilde{M}} \widetilde{B}$.*

Then there is a $(\mathcal{U}(A)\cup\mathcal{G})$-central state on $p\langle\widetilde{M}, \widetilde{B}\rangle p$ which coincides with Tr *on $p\widetilde{M}p$. In particular the von Neumann algebra generated by A and $\mathcal{G}$ is amenable relative to $\widetilde{B}$.*

Proof. By Remark 3.5, we may assume $\mathcal{U}(A) \subset \mathcal{G}$. Recall from Lemma 3.2 that as $\mathcal{M}$-bimodules,

$$L^2(\mathcal{M}) \prec L^2(\mathcal{M}) \otimes_{(N\otimes X)} L^2(\mathcal{M}) = K,$$

and we denote by ν the associated $*$-homomorphism. Let $(\xi_i)_i \subset L^2(\mathcal{M})$ be a net for the given weakly compact action of $\mathcal{G}$ and put a state $\Omega(X) := \mathrm{Lim}_i \langle \nu(X)\xi_i, \xi_i\rangle_{L^2(\mathcal{M})}$ on $\mathrm{C}^*\{\pi_K(\mathcal{M}), \theta_K(\mathcal{M}^\circ)\}$. Observe that it satisfies

(i)′ $\Omega(\pi_K(\pi(x))) = \mathrm{Tr}(pxp)$ for any $x \in \widetilde{M}$;

(ii)′ $\Omega(\pi_K(\pi(a)\theta(\bar{a}))) = 1$ for any $a \in \mathcal{U}(A)$;

(iii)′ $\Omega(\pi_K(\pi(u)\theta(\bar{u}))\theta_K(\pi(u^*)^\circ\theta(u^\circ)^\circ)) = 1$ for any $u \in \mathcal{G}$.

Note that since $\mathcal{J}_{\mathcal{M}}\xi_i = \xi_i$, we also have $\Omega(\theta_K(\pi(x)^\circ)) = \mathrm{Tr}(pxp)$ for any $x \in \widetilde{M}$. Denote by $\nu_{K,H}$ the (not necessarily bounded) $*$-homomorphism for $\mathcal{M}$-bimodules H and K. Here we claim that, using the biexactness of $\widehat{\mathbb{G}}$, the functional $\widetilde{\Omega} := \Omega \circ \nu_{K,H}$ satisfies the following boundedness condition.

Claim. *The functional $\widetilde{\Omega}$ is bounded on $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$.*

Proof of Claim. We first extend Ω on $\mathbb{B}(K)$ by the Hahn–Banach theorem. Then by Lemma 4.4, using assumption (C) and conditions (i)′ and (ii)′, one has

$$\Omega \circ \mathrm{Ad}(\Sigma_{12})\big(\mathbb{K}(L^2(N)) \otimes_{\min} \mathbb{B}(L^2_\ell(\mathbb{R}) \otimes L^2(X) \otimes L^2_r(\mathbb{R}))\big) = 0.$$

Let Θ be a u.c.p. map for biexactness of $\widehat{\mathbb{G}}$ and denote by $\widetilde{\Theta}$ the extension given in Lemma 4.3. Define a state on $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$ by $\widehat{\Omega} := \Omega \circ \widetilde{\Theta}$. Then conclusions of Lemmas 4.3 and 4.4 show that for any $a, b \in N_0$ and $x, y \in \mathbb{R} \ltimes X$,

$$\widehat{\Omega}(\pi_H(xa)\theta_H(b^\circ y^\circ)) = \Omega \circ \widetilde{\Theta}(\pi_H(xa)\theta_H(b^\circ y^\circ)) = \Omega(\pi_K(xa)\theta_K(b^\circ y^\circ)).$$

This means that the functional $\widetilde{\Omega}$ coincides with $\widehat{\Omega}$ on $*$-alg$\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$, and hence it is a state on $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$ since so is $\widehat{\Omega}$. □

We next show that the above boundedness extends partially, using the central weak amenability and a normality of $\widetilde{\Omega}$. This is the second use of the weak amenability. Recall that $\mathcal{M}$ is generated by a copy of $\widetilde{M}$ and $\widetilde{M}^\circ$. We put $\widetilde{M}_0 \subset \mathcal{M}_0$ as the C*-subalgebra generated by $\widetilde{B}$ and N_0, and note that Lemma 4.2 is applied to $\widetilde{M}_0$.

Claim. *The functional $\widetilde{\Omega}$ is bounded on*

$$\mathrm{C}^*\{\pi_H(\widetilde{M}), \pi_H(\widetilde{M}^\circ), \theta_H(\widetilde{M}^\circ), \theta_H(\widetilde{M})\} =: \mathfrak{A},$$

where $\theta_H(\widetilde{M})$ should be understood as $\theta_H((\widetilde{M}^\circ)^\circ)$.

Proof of Claim. Let $(\psi_i)_i$ be a net of finite-rank normal c.c. maps on N as in Theorem 2.9. Up to convex combinations, we may assume $\psi_i \to \mathrm{id}_N$ in the point $*$-strong topology. For each i we put $\psi_i^\circ := J_N \psi_i (J_N \cdot J_N) J_N$ as a normal c.c. map on N°. For each i, since ψ_i commutes with the modular action, one can define a normal c.c. map on $\mathfrak{A}$ by

$$\Psi_i := \mathrm{id}_{L^2_\ell(\mathbb{R})} \otimes \psi_i \otimes \mathrm{id}_{L^2(X)} \otimes \psi_i^\circ \otimes \mathrm{id}_{L^2_r(\mathbb{R})}.$$

Observe that the restriction of Ψ_i on $\pi_H(\widetilde{M})$ defines a normal c.c. map $\tilde{\psi}_i : \widetilde{M} \to \widetilde{M}_0$ (use Lemma 4.2). The same holds for $\theta_H(\widetilde{M}^\circ)$ and define $\tilde{\psi}_i^\circ$ similarly. Then with the formula $\|\pi_H(z)\|_{2,\widetilde{\Omega}} = \|zp\|_{2,\mathrm{Tr}} = \|\theta_H(\bar{z})\|_{2,\widetilde{\Omega}}$ for $z \in \widetilde{M}$ and by the Cauchy–Schwarz inequality, it holds that for any $a, b, x, y \in \widetilde{M}$

$$\begin{aligned}
&\left|\widetilde{\Omega} \circ \Psi_i(\pi_H(ax^\circ)\theta_H(b^\circ y)) - \widetilde{\Omega}(\pi_H(ax^\circ)\theta_H(b^\circ y))\right| \\
&\qquad = \left|\widetilde{\Omega}(\pi_H(\tilde{\psi}_i(a)x^\circ)\theta_H(\tilde{\psi}_i^\circ(b^\circ)y)) - \widetilde{\Omega}(\pi_H(ax^\circ)\theta_H(b^\circ y))\right| \\
&\qquad \le \|\tilde{\psi}_i(a)^* - a^*\|_{2,\mathrm{Tr}}\|x\|_\infty \|b\|_\infty \|y\|_\infty + \|\tilde{\psi}_i(b)^* - b^*\|_{2,\mathrm{Tr}} \|a\|_\infty \|x\|_\infty \|y\|_\infty \\
&\qquad \to 0 \quad \text{as } i \to \infty.
\end{aligned}$$

Hence $\widetilde{\Omega} \circ \Psi_i$ converges pointwisely to $\widetilde{\Omega}$ on the norm-dense $*$-subalgebra $\mathfrak{A}_0 \subset \mathfrak{A}$ generated by $\pi_H(\widetilde{M}), \pi_H(\widetilde{M}^\circ), \theta_H(\widetilde{M}^\circ)$, and $\theta_H(\widetilde{M})$. Observe that $\|\widetilde{\Omega} \circ \Psi_i|_{\mathfrak{A}}\| \le 1$ for all i, since the range of Ψ_i is contained in $\mathrm{C}^*\{\pi_H(\mathcal{M}_0), \theta_H(\mathcal{M}_0^\circ)\}$ and $\widetilde{\Omega}$ is bounded by 1 on this C*-algebra by the previous claim. So we conclude $\|\widetilde{\Omega}|_{\mathfrak{A}}\| \le 1$, as desired. □

Observe that $\widetilde{\Omega}$ is a state, since it is positive on $\mathfrak{A}_0$ by construction, and $\widetilde{\Omega}(1) = 1$. By the Hahn–Banach theorem, we extend $\widetilde{\Omega}$ from $\mathfrak{A}$ to $\mathbb{B}(H)$ and we still denote it by $\widetilde{\Omega}$. By construction, it satisfies that for all $x \in \widetilde{M}$ and $u \in \mathcal{G}$,

$$\widetilde{\Omega}(\pi_H(x)) = \mathrm{Tr}(pxp) \quad \text{and} \quad \widetilde{\Omega}\big(\pi_H(\pi(u)\theta(\bar{u}))\theta_H(\pi(u^*)^\circ\theta(u^\circ)^\circ)\big) = 1.$$

Putting $U(u) := \pi_H(\pi(u)\theta(\bar{u}))\theta_H(\pi(u^*)^\circ\theta(u^\circ)^\circ)$, the second condition implies $\widetilde{\Omega}(Y) = \widetilde{\Omega}(U(u)YU(u)^*)$ for any $u \in \mathcal{G}$ and $Y \in \mathbb{B}(H)$. Recall that since $H = L^2(\mathcal{M}) \otimes_X L^2(\mathcal{M})$, regarding $L^2(\mathcal{M})$ as an $\langle \mathcal{M}, \mathbb{R} \ltimes X\rangle$-$X$-bimodule, the basic construction $\langle \mathcal{M}, \mathbb{R} \ltimes X\rangle$ acts on H on the left, which we again denote by π_H, and its image commutes with $\theta_H(\mathcal{M}^\circ)$. So if $Y \in \langle \mathcal{M}, \mathbb{R} \ltimes X\rangle \cap \theta(\widetilde{M}^\circ)'$, then

$$\widetilde{\Omega}(\pi_H(Y)) = \widetilde{\Omega}(U(u)\pi_H(Y)U(u)^*) = \widetilde{\Omega}(\pi_H(\pi(u))\pi_H(Y)\pi_H(\pi(u))^*)$$

for any $u \in \mathcal{G}$. So the state $\widetilde{\Omega} \circ \pi_H$ is a $\mathcal{G}$-central state on $\langle \mathcal{M}, \mathbb{R} \ltimes X \rangle \cap \theta(\widetilde{M}^\circ)'$. Finally since $\widetilde{M} L^2(\mathbb{R} \ltimes X) \subset L^2(\mathcal{M})$ is dense, the von Neumann subalgebra in $\langle \mathcal{M}, \mathbb{R} \ltimes X \rangle \cap \theta(\widetilde{M}^\circ)'$ generated by $\widetilde{M}$ and $e_{\mathbb{R}\ltimes X} := 1_{L^2(\mathbb{R})} \otimes P_N \otimes 1_X$, where P_N is the 1-dimensional projection onto $\mathbb{C}\Lambda_{\varphi_N}(1_N)$, is canonically identified as $\langle \widetilde{M}, \widetilde{B} \rangle$ (by the fact that $e_{\mathbb{R}\ltimes X}\, a\, e_{\mathbb{R}\ltimes X} = E_{\widetilde{B}}(a) e_{\mathbb{R}\ltimes X}$ for $a \in \widetilde{M}$). Thus the restriction of $\widetilde{\Omega} \circ \pi_H$ on $\langle \widetilde{M}, \widetilde{B} \rangle$ is a $\mathcal{G}$-central state which coincides with Tr on $p\widetilde{M}p$. Using the normality on $p\widetilde{M}p$ and by the Cauchy–Schwarz inequality, we obtain that $\mathcal{G}''$ is amenable relative to $\widetilde{B}$ in $\widetilde{M}$. □

4C. *Proof of Corollary B.*

Proof of Corollary B. Put $M := N \mathbin{\overline{\otimes}} B \supset N_0 \mathbin{\overline{\otimes}} B =: M_0$ and suppose that $A \subset M_0$ is a Cartan subalgebra. We will deduce a contradiction. For this, let R_∞ be the AFD III$_1$ factor and $A_0 \subset R_\infty$ a Cartan subalgebra. Up to exchanging B and A with $B \mathbin{\overline{\otimes}} R_\infty$ and $A \mathbin{\overline{\otimes}} A_0$ respectively, we assume that B is a type III$_1$ factor (see, e.g., Lemma 2.2).

Let ψ_{N_0} and τ_A be faithful normal states on N_0 and A respectively, and E_{N_0} and E_A faithful normal conditional expectations from N to N_0 and from M_0 to A respectively. Put

$$\psi_A := \tau_A \circ E_A, \quad \psi_N := \psi_{N_0} \circ E_{N_0}, \quad \psi := \psi_N \otimes \varphi_B, \quad \varphi := h \otimes \varphi_B$$

and $E_{M_0} := E_{N_0} \otimes \mathrm{id}_B$. Then since all continuous cores are isomorphic, we have $\Pi_{\psi_A \circ E_{M_0}, \psi} : C_\psi(M) \to C_{\psi_A \circ E_{M_0}}(M)$, which restricts to $\Pi_{\psi_A, \psi_{N_0} \otimes \varphi_B} : C_{\psi_{N_0} \otimes \varphi_B}(M_0) \to C_{\psi_A}(M_0)$. Recall that $A \mathbin{\overline{\otimes}} L\mathbb{R} \subset C_{\psi_A}(M_0)$ is a Cartan subalgebra, see, e.g., [Houdayer and Ricard 2011, Proposition 2.6], and hence so is the image

$$\widetilde{A} := \Pi_{\varphi, \psi_A \circ E_{N_0}}(A \mathbin{\overline{\otimes}} L\mathbb{R}) \subset \Pi_{\varphi, \psi_A \circ E_{N_0}}(C_{\psi_A}(M_0)) =: \mathcal{N}.$$

Claim. *There is a conditional expectation $E : \langle C_\varphi(M), C_{\varphi_B}(B) \rangle \to \mathcal{N}$ which is faithful and normal on $C_\varphi(M)$.*

Proof. We first show $A \not\preceq_M B$. Indeed, if $A \preceq_M B$, then we have $A \preceq_{M_0} B$ by Lemma 2.12. So by [Houdayer and Isono 2017, Lemma 4.9], one has $N_0 = B' \cap M_0 \preceq_{M_0} A' \cap M_0 = A$, which is a contradiction. Hence we have $A \not\preceq_M B$.

We apply [Boutonnet et al. 2014, Proposition 2.10] (this holds if A is finite by exactly the same proof) and get $\widetilde{A} \not\preceq_{C_\varphi(M)} C_{\varphi_B}(B)$. Fix any projection $p \in \widetilde{A}$ with $\mathrm{Tr}(p) < \infty$, where Tr is the canonical trace on the core, and observe $p\widetilde{A}p \not\preceq_{C_\varphi(M)} C_{\varphi_B}(B)$ by definition. We apply Theorem A to $p\widetilde{A}p$ and get that $\mathcal{N}_{pC_\varphi(M)p}(p\widetilde{A}p)''$ is amenable relative to $C_{\varphi_B}(B)$. Observe that $\mathcal{N}_{pC_\varphi(M)p}(p\widetilde{A}p)'' = p(\mathcal{N}_{C_\varphi(M)}(\widetilde{A})'')p$; see, e.g., [Houdayer and Ricard 2011, Proposition 2.7]. Combined with [Isono 2017, Remark 3.3], there is a conditional expectation $E_p : p\langle C_\varphi(M), C_{\varphi_B}(B)\rangle p \to p\mathcal{N}p$ which restricts to the Tr-preserving expectation on $pC_\varphi(M)p$. Taking a net $(p_i)_i$ of Tr-finite projections converging to 1 weakly, one can construct a desired conditional expectation by $E(x) := \sigma\text{-weak}\,\mathrm{Lim}_i\, E_{p_i}(p_i x p_i)$ for $x \in \langle C_\varphi(M), C_{\varphi_B}(B)\rangle$. □

We apply [Isono 2017, Theorem 3.2] to the conclusion of the claim and get that M_0 is amenable relative to B in M. Hence there is a conditional expectation $F : \langle M, B \rangle \to M_0$ which is faithful and normal on M. Using the identification $\langle M, B \rangle = \mathbb{B}(L^2(M)) \mathbin{\overline{\otimes}} B$, we can construct a conditional expectation from $\mathbb{B}(L^2(M))$ onto N_0, which means N_0 is injective. This is a contradiction. □

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