Boris Buffoni and Erik Wahlén

STEADY THREE-DIMENSIONAL ROTATIONAL FLOWS: AN APPROACH VIA TWO STREAM FUNCTIONS AND NASH–MOSER ITERATION

msp

STEADY THREE-DIMENSIONAL ROTATIONAL FLOWS: AN APPROACH VIA TWO STREAM FUNCTIONS AND NASH–MOSER ITERATION

BORIS BUFFONI AND ERIK WAHLÉN

We consider the stationary flow of an inviscid and incompressible fluid of constant density in the region $D = (0, L) \times \mathbb{R}^2$. We are concerned with flows that are periodic in the second and third variables and that have prescribed flux through each point of the boundary ∂D. The Bernoulli equation states that the "Bernoulli function" $H := \frac{1}{2}|v|^2 + p$ (where v is the velocity field and p the pressure) is constant along stream lines, that is, each particle is associated with a particular value of H. We also prescribe the value of H on ∂D. The aim of this work is to develop an existence theory near a given constant solution. It relies on writing the velocity field in the form $v = \nabla f \times \nabla g$ and deriving a degenerate nonlinear elliptic system for f and g. This system is solved using the Nash–Moser method, as developed for the problem of isometric embeddings of Riemannian manifolds; see, e.g., the book by Q. Han and J.-X. Hong (2006). Since we can allow H to be nonconstant on ∂D, our theory includes three-dimensional flows with nonvanishing vorticity.

1. Introduction

The Euler equation for an inviscid and incompressible fluid of constant density is given by

$$(v \cdot \nabla)v = -\nabla p, \quad \operatorname{div} v = 0,$$

if in addition the velocity field v is independent of time. As we are concerned with stationary flows on $D = (0, L) \times \mathbb{R}^2$ that are periodic in the second and third variables, it is useful to introduce the cell of the periodic lattice

$$\mathcal{P} = (0, L) \times (0, P_1) \times (0, P_2),$$

where $L > 0$ and the periods $P_1, P_2 > 0$ are given; in particular integrations will mainly be over $\mathcal{P}$ and maxima of continuous functions considered on $\overline{\mathcal{P}}$. Any constant vector field $\bar{v}$ is a solution on D with constant pressure $\bar{p}$. Such a field can always be written in the form $\bar{v} = \nabla \bar{f} \times \nabla \bar{g}$ for some linear functions $\bar{f}, \bar{g}$. If the real-valued functions

$$(x, y, z) \mapsto f_0(x, y, z), \quad (x, y, z) \mapsto g_0(x, y, z), \quad (x, y, z) \in D,$$

are near 0 and (P_1, P_2)-periodic in (y, z), one may try looking for a velocity field of the form

$$v^* = \nabla(\bar{f} + f_0 + f^*) \times \nabla(\bar{g} + g_0 + g^*)$$

for unknown functions f^* and g^* that vanish at the boundaries $x = 0$ and $x = L$. The functions f_0 and g_0 can be interpreted as encoding a perturbation of the boundary conditions at $x = 0$ and $x = L$ given by $\bar{f}$

and $\bar{g}$. If f_0 and g_0 vanish at $x = 0$ and $x = L$, then nothing is gained with respect to the case $f_0 = g_0 = 0$ on D.

In the following theorem, the Sobolev spaces $W^{n,p}_{\mathrm{loc}}(D)$ and $H^n_{\mathrm{loc}}(D)$ consist of functions defined on D such that, when restricted to every bounded open subset $D_b \subset D$, they belong to $W^{n,p}(D_b)$ and $H^n(D_b)$. Note that, in contrast with the usual definition, $\bar{D}_b$ is not required to be included in D. Moreover, $\mathcal{Q}$ is the parallelogram in $\mathbb{R}^2$ spanned by $R P_1 e_1$ and $R P_2 e_2$, where

$$R = \begin{pmatrix} \partial_2 \bar{f} & \partial_3 \bar{f} \\ \partial_2 \bar{g} & \partial_3 \bar{g} \end{pmatrix}$$

is the Jacobian matrix of $(\bar{f}, \bar{g})$ with respect to (y, z) and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Theorem 1.1. *Let $j \in \mathbb{N}_0$ and assume that the first component of $\bar{v}$ does not vanish. Then it is possible to choose $\bar{\epsilon} > 0$ such that if*

- $H_0 \in C^{11+j}(\mathbb{R}^2)$ *is periodic with respect to the lattice in $\mathbb{R}^2$ generated by $R P_1 e_1$ and $R P_2 e_2$ (not necessarily the fundamental periods, this remark holding generally throughout),*

- $c_1, c_2 \in \mathbb{R}$,

- $f_0, g_0 \in H^{13+j}_{\mathrm{loc}}(D) = W^{13+j,2}_{\mathrm{loc}}(D)$, P_1-*periodic in y and P_2-periodic in z,*

- $\|(f_0, g_0)\|^2_{H^{13+j}(\mathcal{P})} + \|H_0\|^2_{C^{11+j}(\bar{\mathcal{Q}})} + |c|^2 < \bar{\epsilon}^2$,

then there exists $(f^*, g^*) \in H^{6+j}_{\mathrm{loc}}(D)$ satisfying

- f^*, g^* *are P_1-periodic in y and P_2-periodic in z,*

- $\qquad\qquad\qquad\qquad f^*, g^*$ *vanish when $x \in \{0, L\}$,* $\qquad\qquad\qquad\qquad$ (1)

- $v^* := \nabla(\bar{f} + f_0 + f^*) \times \nabla(\bar{g} + g_0 + g^*)$ *is a solution to the Euler equation*

$$(v^* \cdot \nabla)v^* = -\nabla p^*, \qquad \mathrm{div}\, v^* = 0 \quad \text{on } D,$$

  with

$$p^* = -\tfrac{1}{2}|v^*|^2 + H(\bar{f} + f_0 + f^*, \bar{g} + g_0 + g^*) \quad \text{and} \quad H(f, g) = c_1 f + c_2 g + H_0(f, g) \quad \text{for all } f, g \in \mathbb{R}. \tag{2}$$

Moreover, there exists a constant $C > 0$ (independent of (f_0, g_0), H_0 and c) such that

$$\|(f^*, g^*)\|_{H^{6+j}(\mathcal{P})} \le C\bar{\epsilon}.$$

The solution is locally unique in the following sense. Let H be as above (but H_0 can be assumed of class C^2 only), $f, g, \tilde{f}, \tilde{g} \in C^3(\bar{D})$ with $(f - \bar{f}, g - \bar{g})$, $(\tilde{f} - \bar{f}, \tilde{g} - \bar{g})$ both (P_1, P_2)-periodic in y and z, and

$$(f(x, y, z), g(x, y, z)) = (\tilde{f}(x, y, z), \tilde{g}(x, y, z)) \quad \text{for all } (x, y, z) \in \{0, L\} \times \mathbb{R}^2.$$

Assume that $v = \nabla f \times \nabla g$ and $\tilde{v} = \nabla \tilde{f} \times \nabla \tilde{g}$ are both solutions to the Euler equation with pressures $-\tfrac{1}{2}|v|^2 + H(f, g)$ and $-\tfrac{1}{2}|\tilde{v}|^2 + H(\tilde{f}, \tilde{g})$, respectively. If $(\nabla f, \nabla g)$ and $(\nabla \tilde{f}, \nabla \tilde{g})$ are in a sufficiently small open convex neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\bar{\mathcal{P}})$ and $\|H_0\|_{C^2(\bar{\mathcal{Q}})}$ is sufficiently small, then $(f, g) = (\tilde{f}, \tilde{g})$ on $[0, L] \times \mathbb{R}^2$.

Remarks. • Observe that $\nabla_{(f,g)}H(\bar{f}+f_0+f^*, \bar{g}+g_0+g^*)$ is P_1-periodic in y and P_2-periodic in z. In general the choice $(f^*, g^*) = -(f_0, g_0)$ is not allowed, as (f^*, g^*) is required to vanish at $x = 0$ and $x = L$, but not (f_0, g_0). When H is constant, the choice $(f^*, g^*) = -(f_0, g_0)$ leads to the constant solution $v^* = \bar{v}$, provided that f_0 and g_0 vanish when $x \in \{0, L\}$. However, when H is not constant, (1) and (2) do not allow to choose $(f^*, g^*) = -(f_0, g_0)$. Indeed, if $(f^*, g^*) = -(f_0, g_0)$, then $v^* = \bar{v}$ and p^* should be constant, which is not compatible with (2) when H is not constant.

• If H_0, f_0 and g_0 are C^∞ smooth, we obtain solutions of arbitrarily high regularity. However, we don't necessarily obtain C^∞ smooth solutions since $\bar{\epsilon}$ depends on j. It might be possible to obtain smooth solutions by applying other versions of the Nash–Moser theorem, for example an analytic version, but that's outside the scope of the paper.

• The uniqueness assertion implies that the solution $(\bar{f}+f_0+f^*, \bar{g}+g_0+g^*)$ only depends on f_0 and g_0 through their boundary values.

• On the other hand, it is possible for two different sets of data to give rise to the same velocity field v (see the Appendix for more details).

The following example illustrates the relationship with Beltrami flows (flows such that, at each point of D, the vorticity is parallel to the velocity) and the role of the boundary conditions at $x = 0$ and $x = L$.

Example. Let $\bar{f}(x, y, z) = y$, $\bar{g}(x, y, z) = z$, $c_1, c_2 = 0$ and $H_0 = 0$, so that $\bar{v} = (1, 0, 0)$. Let $f_0(x, y, z) = \delta x \sin(2\pi z/P_2)$ and $g_0 = 0$, and let (f^*, g^*) be given by Theorem 1.1 (for $|\delta|$ small enough). Remember that f^* and g^* vanish at $x = 0$ and $x = L$. The pointwise flux of v^* at $x = 0$ and $x = L$ is the constant 1:

$$v_1^* = \partial_y(\bar{f} + f_0) \, \partial_z(\bar{g} + g_0) - \partial_z(\bar{f} + f_0) \, \partial_y(\bar{g} + g_0) = 1.$$

Let us prove that v^* is not irrotational by assuming the opposite. Then v_1^* would be a (P_1, P_2)-periodic function in y and z that is harmonic. By the maximum principle, $v_1^* = 1$ and thus (v_2^*, v_3^*) would be x-independent. The functions v_2^* and v_3^* would also be harmonic and thus they would be constant, and v^* would be a constant vector field. Hence the map that sends a fluid parcel when $x = 0$ to its position when $x = L$ would be a translation. But this is impossible because $\bar{f} + f_0 + f^*$ is preserved along every parcel trajectory and its level sets at $x = 0$ (that is, the level sets of $\bar{f} + f_0$ at $x = 0$) cannot be sent by a translation to its level sets at $x = L$. Although v^* is not an irrotational flow, it is a Beltrami flow because $H = 0$. As the flux through the boundaries $x = 0$ and $x = L$ does not vanish, the proportionality factor between the velocity and the vorticity cannot be constant (using also the periodicity in the y- and z-directions). Beltrami flows have been considered in many papers, for example in [Enciso and Peralta-Salas 2015] (Beltrami flows with constant proportionality factors) and [Kaiser et al. 2000] (with nonconstant proportionality factors).

The representation $v = \nabla f \times \nabla g$ can be seen as a generalization of the stream function representation $v = \nabla^\perp \psi$ for planar divergence-free stationary flows, in which the stream function ψ is replaced by a pair of functions f and g (note that f and g are constant on stream lines). This representation always holds locally near regular points of the velocity field; see, e.g., [Barbarosie 2011]. For the reader's convenience,

we give in the Appendix a self-contained proof when v_1 is nonvanishing that the representation holds globally in D with additional (P_1, P_2)-periodicity with respect to y and z for ∇f and ∇g.

In this formulation, the Euler equation has a particularly helpful variational structure [Keller 1996]; see also [Buffoni 2012]. Namely, the pair of functions (f, g) will be called admissible for the present purpose if

- f and g are of class $C^2(\overline{D})$,

- ∇f and ∇g are P_1-periodic in y and P_2-periodic in z,

- $(f(x, y, z), g(x, y, z)) = (\tilde{f}_0(x, y, z), \tilde{g}_0(x, y, z))$ for all $(x, y, z) \in \{0, L\} \times \mathbb{R}^2$,

where $\tilde{f}_0$ and $\tilde{g}_0$ are two fixed functions of class $C^2(\overline{D})$ such that $\nabla \tilde{f}_0$ and $\nabla \tilde{g}_0$ are P_1-periodic in y and P_2-periodic in z. Under these conditions, $v = \nabla f \times \nabla g$ is divergence-free and the first component

$$v_1 = (\nabla f \times \nabla g) \cdot (1, 0, 0) = \partial_y f \, \partial_z g - \partial_y g \, \partial_z f = \partial_y \tilde{f}_0 \, \partial_z \tilde{g}_0 - \partial_y \tilde{g}_0 \, \partial_z \tilde{f}_0$$

of v is prescribed on $\{0, L\} \times \mathbb{R}^2$. In order to get a better insight into the set of admissible (f, g), note that $f(x, y, z) - a_1 y - a_2 z$ and $g(x, y, z) - a_3 y - a_4 z$ are P_1-periodic in y and P_2-periodic in z for some constants $a_1, a_2, a_3, a_4 \in \mathbb{R}$. The boundary condition ensures that $a_1, a_2, a_3, a_4 \in \mathbb{R}$ do not depend on the particular admissible pair of functions (f, g).

We also assume that the function $H : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 and that $\partial_f H$ and $\partial_g H$ composed with every admissible pair (f, g) are (P_1, P_2)-periodic in y and z. The latter is equivalent to requiring that $\nabla_{(f,g)} H$ is periodic with respect to the lattice generated by $P_1(a_1, a_3)$ and $P_2(a_2, a_4)$.

Let $(\tilde{f}, \tilde{g})$ be admissible and assume that $(\tilde{f}, \tilde{g})$ is a critical point of the integral functional

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{2} |\nabla f \times \nabla g|^2 + H(f, g) \right\} dx \, dy \, dz \tag{3}$$

defined on the set of admissible pairs (f, g). Let us check that $\tilde{v} := \nabla \tilde{f} \times \nabla \tilde{g}$ is a solution to the Euler equation with $\tilde{p} = -\tfrac{1}{2} |\tilde{v}|^2 + H(\tilde{f}, \tilde{g})$. We consider admissible variations (f_s, g_s), that is, maps $(s, x, y, z) \to (f_s(x, y, z), g_s(x, y, z))$ of class $C^2([-1, 1] \times \overline{D})$ such that $(f_0, g_0) = (\tilde{f}, \tilde{g})$, (f_1, g_1) is admissible and

$$(f_s, g_s) = ((1 - s) f_0 + s f_1, \, (1 - s) g_0 + s g_1) \quad \text{for all } s \in (-1, 1).$$

The meaning of critical point is that the integral functional at (f_s, g_s) as a function of s has a vanishing derivative at $s = 0$ for every admissible variation (f_s, g_s). If in addition we assume that $(f_1 - f_0, g_1 - g_0)$ is compactly supported in $\mathcal{P}$, we get the Euler–Lagrange equation

$$\begin{pmatrix} -\operatorname{div}(\nabla \tilde{g} \times (\nabla \tilde{f} \times \nabla \tilde{g})) + \partial_f H(\tilde{f}, \tilde{g}) \\ -\operatorname{div}((\nabla \tilde{f} \times \nabla \tilde{g}) \times \nabla \tilde{f})) + \partial_g H(\tilde{f}, \tilde{g}) \end{pmatrix} = 0. \tag{4}$$

Because of the periodicity assumption on $\nabla \tilde{f}$ and $\nabla \tilde{g}$, more general admissible variations (f_s, g_s) do not provide additional knowledge and, thanks to the periodicity condition on $\partial_f H(\tilde{f}, \tilde{g})$ and $\partial_g H(\tilde{f}, \tilde{g})$, (4)

holds true on all of D. Equation (4) can also be written

$$\nabla \tilde{g} \cdot \operatorname{rot} \tilde{v} + \partial_f H(\tilde{f}, \tilde{g}) = 0 \quad \text{and} \quad -\operatorname{rot} \tilde{v} \cdot \nabla \tilde{f} + \partial_g H(\tilde{f}, \tilde{g}) = 0, \quad \text{with } \tilde{v} = \nabla \tilde{f} \times \nabla \tilde{g}. \quad (5)$$

It then follows that

$$\tilde{v} \times \operatorname{rot} \tilde{v} = (\nabla \tilde{f} \times \nabla \tilde{g}) \times \operatorname{rot} \tilde{v} = (\nabla \tilde{f} \cdot \operatorname{rot} \tilde{v}) \nabla \tilde{g} - (\nabla \tilde{g} \cdot \operatorname{rot} \tilde{v}) \nabla \tilde{f}$$

$$= \partial_f H(\tilde{f}, \tilde{g}) \nabla \tilde{f} + \partial_g H(\tilde{f}, \tilde{g}) \nabla \tilde{g} = \nabla_{(x,y,z)} H(\tilde{f}, \tilde{g}). \quad (6)$$

The identity, see, e.g., [Serrin 1959, p. 151],

$$\nabla \left(\tfrac{1}{2} |\tilde{v}|^2 \right) = \tilde{v} \times \operatorname{rot} \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v}$$

gives

$$(\tilde{v} \cdot \nabla) \tilde{v} - \nabla \left(\tfrac{1}{2} |\tilde{v}|^2 \right) + \nabla_{(x,y,z)} H(\tilde{f}, \tilde{g}) = 0,$$

which is equivalent to the classical Euler equation for inviscid, incompressible and time-independent flows

$$(\tilde{v} \cdot \nabla) \tilde{v} + \nabla \tilde{p} = 0 \quad \text{with } \tilde{p} = -\tfrac{1}{2} |\tilde{v}|^2 + H(\tilde{f}, \tilde{g}).$$

$H(\tilde{f}, \tilde{g})$ can be seen as the Bernoulli function, which is preserved by the flow since $\nabla_{(x,y,z)}(H(\tilde{f}, \tilde{g})) \cdot \tilde{v} = 0$ by (6).

The aim of the paper is to develop an existence theory in a small neighborhood of $(\bar{f}, \bar{g}) \in C^\infty(\overline{D})$ when

- $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant, and
- the first component of $\bar{v} = \nabla \bar{f} \times \nabla \bar{g}$ does not vanish.

If we perturb (4) into the equation

$$\begin{pmatrix} -\epsilon (\partial_y^2 \tilde{f} + \partial_z^2 \tilde{f}) - \operatorname{div}(\nabla \tilde{g} \times (\nabla \tilde{f} \times \nabla \tilde{g})) + \partial_f H(\tilde{f}, \tilde{g}) \\ -\epsilon (\partial_y^2 \tilde{g} + \partial_z^2 \tilde{g}) - \operatorname{div}((\nabla \tilde{f} \times \nabla \tilde{g}) \times \nabla \tilde{f}) + \partial_g H(\tilde{f}, \tilde{g}) \end{pmatrix} = 0$$

and then linearize this perturbed equation, the obtained linear problem is coercive [Kohn and Nirenberg 1965], provided that $\epsilon > 0$. The linearization of (4) can thus be described as "degenerate", the x-direction being however nondegenerate [loc. cit.]. In Section 2, we analyze the linear operator obtained from the linearization of (4) and its invertibility, following the classical work [Kohn and Nirenberg 1965] for noncoercive boundary value problems. The analysis of the linearized problem relies on the particular structure of the integral functional (3). The main point is that its quadratic part is positive definite (see Proposition 2.3 for a precise statement). The local uniqueness result is obtained as a corollary.

The Nash–Moser iteration method [Moser 1961; Zehnder 1975] has been applied to noncoercive problems in previous works, like [Kiremidjian 1978; Han and Hong 2006]. The approach we shall follow is the one described in Section 6 of [Han and Hong 2006] for the embedding problem of Riemannian manifolds with nonnegative Gauss curvature. The details are given in Section 3. For simplicity, we have restricted ourselves as in [loc. cit.] to periodicity conditions with respect to (y, z). A key ingredient

are tame estimates for the inverse of the linearization, which are obtained in Section 2 using suitable commutator estimates.

Alber [1992] deals with a closely related setting. The steady Euler equation is considered in a bounded, simply connected, smooth domain $\Omega \subset \mathbb{R}^3$. There are three boundary conditions: (1) the flux through $\partial\Omega$ is given by a function $f \colon \partial\Omega \to \mathbb{R}$, (2) a condition on the vorticity flux through the entrance set $\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) < 0\} := \partial\Omega_-$ and (3) a condition on the Bernoulli function on $\partial\Omega_-$. Under precise assumptions, existence and uniqueness are obtained near a solution v_0 with small vorticity when the boundary conditions (2) and (3) are slightly modified. In the present paper, boundary condition (2) is, roughly speaking, replaced by a condition on the Bernoulli function on the exit set. These more symmetric boundary conditions might be a first step to considering flows which are periodic in x, which is a natural geometry in the study of water waves. Our approach also has the benefit of using a variational structure.

Note that the stationary Euler equation also appears as a model in ideal magnetohydrodynamics, with v replaced by the magnetic field B, the vorticity rot v replaced by the current density J (up to a constant multiple) and the Bernoulli function H replaced by the negative of the fluid pressure p. Grad and Rubin [1958] derived a variational principle for this problem which is rather close to the one considered here, see, e.g., [loc. cit., Theorem 1], although they did not use it to construct solutions. Moreover the above example is related to their Theorems 3 and 5 and to a remark that follows their Theorem 5. A recent work that relies on this variational principle for Euler flows is [Slobodeanu 2015]; it is formulated in a more general geometric framework. An iterative method, not of Nash–Moser type, is developed in [Kaiser et al. 2000] to get Beltrami flows with nonconstant proportionality factors. The boundary conditions there have the same flavor as the ones in [Alber 1992]. Writing a divergence-free velocity field v in the form $v = \nabla f \times \nabla g$ may also be useful for irrotational flows, as it could lead to helpful changes of variables; see [Plotnikov 1980].

2. Linearization

The variational structure of (4) allows one to study its linearization with the help of the quadratic part of the integral functional (3) around an admissible pair (f, g). From now on we shall call a pair (f, g) admissible if

(Ad1) f and g are of class $C^3(\bar{D})$,

(Ad2) ∇f and ∇g are (P_1, P_2)-periodic in y and z.

The quadratic part is given by

$$(F, G) \mapsto \int_{\mathcal{P}} \Big\{ \tfrac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G)$$
$$+ \tfrac{1}{2} \big(\partial_f^2 H(f, g) F^2 + 2 \partial_f \partial_g H(f, g) FG + \partial_g^2 H(f, g) G^2 \big) \Big\} \, dx \, dy \, dz,$$

where (F, G) is assumed admissible in the sense that

(Ad$'$1) F and G are in the Sobolev space $H^1_{\text{loc}}(D)$,

(Ad$'$2) F and G are (P_1, P_2)-periodic in y and z,

(Ad′3) $(F, G) = 0$ on ∂D in the sense of traces.

Condition (Ad′3) is introduced because we shall assume later that the restriction of (f, g) to ∂D is a priori given.

Given an admissible pair (f, g), we shall call H admissible if

(Ad″) $H \in C^2(\mathbb{R}^2)$ and $H''(f, g)$ is (P_1, P_2)-periodic in y and z.

In this section we will mostly think of $H''(f, g)$ as a given function of (x, y, z) rather than a composition. The quadratic part can be written $\frac{1}{2} B_{(f,g)}((F, G), (F, G))$, where $B_{(f,g)}$ is the symmetric bilinear form

$$B_{(f,g)}((F, G), (\delta F, \delta G))$$
$$= \int_{\mathcal{P}} \big\{ (\nabla F \times \nabla g + \nabla f \times \nabla G) \cdot (\nabla \delta F \times \nabla g + \nabla f \times \nabla \delta G)$$
$$+ (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla \delta G) + (\nabla f \times \nabla g) \cdot (\nabla \delta F \times \nabla G)$$
$$+ \partial_f^2 H(f, g) F \delta F + \partial_f \partial_g H(f, g)(F \delta G + G \delta F) + \partial_g^2 H(f, g) G \delta G \big\} \, dx \, dy \, dz.$$

This section contains two kinds of results: firstly, we bound from below the quadratic part and, secondly, we study the regularity of solutions to the linearization of problem (4) at (f, g). A preliminary observation is that the quadratic part is not coercive at (f, g) in the sense that there is no $\alpha > 0$ such that, for all admissible (F, G),

$$\tfrac{1}{2} B_{(f,g)}((F, G), (F, G)) \geq \int_{\mathcal{P}} \{ \alpha(|\nabla F|^2 + |\nabla G|^2) - \alpha^{-1}(F^2 + G^2) \} \, dx \, dy \, dz.$$

For example, taking $G = 0$, the quadratic part becomes

$$F \mapsto \int_{\mathcal{P}} \big(\tfrac{1}{2} |\nabla F \times \nabla g|^2 + \tfrac{1}{2} \partial_f^2 H(f, g) F^2 \big) \, dx \, dy \, dz.$$

In the particular case $f(x, y, z) = y$, $g(x, y, z) = z$, $H = 0$ and $P_1 = P_2 = 1$, the integral reduces to

$$\frac{1}{2} \int_{\mathcal{P}} (F_x^2 + F_y^2) \, dx \, dy \, dz.$$

Choosing F_n of the form

$$F_n(x, y, z) = \phi(x) \cos(2\pi n z),$$

where $\phi \in C^\infty(\mathbb{R}, [0, 1])$ is compactly supported in $(0, 1)$ and takes the value 1 on $\left(\frac{1}{4}, \frac{3}{4}\right)$, we find that the quadratic part and $\|(F_n, G)\|_{L^2(\mathcal{P})}$ have positive constant values along the sequence $\{(F_n, G)\}_{n \geq 1}$. However, $\|(\nabla F_n, \nabla G)\|_{L^2(\mathcal{P})} \to \infty$ and thus α as above cannot exist. For a general pair (f, g), we instead fix $(x_0, y_0, z_0) \in \mathcal{P}$ such that $\nabla g(x_0, y_0, z_0) \neq 0$ and consider F_n which is (P_1, P_2)-periodic in (y, z) and when restricted to $\mathcal{P}$ is given by

$$F_n(x, y, z) = \phi(x, y, z) \cos(n g(x, y, z)),$$

where $\phi \in C^\infty(\overline{\mathcal{P}}, [0, 1])$ is compactly supported in $\mathcal{P}$, with $\phi(x_0, y_0, z_0) = 1$. By choosing n large enough, one again obtains that α cannot exist. In fact, we have made the stronger observation that, for all

$\alpha > 0$, there exists a sequence $\{(F_n, G_n)\}$ of admissible pairs such that

$$\frac{1}{2} B_{(f,g)}((F_n, G_n), (F_n, G_n)) + \alpha^{-1} \int_{\mathcal{P}} (F_n^2 + G_n^2) \, dx \, dy \, dz$$

remains bounded, but $\{(F_n, G_n)\}$ does not have any subsequence converging in $L^2(\mathcal{P})$. This has implications for the regularity of the solutions to the linearized problem, as described below.

Nevertheless, in Theorem 2.1, we bound from below the quadratic part in a rougher way. The term $\int_{\mathcal{P}} \frac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz$ turns out to be rather nice, as shown in the first part of the proof, because it is bounded from below by $\int_{\mathcal{P}} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\} \, dx \, dy \, dz$ (under the simplifying assumption (7), otherwise there is an additional factor). With the help of a Poincaré inequality and thanks to the Dirichlet boundary condition at $x = 0$ and $x = L$, $\int_{\mathcal{P}} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\} \, dx \, dy \, dz$ can in turn be bounded from below by a positive constant times $\|(F, G)\|_{L^2(\mathcal{P})}^2$. In the second and third parts of the proof of Theorem 2.1, we bound from below the second term of the quadratic part, that is, $\int_{\mathcal{P}} (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \, dx \, dy \, dz$: it cannot become too negative with respect to $\int_{\mathcal{P}} \frac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz$. In these estimates, it is assumed that $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$. To get a better feeling for the term $\int_{\mathcal{P}} (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \, dx \, dy \, dz$, observe that it vanishes when v is irrotational because (see the beginning of the second step)

$$\int_{\mathcal{P}} (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \, dx \, dy \, dz = \frac{1}{2} \int_{\mathcal{P}} \operatorname{rot} v \cdot (F \nabla G - G \nabla F) \, dx \, dy \, dz.$$

As we allow v to be slightly rotational, this term needs careful estimates.

As a consequence of Theorem 2.1, the integral functional is strictly convex in a neighborhood of $(\bar{f}, \bar{g})$, which implies local uniqueness of a solution to (4) (but not existence at this stage); see Theorem 2.2.

With the aim to apply the technique of elliptic regularization [Kohn and Nirenberg 1965], we consider for $\epsilon \in [0, 1]$ the regularized quadratic part

$$(F, G) \mapsto \int_{\mathcal{P}} \left\{ \frac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) + \frac{1}{2} \epsilon (|\nabla F|^2 + |\nabla G|^2) \right.$$
$$\left. + \frac{1}{2} \big(\partial_f^2 H(f, g) F^2 + 2 \partial_f \partial_g H(f, g) FG + \partial_g^2 H(f, g) G^2 \big) \right\} dx \, dy \, dz$$
$$:= \frac{1}{2} B_{(f,g)}^\epsilon ((F, G), (F, G)).$$

All the obtained estimates are uniform in $\epsilon \in [0, 1]$, but, in addition, the problem becomes elliptic for $\epsilon \in (0, 1]$.

For every admissible $(f, g) \in C^3(\overline{D})$, we introduce the following system for $(\mu, \nu) \in L^2_{\text{loc}}(D)$ that is (P_1, P_2)-periodic in y and z, and for $(F, G) \in H^2_{\text{loc}}(D)$ admissible in the sense of (Ad′1)–(Ad′3):

$$\mu = -\operatorname{div}\big(\nabla g \times (\nabla F \times \nabla g + \nabla f \times \nabla G) + \nabla G \times (\nabla f \times \nabla g) \big) - \epsilon \Delta F + \partial_f^2 H(f, g) F + \partial_f \partial_g H(f, g) G,$$
$$\nu = -\operatorname{div}\big((\nabla F \times \nabla g + \nabla f \times \nabla G) \times \nabla f + (\nabla f \times \nabla g) \times \nabla F \big) - \epsilon \Delta G + \partial_f \partial_g H(f, g) F + \partial_g^2 H(f, g) G.$$

The right-hand side is the linear operator related to the regularized quadratic part. This system also makes sense in a weak form if, instead of $(F, G) \in H^2_{\text{loc}}(D)$, we ask that $(F, G) \in H^1_{\text{loc}}(D)$. Given (μ, ν) in any higher-order Sobolev space, the main issue of Section 2 is to study the regularity of a solution (F, G),

aiming at estimates of the Sobolev norms, uniformly in $\epsilon \in [0, 1]$. Such a pair (F, G) is easily proved to be unique and its existence for $\epsilon \in (0, 1]$ follows from the fact that the system is elliptic. The same particular case as above gives more insight into this system. Setting $\mu = \nu = 0$, $\epsilon = 0$, $G = 0$, $f(x, y, z) = y$, $g(x, y, z) = z$ and $P_1 = P_2 = 1$, we get

$$- \operatorname{div}(\partial_1 F, \partial_2 F, 0) + \partial_f^2 H(f, g) F = 0,$$
$$- \operatorname{div}(0, -\partial_3 F, 2\partial_2 F) + \partial_f \partial_g H(f, g) F = 0.$$

Keeping only the second-order terms and forgetting the boundary and periodicity conditions, we see that $F(x, y, z) = \cos(z)$ is a solution to both equations. Hence the regularity theory in [Agmon et al. 1964] cannot be used when $\epsilon = 0$, $f(x, y, z) = y$, $g(x, y, z) = z$ and $P_1 = P_2 = 1$.

In Proposition 2.4, we explain how the general system allows one to express $\partial_{11}^2 F$ and $\partial_{11}^2 G$ with respect to the other second-order partial derivatives of F and G, and lower-order terms, involving μ and ν too. After iterative differentiations, this also yields expressions for higher-order derivatives that contain at least two partial derivatives with respect to x. In a more general setting, this is developed in [Kohn and Nirenberg 1965].

For $i \in \{2, 3\}$, multiplying both sides of each equation of the system by $(-1)^r \partial_i^{2r} F$ and $(-1)^r \partial_i^{2r} G$, respectively, summing the two equations and then integrating by parts many times, $B_{(f,g)}(\partial_i^r F, \partial_i^r F)$ arises, with additional bilinear terms in (F, G) that turn out to involve at most r partial derivatives of F and G for each of the two components of each bilinear term. We can make some of these additional terms small if ν is near $\bar{\nu}$ (here, the hypothesis that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant is used; see the remarks following Theorem 2.7). This crucial observation is developed in [Kohn and Nirenberg 1965] in a more general framework, and is presented here in our specific setting in Theorem 2.5. The quadratic part gives then control on the $L^2(\mathcal{P})$-norms of $\partial_i^r F$ and $\partial_i^r G$, but also on the $L^2(\mathcal{P})$-norms of $\partial_1 \partial_i^r F$ and $\partial_1 \partial_i^r G$. Hence the $L^2(\mathcal{P})$-norms of $\partial_i^r F$, $\partial_i^r G$, $\partial_1 \partial_i^r F$ and $\partial_1 \partial_i^r G$ are controlled by the $L^2(\mathcal{P})$-norms of $\partial_i^r \mu$ and $\partial_i^r \nu$ and by a small factor times the $H^r(\mathcal{P})$-norms of F and G. With all these tools, we get the estimate of Theorem 2.8 at the end of Section 2, in which the norm of (f, g) in some Sobolev space also appears, the order of which is under sufficient control. Although we follow ideas from [loc. cit.] (see in particular Theorem 2′), explicit estimates allow one to get explicit regularity results for the solutions obtained by the Nash–Moser procedure. It may be expected that these estimates could be improved and thus also the statements on regularity, but we do not strive in the present work to be optimal. The lack of compactness mentioned above prevents us from proving C^∞ smoothness of the solution using the method behind Theorem 2 in [loc. cit.].

Our first aim is to find conditions that ensure that $B_{(f,g)}$ is positive definite. In [Buffoni 2012], a minimizer of a more general integral functional could be found in some space of general flows, in a very similar spirit as in [Brenier 1999]. Hence it could be expected that, under appropriate conditions, the quadratic part is nonnegative at a solution of (4). In the proof of the following theorem, we also rely on Poincaré's inequality to get the stronger result that the quadratic part is positive definite for (f, g) (not necessarily a solution to (4)) sufficiently close to $(\bar{f}, \bar{g})$ and H'' sufficiently small (see Theorem 2.1). For simplicity, we shall assume in the following statement that

$$|\nabla \bar{f}|^2 + |\nabla \bar{g}|^2 + \sqrt{(|\nabla \bar{f}|^2 + |\nabla \bar{g}|^2)^2 - 4|\bar{\nu}|^2} \leq 2, \quad \bar{\nu} := \nabla \bar{f} \times \nabla \bar{g}. \tag{7}$$

As for (small) $\lambda > 0$, equation (4) remains invariant under the transformation

$$(\tilde{f}, \tilde{g}) \to (\lambda \tilde{f}, \lambda \tilde{g}), \quad H \to \lambda^4 H(\lambda^{-1} \cdot, \lambda^{-1} \cdot),$$

there is no loss of generality.

Theorem 2.1. *Assume that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant, that the first component of $\bar{v}$ does not vanish and that (7) holds true. For admissible (f, g) and (F, G),*

$$B_{(f,g)}((F, G), (F, G))$$

$$\geq \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla F)^2 + \tfrac{1}{16}(v \cdot \nabla G)^2 + (1 - O(\|v'\|_{C(\bar{\mathcal{P}})})) \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{16L^2}(F^2 + G^2) \right.$$

$$\left. + \partial_f^2 H(f, g)F^2 + 2\partial_f \partial_g H(f, g)FG + \partial_g^2 H(f, g)G^2 \right\} dx \, dy \, dz \quad (8)$$

holds if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\bar{\mathcal{P}})$ (independent of H admissible).

Notation. The notation $u = O(v)$ means that the norm (or absolute value) of u is less than a constant times v in the relevant domain. We also use the notation $u \lesssim v$ to indicate that there exists a constant $C > 0$ (independent of u and v) such that $u \leq Cv$.

Remark. It is not essential that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant for this result to hold. The result would still remain true if we instead were to require that $\mathrm{rot}\, \bar{v} = 0$ (the other hypotheses remaining the same) and replace the coefficient $1 - O(\|v'\|_{C(\bar{\mathcal{P}})})$ in (8) by $\exp(-4L\|(v/v_1)'\|_{C(\bar{\mathcal{P}})})$. This might be useful for considering perturbations of other irrotational flows. See, however, the remarks following Theorem 2.7.

Proof. Under the hypotheses of the theorem, we can assume that the first component of the velocity field $v = \nabla f \times \nabla g$ never vanishes (like the one of $\bar{v}$). We study the various terms separately.

<u>First step</u>: Let us first show that

$$\int_{\mathcal{P}} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz \geq \int_{\mathcal{P}} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\} \, dx \, dy \, dz$$

$$\geq (1 - O(\|v'\|_{C(\bar{\mathcal{P}})})) \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{L^2} \int_{\mathcal{P}} (F^2 + G^2) \, dx \, dy \, dz$$

if $(\nabla f, \nabla g)$ is near enough to $(\nabla \bar{f}, \nabla \bar{g})$ in $C^1(\bar{\mathcal{P}})$.

To this end, write

$$\nabla F \times \nabla g + \nabla f \times \nabla G = a\nabla f + b\nabla g + c\nabla f \times \nabla g.$$

By taking the scalar product of both sides with ∇f, ∇g and $\nabla f \times \nabla g$ successively, we get

$$\begin{cases} (\nabla g \times \nabla f) \cdot \nabla F = a|\nabla f|^2 + b\nabla f \cdot \nabla g, \\ (\nabla g \times \nabla f) \cdot \nabla G = a\nabla f \cdot \nabla g + b|\nabla g|^2, \\ (\nabla g \times (\nabla f \times \nabla g)) \cdot \nabla F + ((\nabla f \times \nabla g) \times \nabla f) \cdot \nabla G = c|\nabla f \times \nabla g|^2 \end{cases}$$

and

$$a = \frac{-|\nabla g|^2(v \cdot \nabla F) + (\nabla f \cdot \nabla g)(v \cdot \nabla G)}{|\nabla f|^2|\nabla g|^2 - (\nabla f \cdot \nabla g)^2} = \frac{-|\nabla g|^2(v \cdot \nabla F) + (\nabla f \cdot \nabla g)(v \cdot \nabla G)}{|v|^2},$$

$$b = \frac{-|\nabla f|^2(v \cdot \nabla G) + (\nabla f \cdot \nabla g)(v \cdot \nabla F)}{|v|^2},$$

$$c = \frac{(v \times \nabla f) \cdot \nabla G + (\nabla g \times v) \cdot \nabla F}{|v|^2}.$$

Hence

$$\int_{\mathcal{P}} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz$$

$$\geq \int_{\mathcal{P}} |a\nabla f + b\nabla g|^2 \, dx \, dy \, dz$$

$$= \int_{\mathcal{P}} (a \ \ b) \begin{pmatrix} |\nabla f|^2 & \nabla f \cdot \nabla g \\ (\nabla f \cdot \nabla g) & |\nabla g|^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} dx \, dy \, dz$$

$$= \int_{\mathcal{P}} \frac{1}{|v|^4} (v \cdot \nabla F \ \ v \cdot \nabla G)$$
$$\times \begin{pmatrix} -|\nabla g|^2 & \nabla f \cdot \nabla g \\ \nabla f \cdot \nabla g & -|\nabla f|^2 \end{pmatrix} \begin{pmatrix} |\nabla f|^2 & \nabla f \cdot \nabla g \\ \nabla f \cdot \nabla g & |\nabla g|^2 \end{pmatrix} \begin{pmatrix} -|\nabla g|^2 & \nabla f \cdot \nabla g \\ \nabla f \cdot \nabla g & -|\nabla f|^2 \end{pmatrix} \begin{pmatrix} v \cdot \nabla F \\ v \cdot \nabla G \end{pmatrix} dx \, dy \, dz$$

$$= \int_{\mathcal{P}} \frac{1}{|v|^4} (v \cdot \nabla F \ \ v \cdot \nabla G) \begin{pmatrix} -|\nabla g|^2 & \nabla f \cdot \nabla g \\ \nabla f \cdot \nabla g & -|\nabla f|^2 \end{pmatrix} \begin{pmatrix} -|v|^2 & 0 \\ 0 & -|v|^2 \end{pmatrix} \begin{pmatrix} v \cdot \nabla F \\ v \cdot \nabla G \end{pmatrix} dx \, dy \, dz$$

$$= \int_{\mathcal{P}} \frac{1}{|v|^2} (v \cdot \nabla F \ \ v \cdot \nabla G) \begin{pmatrix} |\nabla g|^2 & -\nabla f \cdot \nabla g \\ -\nabla f \cdot \nabla g & |\nabla f|^2 \end{pmatrix} \begin{pmatrix} v \cdot \nabla F \\ v \cdot \nabla G \end{pmatrix} dx \, dy \, dz$$

$$\geq \int_{\mathcal{P}} \frac{|\nabla f|^2 + |\nabla g|^2 - \sqrt{(|\nabla f|^2 + |\nabla g|^2)^2 - 4|v|^2}}{2|v|^2} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\} \, dx \, dy \, dz$$

because the eigenvalues of

$$\begin{pmatrix} |\nabla g|^2 & -\nabla f \cdot \nabla g \\ -\nabla f \cdot \nabla g & |\nabla f|^2 \end{pmatrix}$$

are $\frac{1}{2}\left(|\nabla f|^2 + |\nabla g|^2 \pm \sqrt{(|\nabla f|^2 + |\nabla g|^2)^2 - 4|v|^2}\right)$. By the simplifying assumption (7),

$$\int_{\mathcal{P}} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz \geq \int_{\mathcal{P}} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\} \, dx \, dy \, dz$$

if $(\nabla f, \nabla g)$ is near enough to $(\nabla \bar{f}, \nabla \bar{g})$ in $C(\overline{\mathcal{P}})$.

To obtain the second inequality of the first step, we now use Poincaré's inequality in one dimension by relying on the fact that F and G vanish on $\{0, L\} \times (0, P_1) \times (0, P_2)$, and then integrate with respect to the two remaining variables. We use again that the first component of $\bar{v}$ does not vanish and that v is in some small neighborhood of $\bar{v}$, so that the first component of v does not vanish either. Given $(\tilde{y}, \tilde{z}) \in \mathbb{R}^2$,

let $\Gamma_{(\tilde{y},\tilde{z})} : [0, L] \to \mathbb{R}^2$ be the function of the variable $\tilde{x} \in [0, L]$ satisfying

$$\Gamma'_{(\tilde{y},\tilde{z})}(\tilde{x}) = \frac{1}{v_1(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x}))}(v_2(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})), v_3(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})))$$

with the initial condition $\Gamma_{(\tilde{y},\tilde{z})}(0) = (\tilde{y}, \tilde{z})$. By Theorem 7.2 of Chapter 1 in [Coddington and Levinson 1955] on the regularity of solutions of ODEs, the map $(\tilde{x}, \tilde{y}, \tilde{z}) \to \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})$ is of class $C^2(\bar{P})$.

Moreover the Jacobian determinant of the map $(\tilde{y}, \tilde{z}) \to \Gamma_{(\tilde{y},\tilde{z})}(s)$ is given by

$$\exp \int_0^s \operatorname{div}_{(y,z)}(v_2/v_1, v_3/v_1)|_{(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x}))} \, d\tilde{x}.$$

Given $\tilde{x} \in (0, L)$, we associate to $(\tilde{x}, \tilde{y}, \tilde{z})$ the point

$$(x, y, z) = (\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})).$$

Observe that $x = \tilde{x}$. We denote by $J(\tilde{x}, \tilde{y}, \tilde{z})$ the Jacobian determinant and obtain

$$J(s, \tilde{y}, \tilde{z}) = \exp \int_0^s \operatorname{div}_{(y,z)}(v_2/v_1, v_3/v_1)|_{(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x}))} \, d\tilde{x} = 1 + O(\|v'\|_{C(\bar{P})})$$

uniformly in $(s, \tilde{y}, \tilde{z}) \in \bar{\mathcal{P}}$ if v is near enough to $\bar{v}$ in $C^1(\bar{\mathcal{P}})$.

Setting

$$\widetilde{F}(\tilde{x}, \tilde{y}, \tilde{z}) = F(x, y, z), \quad \widetilde{G}(\tilde{x}, \tilde{y}, \tilde{z}) = G(x, y, z), \quad \tilde{v}_1(\tilde{x}, \tilde{y}, \tilde{z}) = v_1(x, y, z),$$

we get

$$\partial_1 \widetilde{F}(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{d}{d\tilde{x}} F(\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})) = \nabla F \cdot \begin{pmatrix} 1 \\ v_2/v_1 \\ v_3/v_1 \end{pmatrix} \quad \text{at } (\tilde{x}, \Gamma_{(\tilde{y},\tilde{z})}(\tilde{x})),$$

$$\tilde{v}_1 \partial_1 \widetilde{F} = v \cdot \nabla F, \quad \tilde{v}_1 \partial_1 \widetilde{G} = v \cdot \nabla G$$

and

$$\begin{aligned}
\int_{\mathcal{P}} \{(v \cdot \nabla F)^2 + (v \cdot \nabla G)^2\} \, dx \, dy \, dz &= \int_{\mathcal{P}} \{(\tilde{v}_1 \partial_1 \widetilde{F})^2 + (\tilde{v}_1 \partial_1 \widetilde{G})^2\} J(\tilde{x}, \tilde{y}, \tilde{z}) \, d\tilde{x} \, d\tilde{y} \, d\tilde{z} \\
&\geq \min_{\bar{\mathcal{P}}}(\tilde{v}_1^2 J) \int_{(0,P_1)\times(0,P_2)} \left\{ \int_0^L \{(\partial_1 \widetilde{F})^2 + (\partial_1 \widetilde{G})^2\} \, d\tilde{x} \right\} d\tilde{y} \, d\tilde{z} \\
&\geq \frac{\pi^2 \min_{\bar{\mathcal{P}}} \tilde{v}_1^2 J}{L^2} \int_{(0,P_1)\times(0,P_2)} \left\{ \int_0^L (\widetilde{F}^2 + \widetilde{G}^2) d\tilde{x} \right\} d\tilde{y} \, d\tilde{z} \\
&\geq \frac{\pi^2 \min_{\bar{\mathcal{P}}} \tilde{v}_1^2 J}{L^2 \max_{\bar{\mathcal{P}}} J} \int_{\mathcal{P}} (F^2 + G^2) \, dx \, dy \, dz \\
&\geq (1 - O(\|v'\|_{C(\bar{\mathcal{P}})})) \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{L^2} \int_{\mathcal{P}} (F^2 + G^2) \, dx \, dy \, dz
\end{aligned}$$

if v is in some small neighborhood of $\bar{v}$ in $C^1(\bar{\mathcal{P}})$.

<u>Second step</u>: We now deal with the term $\int_{\mathcal{P}} (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \, dx \, dy \, dz$. Write

$$\operatorname{rot} v = \alpha v + \beta v \times \nabla f + \gamma \nabla g \times v,$$

with

$$\alpha = \frac{\operatorname{rot} v \cdot v}{|v|^2}, \quad \beta = \frac{\operatorname{rot} v \cdot \nabla g}{|v|^2}, \quad \gamma = \frac{\operatorname{rot} v \cdot \nabla f}{|v|^2}.$$

We get

$$\int_{\mathcal{P}} (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \, dx \, dy \, dz = \frac{1}{2} \int_{\mathcal{P}} v \cdot \operatorname{rot}(F \nabla G - G \nabla F) \, dx \, dy \, dz$$

$$= \frac{1}{2} \int_{\mathcal{P}} \operatorname{rot} v \cdot (F \nabla G - G \nabla F) \, dx \, dy \, dz$$

because

$$0 = \int_{\mathcal{P}} \operatorname{div}(v \times (F \nabla G - G \nabla F)) \, dx \, dy \, dz = \int_{\mathcal{P}} (\operatorname{rot} v \cdot (F \nabla G - G \nabla F) - v \cdot \operatorname{rot}(F \nabla G - G \nabla F)) \, dx \, dy \, dz.$$

Hence

$$\int_{\mathcal{P}} (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \, dx \, dy \, dz$$

$$= \frac{1}{2} \int_{\mathcal{P}} (\alpha v + \beta v \times \nabla f + \gamma \nabla g \times v) \cdot (F \nabla G - G \nabla F) \, dx \, dy \, dz$$

$$= \frac{1}{2} \int_{\mathcal{P}} \big\{ \alpha (\nabla F \times \nabla g + \nabla f \times \nabla G) \cdot (G \nabla f - F \nabla g)$$
$$+ (\beta v \times \nabla f + \gamma \nabla g \times v) \cdot (F \nabla G - G \nabla F) \big\} \, dx \, dy \, dz$$

$$\geq \int_{\mathcal{P}} \big\{ -\tfrac{1}{8} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 - \alpha^2 (G^2 |\nabla f|^2 + F^2 |\nabla g|^2)$$
$$+ \tfrac{1}{2} (\beta v \times \nabla f + \gamma \nabla g \times v) \cdot (F \nabla G - G \nabla F) \big\} \, dx \, dy \, dz.$$

The (absolute value of the) first term in this expression does not create problems because it can be controlled by one eighth of the term studied in the first step. Neither does the second term because it can also be controlled by any fraction of the term studied in the first step (as the second term is quadratic in (F, G) and $|\alpha|$ is as small as needed if $\operatorname{rot} v$ is near enough to $\operatorname{rot} \bar{v} = 0$). The aim of the next step is to deal with the last term.

<u>Third step</u>: The aim of this step it to get control of the term

$$\frac{1}{2} \int_{\mathcal{P}} (\beta v \times \nabla f + \gamma \nabla g \times v) \cdot (F \nabla G - G \nabla F) \, dx \, dy \, dz.$$

First, using $\nabla(FG) = G \nabla F + F \nabla G$, we have

$$\frac{1}{2} \int_{\mathcal{P}} (\beta v \times \nabla f) \cdot (F \nabla G - G \nabla F) \, dx \, dy \, dz$$

$$= \frac{1}{2} \int_{\mathcal{P}} (\beta v \times \nabla f) \cdot \nabla(FG) \, dx \, dy \, dz - \int_{\mathcal{P}} (\beta v \times \nabla f) \cdot (G \nabla F) \, dx \, dy \, dz$$

$$= -\frac{1}{2} \int_{\mathcal{P}} FG(\beta \operatorname{rot} v + \nabla \beta \times v) \cdot \nabla f \, dx \, dy \, dz - \int_{\mathcal{P}} (\beta v \times \nabla f) \cdot (G \nabla F) \, dx \, dy \, dz.$$

Similarly, we can rewrite

$$\frac{1}{2}\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G-G\nabla F)\,dx\,dy\,dz$$

$$=-\frac{1}{2}\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot\nabla(FG)\,dx\,dy\,dz+\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G)\,dx\,dy\,dz$$

$$=-\frac{1}{2}\int_{\mathcal{P}}FG(\gamma\,\mathrm{rot}\,v+\nabla\gamma\times v)\cdot\nabla g\,dx\,dy\,dz+\int_{\mathcal{P}}(\gamma\nabla g\times v)\cdot(F\nabla G)\,dx\,dy\,dz.$$

As

$$|-\beta Fv\cdot(\nabla F\times\nabla g+\nabla f\times\nabla G)|\le 2\beta^2 F^2|v|^2+\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2$$

and

$$0=\int_{\mathcal{P}}\mathrm{div}\big(v\times\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)\big)\,dx\,dy\,dz$$

$$=\int_{\mathcal{P}}\mathrm{rot}\,v\cdot\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)\,dx\,dy\,dz-\int_{\mathcal{P}}v\cdot\mathrm{rot}\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)\,dx\,dy\,dz,\quad(9)$$

we have

$$\int_{\mathcal{P}}\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz$$

$$\ge\int_{\mathcal{P}}\{-\beta Fv\cdot(\nabla F\times\nabla g+\nabla f\times\nabla G)-2\beta^2 F^2|v|^2\}\,dx\,dy\,dz$$

$$=\int_{\mathcal{P}}\big\{v\cdot\big(\mathrm{rot}\big(-\beta\tfrac{1}{2}F^2\nabla g+\beta FG\nabla f\big)+\tfrac{1}{2}F^2\nabla\beta\times\nabla g$$
$$-FG\nabla\beta\times\nabla f-\beta G\nabla F\times\nabla f\big)-2\beta^2 F^2|v|^2\big\}\,dx\,dy\,dz$$

$$\overset{(9)}{=}\int_{\mathcal{P}}\big\{\mathrm{rot}\,v\cdot\big(-\beta\tfrac{1}{2}F^2\nabla g+\beta FG\nabla f\big)+\tfrac{1}{2}F^2 v\cdot(\nabla\beta\times\nabla g)$$
$$-FG v\cdot(\nabla\beta\times\nabla f)-\beta G v\cdot(\nabla F\times\nabla f)-2\beta^2 F^2|v|^2\big\}\,dx\,dy\,dz$$

and therefore

$$-\int_{\mathcal{P}}(\beta v\times\nabla f)\cdot(G\nabla F)\,dx\,dy\,dz$$

$$=\int_{\mathcal{P}}\beta G v\cdot(\nabla F\times\nabla f)\,dx\,dy\,dz$$

$$\ge-\int_{\mathcal{P}}\tfrac{1}{8}|\nabla F\times\nabla g+\nabla f\times\nabla G|^2\,dx\,dy\,dz$$

$$+\int_{\mathcal{P}}\big\{\mathrm{rot}\,v\cdot\big(-\tfrac{1}{2}\beta F^2\nabla g+\beta FG\nabla f\big)+\tfrac{1}{2}F^2 v\cdot(\nabla\beta\times\nabla g)$$
$$-FG v\cdot(\nabla\beta\times\nabla f)-2\beta^2 F^2|v|^2\big\}\,dx\,dy\,dz.$$

In the previous computations, substitute f and F by $-g$ and $-G$, g and G by f and F, and β by γ, yielding

$$\int_{\mathcal{P}} (\gamma \nabla g \times v) \cdot (F \nabla G) \, dx \, dy \, dz$$

$$\geq - \int_{\mathcal{P}} \tfrac{1}{8} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \left\{ \operatorname{rot} v \cdot \left(-\tfrac{1}{2} \gamma G^2 \nabla f + \gamma F G \nabla g \right) + \tfrac{1}{2} G^2 v \cdot (\nabla \gamma \times \nabla f) - F G v \cdot (\nabla \gamma \times \nabla g) - 2\gamma^2 G^2 |v|^2 \right\} dx \, dy \, dz.$$

Adding the different contributions, we find that

$$\tfrac{1}{2} \int_{\mathcal{P}} (\beta v \times \nabla f + \gamma \nabla g \times v) \cdot (F \nabla G - G \nabla F) \, dx \, dy \, dz$$

$$\geq - \int_{\mathcal{P}} \tfrac{1}{4} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \left\{ \operatorname{rot} v \cdot \left(-\tfrac{1}{2} \beta F^2 \nabla g + \tfrac{1}{2} F G \beta \nabla f \right) + \tfrac{1}{2} F^2 v \cdot (\nabla \beta \times \nabla g) - \tfrac{1}{2} F G v \cdot (\nabla \beta \times \nabla f) - 2\beta^2 F^2 |v|^2 \right\} dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \left\{ \operatorname{rot} v \cdot \left(-\tfrac{1}{2} \gamma G^2 \nabla f + \tfrac{1}{2} F G \gamma \nabla g \right) + \tfrac{1}{2} G^2 v \cdot (\nabla \gamma \times \nabla f) - \tfrac{1}{2} F G v \cdot (\nabla \gamma \times \nabla g) - 2\gamma^2 G^2 |v|^2 \right\} dx \, dy \, dz.$$

All the absolute values of these terms are controlled by multiples of the term studied in the first step. Moreover $|\nabla \beta|$ and $|\nabla \gamma|$ become small if $(\nabla f, \nabla g)$ is near enough to $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\bar{\mathcal{P}})$.

<u>Last step:</u>

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) \right\} dx \, dy \, dz$$

$$\overset{\text{step 2}}{\geq} \int_{\mathcal{P}} \left\{ \tfrac{3}{8} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 - \alpha(G^2 |\nabla f|^2 + F^2 |\nabla g|^2) \right.$$
$$\left. + \tfrac{1}{2} (\beta v \times \nabla f + \gamma \nabla g \times v) \cdot (F \nabla G - G \nabla F) \right\} dx \, dy \, dz$$

$$\overset{\text{step 3}}{\geq} \int_{\mathcal{P}} \left\{ \tfrac{1}{8} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 - \alpha(G^2 |\nabla f|^2 + F^2 |\nabla g|^2) \right\} dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \left\{ \operatorname{rot} v \cdot \left(-\beta \tfrac{1}{2} F^2 \nabla g + \tfrac{1}{2} F G \beta \nabla f \right) + \tfrac{1}{2} F^2 v \cdot (\nabla \beta \times \nabla g) - \tfrac{1}{2} F G v \cdot (\nabla \beta \times \nabla f) - 2\beta^2 F^2 |v|^2 \right\} dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \left\{ \operatorname{rot} v \cdot \left(-\gamma \tfrac{1}{2} G^2 \nabla f + \tfrac{1}{2} F G \gamma \nabla g \right) + \tfrac{1}{2} G^2 v \cdot (\nabla \gamma \times \nabla f) - \tfrac{1}{2} F G v \cdot (\nabla \gamma \times \nabla g) - 2\gamma^2 G^2 |v|^2 \right\} dx \, dy \, dz$$

$$\overset{\text{step 1}}{\geq} \int_{\mathcal{P}} \tfrac{1}{16} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 \, dx \, dy \, dz$$

$$\overset{\text{step 1}}{\geq} \int_{\mathcal{P}} \left\{ \tfrac{1}{32} (v \cdot \nabla F)^2 + \tfrac{1}{32} (v \cdot \nabla G)^2 + (1 - O(\|v'\|_{C(\bar{\mathcal{P}})})) \frac{\pi^2 \min_{\bar{\mathcal{P}}} v_1^2}{32 L^2} (F^2 + G^2) \right\} dx \, dy \, dz$$

if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\bar{\mathcal{P}})$ (independent of H). $\qquad \square$

Theorem 2.1 implies local uniqueness of solutions (existence will be discussed later).

Theorem 2.2. *Assume that* (f, g) *and* $(\tilde{f}, \tilde{g})$ *and are admissible (see* (Ad1)–(Ad2) *above) such that*

$$(f(x, y, z), g(x, y, z)) = (\tilde{f}(x, y, z), \tilde{g}(x, y, z)) \quad \text{for all } (x, y, z) \in \{0, L\} \times \mathbb{R}^2,$$

and both (f, g) *and* $(\tilde{f}, \tilde{g})$ *are solutions to* (4). *In addition let* $(\bar{f}, \bar{g})$ *be as in* Theorem 2.1 *and H be as in* Theorem 1.1 *(but* H_0 *can be assumed of class* C^2 *only). If* $(\nabla f, \nabla g)$ *and* $(\nabla \tilde{f}, \nabla \tilde{g})$ *are in a sufficiently small open convex neighborhood of* $(\nabla \bar{f}, \nabla \bar{g})$ *in* $C^2(\overline{\mathcal{P}})$ *and* $\|H_0''\|_{C(\overline{\mathcal{Q}})}$ *is sufficiently small, then* $(f, g) = (\tilde{f}, \tilde{g})$ *on* $[0, L] \times \mathbb{R}^2$.

Proof. If they were not equal, we could consider

$$(f_\theta, g_\theta) = \theta(\tilde{f}, \tilde{g}) + (1 - \theta)(f, g)$$

for θ in some slightly larger interval than $[0, 1]$. The map

$$\theta \to \int_{\mathcal{P}} \left\{ \tfrac{1}{2} |\nabla f_\theta \times \nabla g_\theta|^2 + H(f_\theta, g_\theta) \right\} dx \, dy \, dz$$

would be of class C^2, its derivative would vanish at $\theta = 0$ and $\theta = 1$, and its second derivative would be strictly positive on $[0, 1]$ (by Theorem 2.1), which is a contradiction. $\square$

Remark. The proof of Theorem 2.2 relies on the local convexity of the functional (3). It is natural to wonder if local convexity may lead to existence too. Theorem 2.1 shows that the quadratic form $B_{(f,g)}((F, G), (F, G))$ is positive definite if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ (independent of H as long as $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ is sufficiently small). However, as mentioned above, the quadratic form is not coercive at $(f, g) = (\bar{f}, \bar{g})$. This feature creates difficulties in getting good a priori bounds on minimizing sequences. One can hope that they may converge in some weak sense to some kind of weak solution and indeed such kind of results, in a more general setting, are obtained in [Buffoni 2012]. One can also wonder if some kind of regularization of the integral functional followed by a limiting process could lead to regular solutions. If this were feasible, it seems likely that it would rely on a regularity analysis similar to the one that follows. We leave these considerations for further works.

To implement a Nash–Moser iteration, we introduce for $\epsilon \in [0, 1]$ the *regularized quadratic form*

$$(F, G) \mapsto \int_{\mathcal{P}} \Big\{ \tfrac{1}{2} |\nabla F \times \nabla g + \nabla f \times \nabla G|^2 + (\nabla f \times \nabla g) \cdot (\nabla F \times \nabla G) + \tfrac{1}{2}\epsilon (|\nabla F|^2 + |\nabla G|^2)$$
$$+ \tfrac{1}{2} \big(\partial_f^2 H(f, g) F^2 + 2\partial_f \partial_g H(f, g) FG + \partial_g^2 H(f, g) G^2 \big) \Big\} \, dx \, dy \, dz,$$

which is clearly also positive definite if $(\nabla f, \nabla g)$ is in some small neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ and $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ is small enough, uniformly in $\epsilon \in [0, 1]$, and coercive for a fixed $\epsilon \in (0, 1]$. Again, the regularized quadratic form can be written $\tfrac{1}{2} B_{(f,g)}^\epsilon((F, G), (F, G))$, where $B_{(f,g)}^\epsilon$ is the corresponding symmetric bilinear form.

For an admissible $(f, g) \in C^3(\overline{D})$ (see (Ad1)–(Ad2) above), we are interested in the map $(\mu, \nu) \mapsto (F, G)$ defined as follows:

- $(F, G) \in H_{\text{loc}}^1(D)$ is admissible in the sense of (Ad'1)–(Ad'3).
- $(\mu, \nu) \in L_{\text{loc}}^2(D)$ is (P_1, P_2)-periodic in y and z.

- For all $\delta F, \delta G \in H^1_{\mathrm{loc}}(D)$ that are admissible in the sense of (Ad$'$1)–(Ad$'$3)

$$B^\epsilon_{(f,g)}((F, G), (\delta F, \delta G)) = \int_{\mathcal{P}} (\mu \delta F + \nu \delta G)\, dx\, dy\, dz. \tag{10}$$

If (f, g) is admissible and (F, G) is admissible in $H^2_{\mathrm{loc}}(D)$, (10) is equivalent to the system

$$\mu = -\operatorname{div}\big(\nabla g \times (\nabla F \times \nabla g + \nabla f \times \nabla G) + \nabla G \times (\nabla f \times \nabla g)\big) - \epsilon \Delta F + \partial^2_f H(f,g) F + \partial_f \partial_g H(f,g) G,$$
$$\nu = -\operatorname{div}\big((\nabla F \times \nabla g + \nabla f \times \nabla G) \times \nabla f + (\nabla f \times \nabla g) \times \nabla F\big) - \epsilon \Delta G + \partial_f \partial_g H(f,g) F + \partial^2_g H(f,g) G. \tag{11}$$

In particular, if $\epsilon = 0$, then the linear operator related to $B^\epsilon_{(f,g)}$ is the linearization of (4) around (f, g).

Thanks to the fact that the regularized quadratic form is positive definite, (F, G) is uniquely defined by (μ, ν). We leave for later the issue of the existence of (F, G) and its regularity, as dealt with in [Kohn and Nirenberg 1965].

Proposition 2.3. *Assume that $\nabla \bar{f}$ and $\nabla \bar{g}$ are constant, that the first component of $\bar{v}$ does not vanish and that* (7) *holds true. If f, g (admissible) are of class $C^3(\overline{D})$ and H (admissible) is of class $C^2(\mathbb{R}^2)$, $(\nabla f, \nabla g)$ is in some small enough neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ and $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ is small enough, then*

$$B^\epsilon_{(f,g)}((F, G), (F, G)) \geq \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla F)^2 + \tfrac{1}{16}(v \cdot \nabla G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32L^2}(F^2 + G^2) \right\} dx\, dy\, dz. \tag{12}$$

Moreover

$$\|(F, G)\|_{L^2(\mathcal{P})} \leq \frac{32L^2}{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2} \|(\mu, \nu)\|_{L^2(\mathcal{P})} \tag{13}$$

and

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla F)^2 + \tfrac{1}{16}(v \cdot \nabla G)^2 \right\} dx\, dy\, dz \leq \frac{32L^2}{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2} \|(\mu, \nu)\|^2_{L^2(\mathcal{P})}$$

for all periodic $(\mu, \nu) \in L^2_{\mathrm{loc}}(D)$ and all admissible $(F, G) \in H^1_{\mathrm{loc}}(D)$ satisfying (10). *These estimates are uniform in $\epsilon \in [0, 1]$.*

Proof. Assuming $|v'|$ and $|H''(f, g)|$ small enough (as we can), we get in (8)

$$(1 - O(\|v'\|_{C(\overline{\mathcal{P}})})) \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32L^2}(F^2 + G^2) + \tfrac{1}{2}\big(\partial^2_f H(f,g) F^2 + 2\partial_f \partial_g H(f,g) FG + \partial^2_g H(f,g) G^2\big)$$
$$\geq \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{64L^2}(F^2 + G^2)$$

and inequality (12) follows from (8). Applying (10) to $(\delta F, \delta G) = (F, G)$,

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla F)^2 + \tfrac{1}{16}(v \cdot \nabla G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32L^2}(F^2 + G^2) \right\} dx\, dy\, dz \leq B^\epsilon_{(f,g)}((F, G), (F, G))$$
$$\leq \|(\mu, \nu)\|_{L^2(\mathcal{P})} \|(F, G)\|_{L^2(\mathcal{P})},$$

$$\|(F, G)\|_{L^2(\mathcal{P})} \leq \frac{32L^2}{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2} \|(\mu, \nu)\|_{L^2(\mathcal{P})}$$

and

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla F)^2 + \tfrac{1}{16}(v \cdot \nabla G)^2 \right\} dx\, dy\, dz \le \frac{32L^2}{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2} \, \|(\mu, \nu)\|_{L^2(\mathcal{P})}^2. \qquad \square$$

Proposition 2.4. *Assume that the first component of $\bar{v}$ does not vanish and that $(\nabla f, \nabla g)$ is near enough to $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$. Then system* (11) *allows one to express the partial derivatives $\partial_{11}^2 F$ and $\partial_{11}^2 G$ linearly with respect to μ, ν, the other second-order partial derivatives of F and G, the first-order partial derivatives of F and G, and F and G. The coefficients of these two linear expressions are rational functions of f', g', f'', g'', $H''(f, g)$, ϵ (without singularities on $\overline{D}$). More precisely,*

$$\partial_{11}^2 F = a_1\mu + a_2\nu + a_3\partial_{12}^2 F + a_4\partial_{13}^2 F + a_5\partial_{22}^2 F + a_6\partial_{23}^2 F + a_7\partial_{33}^2 F + a_8\partial_{12}^2 G + a_9\partial_{13}^2 G + a_{10}\partial_{22}^2 G$$
$$+ a_{11}\partial_{23}^2 G + a_{12}\partial_{33}^2 G + a_{13}\partial_1 F + a_{14}\partial_2 F + a_{15}\partial_3 F + a_{16}\partial_1 G + a_{17}\partial_2 G + a_{18}\partial_3 G + a_{19}F + a_{20}G,$$

where each a_i, $1 \le i \le 20$, is of the form

$$a_i = \frac{Q_i}{v_1^2 + \epsilon|(\partial_2 f, \partial_3 f, \partial_2 g, \partial_3 g)|^2 + \epsilon^2}$$

for some polynomial

$$Q_i = \begin{cases} Q_i(f', g', \epsilon), & 1 \le i \le 12, \\ Q_i(f'', g''), & 13 \le i \le 18, \\ Q_i(H''), & 19 \le i \le 20. \end{cases}$$

The denominator does not vanish on $\overline{D}$ because $(\nabla f, \nabla g)$ is supposed near enough to $(\nabla \bar{f}, \nabla \bar{g})$ and $\epsilon \in [0, 1]$. Moreover, for all integers $1 \le i \le 20$ and $\ell \ge 0$,

$$\|a_i\|_{C^\ell(\overline{\mathcal{P}})} = \begin{cases} O(\|(f, g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 1 \le i \le 12, \\ O(\|(f, g)\|_{C^{\ell+2}(\overline{\mathcal{P}})} + 1), & 13 \le i \le 18, \\ O(\|H''(f, g)\|_{C^\ell(\overline{\mathcal{P}})} + \|(f, g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 19 \le i \le 20 \end{cases}$$

if all norms are well-defined. Analogous results hold for $\partial_{11}^2 G$ and all the estimates are uniform in $\epsilon \in [0, 1]$.

Proof. If we keep only the second-order terms in (F, G), we get

$$\mu = \nabla g \cdot \mathrm{rot}(\nabla F \times \nabla g + \nabla f \times \nabla G) - \epsilon \Delta F + \cdots,$$
$$\nu = -\mathrm{rot}(\nabla F \times \nabla g + \nabla f \times \nabla G) \cdot \nabla f - \epsilon \Delta G + \cdots.$$

Observe that

$$\mathrm{rot}(\nabla F \times \nabla g) = \Delta g \nabla F - \Delta F \nabla g + F'' \nabla g - g'' \nabla F$$

and thus

$$\mu = \nabla g \cdot ((F'' - \Delta F I)\nabla g) - \nabla g \cdot ((G'' - \Delta G I)\nabla f) - \epsilon \Delta F + \cdots,$$
$$\nu = -\nabla f \cdot ((F'' - \Delta F I)\nabla g) + \nabla f \cdot ((G'' - \Delta G I)\nabla f) - \epsilon \Delta G + \cdots,$$

where I is the identity matrix. To see that this allows one to express $\partial_{11}^2 F$ and $\partial_{11}^2 G$ with respect to μ, ν, the other second-order partial derivatives of F and G, and the first-order partial derivatives of F and G,

and F and G, it is sufficient to study

$$\mu = -\partial_{11}^2 F \nabla g \cdot (J \nabla g) + \partial_{11}^2 G \nabla f \cdot (J \nabla g) - \epsilon \partial_{11}^2 F + \cdots,$$
$$\nu = \partial_{11}^2 F \nabla f \cdot (J \nabla g) - \partial_{11}^2 G \nabla f \cdot (J \nabla f) - \epsilon \partial_{11}^2 G + \cdots,$$

where J is the diagonal matrix with entries $(0, 1, 1)$ on the diagonal and the remainders now also contain the other second-order partial derivatives of F and G. The discriminant of this system for $(\partial_{11}^2 F, \partial_{11}^2 G)$ is

$$(|J\nabla g|^2 + \epsilon)(|J\nabla f|^2 + \epsilon) - ((J\nabla f) \cdot (J\nabla g))^2 = |(J\nabla f) \times (J\nabla g)|^2 + \epsilon|J\nabla f|^2 + \epsilon|J\nabla g|^2 + \epsilon^2$$
$$= v_1^2 + \epsilon|J\nabla f|^2 + \epsilon|J\nabla g|^2 + \epsilon^2.$$

We estimate $\|a_i\|_{C^\ell(\overline{\mathcal{P}})}$, $1 \le i \le 20$, using the inequality

$$\|\xi(u_1, \ldots, u_N)\|_{C^k(\overline{\mathcal{P}})} \le C\|\xi\|_{C^k}(1 + \|u_1\|_{C^k(\overline{\mathcal{P}})} + \cdots + \|u_N\|_{C^k(\overline{\mathcal{P}})}) \tag{14}$$

for $\xi \in C^k([-M, M]^N)$ and $u_j \in C^k(\overline{\mathcal{P}})$ with $\|u_j\|_{C(\overline{\mathcal{P}})} \le M$ for $1 \le j \le N$, which, e.g., follows by interpolation in C^k spaces, see, e.g., of [Hamilton 1982, Theorem 2.2.1, p. 143], and the Faà di Bruno formula. Hence

$$O(\|a_i\|_{C^\ell(\overline{\mathcal{P}})}) = \begin{cases} O(\|(f, g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 1 \le i \le 12, \\ O(\|(f, g)\|_{C^{\ell+2}(\overline{\mathcal{P}})} + 1), & 13 \le i \le 18, \\ O(\|H''(f, g)\|_{C^\ell(\overline{\mathcal{P}})} + \|(f, g)\|_{C^{\ell+1}(\overline{\mathcal{P}})} + 1), & 19 \le i \le 20. \end{cases} \qquad \square$$

We now study to which extent $B^\epsilon_{(f,g)}$ commutes with differentiations in y and z, following the general approach of [Kohn and Nirenberg 1965].

Theorem 2.5. *Let* $(\nabla f, \nabla g)$ *be in any bounded subset of* $C^1(\overline{\mathcal{P}})$, $r \in \{1, 2, 3, \ldots\}$, $(f, g) \in C^{r+2}(\overline{D})$, $H \in C^{r+2}(\mathbb{R}^2)$ *and* $(F, G) \in H_{\mathrm{loc}}^{2r+1}(D)$ *(all admissible). Then, for* $j \in \{2, 3\}$,

$$B^\epsilon_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^\epsilon_{(f,g)}((F, G), (-1)^r(\partial_j^{2r} F, \partial_j^{2r} G))$$
$$= \sum_{p \in \mathcal{S}} \int_{\mathcal{P}} \partial_j^{2r-s_p-t_p} L_p \, \partial_j^{s_p} u_p \, \partial_j^{t_p} v_p \, dx \, dy \, dz + \sum_{p \in \widetilde{\mathcal{S}}} \int_{\mathcal{P}} \partial_j^{r-\tilde{s}_p} \tilde{L}_p \, \partial_j^{\tilde{s}_p} \tilde{u}_p \, \partial_j^r \tilde{v}_p \, dx \, dy \, dz,$$

where, for each p *in some finite sets* $\mathcal{S}$ *and* $\widetilde{\mathcal{S}}$ *of indices,*

$$0 \le s_p \le t_p \le r - 1, \quad 2 \le 2r - s_p - t_p \le r + 1, \quad 0 \le \tilde{s}_p \le r - 1$$

and

$$\{u_p, v_p\} \subset \{\partial_1 F, \partial_2 F, \partial_3 F, \partial_1 G, \partial_2 G, \partial_3 G\}, \quad \{\tilde{u}_p, \tilde{v}_p\} \subset \{F, G\}.$$

For each p, *the coefficient* $L_p(x, y, z)$ *is a polynomial of all partial derivatives of* f *and* g *of order* 1, *while* $\tilde{L}_p$ *is a second-order partial derivative of* H *(with respect to* f *and* g*). Moreover we have the following estimate, where the dependence on* r *is more explicitly stated:*

$$\left\| \sum_{\substack{p \in \mathcal{S}: \\ s_p = t_p = r-1}} \partial_j^{2r-s_p-t_p} L_p \right\|_{C(\overline{\mathcal{P}})} = \left\| \sum_{\substack{p \in \mathcal{S}: \\ s_p = t_p = r-1}} \partial_j^2 L_p \right\|_{C(\overline{\mathcal{P}})} = O(r^2)\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})} \tag{15}$$

(*the function $O(r^2)$ being independent of $f, g, F, G, H''(f, g)$ and ϵ*). *Finally, for the other indices p,*

$$\|\partial_j^{2r-s_p-t_p} L_p\|_{C(\overline{\mathcal{P}})} = O(\|(\nabla f, \nabla g)\|_{C^{2r-s_p-t_p}(\overline{\mathcal{P}})} + 1), \quad p \in \mathcal{S},$$

$$\|\partial_j^{r-\tilde{s}_p} \tilde{L}_p\|_{C(\overline{\mathcal{P}})} = O(\|H''(f, g)\|_{C^{r-\tilde{s}_p}(\overline{\mathcal{P}})}), \qquad p \in \widetilde{\mathcal{S}},$$

(16)

where the constants in the estimates may depend on r.

Remarks. The expression

$$B_{(f,g)}^{\epsilon}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B_{(f,g)}^{\epsilon}((F, G), (-1)^r (\partial_j^{2r} F, \partial_j^{2r} G))$$

would vanish if $(\nabla f, \nabla g)$ and $H''(f, g)$ were independent of y and z, and the statement allows one to estimate its size otherwise. In the statement, we add the property $s_p \leq t_p$. In fact we shall omit this property in the proof, as it is easy to get it by renaming s_p and t_p. The statement would be much easier if we would aim at the weaker inequality $0 \leq s_p \leq t_p \leq r$ (the proof would then rely on straightforward integrations by parts). The crucial regularity gain $s_p, t_p \leq r - 1$ has been explored in a general setting in [Kohn and Nirenberg 1965].

Proof. The typical term of $B_{(f,g)}^{\epsilon}((F, G), (F, G))$ is of either of the form

$$\int_{\mathcal{P}} 2L(x, y, z) u(x, y, z) v(x, y, z) \, dx \, dy \, dz,$$

where

$$\{u, v\} \subset \{\partial_1 F, \partial_2 F, \partial_3 F, \partial_1 G, \partial_2 G, \partial_3 G\}$$

and the coefficient $L(x, y, z)$ can be expressed as a polynomial of the partial derivatives of f and g of order 1, or of the form

$$\int_{\mathcal{P}} 2\tilde{L}(x, y, z) \tilde{u}(x, y, z) \tilde{v}(x, y, z) \, dx \, dy \, dz,$$

where

$$\{\tilde{u}, \tilde{v}\} \subset \{F, G\}$$

and $\tilde{L}$ is equal to $\partial_f^2 H(f, g)$, $2\partial_f \partial_g H(f, g)$ or $\partial_g^2 H(f, g)$. The typical term of

$$B_{(f,g)}^{\epsilon}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B_{(f,g)}^{\epsilon}((F, G), (-1)^r (\partial_j^{2r} F, \partial_j^{2r} G))$$

is therefore either of the form

$$\int_{\mathcal{P}} \left(2L\partial_j^r u \partial_j^r v - (-1)^r L v \partial_j^{2r} u - (-1)^r L u \partial_j^{2r} v\right) dx \, dy \, dz$$

or

$$\int_{\mathcal{P}} \left(2\tilde{L}\partial_j^r \tilde{u} \partial_j^r \tilde{v} - (-1)^r \tilde{L} \tilde{v} \partial_j^{2r} \tilde{u} - (-1)^r \tilde{L} \tilde{u} \partial_j^{2r} \tilde{v}\right) dx \, dy \, dz.$$

We only give the details for the first type of term since the argument for the second is similar but simpler (move r derivatives using integration by parts).

We get as in [Kohn and Nirenberg 1965] (but in a simpler setting)

$$\int_{\mathcal{P}} -(-1)^r Lv \partial_j^{2r} u \, dx \, dy \, dz = \int_{\mathcal{P}} \partial_j^{r+1}(Lv) \partial_j^{r-1} u \, dx \, dy \, dz$$

$$= \int_{\mathcal{P}} \sum_{k=0}^{r+1} \binom{r+1}{k} \partial_j^{r+1-k} L \partial_j^k v \partial_j^{r-1} u \, dx \, dy \, dz$$

$$= \int_{\mathcal{P}} L \partial_j^{r+1} v \partial_j^{r-1} u \, dx \, dy \, dz + \int_{\mathcal{P}} (r+1) \partial_j L \partial_j^r v \partial_j^{r-1} u \, dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \tfrac{1}{2} r(r+1) \partial_j^2 L \partial_j^{r-1} v \partial_j^{r-1} u \, dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \sum_{k=0}^{r-2} \binom{r+1}{k} \partial_j^{r+1-k} L \partial_j^k v \partial_j^{r-1} u \, dx \, dy \, dz$$

and thus, together with the equality one gets by permuting u and v,

$$\int_{\mathcal{P}} \left(2L \partial_j^r u \partial_j^r v - (-1)^r Lv \partial_j^{2r} u - (-1)^r Lu \partial_j^{2r} v \right) dx \, dy \, dz$$

$$= \int_{\mathcal{P}} L \partial_j^2 (\partial_j^{r-1} u \partial_j^{r-1} v) \, dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} (r+1) \partial_j L \partial_j (\partial_j^{r-1} u \partial_j^{r-1} v) \, dx \, dy \, dz + \int_{\mathcal{P}} r(r+1) \partial_j^2 L \partial_j^{r-1} u \partial_j^{r-1} v \, dx \, dy \, dz$$

$$+ \int_{\mathcal{P}} \sum_{k=0}^{r-2} \binom{r+1}{k} \partial_j^{r+1-k} L (\partial_j^k v \partial_j^{r-1} u + \partial_j^k u \partial_j^{r-1} v) \, dx \, dy \, dz$$

$$= r^2 \int_{\mathcal{P}} \partial_j^2 L \partial_j^{r-1} u \partial_j^{r-1} v \, dx \, dy \, dz + \int_{\mathcal{P}} \sum_{k=0}^{r-2} \binom{r+1}{k} \partial_j^{r+1-k} L (\partial_j^k v \partial_j^{r-1} u + \partial_j^k u \partial_j^{r-1} v) \, dx \, dy \, dz.$$

With respect to the j-th variable, L is differentiated at most $r+1$ times, and u and v at most $r-1$ times. Moreover the term containing $\partial_j^{r-1} u \partial_j^{r-1} v$ is given by

$$r^2 \int_{\mathcal{P}} \partial_j^2 L \partial_j^{r-1} u \partial_j^{r-1} v \, dx \, dy \, dz,$$

where

$$\|\partial_j^2 L\|_{C(\overline{\mathcal{P}})} = O(\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})})$$

(using the fact that $(\nabla f, \nabla g)$ is supposed to be in some bounded subset of the algebra $C^1(\overline{\mathcal{P}})$). To get (16), we use (14) with $k = 2r - s_p - t_p$ and $\xi = L$. □

In the two following results, everything is uniform in $\epsilon \in [0, 1]$ and we do not state explicitly the dependence on ϵ.

Proposition 2.6. *If $(f, g, H) \in C^3(\overline{D}) \times C^3(\overline{D}) \times C^3(\mathbb{R}^2)$ is admissible, $(\nabla f, \nabla g)$ is in some small enough neighborhood of $(\nabla \bar{f}, \nabla \bar{g})$ in $C^2(\overline{\mathcal{P}})$ and $\|H''(f, g)\|_{C(\overline{\mathcal{P}})}$ is small enough, then*

$$\|(F, G)\|_{H^1(\mathcal{P})} = O(\|H''(f, g)\|_{C^1(\overline{\mathcal{P}})} + 1)\|(\mu, \nu)\|_{H^1(\mathcal{P})} \tag{17}$$

and

$$\sum_{j \in \{2,3\}} \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j G)^2 \right\} dx \, dy \, dz = O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})} + 1)^2 \|(\mu, v)\|^2_{H^1(\mathcal{P})}$$

for all periodic $(\mu, v) \in H^1_{\mathrm{loc}}(D)$ *and all admissible* $(F, G) \in H^3_{\mathrm{loc}}(D)$ *satisfying* (10).

Proof. In Theorem 2.5, we consider $r = 1$. Applying (10) to $(\delta F, \delta G) = -(\partial_j^2 F, \partial_j^2 G)$ with $j \in \{2, 3\}$ and using Proposition 2.3, we get

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32 L^2}((\partial_j F)^2 + (\partial_j G)^2) \right\} dx \, dy \, dz$$

$$\leq B^\epsilon_{(f,g)}((\partial_j F, \partial_j G), (\partial_j F, \partial_j G))$$

$$= B^\epsilon_{(f,g)}((F, G), -(\partial_j^2 F, \partial_j^2 G)) + \{ B^\epsilon_{(f,g)}((\partial_j F, \partial_j G), (\partial_j F, \partial_j G)) - B^\epsilon_{(f,g)}((F, G), -(\partial_j^2 F, \partial_j^2 G)) \}$$

$$\overset{(10)}{=} \int_{\mathcal{P}} (\partial_j \mu \partial_j F + \partial_j v \partial_j G) \, dx \, dy \, dz + \{ B^\epsilon_{(f,g)}((\partial_j F, \partial_j G), (\partial_j F, \partial_j G)) - B^\epsilon_{(f,g)}((F, G), -(\partial_j^2 F, \partial_j^2 G)) \}$$

$$\overset{(15),(16)}{\leq} \|(\partial_j \mu, \partial_j v)\|_{L^2(\mathcal{P})} \|(\partial_j F, \partial_j G)\|_{L^2(\mathcal{P})} + O(\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})}) \|(F, G)\|^2_{H^1(\mathcal{P})}$$
$$+ O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})}) \|(F, G)\|_{L^2(\mathcal{P})} \|(\partial_j F, \partial_j G)\|_{L^2(\mathcal{P})}$$

$$\leq \|(\partial_j \mu, \partial_j v)\|_{L^2(\mathcal{P})} \|(\partial_j F, \partial_j G)\|_{L^2(\mathcal{P})} + O(\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})}) \|(F, G)\|^2_{H^1(\mathcal{P})}$$
$$+ \delta^{-1} O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})})^2 \|(F, G)\|^2_{L^2(\mathcal{P})} + \delta \|(\partial_j F, \partial_j G)\|^2_{L^2(\mathcal{P})}.$$

If, in addition,

$$\|(\partial_2 \nabla f, \partial_3 \nabla f, \partial_2 \nabla g, \partial_3 \nabla g)\|_{C^1(\overline{\mathcal{P}})} < \delta$$

and $\delta > 0$ is small enough, we get (note that the coefficient 32 is replaced by 64, and later by 128)

$$\sum_{j \in \{2,3\}} \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{64 L^2}((\partial_j F)^2 + (\partial_j G)^2) \right\} dx \, dy \, dz$$

$$\lesssim \|(\mu, v)\|^2_{H^1(\mathcal{P})} + \delta^{-1}(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})} + 1)^2 \|(F, G)\|^2_{L^2(\mathcal{P})} + \delta \|(\partial_1 F, \partial_1 G)\|^2_{L^2(\mathcal{P})}$$

$$\overset{(13)}{\lesssim} \|(\mu, v)\|^2_{H^1(\mathcal{P})} + \delta^{-1}(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})} + 1)^2 \|(\mu, v)\|^2_{L^2(\mathcal{P})} + \delta \|(\partial_1 F, \partial_1 G)\|^2_{L^2(\mathcal{P})}.$$

Using the last inequality in Proposition 2.3 to estimate $\|\partial_1 F\|^2_{L^2(\mathcal{P})}$ and $\|\partial_1 G\|^2_{L^2(\mathcal{P})}$ (using also the fact that the first component of v never vanishes), we obtain

$$\|(\partial_1 F, \partial_1 G)\|^2_{L^2(\mathcal{P})} = O(\|(\mu, v, \partial_2 F, \partial_2 G, \partial_3 F, \partial_3 G)\|^2_{L^2(\mathcal{P})})$$

and

$$\sum_{j \in \{2,3\}} \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j G)^2 \right\} dx \, dy \, dz + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{128 L^2} \|(\nabla F, \nabla G)\|^2_{L^2(\mathcal{P})}$$

$$= O(\|H''(f,g)\|_{C^1(\overline{\mathcal{P}})} + 1)^2 \|(\mu, v)\|^2_{H^1(\mathcal{P})}.$$

We get (17) by combining this with (13). $\qquad \square$

By induction, we get the following theorem.

Theorem 2.7. *Let $r \geq 1$ be an integer, $(f, g) \in H^{r+4}_{\text{loc}}(D)$ (admissible) be in some small enough neighborhood of $(\bar{f}, \bar{g})$ in $H^5(\mathcal{P})$, $H \in C^2(\mathbb{R}^2)$ be admissible, $H''(f, g) \in C^r(\overline{\mathcal{P}})$ and $H''(f, g)$ be small enough in $C(\overline{\mathcal{P}})$. There exists a constant $C_r > 0$ such that, if*

$$\|(\partial_2 \nabla f, \partial_3 \nabla f, \partial_2 \nabla g, \partial_3 \nabla g)\|_{C^1(\overline{\mathcal{P}})} < C_r^{-1}, \tag{18}$$

then

$$\sum_{j \in \{2,3\}} \int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j^r F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j^r G)^2 \right\} dx \, dy \, dz + \|(F, G)\|^2_{H^r(\mathcal{P})}$$

$$\leq C_r \|(\mu, v)\|^2_{H^r(\mathcal{P})} + C_r \|(\mu, v)\|^2_{H^1(\mathcal{P})} (\|(f, g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f, g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2 \tag{19}$$

for all periodic $(\mu, v) \in H^r(\mathcal{P})$ and all admissible $(F, G) \in H^{2r+1}_{\text{loc}}(D)$ satisfying (10).

Remarks. • In (18), all terms in the norm are differentiated at least once with respect to y or z. In the first sentence of the statement, the small neighborhood and the small bound on the size of $H''(f, g)$ in $C(\overline{\mathcal{P}})$ are independent of $r \geq 1$. The constant C_r can depend on them, on r, $\bar{f}$ and $\bar{g}$, but not on H, f and g.

• The r-dependence in (18) is due to the appearance of r in the estimate (15) in Theorem 2.5; see also (23) below.

• Unlike Theorem 2.1 where the constancy of $\bar{v}$ was not essential it really does matter here; see (18).

Proof. As the result is already known for $r = 1$ (see Proposition 2.6) let us assume that $r \geq 2$.

First step: We first bound from above

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j^r F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j^r G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32 L^2} ((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx \, dy \, dz$$

for $j \in \{2, 3\}$. We shall deal with $\partial_1^r F$ and $\partial_1^r G$ in the third and fourth steps. Applying (10) to $(\delta F, \delta G) = (-1)^r (\partial_j^{2r} F, \partial_j^{2r} G)$ with $j \in \{2, 3\}$, and using Proposition 2.3 we get

$$\int_{\mathcal{P}} \left\{ \tfrac{1}{16}(v \cdot \nabla \partial_j^r F)^2 + \tfrac{1}{16}(v \cdot \nabla \partial_j^r G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32 L^2} ((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx \, dy \, dz$$

$$\leq B^\epsilon_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G))$$

$$= B^\epsilon_{(f,g)}((F, G), (-1)^r (\partial_j^{2r} F, \partial_j^{2r} G))$$

$$+ \{B^\epsilon_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^\epsilon_{(f,g)}((F, G), (-1)^r (\partial_j^{2r} F, \partial_j^{2r} G))\}$$

$$\overset{(10)}{=} \int_{\mathcal{P}} (\partial_j^r \mu \partial_j^r F + \partial_j^r v \partial_j^r G) \, dx \, dy \, dz$$

$$+ \{B^\epsilon_{(f,g)}((\partial_j^r F, \partial_j^r G), (\partial_j^r F, \partial_j^r G)) - B^\epsilon_{(f,g)}((F, G), (-1)^r (\partial_j^{2r} F, \partial_j^{2r} G))\}.$$

By Theorem 2.5,

$$\int_{\mathcal{P}} \left\{ \frac{1}{16}(v \cdot \nabla \partial_j^r F)^2 + \frac{1}{16}(v \cdot \nabla \partial_j^r G)^2 + \frac{\pi^2 \min_{\overline{\mathcal{P}}} v_1^2}{32 L^2}((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx\, dy\, dz$$

$$\leq \|(\partial_j^r \mu, \partial_j^r \nu)\|_{L^2(\mathcal{P})} \|(\partial_j^r F, \partial_j^r G)\|_{L^2(\mathcal{P})} + O(r^2)\|(\partial_j \nabla f, \partial_j \nabla g)\|_{C^1(\overline{\mathcal{P}})} \|(F, G)\|_{H^r(\mathcal{P})}^2$$

$$+ \sum O(\|(f, g)\|_{H^{k_1+3}(\mathcal{P})} + 1)\|(F, G)\|_{H^{k_2+1}(\mathcal{P})} \|(F, G)\|_{H^{k_3+1}(\mathcal{P})}$$

$$+ \sum O(\|H''(f, g)\|_{C^{r-k_4}(\overline{\mathcal{P}})}) \|(F, G)\|_{H^{k_4}(\mathcal{P})} \|(F, G)\|_{H^r(\mathcal{P})}, \quad (20)$$

where the sums are over all integers $k_1, k_2, k_3 \geq 0$ such that

$$k_1 + k_2 + k_3 = 2r, \quad k_1 \leq r+1, \quad k_2 \leq k_3 \leq r-1, \quad k_2 + k_3 < 2r - 2$$

(this implies $k_1 > 2$ and, as $r \geq 2$, $k_2 + k_3 > 0$) and $0 \leq k_4 \leq r-1$. Here and in the following estimates, we only indicate the r-dependence in the coefficients of $\|(F, G)\|_{H^r(\mathcal{P})}$. We don't keep track of the r-dependence of the lower-order terms.

By standard interpolation in Sobolev spaces based on the equality

$$k_j + 1 = \frac{r-1-k_j}{r-1} \cdot 1 + \frac{k_j}{r-1} \cdot r, \quad j = 2, 3,$$

see, e.g., [Han and Hong 2006, Section 4.3], the first sum can be estimated by

$$\sum O(\|(f, g)\|_{H^{k_1+3}(\mathcal{P})} + 1)\|(F, G)\|_{H^1(\mathcal{P})}^{(k_1-2)/(r-1)} \|(F, G)\|_{H^r(\mathcal{P})}^{(2r-k_1)/(r-1)}$$

$$= \sum \{O(\|(f, g)\|_{H^{k_1+3}(\mathcal{P})} + 1)^{(2(r-1))/(k_1-2)} \delta^{-(2r-k_1)/(k_1-2)} \|(F, G)\|_{H^1(\mathcal{P})}^2 \}^{(k_1-2)/(2(r-1))}$$

$$\times \{\delta \|(F, G)\|_{H^r(\mathcal{P})}^2 \}^{(2r-k_1)/(2(r-1))},$$

where $\delta > 0$ will be chosen as small as needed. The choice of $\delta > 0$ can depend on r, $\bar{f}$ and $\bar{g}$, but not on (F, G), (μ, ν), H, f and g. In what follows, we write explicitly some negative powers of δ, even when they can be merged with other positive factors, for example those referred to in the notation $\lesssim$ (possibly depending on r, $\bar{f}$ and $\bar{g}$). By Young's inequality for products, $xy \leq p^{-1}x^p + q^{-1}y^q$ with $p = 2(r-1)/(k_1-2)$, $q = 2(r-1)/(2r-k_1)$, and interpolation based on the equality

$$k_1 + 3 = \frac{r+1-k_1}{r-1} \cdot 5 + \frac{k_1-2}{r-1} \cdot (r+4),$$

this can in turn be estimated by

$$\delta \|(F, G)\|_{H^r(\mathcal{P})}^2 + \sum \delta^{-(2r-k_1)/(k_1-2)} O(\|(f, g)\|_{H^{k_1+3}(\mathcal{P})} + 1)^{2(r-1)/(k_1-2)} \|(F, G)\|_{H^1(\mathcal{P})}^2$$

$$\lesssim \delta \|(F, G)\|_{H^r(\mathcal{P})}^2 + \sum \delta^{-(2r-k_1)/(k_1-2)}$$

$$\times (\|(f, g)\|_{H^5(\mathcal{P})} + 1)^{2(r+1-k_1)/(k_1-2)} (\|(f, g)\|_{H^{r+4}(\mathcal{P})} + 1)^2 \|(F, G)\|_{H^1(\mathcal{P})}^2.$$

By Proposition 2.6, the sum is thus estimated above:

$$\sum (\|(f, g)\|_{H^{k_1+3}(\mathcal{P})} + 1)\|(F, G)\|_{H^{k_2+1}(\mathcal{P})} \|(F, G)\|_{H^{k_3+1}(\mathcal{P})}$$

$$\lesssim \delta^{-2r}(\|(f, g)\|_{H^{r+4}(\mathcal{P})} + 1)^2 \|(\mu, \nu)\|_{H^1(\mathcal{P})}^2 + \delta \|(F, G)\|_{H^r(\mathcal{P})}^2. \quad (21)$$

We have also used that, by assumption, (f, g) is in some small enough neighborhood of $(\bar{f}, \bar{g})$ in $H^5(\mathcal{P})$.

The second sum can similarly be estimated as follows:

$$\sum \|H''(f,g)\|_{C^{r-k_4}(\overline{\mathcal{P}})}\|(F,G)\|_{H^{k_4}(\mathcal{P})}\|(F,G)\|_{H^r(\mathcal{P})}$$

$$\lesssim \sum \|H''(f,g)\|_{C(\overline{\mathcal{P}})}^{k_4/r}\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}^{(r-k_4)/r}\|(F,G)\|_{L^2(\mathcal{P})}^{(r-k_4)/r}\|(F,G)\|_{H^r(\mathcal{P})}^{(r+k_4)/r}$$

$$\overset{(13)}{\lesssim} \delta^{-2r}\|H''(f,g)\|_{C^r(\overline{\mathcal{P}})}^2\|(\mu,\nu)\|_{L^2(\mathcal{P})}^2 + \delta\|(F,G)\|_{H^r(\mathcal{P})}^2. \tag{22}$$

Let us now choose

$$\|(\partial_2\nabla f,\partial_3\nabla f,\partial_2\nabla g,\partial_3\nabla g)\|_{C^1(\overline{\mathcal{P}})} < r^{-2}\delta. \tag{23}$$

If δ is small enough (this is allowed by assumption (18)), then, by (20)–(22) (note that the coefficient 32 is replaced by 64),

$$\sum_{j\in\{2,3\}} \int_{\mathcal{P}} \left\{ \frac{1}{16}(v\cdot\nabla\partial_j^r F)^2 + \frac{1}{16}(v\cdot\nabla\partial_j^r G)^2 + \frac{\pi^2\min_{\overline{\mathcal{P}}}v_1^2}{64L^2}((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx\,dy\,dz + \|(F,G)\|_{L^2(\mathcal{P})}^2$$

$$\lesssim \|(F,G)\|_{L^2(\mathcal{P})}^2 + \delta^{-1}\|(\mu,\nu)\|_{H^r(\mathcal{P})}^2 + \delta\|(F,G,\partial_1 F,\partial_1 G)\|_{H^{r-1}(\mathcal{P})}^2$$

$$+ \delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f,g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2\|(\mu,\nu)\|_{H^1(\mathcal{P})}^2$$

because, for $\hat{r}=r$,

$$\sum_{|\alpha_2|+|\alpha_3|\leq\hat{r}} \|(\partial^\alpha F,\partial^\alpha G)\|_{L^2(\mathcal{P})}^2 \lesssim \|(F,G)\|_{L^2(\mathcal{P})}^2 + \sum_{j\in\{2,3\}} \|(\partial_j^{\hat{r}} F,\partial_j^{\hat{r}} G))\|_{L^2(\mathcal{P})}^2, \tag{24}$$

where the sum is over all multi-indices $\alpha = (\alpha_2,\alpha_3)\in\mathbb{N}_0^2$ such that $|\alpha_2|+|\alpha_3|\leq\hat{r}$ and ∂^α is the corresponding partial derivative with respect to the variables (y,z). Thanks to the induction hypothesis,

$$\sum_{j\in\{2,3\}} \int_{\mathcal{P}} \left\{ \frac{1}{16}(v\cdot\nabla\partial_j^r F)^2 + \frac{1}{16}(v\cdot\nabla\partial_j^r G)^2 + \frac{\pi^2\min_{\overline{\mathcal{P}}}v_1^2}{64L^2}((\partial_j^r F)^2 + (\partial_j^r G)^2) \right\} dx\,dy\,dz + \|(F,G)\|_{L^2(\mathcal{P})}^2$$

$$\lesssim \delta^{-1}\|(\mu,\nu)\|_{H^r(\mathcal{P})}^2 + \delta\|(\partial_1 F,\partial_1 G)\|_{H^{r-1}(\mathcal{P})}^2$$

$$+ \delta^{-2r}(\|(f,g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f,g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2\|(\mu,\nu)\|_{H^1(\mathcal{P})}^2. \tag{25}$$

<u>Second step</u>: Let us now deal with the terms containing only one partial derivative with respect to x and $r-1$ partial derivatives with respect to y or z. By induction, we know that

$$\sum_{j\in\{2,3\}} \int_{\mathcal{P}} \left\{ \frac{1}{16}(v\cdot\nabla\partial_j^{r-1} F)^2 + \frac{1}{16}(v\cdot\nabla\partial_j^{r-1} G)^2 \right\} dx\,dy\,dz + \|(F,G)\|_{H^{r-1}(\mathcal{P})}^2$$

$$\leq C_{r-1}\|(\mu,\nu)\|_{H^{r-1}(\mathcal{P})}^2 + C_{r-1}\|(\mu,\nu)\|_{H^1(\mathcal{P})}^2(\|(f,g)\|_{H^{r+3}(\mathcal{P})} + \|H''(f,g)\|_{C^{r-1}(\overline{\mathcal{P}})} + 1)^2$$

and thus

$$\sum_{j\in\{2,3\}} \|(\partial_1\partial_j^{r-1} F,\partial_1\partial_j^{r-1} G)\|_{L^2(\mathcal{P})}^2$$

$$\lesssim \sum_{j\in\{2,3\}} \|(\partial_2\partial_j^{r-1} F,\partial_2\partial_j^{r-1} G,\partial_3\partial_j^{r-1} F,\partial_3\partial_j^{r-1} G)\|_{L^2(\mathcal{P})}^2$$

$$+ \|(\mu,\nu)\|_{H^{r-1}(\mathcal{P})}^2 + \|(\mu,\nu)\|_{H^1(\mathcal{P})}^2(\|(f,g)\|_{H^{r+3}(\mathcal{P})} + \|H''(f,g)\|_{C^{r-1}(\overline{\mathcal{P}})} + 1)^2$$

because the first component of v never vanishes. Together with the first step and thanks to (24) with $\hat{r} = r$, this gives

$$\|(F, G)\|_{L^2(\mathcal{P})}^2 + \sum_{j \in \{2,3\}} \|(\partial_1 \partial_j^{r-1} F, \partial_1 \partial_j^{r-1} G)\|_{L^2(\mathcal{P})}^2$$
$$\lesssim \delta^{-1} \|(\mu, v)\|_{H^r(\mathcal{P})}^2 + \delta \|(\partial_1 F, \partial_1 G)\|_{H^{r-1}(\mathcal{P})}^2$$
$$+ \delta^{-2r} (\|(f, g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f, g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2 \|(\mu, v)\|_{H^1(\mathcal{P})}^2.$$

Applying (24) to $\hat{r} = r - 1$ and to $(\partial_1 F, \partial_1 G)$, we obtain for small enough δ

$$\|(F, G)\|_{L^2(\mathcal{P})}^2 + \|(\partial_1 F, \partial_1 G)\|_{L^2(\mathcal{P})}^2 + \sum_{j \in \{2,3\}} \|(\partial_j^{r-1} \partial_1 F, \partial_j^{r-1} \partial_1 G)\|_{L^2(\mathcal{P})}^2$$
$$\lesssim \delta^{-1} \|(\mu, v)\|_{H^r(\mathcal{P})}^2 + \delta \|(\partial_1^2 F, \partial_1^2 G)\|_{H^{r-2}(\mathcal{P})}^2$$
$$+ \delta^{-2r} (\|(f, g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f, g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2 \|(\mu, v)\|_{H^1(\mathcal{P})}^2. \quad (26)$$

Third step: We now deal with partial derivatives in which F and G are differentiated at least twice with respect to x. We estimate these using induction on the number of partial derivatives with respect to x for a fixed r. In the special case $r = 2$ there is only one second-order partial derivative to estimate, and we simply note directly using Proposition 2.4 that

$$\|(\partial_1^2 F, \partial_1^2 G)\|_{L^2(\mathcal{P})} \lesssim \|(\mu, v)\|_{L^2(\mathcal{P})} + \|(\partial_2 \nabla F, \partial_2 \nabla G, \partial_3 \nabla F, \partial_3 \nabla G)\|_{L^2(\mathcal{P})} + \|(F, G)\|_{H^1(\mathcal{P})}$$
$$\overset{(17)}{\lesssim} (\|H''(f, g)\|_{C^1(\overline{\mathcal{P}})} + 1)\|(\mu, v)\|_{H^1(\mathcal{P})} + \|(\partial_2 \nabla F, \partial_2 \nabla G, \partial_3 \nabla F, \partial_3 \nabla G)\|_{L^2(\mathcal{P})}.$$

Next, let $r > 2$ and B_s be a differential operator of order $r - 2$ in (x, y, z) that consists of an iteration of $r - 2$ partial derivatives, exactly s of which are with respect to x $(0 \leq s \leq r - 2)$. Differentiating $r - 2$ times the expressions for $\partial_1^2 F$ and $\partial_1^2 G$ in Proposition 2.4, we get

$$\|(B_s \partial_1^2 F, B_s \partial_1^2 G)\|_{L^2(\mathcal{P})} \lesssim \sum_{k=0}^{r-2} (\|(f, g)\|_{H^{r+1-k}(\mathcal{P})} + 1)\|(\mu, v)\|_{H^k(\mathcal{P})}$$
$$+ \sum_{k=0}^{r-2} (\|(f, g)\|_{H^{r+2-k}(\mathcal{P})} + 1)\|(F, G)\|_{H^{k+1}(\mathcal{P})}$$
$$+ \sum_{k=0}^{r-2} \|H''\|_{C^{r-2-k}(\overline{\mathcal{P}})} \|(F, G)\|_{H^k(\mathcal{P})}$$
$$+ (\|(f, g)\|_{H^3(\mathcal{P})} + 1)(\|D_s(\partial_2 F, \partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F, \partial_3 G)\|_{L^2(\mathcal{P})}),$$

where D_s and E_s are matricial differential operators of order $r - 1$ in (x, y, z), but at most of order $s + 1$ when seen as differential operators in x (their coefficients being constants). The terms involving E_s and D_s come from applying B_s to the terms in Proposition 2.4 involving $\partial_{\alpha\beta}^2 F$ or $\partial_{\alpha\beta}^2 G$ with $(\alpha, \beta) \neq (1, 1)$. The last inequality allows one to estimate differential expressions of order $s + 2$ with respect to x by differential expressions of orders at most $s + 1$ with respect to x.

We get again by interpolation and Young's inequality

$$
\begin{aligned}
\|(B_s\partial_1^2 F, B_s\partial_1^2 G)\|_{L^2(\mathcal{P})} & \\
&\lesssim (\|(f,g)\|_{H^{r+1}(\mathcal{P})} + 1)\|(\mu,\nu)\|_{L^2(\mathcal{P})} + (\|(f,g)\|_{H^3(\mathcal{P})} + 1)\|(\mu,\nu)\|_{H^{r-2}(\mathcal{P})} \\
&\quad + (\|(f,g)\|_{H^{r+3}(\mathcal{P})} + 1)\|(F,G)\|_{L^2(\mathcal{P})} + (\|(f,g)\|_{H^4(\mathcal{P})} + 1)\|(F,G)\|_{H^{r-1}(\mathcal{P})} \\
&\quad + \|H''(f,g)\|_{C^{r-2}(\overline{\mathcal{P}})}\|(F,G)\|_{L^2(\mathcal{P})} + \|H''(f,g)\|_{C(\overline{\mathcal{P}})}\|(F,G)\|_{H^{r-2}(\mathcal{P})} \\
&\quad + (\|(f,g)\|_{H^3(\mathcal{P})} + 1)(\|D_s(\partial_2 F, \partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F, \partial_3 G)\|_{L^2(\mathcal{P})}) \\
&\lesssim (\|(f,g)\|_{H^{r+3}(\mathcal{P})} + \|H''(f,g)\|_{C^{r-2}(\overline{\mathcal{P}})} + 1)\|(\mu,\nu)\|_{L^2(\mathcal{P})} \\
&\quad + \|(\mu,\nu)\|_{H^{r-2}(\mathcal{P})} + \|(F,G)\|_{H^{r-1}(\mathcal{P})} + \|D_s(\partial_2 F, \partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F, \partial_3 G)\|_{L^2(\mathcal{P})} \\
&\lesssim (\|(f,g)\|_{H^{r+3}(\mathcal{P})} + \|H''(f,g)\|_{C^{r-1}(\overline{\mathcal{P}})} + 1)\|(\mu,\nu)\|_{H^1(\mathcal{P})} \\
&\quad + \|(\mu,\nu)\|_{H^{r-1}(\mathcal{P})} + \|D_s(\partial_2 F, \partial_2 G)\|_{L^2(\mathcal{P})} + \|E_s(\partial_3 F, \partial_3 G)\|_{L^2(\mathcal{P})},
\end{aligned}
$$

where we've used the induction hypothesis (19) with r replaced by $r-1$ in the last step. By induction on s, we get the estimate

$$
\begin{aligned}
\|(B_s\partial_1^2 F, B_s\partial_1^2 G)\|_{L^2(\mathcal{P})} &\lesssim (\|(f,g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f,g)\|_{C^r(\overline{\mathcal{P}})} + 1)\|(\mu,\nu)\|_{H^1(\mathcal{P})} \\
&\quad + \|(\mu,\nu)\|_{H^r(\mathcal{P})} + \sum_{j\in\{2,3\}} \|(\partial_j^{r-1}\partial_1 F, \partial_j^{r-1}\partial_1 G)\|_{L^2(\mathcal{P})} + \delta\|(\partial_1^2 F, \partial_1^2 G)\|_{H^{r-2}(\mathcal{P})},
\end{aligned}
$$

thanks to (24) applied to (F,G) and $(\partial_1 F, \partial_1 G)$, and to (25). Hence, choosing δ sufficiently small

$$
\begin{aligned}
\|(\partial_1^2 F, \partial_1^2 G)\|_{H^{r-2}(\mathcal{P})} &\lesssim (\|(f,g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f,g)\|_{C^r(\overline{\mathcal{P}})} + 1)\|(\mu,\nu)\|_{H^1(\mathcal{P})} \\
&\quad + \|(\mu,\nu)\|_{H^r(\mathcal{P})} + \sum_{j\in\{2,3\}} \|(\partial_j^{r-1}\partial_1 F, \partial_j^{r-1}\partial_1 G)\|_{L^2(\mathcal{P})}. \quad (27)
\end{aligned}
$$

Combining (27) with (26) and again choosing δ sufficiently small allows us to estimate all partial derivatives of order r with precisely one derivative with respect to x. Substitution of the resulting estimate into (27) gives us control of all derivatives with at least two derivatives with respect to x.

Conclusion: The estimate of the statement follows from the three steps. $\qquad\square$

Let us deal with the case $\epsilon = 0$ with the help of the technique of elliptic regularization introduced and well explained in [Kohn and Nirenberg 1965]; see, e.g, p. 449, the beginning of the proof of Theorem 2 and the proof of Theorem 2′ in that work. Firstly, when $\epsilon > 0$, one deduces from this a priori estimate the existence of an admissible solution $(F,G) \in H^r(\mathcal{P})$ given any $(\mu,\nu) \in H^r(\mathcal{P})$, by approximating (f,g), $H''(f,g)$ itself and (μ,ν) by smooth functions. The existence of (F,G) is ensured because the problem is elliptic in this case. Secondly, as the above estimate holds uniformly in $\epsilon \in (0,1]$, the existence persists when taking the limit $\epsilon \to 0$. Thus we get the following theorem.

Theorem 2.8. *Let $\epsilon = 0$, $r \geq 1$ be an integer, $(f,g) \in H_{\mathrm{loc}}^{r+4}(D)$ (admissible) be in some small enough neighborhood of $(\bar{f}, \bar{g})$ in $H^5(\mathcal{P})$, $H \in C^2(\mathbb{R}^2)$ be admissible, $H''(f,g) \in C^r(\overline{D})$ and $H''(f,g)$ be small*

enough in $C(\overline{\mathcal{P}})$. *There exists a constant* $C_r > 0$ *such that if*

$$\|(\partial_2 \nabla f, \partial_3 \nabla f, \partial_2 \nabla g, \partial_3 \nabla g)\|_{C^1(\overline{\mathcal{P}})} < C_r^{-1},$$

then for any periodic $(\mu, \nu) \in H_{\mathrm{loc}}^r(D)$ *there exists an admissible* $(F, G) \in H_{\mathrm{loc}}^r(D)$ *satisfying* (10) (*with* $\epsilon = 0$) *and*

$$\|(F, G)\|_{H^r(\mathcal{P})}^2 \leq C_r \|(\mu, \nu)\|_{H^r(\mathcal{P})}^2 + C_r \|(\mu, \nu)\|_{H^1(\mathcal{P})}^2 (\|(f, g)\|_{H^{r+4}(\mathcal{P})} + \|H''(f, g)\|_{C^r(\overline{\mathcal{P}})} + 1)^2.$$

This result remains true without the simplifying hypothesis (7).

3. A solution by the Nash–Moser method

In this section we shall take $\bar{f}$ and $\bar{g}$ to be some fixed linear functions and let R be the corresponding Jacobian matrix with respect to (y, z) as in the Introduction.

Let us define three decreasing sequences of Banach spaces.

Definition of the Banach spaces $\mathcal{U}_k$. For each integer $k \geq 2$, let $\mathcal{U}_k$ be the real linear space of all (F, G) in $H_{\mathrm{loc}}^k(D)$ satisfying (Ad'2) and (Ad'3). We define the norm $\|\cdot\|_k$ on $\mathcal{U}_k$ as

$$\|(F, G)\|_k^2 = \|F\|_{H^k(\mathcal{P})}^2 + \|G\|_{H^k(\mathcal{P})}^2.$$

Definition of the Banach spaces $\mathcal{V}_k$. For each integer $k \geq 0$, let $\mathcal{V}_k$ be the real linear space of all (μ, ν) in $H_{\mathrm{loc}}^k(D)$ that satisfy the periodicity condition (Ad'2) almost everywhere. We define the norm $\|\cdot\|_k$ on $\mathcal{V}_k$ by

$$\|(\mu, \nu)\|_k^2 = \|\mu\|_{H^k(\mathcal{P})}^2 + \|\nu\|_{H^k(\mathcal{P})}^2.$$

Definition of the Banach spaces $\mathcal{W}_k$. For each integer $k \geq 4$, let $\mathcal{W}_k$ be the real linear space of (f_0, g_0, H_0, c) such that

(i) $f_0, g_0 \in H_{\mathrm{loc}}^k(D)$ satisfy the periodicity condition (Ad'2),

(ii) $H_0 \in C^{k-2}(\mathbb{R}^2)$ is periodic with respect to the lattice generated by $R P_1 e_1$ and $R P_2 e_2$, and $c \in \mathbb{R}^2$.

Note that (ii) ensures that $H_0(\bar{f} + f_0 + f_1, \bar{g} + g_0 + g_1)$ satisfies (Ad'2) for all $(f_1, g_1) \in \mathcal{U}_k$.

We define the norm $\|\cdot\|_k$ on $\mathcal{W}_k$ by

$$\|(f_0, g_0, H_0, c)\|_k^2 = \|f_0\|_{H^k(\mathcal{P})}^2 + \|g_0\|_{H^k(\mathcal{P})}^2 + \|H_0\|_{C^{k-2}(\overline{\mathcal{Q}})}^2 + |c|^2.$$

Given $(f_0, g_0, H_0, c) \in \mathcal{W}_4$, with $H_0 \in C^3(\mathbb{R}^2)$, we define the map $\mathcal{F} : \mathcal{U}_4 \to \mathcal{V}_2$ by

$$\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \to \mathcal{F} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} -\operatorname{div}(\nabla g \times (\nabla f \times \nabla g)) + \partial_f H(f, g) \\ -\operatorname{div}((\nabla f \times \nabla g) \times \nabla f)) + \partial_g H(f, g) \end{pmatrix}$$

with $f = \bar{f} + f_0 + f_1$, $g = \bar{g} + g_0 + g_1$ and $H(f, g) = c_1 f + c_2 g + H_0(f, g)$.

The following theorem results directly from Theorem 2.8 and (14) (with $\xi = H$).

Theorem 3.1. *Let $k \geq 1$ be an integer and suppose that $(f_0, g_0, H_0, c) \in \mathcal{W}_{k+4}$, $(f_1, g_1) \in \mathcal{U}_{k+4}$, $\|H_0''\|_{C(\bar{Q})}$ is small enough, and (f, g) is in some small enough neighborhood of $(\bar{f}, \bar{g})$ in $H^5(\mathcal{P})$, with*

$$f = \bar{f} + f_0 + f_1, \quad g = \bar{g} + g_0 + g_1 \quad and \quad H(f, g) = c_1 f + c_2 g + H_0(f, g).$$

There exists a constant $M_k > 0$ such that if

$$\|(\partial_2 \nabla f, \partial_3 \nabla f, \partial_2 \nabla g, \partial_3 \nabla g)\|_{C^1(\bar{\mathcal{P}})} < M_k^{-1},$$

we get the following. Given any $(\mu, \nu) \in \mathcal{V}_k$, there exists a unique $(F, G) \in \mathcal{U}_k$ satisfying (10) with $\epsilon = 0$. It also satisfies

$$\|(F, G)\|_k \leq M_k \|(\mu, \nu)\|_k + M_k \|(\mu, \nu)\|_1 (\|(f_1, g_1)\|_{H^{k+4}(\mathcal{P})} + 1)$$

and

$$\|(F, G)\|_0 \leq M_0 \|(\mu, \nu)\|_0$$

for some constant $M_0 > 0$ independent of k.

Remark. The constants M_k in Theorem 3.1 can also depend on (f_0, g_0, H_0, c) and $(\bar{f}, \bar{g})$.

Let us state Theorem 6.3.1 in [Han and Hong 2006]. There Ω is a smooth domain in $\mathbb{R}^n$ or a rectangle with the sides parallel to the coordinate axes and with periodic boundary conditions with respect to $n - 1$ coordinates. The corresponding Sobolev spaces are simply denoted by H^k.

Theorem 3.2. *Suppose $\mathcal{F}(w)$ is a nonlinear differential operator of order m in a domain $\Omega \subset \mathbb{R}^n$, given by*

$$\mathcal{F}(w) = \Gamma(x, w, \partial w, \dots, \partial^m w),$$

where Γ is smooth (see, however, the remark below).

Suppose that d_0, d_1, d_2, d_3, s_0 and $\tilde{s}$ are nonnegative integers with

$$d_0 \geq m + [n/2] + 1$$

and

$$\tilde{s} \geq \max\{3m + 2d_* + [n/2] + 2, m + d_* + d_0 + 1, m + d_2 + d_3 + 1\},$$

where $d_* = \max\{d_1, d_3 - s_0 - 1\}$. Assume that, for any $h \in H^{\tilde{s}+d_1} = H^{\tilde{s}+d_1}(\Omega)$ and $w \in H^{\tilde{s}+d_2}$ with

$$\|w\|_{H^{d_0}} \leq r_0 := 1,$$

the linear equation

$$\mathcal{F}'(w)\rho = h \tag{28}$$

admits a solution $\rho \in H^{\tilde{s}}$ satisfying for any $s = 0, 1, \dots, \tilde{s}$

$$\|\rho\|_{H^s} \leq c_s(\|h\|_{H^{s+d_1}} + (s - s_0)^+ (\|w\|_{H^{s+d_2}} + 1)\|h\|_{H^{d_3}}),$$

where c_s is a positive constant independent of h, w and ρ. Then there exists a positive constant μ_*, depending only on $\Omega, c_s, m, d_0, d_1, d_2, d_3, s_0$ and $\tilde{s}$, such that if

$$\|\mathcal{F}(0)\|_{H^{\tilde{s}-m}} \leq \mu_*^2, \tag{29}$$

the equation $\mathcal{F}(w) = 0$ admits an $H^{\tilde{s}-m-d_*-1}$ solution w in Ω.

Remarks. • By inspecting the proof in [Han and Hong 2006], we see that it holds as well for systems of $N \geq 1$ differential equations. Moreover the constant $r_0 = 1$ can be replaced by any fixed value $r_0 > 0$ by multiplying appropriately functions by constant factors.

• Also the solution w is the limit in $H^{\tilde{s}-m-d_*-1}$ of sums of solutions in $H^{\tilde{s}}$ to linear equations of type (28). See in [loc. cit.] equations (6.3.14) and (6.3.15), and the proof of Theorem 6.3.1 on p. 103.

• We can relax the condition that Γ is smooth. Let $\hat{c} > 0$ be such that, for all $w \in H^{d_0}$ with $\|w\|_{H^{d_0}} \leq r_0$, we have

$$\|w\|_{C^m(\bar{\Omega})} \leq \hat{c},$$

and define $\Sigma \subset \mathbb{R}^{N+Nn+Nn^2+\cdots+Nn^m}$ as the ball of radius $\hat{c}$ centered at the origin. In the proof, the map Γ appears in the various estimates via $\|\mathcal{F}(0)\|_{H^{\tilde{s}-m}}$ and via "constants" depending on

$$\|\partial_\alpha \partial_\beta \Gamma\|_{C^{\tilde{s}-m}(\bar{\Omega} \times \bar{\Sigma})},$$

where ∂_α and ∂_β are all possible partial derivatives with respect to $w, \ldots, \partial^m w$. See (14) and, in [loc. cit.], the proof of $(P_3)_{\ell+1}$ on p. 101. It therefore suffices to assume that Γ is of class $C^{\tilde{s}-m+2}$.

• From [loc. cit.] it follows that there exists a constant $C > 0$ such that $\|w\|_{H^{\tilde{s}-m-d_*-1}} \leq C\mu_*^2$. More precisely, see in [loc. cit.] the last estimate in the proof of $(P_1)_{l+1}$ on p. 100, (6.3.31) and the proof of Theorem 6.3.1 on p. 103.

To apply this theorem, we need to check (29). For this reason, we shall stay near a solution (namely $(f_1, g_1) = 0$) to an unperturbed problem (namely $(f_0, g_0) = 0$ and $H = 0$), so that (29) is satisfied, and rely on the fact that all relevant "constants" (in particular μ^*) for the perturbed problem can be chosen equal to those of the unperturbed problem.

Theorem 3.3. *Let $j \geq 0$ be an integer, $R > 0$ arbitrary and $\delta > 0$ sufficiently small and assume that $(f_0, g_0, H_0, c) \in \mathcal{W}_{13+j}$ with $\|(f_0, g_0, H_0, c)\|_{13+j} < R$ and $\|(f_0, g_0, H_0, 0)\|_5 < \delta$. It is possible to choose $\epsilon > 0$ (independent of (f_0, g_0, H_0, c), but depending on $(\bar{f}, \bar{g})$, j, R and δ) such that if $\|\mathcal{F}(0, 0)\|_{7+j} < \epsilon$ then there exists $(f^*, g^*) \in \mathcal{U}_{6+j}$ satisfying $\mathcal{F}(f^*, g^*) = 0$.*

Proof. We choose $r_0 > 0$ small enough so that Theorem 3.1 with $k = 9 + j$ can be applied for all $(f_1, g_1) \in \mathcal{U}_5$ in the closed ball of radius r_0 centered at the origin. Let $\hat{c} > 0$ be such that

$$\|(f_1, g_1)\|_{C^2(\bar{\mathcal{P}})} \leq \hat{c}$$

for all $(f_1, g_1) \in \mathcal{U}_5$ in this ball, and define $\Sigma \subset \mathbb{R}^{2+6+18}$ as the ball of radius $\hat{c}$ centered at the origin.

We apply Theorem 3.2 with $m = 2$, $\Omega = \mathcal{P} \subset \mathbb{R}^n$, $n = 3$, $d_0 = 5$, $d_1 = 0$, $d_2 = 4$, $d_3 = 1$, $s_0 = 1$, $d_* = 0$ and $\tilde{s} = 9 + j$. We get

$$\tilde{s} + d_1 = 9 + j, \quad \tilde{s} + d_2 = 13 + j, \quad \tilde{s} - m = 7 + j, \quad \tilde{s} - m - d_* - 1 = 6 + j$$

and a solution $(f^*, g^*) \in H^{6+j}(\mathcal{P})$. Let the map $\Gamma : \mathcal{P} \times \mathbb{R}^{1+1+3+3+9+9} \to \mathbb{R}^2$ be such that

$$\mathcal{F}(f_1, g_1) = \Gamma(x, y, z, f_1, g_1, f_1', g_1', f_1'', g_1'').$$

It appears in the various estimates also via "constants" depending on $\|\partial_\alpha \partial_\beta \Gamma\|_{C^{\tilde{s}-m}(\overline{\mathcal{P}} \times \overline{\Sigma})}$, where ∂_α and ∂_β are all possible partial derivatives with respect to f_1, g_1, f_1', g_1', f_1'' or g_1''. Observe that $(f_0, g_0, H_0, c) \in \mathcal{W}_{13+j}$ implies $(f_0, g_0, H_0, c) \in C^{\tilde{s}+2}(\overline{\mathcal{P}}) \times C^{\tilde{s}+2}(\overline{\mathcal{P}}) \times C^{\tilde{s}+2}(\overline{\mathcal{Q}}) \times \mathbb{R}^2$ and $\partial_\alpha \partial_\beta \Gamma \in C^{\tilde{s}-m}(\overline{\mathcal{P}} \times \overline{\Sigma})$. As (f^*, g^*) is the limit in $H^{6+j}(\mathcal{P})$ of sums of solutions in $\mathcal{U}_{9+j}$ to equations of type (10) (with $\epsilon = 0$), it satisfies (Ad′3) and thus belongs to $\mathcal{U}_{6+j}$. □

As a corollary, we get the following simplified statement.

Theorem 3.4. *Assume that $H_0 \in C^{11+j}$ and $f_0, g_0 \in H^{13+j}$. It is possible to choose $\bar{\epsilon} > 0$ such that if $\|(f_0, g_0, H_0, c)\|_{13+j} < \bar{\epsilon}$, then there exists $(f^*, g^*) \in \mathcal{U}_{6+j}$ satisfying $\mathcal{F}(f^*, g^*) = 0$.*

Theorem 1.1 is a reformulation of this last result and Theorem 2.2.

Appendix: Representation of divergence-free vector fields

The fact that the vector field $\nabla f \times \nabla g$ is divergence-free if f and g are C^2 is easily checked using the formula $\mathrm{div}(u \times v) = v \cdot \mathrm{rot}\, u - u \cdot \mathrm{rot}\, v$. A local converse near points where v is nonzero has been known for a long time; see, e.g., [Barbarosie 2011; Cartan 1967, Chapter 3, Exercise 14]. A local converse that can be seen as a global converse under additional conditions can be found in Appendix I in [Grad and Rubin 1958]. In the present appendix, we give for the reader's convenience a self-contained proof that a divergence-free vector field $v \in C^2(\overline{D})$ can be represented globally in this form if v is periodic in y and z and $v_1 \neq 0$ in $\overline{D}$, and that f and g can be chosen to be of the form "linear plus periodic". Our argument is essentially a simple version of an elementary proof of global equivalence of volume forms on compact connected manifolds due to [Moser 1965].

For a given point $(x, y, z) \in \overline{D}$ we solve the system of ODEs $\phi' = v(\phi)$, with $\phi(0) = (x, y, z)$, and let $T = T(x, y, z)$ be the unique time such that $\phi_1(-T; x, y, z) = 0$ (here we use that $\inf_{\overline{D}} |v_1| > 0$ and $\sup_{\overline{D}} |v| < \infty$). We define the C^2 functions $Y, Z \colon \overline{D} \to \mathbb{R}^2$ by

$$Y \colon (x, y, z) \mapsto \phi_2(-T; x, y, z) \quad \text{and} \quad Z \colon (x, y, z) \mapsto \phi_3(-T; x, y, z).$$

The functions Y and Z are invariants of the vector field v and therefore $\nabla Y \times \nabla Z = \lambda v$ for some function λ. Using the fact that v is divergence-free, it is easily established that λ is another invariant and therefore

$$\nabla Y \times \nabla Z = \frac{1}{v_1(0, Y, Z)} v$$

in view of the relations $Y(0, y, z) = y$ and $Z(0, y, z) = z$. If $F, G \colon \mathbb{R}^2 \to \mathbb{R}^2$ and

$$f(x, y, z) = F(Y(x, y, z), Z(x, y, z)), \quad g(x, y, z) = G(Y(x, y, z), Z(x, y, z)),$$

then

$$\nabla f \times \nabla g = (\partial_1 F\, \partial_2 G - \partial_2 F\, \partial_1 G) \nabla Y \times \nabla Z.$$

Thus in order to have $\nabla f \times \nabla g = v$ we must find F and G with

$$\partial_1 F(Y, Z)\, \partial_2 G(Y, Z) - \partial_2 F(Y, Z)\, \partial_1 G(Y, Z) = v_1(0, Y, Z).$$

If it weren't for the periodicity conditions, this would be trivial. We describe next how to make a choice which respects these conditions (the choice is not unique).

Note that $v_1(0, Y, Z)$ is P_1-periodic in Y and P_2-periodic in Z. Let

$$\alpha = \frac{1}{P_1 P_2} \int_0^{P_1} \int_0^{P_2} v_1(0, Y, Z) \, dY \, dZ$$

and write $v_1(0, Y, Z) = a(Y) \, b(Y, Z)$, where

$$a(Y) = \frac{1}{P_2} \int_0^{P_2} v_1(0, Y, Z) \, dZ \quad \text{and} \quad b(Y, Z) = \frac{v_1(0, Y, Z)}{a(Y)},$$

so that

$$\frac{1}{P_1} \int_0^{P_1} a(Y) \, dY = \alpha \quad \text{and} \quad \frac{1}{P_2} \int_0^{P_2} b(Y, Z) \, dZ = 1.$$

We choose

$$F(Y) = \int_0^Y a(s) \, ds \quad \text{and} \quad G(Y, Z) = \int_0^Z b(Y, s) \, ds.$$

Note that F and G (and hence f and g) are C^2 and that the map

$$\Psi : (Y, Z) \mapsto (F(Y), G(Y, Z))$$

from $\mathbb{R}^2$ to itself is bijective. It is easily verified that

$$\partial_1 F(Y) \, \partial_2 G(Y, Z) = a(Y) \, b(Y, Z) = v_1(0, Y, Z),$$

that $F(Y) - \alpha Y$ is P_1-periodic and that $G(Y, Z) - Z$ is (P_1, P_2)-periodic. Finally, by the periodicity of v and standard ODE theory, it follows that $(Y(x, y, z), Z(x, y, z)) - (y, z)$ is P_1 periodic in y and P_2-periodic in z, and therefore so is $(f(x, y, z), g(x, y, z)) - (\alpha y, z)$. This concludes the proof.

As mentioned above, the representation $v = \nabla f \times \nabla g$ is not unique. Indeed, if $\Phi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ satisfies

$$\det \Phi' = \partial_1 \Phi_1 \partial_2 \Phi_2 - \partial_2 \Phi_1 \partial_1 \Phi_2 = 1,$$

then $(\tilde{f}, \tilde{g}) = \Phi(f, g)$ also satisfies $\nabla \tilde{f} \times \nabla \tilde{g} = v$. Moreover, $(\tilde{f}, \tilde{g})$ is also linear plus (P_1, P_2)-periodic in (y, z) if $\Phi(f, g) = T(f, g) + \Phi_0(f, g)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and Φ_0 is $(\alpha P_1, P_2)$-periodic.

Note that T is bijective, since otherwise one could find a nonzero linear functional ℓ annihilating its range. This would cause $\ell \circ \Phi$ to be periodic, and thus $\ell \circ \Phi$ would have a critical point at which $\det \Phi'$ would vanish. As T is bijective, Φ is proper and hence bijective by the global inversion theorem (using again $\det \Phi' = 1$).

Conversely, if $v = \nabla \tilde{f} \times \nabla \tilde{g}$ for some C^2 functions $\tilde{f}$ and $\tilde{g}$, then $\tilde{f}$ and $\tilde{g}$ are constant along the streamlines of v. Hence

$$(\tilde{f}(x, y, z), \tilde{g}(x, y, z)) = (\tilde{f}(0, Y, Z), \tilde{g}(0, Y, Z))$$

with $(Y, Z) = (Y(x, y, z), Z(x, y, z))$ as above, and we obtain $(\tilde{f}, \tilde{g}) = \Phi(f, g)$, where $\Phi = (\tilde{f}, \tilde{g})|_{x=0} \circ \Psi^{-1}$ is C^2. Moreover, Φ is linear plus $(\alpha P_1, P_2)$-periodic and $\det \Phi' = 1$.

Let us finally note that the Bernoulli function $H = \frac{1}{2}|v|^2 + P$ can clearly be written as a function of (f, g) since it is constant on streamlines. Denoting this function also by $H(f, g)$, we find that if (f, g) is transformed to $(\tilde{f}, \tilde{g}) = \Phi(f, g)$ with Φ as above, then H is transformed to $H \circ \Phi^{-1}$.

Acknowledgments

References

[Agmon et al. 1964] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II", *Comm. Pure Appl. Math.* **17** (1964), 35–92. MR Zbl

[Alber 1992] H.-D. Alber, "Existence of three-dimensional, steady, inviscid, incompressible flows with nonvanishing vorticity", *Math. Ann.* **292**:3 (1992), 493–528. MR Zbl

[Barbarosie 2011] C. Barbarosie, "Representation of divergence-free vector fields", *Quart. Appl. Math.* **69**:2 (2011), 309–316. MR Zbl

[Brenier 1999] Y. Brenier, "Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations", *Comm. Pure Appl. Math.* **52**:4 (1999), 411–452. MR Zbl

[Buffoni 2012] B. Buffoni, "Generalized flows satisfying spatial boundary conditions", *J. Math. Fluid Mech.* **14**:3 (2012), 501–528. MR Zbl

[Cartan 1967] H. Cartan, *Calcul différentiel*, Hermann, Paris, 1967. MR Zbl

[Coddington and Levinson 1955] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955. MR Zbl

[Enciso and Peralta-Salas 2015] A. Enciso and D. Peralta-Salas, "Existence of knotted vortex tubes in steady Euler flows", *Acta Math.* **214**:1 (2015), 61–134. MR Zbl

[Grad and Rubin 1958] H. Grad and H. Rubin, "Hydromagnetic equilibria and force-free fields", pp. 190–197 in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, Vol. 31: Theoretical and experimental aspects of controlled nuclear fusion* (Geneva, 1958), edited by J. H. Martens et al., United Nations, Geneva, 1958.

[Hamilton 1982] R. S. Hamilton, "The inverse function theorem of Nash and Moser", *Bull. Amer. Math. Soc.* (*N.S.*) **7**:1 (1982), 65–222. MR Zbl

[Han and Hong 2006] Q. Han and J.-X. Hong, *Isometric embedding of Riemannian manifolds in Euclidean spaces*, Mathematical Surveys and Monographs **130**, American Mathematical Society, Providence, RI, 2006. MR Zbl

[Kaiser et al. 2000] R. Kaiser, M. Neudert, and W. von Wahl, "On the existence of force-free magnetic fields with small nonconstant α in exterior domains", *Comm. Math. Phys.* **211**:1 (2000), 111–136. MR Zbl

[Keller 1996] J. J. Keller, "A pair of stream functions for three-dimensional vortex flows", *Z. Angew. Math. Phys.* **47**:6 (1996), 821–836. MR Zbl

[Kiremidjian 1978] G. K. Kiremidjian, "A Nash–Moser-type implicit function theorem and nonlinear boundary value problems", *Pacific J. Math.* **74**:1 (1978), 105–132. MR Zbl

[Kohn and Nirenberg 1965] J. J. Kohn and L. Nirenberg, "Non-coercive boundary value problems", *Comm. Pure Appl. Math.* **18** (1965), 443–492. MR Zbl

[Moser 1961] J. Moser, "A new technique for the construction of solutions of nonlinear differential equations", *Proc. Nat. Acad. Sci. U.S.A.* **47** (1961), 1824–1831. MR Zbl

[Moser 1965] J. Moser, "On the volume elements on a manifold", *Trans. Amer. Math. Soc.* **120** (1965), 286–294. MR Zbl

[Plotnikov 1980] P. I. Plotnikov, "Solvability of the problem of spatial gravitational waves on the surface of an ideal liquid", *Dokl. Akad. Nauk SSSR* **251**:3 (1980), 591–594. In Russian; translated in *Sov. Phys. Dokl.* **25** (1980), 170–171. MR Zbl

[Serrin 1959] J. Serrin, "Mathematical principles of classical fluid mechanics", pp. 125–263 in *Handbuch der Physik,* **8/1**: *Strömungsmechanik I*, edited by S. Flügge and C. Truesdell, Springer, Berlin, 1959. MR

[Slobodeanu 2015] R. Slobodeanu, "Steady Euler flows and the Faddeev–Skyrme model with mass term", *J. Math. Phys.* **56**:2 (2015), 023102, 12. MR Zbl

[Zehnder 1975] E. Zehnder, "Generalized implicit function theorems with applications to some small divisor problems, I", *Comm. Pure Appl. Math.* **28** (1975), 91–140. MR Zbl

BORIS BUFFONI: boris.buffoni@epfl.ch
Institut de mathématiques, Station 8, Ecole Polytechnique Fédérale de Lausanne, Lausanne, Switzerland

ERIK WAHLÉN: erik.wahlen@math.lu.se
Centre for Mathematical Sciences, Lund University, Lund, Sweden

Analysis & PDE

msp.org/apde

Analysis & PDE

Volume 12   No. 5   2019