ANALYSIS & PDE

Volume 11 No. 5 2018

MICHAEL GOLDMAN, MATTEO NOVAGA AND BERARDO RUFFINI

ON MINIMIZERS OF AN ISOPERIMETRIC PROBLEM WITH LONG-RANGE INTERACTIONS UNDER A CONVEXITY CONSTRAINT

msp

ANALYSIS AND PDE
Vol. 11, No. 5, 2018
dx.doi.org/10.2140/apde.2018.11.1113

ON MINIMIZERS OF AN ISOPERIMETRIC PROBLEM WITH LONG-RANGE INTERACTIONS UNDER A CONVEXITY CONSTRAINT

MICHAEL GOLDMAN, MATTEO NOVAGA AND BERARDO RUFFINI

We study a variational problem modeling the behavior at equilibrium of charged liquid drops under a convexity constraint. After proving the well-posedness of the model, we show $C^{1,1}$-regularity of minimizers for the Coulombic interaction in dimension two. As a by-product we obtain that balls are the unique minimizers for small charge. Eventually, we study the asymptotic behavior of minimizers, as the charge goes to infinity.

1. Introduction

We are interested in the existence and regularity of minimizers of the problem

$$\min\{\mathcal{F}_{Q,\alpha}(E) : E \subset \mathbb{R}^N \text{ convex body},\ |E| = V\}, \tag{1-1}$$

where, for $E \subset \mathbb{R}^N$, $V, Q > 0$ and $\alpha \in [0, N)$, we have set

$$\mathcal{F}_{Q,\alpha}(E) := P(E) + Q^2 \mathcal{I}_\alpha(E). \tag{1-2}$$

Here $P(E) := \mathcal{H}^{N-1}(\partial E)$ stands for the perimeter of E and, letting $\mathcal{P}(E)$ be the set of probability measures supported on the closure of E, we set for $\alpha \in (0, N)$,

$$\mathcal{I}_\alpha(E) := \inf_{\mu \in \mathcal{P}(E)} \int_{E \times E} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}, \tag{1-3}$$

and for $\alpha = 0$,

$$\mathcal{I}_0(E) := \inf_{\mu \in \mathcal{P}(E)} \int_{E \times E} \log\left(\frac{1}{|x-y|}\right) d\mu(x)\, d\mu(y). \tag{1-4}$$

Notice that, up to rescaling, we can assume, as we shall do for the rest of the paper, that $V = 1$.

Starting from the seminal work [Strutt (Lord Rayleigh) 1882] (in the Coulombic case $N = 3$, $\alpha = 1$), the functional (1-2) has been extensively studied in the physical literature to model the shape of charged liquid drops; see [Goldman et al. 2015]. In particular, it is known that the ball is a *linearly* stable critical point for (1-1) if the charge Q is not too large; see for instance [Fontelos and Friedman 2004]. However, quite surprisingly, the authors showed in [Goldman et al. 2015] that, without the convexity constraint, (1-2) never admits minimizers under a volume constraint for any $Q > 0$ and $\alpha < N - 1$. In particular, this implies that in this model a charged drop is always *nonlinearly* unstable. This result is in sharp contrast with experiments, see for instance [Zeleny 1917; Taylor 1964], where there is evidence of stability of the

MSC2010: 49J30, 49J45, 49S05.
Keywords: nonlocal isoperimetric problem, convexity constraint.

ball for small charges. This suggests that the energy $\mathcal{F}_{Q,\alpha}(E)$ does not include all the physically relevant contributions.

As shown in [Goldman et al. 2015], a possible way to gain well-posedness of the problem is requiring some extra regularity of the admissible sets. In this paper, we consider an alternative type of constraint, namely the convexity of admissible sets. This assumption seems reasonable as long as the minimizers remain strictly convex, that is, for small enough charges. Let us point out that in [Muratov and Novaga 2016], still another regularizing mechanism is proposed. There, well-posedness is obtained by adding an entropic term which prevents charges from concentrating too much on the boundary of E. We point out that it has been recently shown in [Muratov et al. 2016] that in the borderline case $\alpha = 1$, $N = 2$ such a regularization is not needed for the model to be well-posed. For a more exhaustive discussion about the physical motivations and the literature on related problems we refer to the papers [Muratov and Novaga 2016; Goldman et al. 2015].

Using the compactness properties of convex sets, our first result is the existence of minimizers for every charge $Q > 0$.

Theorem 1.1. *For every $\alpha \in [0, N)$ and every Q, (1-1) admits a minimizer.*

We then study the regularity of minimizers. As often in variational problems with convexity constraints, regularity (or singularity) of minimizers is hard to deal with in dimensions larger than two; see [Lamboley et al. 2012, 2016]. We thus restrict ourselves to $N = 2$. Since our analysis strongly uses the regularity of *equilibrium measures*, i.e., the minimizer of (1-3), we are further reduced to studying the case $\alpha = N - 2$ (that is, $\alpha = 0$ in this case). The second main result of the paper is then:

Theorem 1.2. *Let $N = 2$ and $\alpha = 0$. Then for every $Q > 0$, the minimizers of (1-1) are of class $C^{1,1}$.*

Since we are able to prove uniform $C^{1,1}$ estimates as Q goes to zero, building upon our previous stability results established in [Goldman et al. 2015], we get:

Corollary 1.3. *If $N = 2$ and $\alpha = 0$, for Q small enough, the only minimizers of (1-1) are balls.*

The proof of Theorem 1.2 is based on the natural idea of comparing the minimizers with a competitor made by "cutting out the angles". However, the nonlocal nature of the problem makes the estimates nontrivial. As already mentioned, a crucial point is an estimate on the integrability of the equilibrium measures. This is obtained by drawing a connection with harmonic measures (see Section 3). Let us point out[1] that, up to proving the regularity of the shape functional $\mathcal{I}_0$ and computing its shape derivative, one could have obtained a proof of Theorem 1.2 by applying the abstract regularity result of [Lamboley et al. 2012]. Nevertheless, since our proof has a nice geometrical flavor and since regularity of $\mathcal{I}_0$ is not known in dimension two (see for instance [Jerison 1996; Crasta et al. 2005; Novaga and Ruffini 2015] for the proof in higher dimensions), we decided to keep it.

We remark that, differently from the two-dimensional case, when $N = 3$ we expect the onset of singularities at a critical value $Q_c > 0$, with the shape of a spherical cone with a prescribed angle. Such singularities are also observed in experiments and are usually called *Taylor cones*; see [Taylor 1964;

[1] This was suggested to us by J. Lamboley.

Zeleny 1917]. At the moment we are not able to show the presence of such singularities in our model, and this will be the subject of future research.

Eventually, in Section 6, we study the behavior of the optimal sets when the charge goes to infinity. Even though this regime is less significant from the point of view of the applications, we believe that it is still mathematically interesting. Building on Γ-convergence results, we prove:

Theorem 1.4. *Let $\alpha \in [0, 1)$ and $N \geq 2$. Then, every minimizer E_Q of (1-1) satisfies (up to a rigid motion)*

$$Q^{-\frac{2N(N-1)}{1+(N-1)\alpha}} E_Q \to [0, L_{N,\alpha}] \times \{0\}^{N-1},$$

where the convergence is in the Hausdorff topology and where

$$L_{N,\alpha} := \left(\frac{\alpha(N-1)\mathcal{I}_\alpha([0,1])}{N^{\frac{N-2}{N-1}} \omega_{N-1}^{\frac{1}{N-1}}} \right)^{\frac{(N-1)}{1+\alpha(N-1)}} \quad \text{for } \alpha \in (0,1) \qquad \text{and} \qquad L_{N,0} := \frac{(N-1)^{N-1}}{\omega_{N-1} N^{N-2}},$$

ω_N being the volume of the unit ball in $\mathbb{R}^N$. For $\alpha = 1$ and $N = 2, 3$, we have

$$Q^{-\frac{2(N-1)}{N}} (\log Q)^{-1+\frac{1}{N}} E_Q \to [0, L_{N,1}] \times \{0\}^{N-1},$$

where

$$L_{N,1} := \left(\frac{4(N-1)}{N^{\frac{N-2}{N-1}} \omega_{N-1}^{\frac{1}{N-1}}} \right)^{\frac{N-1}{N}}.$$

An obvious consequence of this result is that the ball cannot be a minimizer for Q large enough. For a careful analysis of the loss of linear stability of the ball we refer to [Fontelos and Friedman 2004].

2. Existence of minimizers

We now show that the minimum in (1-1) is achieved. We begin with a simple lemma linking estimates on the energy with estimates on the size of the convex body.

Lemma 2.1. *Let $N \geq 2$, and $\lambda_1, \dots, \lambda_N > 0$. Letting $E := \prod_{i=1}^N [0, \lambda_i]$, $V := |E|$ and $\Phi := V^{-\frac{N-2}{N-1}} P(E)$, it holds that*[2]

$$\max_i \lambda_i \lesssim \Phi^{N-1} \quad \text{and} \quad \min_i \lambda_i \sim V^{\frac{1}{N-1}} \Phi^{-1}, \tag{2-1}$$

where the involved constants depend only on the dimension. Moreover, letting $i_{\max}$ be such that $\lambda_{i_{\max}} = \max_i \lambda_i$, it holds for $\alpha > 0$ that

$$\lambda_{i_{\max}} \gtrsim \mathcal{I}_\alpha(E)^{-\frac{1}{\alpha}} \qquad \text{and} \qquad \lambda_i \lesssim \mathcal{I}_\alpha(E)^{\frac{1}{\alpha}} \Phi^{N-2} V^{\frac{1}{N-1}} \quad \text{for } i \neq i_{\max}, \tag{2-2}$$

and for $\alpha = 0$,

$$\lambda_{i_{\max}} \gtrsim \exp(-\mathcal{I}_0(E)) \qquad \text{and} \qquad \lambda_i \lesssim \exp(\mathcal{I}_0(E)) \Phi^{N-2} V^{\frac{1}{N-1}} \quad \text{for } i \neq i_{\max}, \tag{2-3}$$

where the constants implicitly appearing in (2-2) and (2-3) depend only on N and α.

[2] Here and in the rest of the paper, we write $f \lesssim g$ if there exists $C > 0$ such that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$, we will simply write $f \sim g$.

Proof. Without loss of generality, we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. Then, since $V = \prod_{i=1}^N \lambda_i$ and $P(E) \lesssim \prod_{i=1}^{N-1} \lambda_i$, taking the ratio of these two quantities, we obtain $\lambda_N \gtrsim VP(E)^{-1} = V^{\frac{1}{N-1}} \Phi^{-1}$. Now, since the λ_i are decreasing (in particular $\lambda_i \geq \lambda_N$ for all i), this implies

$$\Phi \gtrsim V^{-\frac{N-2}{N-1}} \prod_{i=1}^{N-1} \lambda_i = V^{-\frac{N-2}{N-1}} \lambda_1 \prod_{i=2}^{N-1} \lambda_i \gtrsim V^{-\frac{N-2}{N-1}} \lambda_1 V^{\frac{N-2}{N-1}} \Phi^{-(N-2)},$$

yielding (2-1).

Assume now that $\alpha > 0$. Then, from $\operatorname{diam}(E) \sim \lambda_1$, we get $\mathcal{I}_\alpha(E) \gtrsim \lambda_1^{-\alpha}$. If $N = 2$, together with $\lambda_1 \lambda_2 = V$, this implies (2-2). If $N \geq 3$, we infer as above that

$$\Phi \gtrsim V^{-\frac{N-2}{N-1}} \lambda_1 \lambda_2 \prod_{i=3}^{N-1} \lambda_i \gtrsim V^{-\frac{N-2}{N-1}} \mathcal{I}_\alpha(E)^{-\frac{1}{\alpha}} \lambda_2 V^{\frac{N-3}{N-1}} \Phi^{-(N-3)} \gtrsim V^{-\frac{1}{N-1}} \Phi^{-(N-3)} \mathcal{I}_\alpha(E)^{-\frac{1}{\alpha}} \lambda_2.$$

This gives (2-2). The case $\alpha = 0$ follows analogously, using the fact that $\mathcal{I}_0(E) \geq C - \log \lambda_1$. □

The next result follows directly from John's lemma [1948].

Lemma 2.2. *There exists a dimensional constant $C_N > 0$ such that for every convex body $E \subset \mathbb{R}^N$, up to a rotation and a translation, there exists $\mathcal{R} := \prod_{i=1}^N [0, \lambda_i]$ such that*

$$\mathcal{R} \subseteq E \subseteq C_N \mathcal{R}.$$

As a consequence $\operatorname{diam}(E) \sim \operatorname{diam}(\mathcal{R})$, $|E| \sim |\mathcal{R}|$, $P(E) \sim P(\mathcal{R})$ *and* $\mathcal{I}_\alpha(E) \sim \mathcal{I}_\alpha(\mathcal{R})$ *for* $\alpha > 0$ *(and* $\exp(-\mathcal{I}_0(E)) \sim \exp(-\mathcal{I}_0(\mathcal{R}))$*).*

With these two preliminary results at hand, we can prove existence of minimizers for (1-1).

Theorem 2.3. *For every $Q > 0$ and $\alpha \in [0, N)$, (1-1) has a minimizer.*

Proof. Let E_n be a minimizing sequence and let us prove that $\operatorname{diam}(E_n)$ is uniformly bounded. Let $\mathcal{R}_n$ be the parallelepipeds given by Lemma 2.2. Since $\operatorname{diam}(E_n) \sim \operatorname{diam}(\mathcal{R}_n)$, it is enough to estimate $\operatorname{diam}(\mathcal{R}_n)$ from above. Let us begin with the case $\alpha > 0$. In this case, since $\mathcal{I}_\alpha(\mathcal{R}_n) \geq 0$, by (2-1), applied with $V = 1$, we get

$$\operatorname{diam}(\mathcal{R}_n) \lesssim P(\mathcal{R}_n)^{N-1} \lesssim \mathcal{F}_{Q,\alpha}(E_n)^{N-1}.$$

In the case $\alpha = 0$, from (2-1) and (2-3) applied to $V = 1$, we get

$$P(\mathcal{R}_n) \gtrsim \exp\left(-\frac{\mathcal{I}_0(\mathcal{R}_n)}{N-1}\right)$$

so that

$$\mathcal{F}_{Q,0}(\mathcal{R}_n) \gtrsim \exp\left(-\frac{\mathcal{I}_0(\mathcal{R}_n)}{N-1}\right) + Q^2 \mathcal{I}_0(\mathcal{R}_n).$$

From this we obtain that $|\mathcal{I}_0(\mathcal{R}_n)|$ is bounded and thus also $P(\mathcal{R}_n)$ is bounded, whence, arguing as above, we obtain a uniform bound on $\operatorname{diam}(\mathcal{R}_n)$.

Since the E_n are convex sets, up to a translation, we can extract a subsequence which converges in the Hausdorff (and L^1) topology to some convex body E of volume 1. Since the perimeter functional is lower semicontinuous with respect to the L^1 convergence, and the Riesz potential $\mathcal{I}_\alpha$ is lower semicontinuous with respect to the Hausdorff convergence, see [Landkof 1972; Saff and Totik 1997; Goldman et al. 2015, Proposition 2.2], we get that E is a minimizer of (1-1). $\square$

3. Regularity of the planar charge distribution for the logarithmic potential

We now focus on the case $N = 2$ and $\alpha = 0$. Relying on classical results on harmonic measures, we show that for every convex set E, the corresponding optimal measure μ for $\mathcal{I}_0(E)$ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \partial E$ with L^p estimates. Upon making that connection between μ and harmonic measures, this fact is fairly classical. However, since we could not find a proper reference, we recall (and slightly adapt) a few useful results. Let us point out that most definitions and results of this section extend to the case $N \geq 3$ and $\alpha = N - 2$, and to more general classes of sets. In particular, for bounded Lipschitz sets, the fact that harmonic measures are absolutely continuous with respect to the surface measure with L^p densities for $p > 2$ was established in [Dahlberg 1977], and extended later to more general domains; see for instance [Kenig and Toro 1997; 1999; Jerison and Kenig 1982]. The interest for harmonic measures stems from the fact that they bear a lot of geometric information; see in particular [Alt and Caffarelli 1981; Kenig and Toro 1999]. The main result of this section is the following.

Theorem 3.1. *Let E_n be a sequence of compact convex bodies converging to a convex body E and let μ_n be the associated equilibrium measures. Then, $\mu_n = f_n \mathcal{H}^1 \llcorner \partial E_n$ and there exists $p > 2$ and $M > 0$ (depending only on E) such that $f_n \in L^p(\partial E_n)$ with*

$$\|f_n\|_{L^p(\partial E_n)} \leq M.$$

Moreover, if E is smooth, then p can be taken arbitrarily large.

Remark 3.2. By applying the previous result with $E_n = E$, we get that the equilibrium measure of a convex set is always in some $L^p(\partial E)$ with $p > 2$. We stress also that the exponent p and the bound on the L^p norm of its equilibrium measure depend indeed on the set: for instance, a sequence of convex sets with smooth boundaries converging to a square cannot have equilibrium measures with densities uniformly bounded in L^p for $p > 4$.

We will define here $\Omega := E^c$. Let us recall the definition of harmonic measures; see [Garnett and Marshall 2005; Kenig and Toro 1999].

Definition 3.3. Let Ω be a Lipschitz open set (bounded or unbounded) such that $\mathbb{R}^2 \setminus \partial\Omega$ has two connected components, and let $X \in \Omega$. We denote by G_Ω^X the Green function of Ω with pole at X, i.e., the unique distributional solution of

$$-\Delta G_\Omega^X = \delta_X \quad \text{in } \Omega \qquad \text{and} \qquad G_\Omega^X = 0 \quad \text{on } \partial\Omega,$$

and by ω_Ω^X the *harmonic measure of* Ω *with pole at* X, that is, the unique (positive) measure such that for every $f \in C^0(\partial\Omega)$, the solution u of

$$-\Delta u = 0 \quad \text{in } \Omega \qquad \text{and} \qquad u = f \quad \text{on } \partial\Omega$$

satisfies

$$u(X) = \int_{\partial\Omega} f(y)\, d\omega_\Omega^X(y).$$

If Ω is unbounded with $\partial\Omega$ bounded and $0 \in \overline{\Omega}^c$, we call ω_Ω^∞ the *harmonic measure of* Ω *with pole at infinity*, that is, the unique probability measure on $\partial\Omega$ satisfying

$$\int_{\partial\Omega} \phi\, d\omega^\infty = \int_\Omega u\Delta\phi \quad \text{for all } \phi \in C_c^\infty(\mathbb{R}^2),$$

where u is the solution of

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim_{|z|\to+\infty}\{u(z) - \frac{1}{2\pi}\log|z|\} \text{ exists and is finite.} \end{cases} \tag{3-1}$$

When it is clear from the context, we omit the dependence of G^X, ω^X or ω^∞ on the domain Ω.

Remark 3.4. For smooth domains, it is not hard to check that $\omega^X = \partial_\nu G^X \mathcal{H}^1 \llcorner \partial\Omega$, and that $\omega^\infty = \partial_\nu u \mathcal{H}^1 \llcorner \partial\Omega$, where ν is the inward unit normal to Ω. Moreover, for Ω unbounded, if h^∞ is the harmonic function in Ω with $h^\infty(z) = -\frac{1}{2\pi}\log|z|$ on $\partial\Omega$, then the function u from (3-1) can also be defined by $u(z) = \frac{1}{2\pi}\log|z| + h^\infty(z)$.

We may now make the connection between harmonic measures and equilibrium measures. For E a Lipschitz bounded open set containing 0, let μ be the optimal measure for $\mathcal{I}_0(E)$ and let

$$v(x) := \int_{\partial E} -\log(|x-y|)\, d\mu(y).$$

Since

$$-\Delta v = 2\pi\mu \quad \text{in } \mathbb{R}^2, \qquad v < \mathcal{I}_0(E) \quad \text{in } E^c \qquad \text{and} \qquad v = \mathcal{I}_0(E) \quad \text{on } \partial E,$$

if we let $u := (2\pi)^{-1}(\mathcal{I}_0(E) - v)$, we see that it satisfies (3-1) for $\Omega = E^c$. Therefore, $\mu = \omega_{E^c}^\infty$ (recall that $\mu(\partial E) = 1$). For Lipschitz sets Ω, it is well known that ω^∞ is absolutely continuous with respect to $\mathcal{H}^1 \llcorner \partial\Omega$ with density in $L^p(\partial\Omega)$ for some $p > 1$; see [Garnett and Marshall 2005, Theorem 4.2]. However, we will need a stronger result, namely that it is in $L^p(\partial\Omega)$ for some $p > 2$, with estimates on the L^p norm depending only on the geometry of Ω.

Given a convex body E and a point $x \in \partial E$, we call the *angle* of ∂E at x the angle spanned by the tangent cone $\bigcup_{\lambda>0} \lambda(E - x)$.

We now state a crucial lemma which relates in a quantitative way the regularity of E with the integrability properties of the corresponding harmonic measure. This result is a slight adaptation of [Warschawski and Schober 1966, Theorem 2].

Lemma 3.5. *Let E be a convex body containing the origin in its interior, let $\bar\zeta \in (0, \pi]$ be the minimal angle of ∂E, and let $p_c := \pi/(\pi - \bar\zeta) + 1$ if $\bar\zeta < \pi$ and $p_c := +\infty$ if $\bar\zeta = \pi$. Let also E_n be a sequence of convex bodies converging to E in the Hausdorff topology. Then, for every $1 \le p < p_c$, there exists*

$C(p,\partial E)$ such that for n large enough (depending on p), every conformal map $\psi_n : E_n^c \to B_1$ with $\psi_n(\infty)=0$ satisfies

$$\int_{\partial E_n} |\psi_n'|^p \le C(p,\partial E), \tag{3-2}$$

where we indicate by $|\psi_n'|$ the absolute value of the derivative of ψ_n (seen as a complex function). In particular, for n large enough, $\psi_n' \in L^p(\partial E_n)$ for some $p>2$.

Proof. The scheme of the proof follows that of [Warschawski and Schober 1966, Theorem 2, Equation (9)]; thus we limit ourselves to pointing out the main differences. We begin by noticing that although Theorem 2 of that paper is written for bounded sets, up to composing with the map $z \to z^{-1}$, this does not create any difficulties.

We first introduce some notation from [Warschawski and Schober 1966]. Given a convex body E we let $\partial E = \{\gamma(s) : s \in [0,L]\}$ be an arc-length parametrization of ∂E. Notice that, for every s, the left and right derivatives $\gamma_\pm'(s)$ exist and the angle $v(s)$ between $\gamma'(s)$ and a fixed direction, say e_1, is a function of bounded variation. Up to changing the orientation of ∂E, we can assume that v is increasing. We then let

$$\bar\eta := \max_s [v(s^+) - v(s^-)] \ge 0,$$

where $v(s^\pm)$ are the left and right limits at s of v. Notice that $\bar\xi = \pi - \bar\eta$ is the minimal angle of ∂E.

Letting $\varphi_n := \psi_n^{-1}$, we want to prove that there exists $C(p,\partial E)$ such that

$$\int_{\partial B_1} |\varphi_n'|^{-p} \le C(p,\partial E)$$

for n large enough and for $p < \pi/\bar\eta$. By a change of variables, this yields (3-2). Let $p < p' < \pi/\bar\eta$, and let as in [Warschawski and Schober 1966],

$$h := \frac{1}{2\pi}(p\bar\eta + \pi) \quad \text{and} \quad h' := \frac{1}{2\pi}(p'\bar\eta + \pi),$$

so that

$$\frac{\pi h}{p} > \frac{\pi h'}{p'} > \bar\eta.$$

Let now v^n (respectively v) be the angle functions corresponding to the sets E_n (respectively E). As in [Warschawski and Schober 1966], there exists $\delta > 0$ such that for $s - s' \le \delta$,

$$v(s) - v(s') \le \frac{\pi h'}{p'}.$$

By the convexity of E_n and by the convergence of E_n to E, for n large enough and for $s - s' \le \delta$ we get that

$$v^n(s) - v^n(s') \le \frac{\pi h}{p}.$$

Let $L_n := \mathcal{H}^1(\partial E_n)$ and let us extend v^n to $\mathbb{R}$ by letting for $s \ge 0$,

$$v^n(s) := v^n\Big(L_n\Big\lfloor \frac{s}{L_n}\Big\rfloor\Big) + v^n\Big(s - L_n\Big\lfloor \frac{s}{L_n}\Big\rfloor\Big),$$

and similarly for $s \le 0$, so that v^n is an increasing function with $(v^n)'$ periodic of period L_n. Let now $k_n := \lceil L_n/\delta \rceil \in \mathbb{N}$ and $\delta_n := L/k_n$. By the convergence of E_n to E, we have k_n and δ_n are uniformly bounded from above and below. For $t \in [0, \delta_n]$, and $0 \le j \le k_n$, let $s_j^t := t + j\delta_n$. Since

$$\begin{aligned}\int_0^{\delta_n} \sum_{j=0}^{k_n-1} \int_{s_j^t}^{s_{j+1}^t} \frac{v^n(s) - v^n(s_j^t)}{s - s_j^t}\, ds\, dt &= \sum_{j=0}^{k_n-1} \int_0^{\delta_n}\int_0^{\delta_n} \frac{v^n(s+t+j\delta_n) - v^n(t+j\delta_n)}{s}\, dt\, ds \\ &= \int_0^{\delta_n} \frac{1}{s} \sum_{j=0}^{k_n-1} \int_0^{\delta_n} v^n(s+t+j\delta_n) - v^n(t+j\delta_n)\, dt\, ds \\ &= \int_0^{\delta_n} \frac{1}{s}\left(\int_{L_n}^{L_n+s} v^n(t)\, dt - \int_0^s v^n(t) dt\right) ds \\ &\le 2\delta_n \sup_{[0,2L_n]} |v^n| \lesssim \delta_n \|v\|_\infty,\end{aligned}$$

we can find $\bar{t} \in (0, \delta_n)$ such that

$$\sum_{j=0}^{k_n-1} \int_{s_j^{\bar{t}}}^{s_{j+1}^{\bar{t}}} \frac{v^n(s) - v^n(s_j^{\bar{t}})}{s - s_j^{\bar{t}}}\, ds \lesssim \|v\|_\infty.$$

For notational simplicity, let us simply define $s_j := s_j^{\bar{t}}$. Arguing as above, we can further assume that

$$\sum_{j=0}^{k_n-1} \int_{s_j}^{s_{j+1}} \frac{v^n(s_{j+1}) - v^n(s)}{s_{j+1} - s}\, ds \lesssim \|v\|_\infty.$$

The proof then follows almost exactly as in [Warschawski and Schober 1966, Theorem 2], by replacing the pointwise quantity

$$G_j^n := \sup_{s_j < s < s_{j+1}} \frac{v^n(s) - v^n(s_j)}{s - s_j}$$

by the integral ones. There is just one additional change in the proof: letting

$$0 \le \lambda_j^n := v^n(s_{j+1}) - v^n(s_j) \le \frac{\pi h}{p},$$

we see that in the estimates of [Warschawski and Schober 1966, Theorem 2], the quantity $\max_{\lambda_j^n \neq 0} 1/\lambda_j^n$ appears and could be unbounded in n. Let $\gamma_n(s)$ be the arc-length parametrization of ∂E_n and let $\theta_n(s)$ be such that $\gamma_n(s) = \varphi_n(e^{i\theta_n(s)})$. For $0 < r < 1$ and $j \in [0, k_n - 1]$, if $\lambda_j^n \neq 0$, we have

$$\frac{1}{\lambda_j^n} \int_{s_j}^{s_{j+1}} dv^n(s) \int_{\theta_n(s_j)}^{\theta_n(s_{j+1})} \frac{dt}{|e^{i\theta_n(s)} - re^{it}|^h} \lesssim \frac{1}{1-h}.$$

Using this estimate, the proof concludes exactly as in [Warschawski and Schober 1966, Theorem 2]. □

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Without loss of generality we can assume that the sets E_n and E contain the origin in their interior. As observed above, we then have $\mu_n = \omega_{E_n^c}^\infty$. Let ψ_n be a conformal mapping from E_n^c

to B_1 with $\psi_n(\infty)=0$. We have

$$\mu_n = \omega^{\infty}_{E_n^c} = (\psi_n^{-1})_\sharp\, \omega^0_{B_1} = (\psi_n^{-1})_\sharp \frac{\mathcal{H}^1 \llcorner \partial B_1}{2\pi} = \frac{|\psi_n'|}{2\pi} \mathcal{H}^1 \llcorner \partial E_n.$$

Then, Lemma 3.5 gives the desired estimate. □

We will also need a similar estimate for $C^{1,\beta}$ sets.

Lemma 3.6. *Let E be a convex set with boundary of class $C^{1,\beta}$. Then, the optimal charge distribution μ is of class $C^{0,\beta}$ and in particular it is in $L^\infty(\partial E)$. Moreover, $\|\mu\|_{C^{0,\beta}}$ depends only on the $C^{1,\beta}$ norm of ∂E.*

Proof. Up to translation we can assume that $0 \in E$ with $\operatorname{dist}(0,\partial E) \ge c$ (with c depending only on the $C^{1,\beta}$ character of ∂E). By [Pommerenke 1992, Theorem 3.6], there exists a conformal mapping ψ of class $C^{1,\beta}$ which maps E^c into B_1 with $\psi(\infty)=0$ and $\|\psi\|_{C^{1,\beta}(E^c)}$ controlled by the $C^{1,\beta}$ character of ∂E. Since, as before, $\mu = (\psi^{-1})_\sharp \omega^0_{B_1}$, the claim follows by Lemma 3.5. □

4. $C^{1,1}$-regularity of minimizers for $N=2$ and $\alpha=0$

We now show that any minimizer of (1-1) has boundary of class $C^{1,1}$. We begin by showing that we can drop the volume constraint by adding a volume penalization to the functional. This penalization is commonly used in isoperimetric-type problems; see for instance [Esposito and Fusco 2011; Goldman and Novaga 2012]. Let Λ be a positive number and define the functional

$$\mathcal{G}_\Lambda(E) := P(E) + Q^2 \mathcal{I}_0(E) + \Lambda \big| |E| - 1 \big|.$$

Lemma 4.1. *For every $Q_0>0$, there exists $\bar\Lambda>0$ such that, if $\Lambda>\bar\Lambda$ and $Q\le Q_0$, the minimizers of*

$$\min_{E\subseteq\mathbb{R}^2,\, E \text{ convex}} \mathcal{G}_\Lambda(E) \tag{4-1}$$

are also minimizers of (1-1) and vice versa. Furthermore, the diameter of the minimizers of (4-1) is uniformly bounded by a constant depending only on Q_0.

Proof. Let us fix $Q_0>0$ and let $Q<Q_0$. Let B be a ball with $|B|=1$. Then for any $E\subset\mathbb{R}^2$ such that $\mathcal{G}_\Lambda(E)\le\mathcal{G}_\Lambda(B)$ we have

$$\operatorname{diam}(E) - Q^2\log(\operatorname{diam}(E)) \le \mathcal{G}_\Lambda(E) \le \mathcal{G}_\Lambda(B) = \mathcal{F}_{Q,0}(B) \lesssim 1,$$

where the constant involved depends only on Q_0. For such sets, $\operatorname{diam}(E)$ is bounded by a constant R depending only on Q_0, and thus $I_0(E)\ge I_0(B_R)$. This implies that every minimizing sequence is uniformly bounded so that, up to passing to a subsequence, it converges in Hausdorff distance to a minimizer of $\mathcal{G}_\Lambda$ whose diameter is bounded by R. Moreover, for

$$\Lambda > \bar\Lambda := P(B) + Q_0^2(\mathcal{I}_0(B) + |\mathcal{I}_0(B_R)|)$$

we have that $|E|>0$. Indeed, for $|E|=0$ the inequality $\mathcal{G}_\Lambda(E)\le\mathcal{G}_\Lambda(B)$ implies $\Lambda\le\bar\Lambda$.

Notice that the minimum in (4-1) is always less than or equal to the minimum in (1-1). We are thus left to prove the opposite inequality. Assume that E is not a minimizer for $\mathcal{F}_{Q,0}$. In this case we get

$$\sigma := \big||E|-1\big| > 0.$$

From the uniform bound on the diameter of E we deduce that $\Lambda\sigma$ is itself also bounded by a constant (again depending only on Q_0). From now on we assume that $|E|<1$, or equivalently, $|E| = 1-\sigma$, since the other case is analogous. Let us define

$$F := \frac{1}{(1-\sigma)^{\frac{1}{2}}} E,$$

so that $|F| = 1$. Then, by the minimality of E, the homogeneity of the perimeter and recalling that

$$\mathcal{I}_0(\lambda E) = \mathcal{I}_0(E) - \log(\lambda),$$

a Taylor expansion gives

$$\begin{aligned}\Lambda\sigma &= \mathcal{G}_\Lambda(E) - \mathcal{F}_{Q,0}(E) \le \mathcal{G}_\Lambda(F) - \mathcal{F}_{Q,0}(E)\\ &= P(E)(1-\sigma)^{-\frac{1}{2}} + Q^2\mathcal{I}_0(E) + \tfrac{1}{2}\log(1-\sigma) - \mathcal{F}_{Q,0}(E)\\ &\le P(E)((1-\sigma)^{-\frac{1}{2}}-1) \le \tfrac{1}{2}P(E)\sigma,\end{aligned}$$

so that $\Lambda \le \frac{1}{2}P(E) \lesssim 1$. Therefore, if Λ is large enough, we must have $\sigma = 0$ or equivalently that E is also a minimizer of $\mathcal{F}_{Q,0}$. $\square$

Let now E be a minimizer of (4-1). In order to prove the regularity of E, we shall construct a competitor in the following way: Since E is a convex body, there exists ε_0 such that for $\varepsilon \le \varepsilon_0$, and every $x_0 \in \partial E$, we have $\partial E \cap \partial B_\varepsilon(x_0) = \{x_1^\varepsilon, x_2^\varepsilon\}$ (in particular $|x_0 - x_i^\varepsilon| = \varepsilon$). Let us fix x_0. For $\varepsilon \le \varepsilon_0$, let x_1^ε, x_2^ε be given as above and let L_ε be the line joining x_1^ε to x_2^ε. Denote by H_ε^+ the half-space with boundary L_ε containing x_0 (and H_ε^- be its complementary). We then define our competitor as

$$E_\varepsilon := E \cap H_\varepsilon^-.$$

Let us fix some further notation (see Figure 1):

- We denote by $\Pi : \partial E \cap H_\varepsilon^+ \to L_\varepsilon$ the projection of the cap of ∂E inside H_ε^+, on L_ε. We shall extend Π to the whole ∂E as the identity, outside $\partial E \cap H_\varepsilon^+$.
- If $f\mathcal{H}^1 \llcorner \partial E$ is the optimal measure for $\mathcal{I}_0(E)$, we let $f_\varepsilon := \Pi_\sharp f$ (which is defined on ∂E_ε) so that $\mu_\varepsilon := f_\varepsilon H^1 \llcorner \partial E_\varepsilon$ is a competitor for $\mathcal{I}_0(E_\varepsilon)$.
- For $x, y \in \partial E$, we denote by $\gamma_\varepsilon(x,y)$ the acute angle between the line $L_{x,y}$ joining x to y and L_ε (if $L_{x,y}$ is parallel to L_ε, we set $\gamma_\varepsilon(x,y) = 0$).
- If $y = x_0$, then we define $\gamma_\varepsilon(x) := \gamma_\varepsilon(x, x_0)$.
- We let $\gamma_\varepsilon := \gamma_\varepsilon(x_1^\varepsilon) = \gamma_\varepsilon(x_2^\varepsilon)$.
- We let $\partial B_{3\varepsilon}(x_0) \cap \partial E = \{x_1^{3\varepsilon}, x_2^{3\varepsilon}\}$. As before, we define $H_{3\varepsilon}^+$ as the half-space bounded by $L_{x_1^{3\varepsilon}, x_2^{3\varepsilon}}$ containing x_0 and $H_{3\varepsilon}^-$ as its complement. Let then $\Sigma_\varepsilon := \partial E \cap H_\varepsilon^+$, $\Sigma_{3\varepsilon} := \partial E \cap H_{3\varepsilon}^+$ and $\Gamma_\varepsilon := \partial E \cap H_{3\varepsilon}^-$.

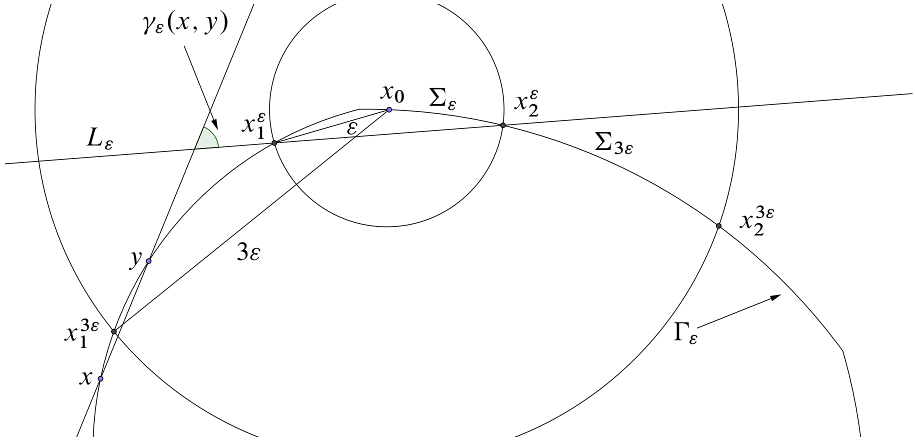

Figure 1

- We let $\Delta V := |E| - |E_\varepsilon|$, $\Delta P := P(E) - P(E_\varepsilon)$ and $\Delta\mathcal{I}_0 := \mathcal{I}_0(E_\varepsilon) - \mathcal{I}_0(E)$.

We point out some simple remarks:

- Thanks to Theorem 3.1 we have that the optimal measure f satisfies $f \in L^p(\partial E)$ for some $p = p(E) > 2$.
- If E is a convex body then γ_ε is bounded away from $\frac{\pi}{2}$ and $|x_1^{3\varepsilon} - x_1^\varepsilon| \sim |x_2^{3\varepsilon} - x_2^\varepsilon| \sim \varepsilon$.
- The quantities ΔV, ΔP and $\Delta\mathcal{I}_0$ are nonnegative by definition.
- All the constants involved up to now depend only on the Lipschitz character of ∂E. In particular, if E_n is a sequence of convex bodies converging to a convex body E, then these constants depend only on the geometry of E.

Before stating the main result of this section, we prove two regularity lemmas.

Lemma 4.2. *Let $0 < \beta \le 1$ and $C, \varepsilon_0 > 0$ be given. Then, every convex body E such that for every $x_0 \in \partial E$ and every $\varepsilon \le \varepsilon_0$,*

$$\Delta V \le C\varepsilon^{2+\beta}, \tag{4-2}$$

is $C^{1,\beta}$ with $C^{1,\beta}$-norm depending only on the Lipschitz character of ∂E, ε_0 and C.

Proof. Let $x_0 \in \partial E$ be fixed. Since E is convex, there exist $R > 0$ and a convex function $u : I \to \mathbb{R}$ such that $\partial E \cap B_R(x_0) = \{(t, u(t)) : t \in I\}$ for some interval $I \subset \mathbb{R}$. Furthermore, $\|u'\|_{L^\infty} \lesssim 1$. Let $\bar{x} \in \partial E \cap B_R(x_0)$. Without loss of generality, we can assume that $\bar{x} = 0 = (0, u(0))$. By the convexity of u, up to adding a linear function, we can further assume that $u \ge 0$ in I. Thanks to the Lipschitz bound on u, for $x = (t, u(t)) \in \partial E \cap B_R(x_0)$, we have

$$|x| = (t^2 + |u(t)|^2)^{\frac{1}{2}} \sim t. \tag{4-3}$$

Let now $\varepsilon > 0$. For $\delta > 0$, let $-1 \ll t_1^\delta < 0 < t_2^\delta \ll 1$ such that $x_i^\delta = (t_i^\delta, u(t_i^\delta))$ for $i = 1, 2$ (see the notation above). By (4-3), there exists $\lambda > 0$, depending only on the Lipschitz character of u, such that

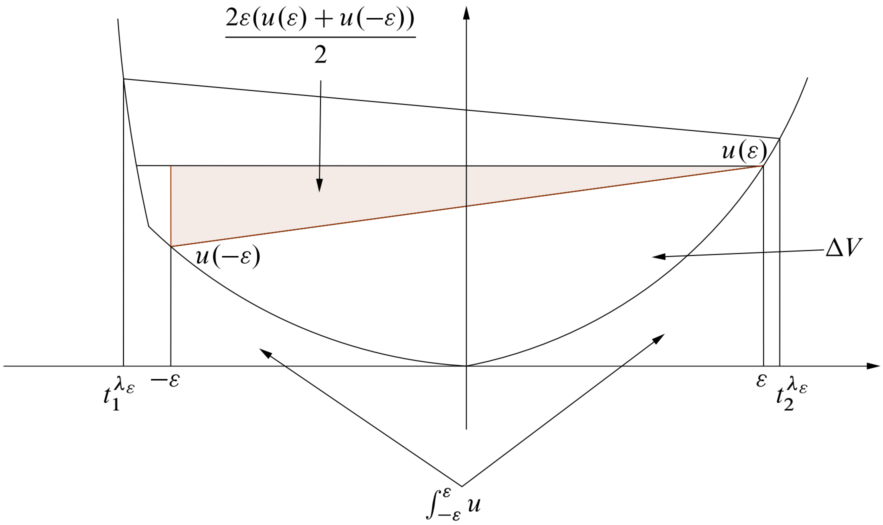

Figure 2

$|t_i^{\lambda\varepsilon}| \geq \varepsilon$. Without loss of generality, we can now assume that $u(-\varepsilon) \leq u(\varepsilon)$. In particular, considering the ΔV associated to $\lambda\varepsilon$, we have that (see Figure 2)

$$\begin{aligned}\Delta V &\geq 2\varepsilon u(\varepsilon) - \frac{2\varepsilon(u(\varepsilon)-u(-\varepsilon))}{2} - \int_{-\varepsilon}^{\varepsilon} u(t)\,dt \\ &= \varepsilon(u(\varepsilon)+u(-\varepsilon)) - \int_{-\varepsilon}^{\varepsilon} u(t)\,dt.\end{aligned}$$

Since u is decreasing in $[-\varepsilon, 0]$ and increasing in $[0, \varepsilon]$, this means that both

$$\varepsilon u(\varepsilon) - \int_0^{\varepsilon} u \lesssim \varepsilon^{2+\beta} \quad \text{and} \quad \varepsilon u(-\varepsilon) - \int_{-\varepsilon}^{0} u \lesssim \varepsilon^{2+\beta} \tag{4-4}$$

hold. Let us prove that this implies that for $|t|$ small enough

$$u(t) \lesssim |t|^{1+\beta}. \tag{4-5}$$

We can assume without loss of generality that $t > 0$. By (4-4) and the monotonicity of u,

$$tu(t) \leq Ct^{2+\beta} + \int_0^{\frac{t}{2}} u + \int_{\frac{t}{2}}^{t} u \leq Ct^{2+\beta} + \tfrac{1}{2}t\big(u\big(\tfrac{1}{2}t\big) + u(t)\big),$$

from which we obtain

$$u(t) - u\big(\tfrac{1}{2}t\big) \lesssim t^{1+\beta}.$$

Applying this for $k \geq 0$ to $t_k = 2^{-k}t$ and summing over k we obtain

$$u(t) \lesssim \sum_{k=0}^{\infty} (2^{-k}t)^{1+\beta} \lesssim t^{1+\beta},$$

that is, (4-5).

In other words, we have proven that u is differentiable in zero with $u'(0) = 0$ and that for $|t|$ small enough,

$$|u(t) - u(0) - u'(0)t| \lesssim |t|^{1+\beta}.$$

Since the point zero was arbitrarily chosen, this yields that u is differentiable everywhere and that for $t, s \in I$ with $|t - s|$ small enough,

$$|u(t) - u(s) - u'(s)(t - s)| \lesssim |t - s|^{\beta+1},$$

which is equivalent to the $C^{1,\beta}$ regularity of ∂E.[3] $\square$

Lemma 4.3. *Suppose that the minimizer E for (4-1) has boundary of class $C^{1,\beta}$ for some $0 < \beta < 1$. Then, there exists $R > 0$ (depending only on the $C^{1,\beta}$ character of ∂E) such that for every $x_0 \in \partial E$, $x \in \Sigma_\varepsilon$ and $y \in B_R(x_0)$,*

$$\gamma_\varepsilon(x, y) \lesssim \varepsilon^\beta + |x - y|^\beta. \tag{4-6}$$

Proof. Without loss of generality, we can assume that $x_0 = 0$. As in the proof of Lemma 4.2, since E is convex and of class $C^{1,\beta}$ in the ball $B_R(0)$ for a small enough R, we know that ∂E is a graph over its tangent of a $C^{1,\beta}$ function u. Up to a rotation, we can further assume that this tangent is horizontal so that for some interval $I \subset \mathbb{R}$, we have $\partial E \cap B_R(0) = \{(t, u(t)) : t \in I\}$. In particular, if $x = (t, u(t)) \in \partial E \cap B_R(0)$, $|u(t)| \lesssim |t|^{1+\beta}$ and $|u'(t)| \lesssim |t|^\beta$.

For $x = (t, u(t)) \in \Sigma_\varepsilon$ and $y = (s, u(s)) \in B_R(0)$, let $\tilde{\gamma}_\varepsilon(x, y)$ be the angle between $L_{x,y}$ and the horizontal line; i.e., $\tan(\tilde{\gamma}_\varepsilon(x, y)) = |u(t) - u(s)|/|t - s|$. Let us begin by estimating $\tilde{\gamma}_\varepsilon$. First, if $|x - y| \lesssim \varepsilon$ (which thanks to (4-3) amounts to $|t - s| \lesssim \varepsilon$ and thus $|t| + |s| \lesssim \varepsilon$ since $x \in \Sigma_\varepsilon$),

$$\tilde{\gamma}_\varepsilon(x, y) \sim \frac{|u(t) - u(s)|}{|t - s|} \le \sup_{r \in [s,t]} |u'(r)| \lesssim \varepsilon^\beta.$$

Otherwise, if $|x - y| \gg \varepsilon$, since $|x| \lesssim \varepsilon$, we have $|x - y| \sim |y| \sim |s|$ and thus

$$\tilde{\gamma}_\varepsilon(x, y) \lesssim \frac{|u(t)| + |u(s)|}{|t - s|} \lesssim \frac{\varepsilon^{1+\beta} + |s|^{1+\beta}}{|s|} \lesssim |s|^\beta \lesssim |x - y|^\beta.$$

Putting these estimates together, we find

$$\tilde{\gamma}_\varepsilon(x, y) \lesssim \varepsilon^\beta + |x - y|^\beta. \tag{4-7}$$

Let ξ_ε be the angle between L_ε and the horizontal line (see Figure 3). Since $\gamma_\varepsilon(x, y) = \tilde{\gamma}_\varepsilon \pm \xi_\varepsilon$, (4-6) holds provided that we can show

$$\xi_\varepsilon \lesssim \varepsilon^\beta. \tag{4-8}$$

Let $t_1^\varepsilon, t_2^\varepsilon \sim \varepsilon$ be such that $x_1^\varepsilon = (-t_1^\varepsilon, u(-t_1^\varepsilon))$ and $x_2^\varepsilon = (t_2^\varepsilon, u(t_2^\varepsilon))$. We see that ξ_ε is maximal if $u(-t_1^\varepsilon) = 0$, and then $t_1^\varepsilon = \varepsilon$. In that case, $\tan \xi_\varepsilon = u(t_2^\varepsilon)/(\varepsilon + t_2^\varepsilon)$.

[3] Indeed, for $|s - t| \le \varepsilon_1$, we have $|u'(t) - u'(s)| \le |t - s|^{-1}(|u(t) - u(s) - u'(s)(t - s)| + |u(s) - u(t) - u'(t)(s - t)|) \lesssim |t - s|^\beta$.

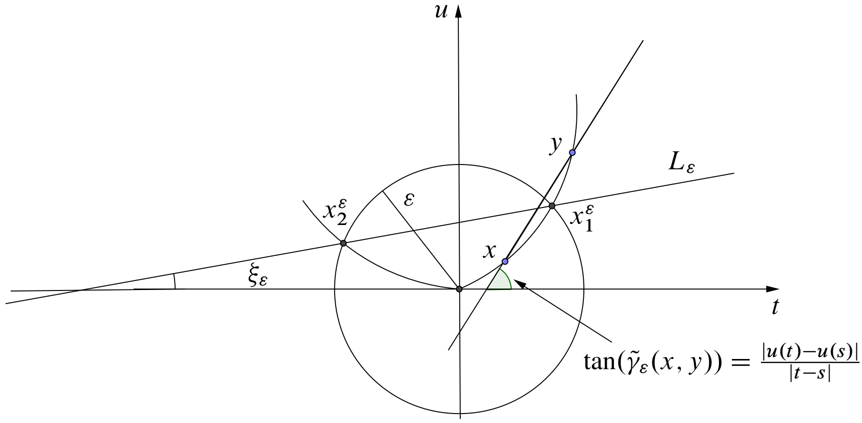

Figure 3

Since $u(t_2^\varepsilon) \lesssim \varepsilon^{1+\beta}$, and $t_\varepsilon^2 \lesssim \varepsilon$, we obtain

$$\xi_\varepsilon \sim \tan \xi_\varepsilon \lesssim \frac{\varepsilon^{1+\beta}}{\varepsilon} = \varepsilon^\beta,$$

proving (4-8). This concludes the proof of (4-6). $\square$

We pass now to the main result of this section.

Theorem 4.4. *Every minimizer of (4-1) is $C^{1,1}$. Moreover, for every Q_0 and every $Q \le Q_0$, the $C^{1,1}$ character of ∂E depends only on Q_0, the Lipschitz character of ∂E and $\|f\|_{L^p(\partial E)}$.*

Proof. Let E be a minimizer of (4-1), $x_0 \in \partial E$ be fixed and let $\varepsilon \le \varepsilon_0$. With the above notation in force, we begin by observing that using E_ε as a competitor, by the minimality of E for (4-1), we have

$$Q^2 \Delta \mathcal{I}_0 \ge \Delta P - \Lambda \Delta V. \tag{4-9}$$

We are thus going to estimate $\Delta \mathcal{I}_0$, ΔP and ΔV in terms of ε and γ_ε. This will give us a quantitative decay estimate for γ_ε. This in turn, in light of (4-10) below and Lemma 4.2, will provide the desired regularity of E.

Step 1 (*volume estimate*): In this first step, we prove that

$$\Delta V \sim \varepsilon^2 \gamma_\varepsilon. \tag{4-10}$$

By construction, we have

$$\Delta V = |E| - |E_\varepsilon| = |E \cap H_\varepsilon^+|.$$

By convexity, we first have that the triangle with vertices x_0, x_1^ε, x_2^ε is contained inside $E \cap H_\varepsilon^+$. By convexity again, letting $\bar{x}_1^\varepsilon$ be the point of $\partial B_\varepsilon(x_0)$ diametrically opposed to x_1^ε (and similarly for $\bar{x}_2^\varepsilon$), we get that $E \cap H_\varepsilon^+$ is contained in the union of the triangles of vertices x_1^ε, x_2^ε, $\bar{x}_1^\varepsilon$ and x_1^ε, x_2^ε, $\bar{x}_2^\varepsilon$ (see Figure 4).

Therefore, we obtain

$$\Delta V \sim \varepsilon^2 \cos \gamma_\varepsilon \sin \gamma_\varepsilon \sim \varepsilon^2 \gamma_\varepsilon.$$

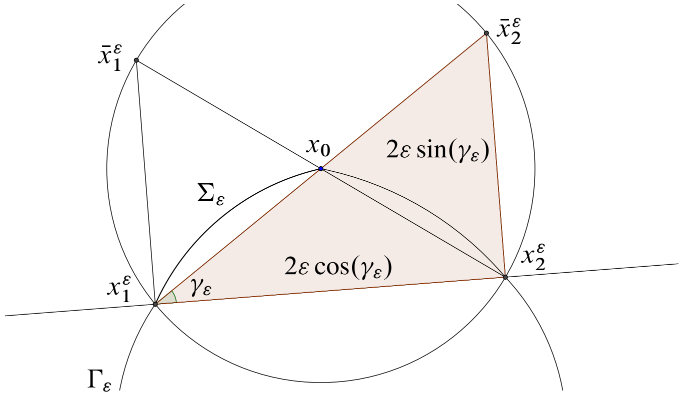

Figure 4. ΔV is contained in the union of the triangles of vertices $x_1^\varepsilon, x_2^\varepsilon, \bar{x}_1^\varepsilon$ and $x_1^\varepsilon, x_2^\varepsilon, \bar{x}_2^\varepsilon$.

Step 2 (*perimeter estimate*): Since the triangle with vertices $x_0, x_1^\varepsilon, x_2^\varepsilon$ is contained inside $E \cap H_\varepsilon^+$, it holds that

$$\Delta P = P(E) - P(E_\varepsilon) \geq 2\varepsilon(1-\cos\gamma_\varepsilon) \gtrsim \varepsilon\gamma_\varepsilon^2. \tag{4-11}$$

Step 3 (*nonlocal energy estimate*): We now estimate $\Delta\mathcal{I}_0$. Since μ_ε is a competitor for $\mathcal{I}_0(E_\varepsilon)$, recalling that Π is the identity outside Σ_ε, we have

$$\begin{aligned}
\Delta\mathcal{I}_0 &= \mathcal{I}_0(E_\varepsilon) - \mathcal{I}_0(E)\\
&\leq \int_{\partial E_\varepsilon\times\partial E_\varepsilon} f_\varepsilon(x) f_\varepsilon(y)\log\left(\frac{1}{|x-y|}\right) - \int_{\partial E\times\partial E} f(x)f(y)\log\left(\frac{1}{|x-y|}\right)\\
&= \int_{\partial E\times\partial E} f(x)f(y)\log\left(\frac{1}{|\Pi(x)-\Pi(y)|}\right) - \int_{\partial E\times\partial E} f(x)f(y)\log\left(\frac{1}{|x-y|}\right)\\
&= \int_{\partial E\times\partial E} f(x)f(y)\log\left(\frac{|x-y|}{|\Pi(x)-\Pi(y)|}\right).
\end{aligned}$$

Since for $x, y \in \Sigma_\varepsilon^c$, we have $|\Pi(x)-\Pi(y)| = |x-y|$, we get

$$\begin{aligned}
\Delta\mathcal{I}_0 &\leq \int_{\Sigma_{3\varepsilon}\times\Sigma_{3\varepsilon}} f(x)f(y)\log\left(\frac{|x-y|}{|\Pi(x)-\Pi(y)|}\right) + 2\int_{\Sigma_\varepsilon}\int_{\Gamma_\varepsilon} f(x)f(y)\log\left(\frac{|x-y|}{|\Pi(x)-y|}\right)\\
&=: I_1 + 2I_2.
\end{aligned}$$

We first estimate I_1:

$$\begin{aligned}
I_1 &= \int_{\Sigma_{3\varepsilon}\times\Sigma_{3\varepsilon}} f(x)f(y)\log\left(1 + \frac{|x-y|-|\Pi(x)-\Pi(y)|}{|\Pi(x)-\Pi(y)|}\right)\\
&\leq \int_{\Sigma_{3\varepsilon}\times\Sigma_{3\varepsilon}} f(x)f(y)\frac{|x-y|-|\Pi(x)-\Pi(y)|}{|\Pi(x)-\Pi(y)|}.
\end{aligned}$$

Since for any $x, y \in \Sigma_{3\varepsilon}$ we have (with equality if $x, y \in \Sigma_\varepsilon$),

$$\cos(\gamma_\varepsilon(x,y))|x-y| \leq |\Pi(x)-\Pi(y)|,$$

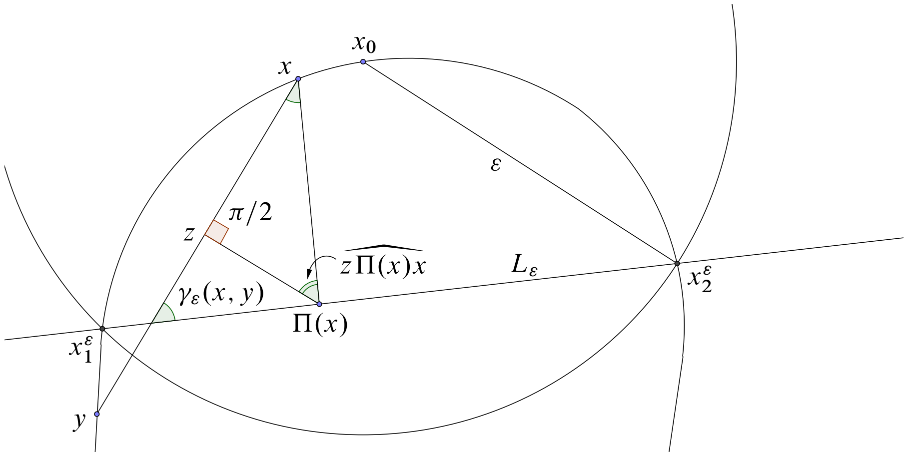

Figure 5. The angle $\widehat{z\Pi(x)x}$ equals $\gamma_\varepsilon(x,y)$.

we get

$$I_1 \le \int_{\Sigma_{3\varepsilon}\times\Sigma_{3\varepsilon}} f(x)f(y)\left(\frac{1}{\cos(\gamma_\varepsilon(x,y))}-1\right) \lesssim \int_{\Sigma_{3\varepsilon}\times\Sigma_{3\varepsilon}} \gamma_\varepsilon^2(x,y)f(x)f(y). \tag{4-12}$$

Using then Hölder's inequality (recall that $f\in L^p(\partial E)$ for some $p>2$) to get

$$\int_{\Sigma_{3\varepsilon}} f \le \left(\int_{\Sigma_{3\varepsilon}} f^p\right)^{\frac{1}{p}} \mathcal{H}^1(\Sigma_{3\varepsilon})^{\frac{p-1}{p}} \lesssim \varepsilon^{\frac{p-1}{p}}, \tag{4-13}$$

and $\gamma_\varepsilon(x,y)\lesssim 1$, we obtain

$$I_1 \lesssim \varepsilon^{2\frac{p-1}{p}}. \tag{4-14}$$

We can now estimate I_2:

$$\begin{aligned} I_2 &= \int_{\Sigma_\varepsilon}\int_{\Gamma_\varepsilon} f(x)f(y)\log\left(1+\left(\frac{|x-y|-|\Pi(x)-y|}{|\Pi(x)-y|}\right)\right) \\ &\le \int_{\Sigma_\varepsilon}\int_{\Gamma_\varepsilon} f(x)f(y)\left(\frac{|x-y|-|\Pi(x)-y|}{|\Pi(x)-y|}\right). \end{aligned}$$

Denote by z the projection of $\Pi(x)$ on the line containing x and y. Then, since the projection is a 1-Lipschitz function, it holds that $|z-y|\le|\Pi(x)-y|$. Thus,

$$|x-y|-|y-\Pi(x)| = |x-z|+|z-y|-|y-\Pi(x)| \le |x-z|.$$

Arguing as in Step 1, we get $|x-\Pi(x)|\le|\bar{x}_2^\varepsilon - x_2^\varepsilon|\lesssim \varepsilon\gamma_\varepsilon$. Furthermore, the angle $\widehat{z\Pi(x)x}$ equals $\gamma_\varepsilon(x,y)$ (see Figure 5), so that

$$|x-y|-|y-\Pi(x)| \le |x-z| = |x-\Pi(x)|\sin(\gamma_\varepsilon(x,y)) \lesssim \varepsilon\gamma_\varepsilon\gamma_\varepsilon(x,y).$$

On the other hand, since $|y-x|\geq 2\varepsilon$ (indeed $|x-x_0|\leq\varepsilon$ and $|y-x_0|\geq 3\varepsilon$), we have

$$|y-\Pi(x)|\geq|y-x|-|x-\Pi(x)|\gtrsim|y-x|-\varepsilon\gtrsim|y-x|.$$

Therefore,

$$I_2\lesssim\varepsilon\gamma_\varepsilon\int_{\Sigma_\varepsilon}\int_{\Gamma_\varepsilon}\frac{f(x)f(y)\gamma_\varepsilon(x,y)}{|y-x|}. \tag{4-15}$$

There exists $M>0$ which depends only on the Lipschitz character of ∂E such that for $x\in\Sigma_\varepsilon$ and $y\in\Gamma_\varepsilon\cap B_M(x_0)$,

$$|y-x|\geq\min_{i=1,2}|y-x_i^\varepsilon|.$$

Let $\Gamma_\varepsilon^N:=\Gamma_\varepsilon\cap B_M(x_0)$ and $\Gamma_\varepsilon^F:=\Gamma_\varepsilon\cap B_M^c(x_0)$. We then have

$$\begin{aligned}I_2&\lesssim\varepsilon\gamma_\varepsilon\biggl(\int_{\Sigma_\varepsilon\times\Gamma_\varepsilon^N}\frac{f(x)f(y)\gamma_\varepsilon(x,y)}{\min_i|y-x_i^\varepsilon|}+\int_{\Sigma_\varepsilon\times\Gamma_\varepsilon^F}f(x)f(y)\gamma_\varepsilon(x,y)\biggr)\\&=:I_2^N+I_2^F.\end{aligned}$$

We begin by estimating I_2^F. Since $\gamma_\varepsilon(x,y)\lesssim 1$, using Hölder's inequality we find

$$\begin{aligned}I_2^F&\lesssim\varepsilon\gamma_\varepsilon\biggl(\int_{\Gamma_\varepsilon}f\biggr)\biggl(\int_{\Sigma_\varepsilon}f\biggr)\\&\leq\varepsilon\gamma_\varepsilon\|f\|_{L^p}\mathcal{H}^1(\Gamma_\varepsilon)^{1-\frac{1}{p}}\|f\|_{L^p}\mathcal{H}^1(\Sigma_\varepsilon)^{1-\frac{1}{p}}\\&\lesssim\varepsilon\gamma_\varepsilon\mathcal{H}^1(\Sigma_\varepsilon)^{1-\frac{1}{p}}\\&\lesssim\varepsilon^{2-\frac{1}{p}}\gamma_\varepsilon.\end{aligned} \tag{4-16}$$

We can now estimate I_2^N. Recall that

$$I_2^N:=\varepsilon\gamma_\varepsilon\int_{\Sigma_\varepsilon\times\Gamma_\varepsilon^N}\frac{f(x)f(y)\gamma_\varepsilon(x,y)}{\min_i|y-x_i^\varepsilon|}. \tag{4-17}$$

As before, we use $\gamma_\varepsilon(x,y)\lesssim 1$ together with Hölder's inequality applied twice to get

$$\int_{\Sigma_\varepsilon\times\Gamma_\varepsilon^N}\frac{f(x)f(y)\gamma_\varepsilon(x,y)}{\min_i|y-x_i^\varepsilon|}\lesssim\varepsilon^{1-\frac{1}{p}}\biggl(\int_{\Gamma_\varepsilon^N}\frac{1}{\min_i|y-x_i^\varepsilon|^{\frac{p}{p-1}}}\biggr)^{\frac{p-1}{p}}.$$

Since E is convex, its boundary can be locally parametrized by Lipschitz functions so that, if M is small enough (depending only on the Lipschitz regularity of ∂E), then for $y\in\Gamma_\varepsilon^N$, we have

$$\min_i\ell(y,\tilde{x}_i^\varepsilon)\sim\min_i|y-\tilde{x}_i^\varepsilon|$$

(where $\ell(x,y)$ denotes the geodesic distance on ∂E). From this we get

$$\int_{\Gamma_\varepsilon^N}\frac{1}{\min_i|y-x_i^\varepsilon|^{\frac{p}{p-1}}}\lesssim\varepsilon^{-\frac{1}{p-1}}.$$

From this we conclude that

$$I_2^N \lesssim \gamma_\varepsilon \varepsilon^{2-\frac{2}{p}}. \tag{4-18}$$

Step 4 ($C^{1,\beta}$ *regularity*): We now prove that E has boundary of class $C^{1,\beta}$. To this aim, we can assume that $\Delta V \ll \Delta P$. Indeed, if $\Delta V \gtrsim \Delta P$, thanks to (4-10) and (4-11), we would get $\gamma_\varepsilon \lesssim \varepsilon$ and thus $\Delta V \lesssim \varepsilon^3$, which by Lemma 4.2 would already ensure the $C^{1,1}$ regularity of ∂E. Using (4-9), (4-11), (4-14), (4-16) and (4-18), we get

$$Q^2(\varepsilon^{1-\frac{2}{p}} + \gamma_\varepsilon(\varepsilon^{1-\frac{1}{p}} + \varepsilon^{1-\frac{2}{p}})) \gtrsim \gamma_\varepsilon^2. \tag{4-19}$$

Now since $\varepsilon^{1-\frac{1}{p}} \lesssim \varepsilon^{1-\frac{2}{p}}$, this reduces further to

$$Q^2(\varepsilon^{1-\frac{2}{p}} + \gamma_\varepsilon \varepsilon^{1-\frac{2}{p}}) \gtrsim \gamma_\varepsilon^2. \tag{4-20}$$

We can now distinguish two cases. Either $Q^2\varepsilon^{2(\frac{1}{2}-\frac{1}{p})} \gtrsim \gamma_\varepsilon^2$ and then $\gamma_\varepsilon \lesssim Q\varepsilon^{(\frac{1}{2}-\frac{1}{p})}$ or $Q^2\gamma_\varepsilon\varepsilon^{1-\frac{2}{p}} \gtrsim \gamma_\varepsilon^2$ and then $\gamma_\varepsilon \lesssim Q^2\varepsilon^{1-\frac{2}{p}}$. Thus in both cases, since $p > 2$, we find $\gamma_\varepsilon \lesssim Q\varepsilon^\beta$ for some $\beta > 0$ and we can conclude, by means of (4-10) and Lemma 4.2, that ∂E is $C^{1,\beta}$.

Step 5 ($C^{1,1}$ *regularity*): Thanks to Lemma 3.6, we get that $f \in L^\infty$ with $\|f\|_{L^\infty}$ depending only on the Lipschitz character of ∂E and on $\|f\|_{L^p}$. Using this new information, we can improve (4-14), (4-16) and (4-18) to

$$I_1 \lesssim \varepsilon^2, \quad I_2^F \lesssim \gamma_\varepsilon \varepsilon^2, \quad \text{and} \quad I_2^N \lesssim \gamma_\varepsilon \varepsilon^2 |\log \varepsilon|. \tag{4-21}$$

Arguing as in Step 4, we find $\gamma_\varepsilon \lesssim Q\varepsilon^{\frac{1}{2}}$ and thus ∂E is of class $C^{1,\frac{1}{2}}$. In order to get higher regularity, we need to get a better estimate on $\gamma_\varepsilon(x, y)$.

Going back to (4-12) and using (4-6) with $\beta = \frac{1}{2}$, we find the improved estimate

$$I_1 \lesssim \varepsilon^3. \tag{4-22}$$

If we also use (4-6) in (4-17), we obtain

$$\begin{aligned}
I_2^N &\lesssim \varepsilon\gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}} + |x-y|^{\frac{1}{2}}}{\min_i |y - \tilde{x}_i^\varepsilon|} \\
&\lesssim \varepsilon\gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}} + \min_i \{|x - \tilde{x}_i^\varepsilon|^{\frac{1}{2}} + |y - \tilde{x}_i^\varepsilon|^{\frac{1}{2}}\}}{\min_i |y - \tilde{x}_i^\varepsilon|} \\
&\lesssim \varepsilon\gamma_\varepsilon \int_{\Sigma_\varepsilon \times \Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}} + \min_i |y - \tilde{x}_i^\varepsilon|^{\frac{1}{2}}}{\min_i |y - \tilde{x}_i^\varepsilon|} \\
&\lesssim \varepsilon^2\gamma_\varepsilon \int_{\Gamma_\varepsilon^N} \frac{\varepsilon^{\frac{1}{2}}}{\min_i |y - \tilde{x}_i^\varepsilon|} + \frac{1}{\min_i |y - \tilde{x}_i^\varepsilon|^{\frac{1}{2}}} \\
&\lesssim \varepsilon^2\gamma_\varepsilon(\varepsilon^{\frac{1}{2}}|\log \varepsilon| + 1) \lesssim \varepsilon^2\gamma_\varepsilon.
\end{aligned}$$

As in the beginning of Step 4, we can assume that $\Delta V \ll \Delta P$, so that by (4-9) and (4-11) we have $Q^2\Delta\mathcal{I}_0 \gtrsim \Delta P \gtrsim \varepsilon\gamma_\varepsilon^2$. By the previous estimate for I_2^N, (4-22) and the second inequality in (4-21) we eventually get

$$Q^2\varepsilon^2\gamma_\varepsilon \sim Q^2(\varepsilon^3+\varepsilon^2\gamma_\varepsilon) \gtrsim \varepsilon\gamma_\varepsilon^2,$$

which leads to $\gamma_\varepsilon \lesssim Q^2\varepsilon$. By using again Lemma 4.2, the proof is concluded. □

5. Minimality of the ball for $N = 2$ and Q small

We now use the regularity result obtained in Section 4 to prove that for small charges, the only minimizers of $\mathcal{F}_{Q,0}$ in dimension two are balls.

Theorem 5.1. *Let $N = 2$ and $\alpha = 0$. There exists $Q_0 > 0$ such that for $Q < Q_0$, up to translations, the only minimizer of (1-1) is the ball.*

Proof. Let E_Q be a minimizer of $\mathcal{F}_{Q,0}$ and let B be a ball of measure 1. By the minimality of E_Q, we have

$$P(E_Q)-P(B) \le Q^2(\mathcal{I}_0(B)-\mathcal{I}_0(E_Q)) \le Q^2(\mathcal{I}_0(B)+|\mathcal{I}_0(E_Q)|). \tag{5-1}$$

By Lemma 4.1 the diameter of E_Q is uniformly bounded and so is $|\mathcal{I}_0(E_Q)|$. Using the quantitative isoperimetric inequality, see [Fusco et al. 2008], we infer

$$|E_Q\Delta B|^2 \lesssim P(E_Q)-P(B) \le Q^2(\mathcal{I}_0(B)+|\mathcal{I}_0(E_Q)|).$$

This implies that E_Q converges to B in L^1 as $Q \to 0$. From the convexity of E_Q, this implies the convergence also in the Hausdorff metric. Since the sets E_Q are all uniformly bounded and of fixed volume, they are uniformly Lipschitz. Theorem 4.4 then implies that ∂E_Q are $C^{1,1}$-regular sets with $C^{1,1}$ norm uniformly bounded. Therefore, thanks to the Arzelà–Ascoli theorem, we can write

$$\partial E_Q = \{(1+\varphi_Q(x))x : x \in \partial B\},$$

with $\|\varphi_Q\|_{C^{1,\beta}}$ converging to 0 as $Q \to 0$ for every $\beta < 1$. From Lemma 3.6 we infer that the optimal measures μ_Q for E_Q are uniformly $C^{0,\beta}$ and in particular are uniformly bounded. Using now [Goldman et al. 2015, Proposition 6.3], we get that for small enough Q,

$$\|\mu_Q\|^2_{L^\infty}(P(E_Q)-P(B)) \gtrsim \mathcal{I}_0(B)-\mathcal{I}_0(E_Q).$$

Putting this into (5-1), we then obtain

$$P(E_Q)-P(B) \lesssim Q^2(P(E_Q)-P(B)),$$

from which we deduce that for Q small enough, $P(E_Q) = P(B)$. Since, up to translations, the ball is the unique solution of the isoperimetric problem, this implies $E_Q = B$. □

6. Asymptotic behavior as $Q \to +\infty$

We now characterize the limit shape of (suitably rescaled) minimizers of $\mathcal{F}_{Q,\alpha}$, with $\alpha \in [0, 1]$, as the charge Q tends to $+\infty$. For this, we fix a sequence $Q_n \to +\infty$.

The case $\alpha \in [0,1)$. For $n \in \mathbb{N}$, we let $V_n := Q_n^{-\frac{2N(N-1)}{1+(N-1)\alpha}}$ (so that $V_n \to 0$ as $n \to +\infty$) and

$$\mathcal{A}_{n,\alpha} := \{E \subset \mathbb{R}^N \text{ convex body}, |E| = V_n\},$$
$$\widehat{\mathcal{F}}_{n,\alpha}(E) := V_n^{-\frac{N-2}{N-1}} P(E) + \mathcal{I}_\alpha(E) \quad \text{for } E \in \mathcal{A}_{n,\alpha}.$$

It is straightforward to check that if E is a minimizer of (1-1), then the rescaled set

$$\widehat{E} := Q_n^{-\frac{2(N-1)}{1+(N-1)\alpha}} E$$

is a minimizer of $\widehat{\mathcal{F}}_{n,\alpha}$ in the class $\mathcal{A}_{n,\alpha}$.

We begin with a compactness result for a sequence of sets of equibounded energy.

Proposition 6.1. *Let $\alpha \in [0,1)$ and let $E_n \in \mathcal{A}_{n,\alpha}$ be such that*

$$\sup_n \widehat{\mathcal{F}}_{n,\alpha}(E_n) < +\infty.$$

Then, up to extracting a subsequence and up to rigid motions, the sets E_n converge in the Hausdorff topology to the segment $[0,L] \times \{0\}^{N-1}$ for some $L \in (0,+\infty)$.

Proof. The bound on $\mathcal{I}_\alpha(E_n)$ directly implies with (2-2) (or (2-3) in the case $\alpha = 0$) that the diameter of E_n is uniformly bounded from below.

Let us show that the diameter of E_n is also uniformly bounded from above. Arguing as in Theorem 2.3, let $\mathcal{R}_n = \prod_{i=1}^N [0, \lambda_i^n]$ be the parallelepipeds given by Lemma 2.2, and assume without loss of generality that $\lambda_1^n \geq \lambda_2^n \geq \cdots \geq \lambda_N^n$. In the case $\alpha > 0$, (2-1) directly gives the bound, while for $\alpha = 0$, we get using (2-1) and (2-3), that $|\mathcal{I}_0(\mathcal{R}_n)|$ is uniformly bounded, from which the bound on the diameter follows, using once again (2-1). Moreover, from (2-2) and (2-3), we obtain that $\lambda_i^n \sim V_n^{\frac{1}{N-1}}$ (where the constants depend on $\widehat{\mathcal{F}}_{n,\alpha}(E_n)$) for $i = 2, \ldots, N$. The convex bodies E_n are therefore compact in the Hausdorff topology and any limit set is a nontrivial segment of length $L \in (0,+\infty)$. $\square$

In the proof of the Γ-convergence result we will use the following result.

Lemma 6.2. *Let $0 < \gamma < \beta$ with $\beta \geq 1$, $V > 0$ and $L > 0$, then*

$$\min\left\{\int_0^L f^\gamma : \int_0^L f^\beta = V,\ f \text{ concave and } f \geq 0\right\} = \frac{(\beta+1)^{\frac{\gamma}{\beta}}}{\gamma+1} L^{1-\frac{\gamma}{\beta}} V^{\frac{\gamma}{\beta}}. \tag{6-1}$$

Proof. For $L, V > 0$, let

$$M(L,V) := \min\left\{\int_0^L f^\gamma : \int_0^L f^\beta = V,\ f \text{ concave and } f \geq 0\right\}.$$

Let us now prove (6-1). By scaling, we can assume that $L = V = 1$. Thanks to the concavity and positivity constraints, existence of a minimizer for (6-1) follows. Let f be such a minimizer. Let us prove that we can assume that f is nonincreasing. Notice first that by definition, it holds that

$$M(1,1) = \int_0^1 f^\gamma.$$

Up to a rearrangement, we can assume that f is symmetric around the point $\frac{1}{2}$, so that f is nonincreasing in $\left[\frac{1}{2}, 1\right]$ and

$$\int_{\frac{1}{2}}^{1} f^{\gamma} = \tfrac{1}{2} M(1,1) = M\left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Finally letting $\hat{f}(x) := f\left(\frac{1}{2}\left(x + \frac{1}{2}\right)\right)$ for $x \in [0,1]$, we have that $\hat{f}$ is nonincreasing, admissible for (6-1) and

$$\int_0^1 \hat{f}^{\gamma} = 2 \int_{\frac{1}{2}}^{1} f^{\gamma} = M(1,1),$$

so that $\hat{f}$ is also a minimizer for (6-1).

Assume now that f is not affine in $(0,1)$. Then there is $\bar{x} > 0$ such that for all $0 < x \leq \bar{x}$

$$f(x) > f(0) - (f(0) - f(1))x.$$

Let $\tilde{f} := \lambda - (\lambda - f(1))x$ with $\lambda > f(0)$ chosen so that

$$\int_0^1 f^{\beta-1} \tilde{f} = \int_0^1 f^{\beta}. \tag{6-2}$$

Now, let $g := \tilde{f} - f$. Since $f + g = \tilde{f}$ is concave, for every $0 \leq t \leq 1$, we have $f + tg$ is a concave function. For $\delta \in \mathbb{R}$, let $f_{t,\delta} := f + t(g + \delta(1-x))$. Let finally δ_t be such that

$$\int_0^1 f_{t,\delta_t}^{\beta} = \int_0^1 f^{\beta}.$$

Thanks to (6-2) and since $\beta \geq 1$, we have $|\delta_t| = O(t)$. Since f_{t,δ_t} is concave, by the minimality of f we get

$$\int_0^1 f_{t,\delta_t}^{\gamma} - \int_0^1 f^{\gamma} \geq 0.$$

Dividing by t and taking the limit as t goes to zero, we obtain

$$\int_0^1 f^{\gamma-1} g \geq 0.$$

Let $z \in (0,1)$ be the unique point such that $\tilde{f}(z) = f(z)$ (so that $\tilde{f}(x) > f(x)$ for $x < z$ and $\tilde{f}(x) < f(x)$ for $x > z$). We then have

$$\begin{aligned}
0 &\leq \int_0^1 f^{\beta-1} \frac{\tilde{f} - f}{f^{\beta-\gamma}} \\
&= \int_0^z f^{\beta-1} \frac{\tilde{f} - f}{f^{\beta-\gamma}} + \int_z^1 f^{\beta-1} \frac{\tilde{f} - f}{f^{\beta-\gamma}} \\
&< \frac{1}{f^{\beta-\gamma}(z)} \left(\int_0^z f^{\beta-1}(\tilde{f} - f) + \int_z^1 f^{\beta-1}(\tilde{f} - f) \right) \\
&= \frac{1}{f^{\beta-\gamma}(z)} \int_0^1 f^{\beta-1}(\tilde{f} - f),
\end{aligned}$$

which contradicts (6-2).

We are left to study the case when f is linear. Assume that $f(1)>0$ and let

$$\delta := \frac{\int_0^1 f^{\beta-1}}{\int_0^1 x f^{\beta-1}} > 1,$$

so that in particular, $\int_0^1 f^{\beta-1}(1-\delta x)=0$. Up to adjusting the volume as in the previous case, for $t>0$ small enough, $f+t(1-\delta x)$ is admissible. From this, arguing as above, we find that

$$\int_0^1 f^{\gamma-1}(1-\delta x) \geq 0.$$

By splitting the integral around the point $\bar{z}=\delta^{-1}\in(0,1)$ and proceeding as above, we get again a contradiction. As a consequence, we obtain that $f(x)=\lambda(1-x)$, with $\lambda=(\beta+1)^{\frac{1}{\beta}}$ so that the volume constraint is satisfied. This concludes the proof of (6-1). □

We now prove the following Γ-convergence result.

Theorem 6.3. *For $\alpha\in[0,1)$, the functionals $\widehat{\mathcal{F}}_{n,\alpha}$ Γ–converge in the Hausdorff topology, as $n\to+\infty$, to the functional*

$$\widehat{\mathcal{F}}_\alpha(E) := \begin{cases} C_N\, L^{\frac{1}{N-1}} + \mathcal{I}_\alpha([0,1])/L^\alpha & \text{if } E \simeq [0,L]\times\{0\}^{N-1} \text{ and } \alpha>0,\\ C_N\, L^{\frac{1}{N-1}} + \mathcal{I}_0([0,1]) - \log L & \text{if } E \simeq [0,L]\times\{0\}^{N-1} \text{ and } \alpha=0,\\ +\infty & \text{otherwise,}\end{cases}$$

where $E\simeq F$ means that $E=F$ up to a rigid motion, and $C_N := \omega_{N-1}^{\frac{1}{N-1}} N^{\frac{N-2}{N-1}}$ with ω_N the volume of the ball of radius 1 in $\mathbb{R}^N$ (so that for $N=2$ we have $C_2=2$).

Proof. By Proposition 6.1 we know that the Γ-limit is $+\infty$ on the sets which are not segments.

Let us first prove the Γ-limsup inequality. Given $L\in(0,+\infty)$, we are going to construct E_n symmetric with respect to the hyperplane $\{0\}\times\mathbb{R}^{N-1}$. For $t\in\left[0,\frac{L}{2}\right]$, we let

$$r(t) := \left(\frac{N V_n}{\omega_{N-1} L}\right)^{\frac{1}{N-1}} \left(1-\frac{2t}{L}\right)$$

and then

$$E_n \cap (\mathbb{R}^+\times\mathbb{R}^{N-1}) := \left\{(t, B^{N-1}_{r(t)}) : t\in\left[0,\tfrac{L}{2}\right]\right\},$$

where $B^{N-1}_{r(t)}$ is the ball of radius $r(t)$ in $\mathbb{R}^{N-1}$. With this definition, $|E_n|=V_n$, so that $E_n\in\mathcal{A}_{n,\alpha}$. We then compute

$$\begin{aligned} P(E_n) &= 2\int_0^{\frac{L}{2}} \mathcal{H}^{N-2}(\mathbb{S}^{N-2}) r(t)^{N-2}\sqrt{1+|r'|^2}\\ &= 2(N-1)\,\omega_{N-1}\left(\frac{N V_n}{\omega_{N-1} L}\right)^{\frac{N-2}{N-1}} \int_0^{\frac{L}{2}} \left(1-\frac{2t}{L}\right)^{N-2}\left(1+\frac{c_N}{L^2}\left(\frac{V_n}{L}\right)^{\frac{2}{N-1}}\right)^{\frac{1}{2}}\\ &= C_N V_n^{\frac{N-2}{N-1}} L^{\frac{1}{N-1}} + o(V_n^{\frac{N-2}{N-1}}). \end{aligned}$$

Letting μ_α be the optimal measure for $\mathcal{I}_\alpha\bigl(\bigl[-\frac{L}{2},\frac{L}{2}\bigr]\bigr)$, we then have

$$\widehat{\mathcal{F}}_{n,\alpha}(E_n) \le C_N L^{\frac{1}{N-1}} + \mathcal{I}_\alpha([0,L]) + o(1),$$

which gives the Γ-limsup inequality.

We now turn to the Γ-liminf inequality. Let $E_n \in \mathcal{A}_{n,\alpha}$ be such that $E_n \to [0,L]\times\{0\}^{N-1}$ in the Hausdorff topology. Since $\mathcal{I}_\alpha$ is continuous under Hausdorff convergence, it is enough to prove that

$$\liminf_{n\to+\infty} V_n^{-\frac{N-2}{N-1}} P(E_n) \ge C_N\, L^{\frac{1}{N-1}}. \tag{6-3}$$

Let $L_n := \operatorname{diam}(E_n)$. By Hausdorff convergence, we have that $L_n \to L$. Moreover, up to a rotation and a translation, we can assume that $[0,L_n]\times\{0\}^{N-1}\subset E_n$. For $N=2$, we directly obtain $P(E_n)\ge 2L_n$, which gives (6-3). We thus assume from now on that $N\ge 3$. Let $\widetilde{E}_n$ be the set obtained from E_n after a Schwarz symmetrization around the axis $\mathbb{R}\times\{0\}^{N-1}$. By Brunn's principle [1887], $\widetilde{E}_n$ is still a convex set with $P(E_n)\ge P(\widetilde{E}_n)$ and $|E_n|=|\widetilde{E}_n|$. We thus have

$$\widetilde{E}_n = \bigcup_{t\in[0,L_n]} \{t\}\times B^{N-1}_{r(t)}$$

for an appropriate function $r(t)$, and, by Fubini's theorem,

$$\int_0^{L_N} r(t)^{N-1} = \frac{V_n}{\omega_{N-1}}.$$

By the coarea formula [Ambrosio et al. 2000, Theorem 2.93], we then get

$$P(\widetilde{E}_n) \ge \mathcal{H}^{N-2}(\mathbb{S}^{N-2}) \int_0^{L_n} r(t)^{N-2}\sqrt{1+|r'(t)|^2} \ge \mathcal{H}^{N-2}(\mathbb{S}^{N-2}) \int_0^{L_n} r(t)^{N-2}.$$

Applying then Lemma 6.2 with $\gamma = N-2$ and $\beta = N-1$, we obtain (6-3). □

Remark 6.4. For $\alpha\in[0,1)$ and $N\ge 2$, it is easy to optimize $\widehat{F}_\alpha$ in L and obtain the values $L_{N,\alpha}$ given in Theorem 1.4.

From Proposition 6.1, Theorem 6.3 and the uniqueness of the minimizers for $\widehat{F}_\alpha$, we directly obtain the following asymptotic result for minimizers of (1-1).

Corollary 6.5. *Let $\alpha\in[0,1)$ and $N\ge 2$. Then, up to rescalings and rigid motions, every sequence E_n of minimizers of (1-1) converges in the Hausdorff topology to $[0,L_{N,\alpha}]\times\{0\}^{N-1}$.*

The case $N=2,3$ and $\alpha=1$. In the case $\alpha\ge 1$, the energy $\mathcal{I}_\alpha$ is infinite on segments and thus a Γ-limit of the same type as the one obtained in Theorem 6.3 cannot be expected. Nevertheless in the Coulombic case $N=3$, $\alpha=1$ we can use a dual formulation of the nonlocal part of the energy to obtain the Γ-limit. As a by-product, we can also treat the case $N=2$, $\alpha=1$.

For $N=2,3$ and $n\in\mathbb{N}$, we let

$$\mathcal{A}_{n,1} := \{E\subset\mathbb{R}^3 \text{ convex body}, |E| = Q_n^{-2(N-1)}(\log Q_n)^{-(N-1)}\},$$

$$\widehat{\mathcal{F}}_{n,1}(E) := Q_n^{2(N-2)}(\log Q_n)^{N-2}\, P(E) + \frac{\mathcal{I}_1(E)}{\log Q_n} \quad \text{for } E\in\mathcal{A}_{n,1}.$$

As before, if E is a minimizer of (1-1), then the rescaled set

$$\widehat{E} := Q_n^{-\frac{2(N-1)}{N}} (\log Q_n)^{-\frac{(N-1)}{N}} E$$

is a minimizer of $\widehat{\mathcal{F}}_{n,1}$ in $\mathcal{A}_{n,1}$.

Let $C_\varepsilon := [0,1] \times B_\varepsilon \subset \mathbb{R}^3$ be a narrow cylinder of radius $\varepsilon > 0$ (where B_ε denotes a two-dimensional ball of radius ε). We begin by proving the following estimate on $\mathcal{I}_1(C_\varepsilon)$:

Proposition 6.6. *It holds that*

$$\lim_{\varepsilon \to 0} \frac{\mathcal{I}_1(C_\varepsilon)}{|\log \varepsilon|} = 2. \tag{6-4}$$

As a consequence, for every $L > 0$,

$$\lim_{\varepsilon \to 0} \frac{\mathcal{I}_1([0,L] \times B_\varepsilon)}{|\log \varepsilon|} = \frac{2}{L}. \tag{6-5}$$

Proof. The equality in (6-4) is well known; see for instance [Maxwell 1877]. We include here a proof for the reader's convenience.

To show that

$$\lim_{\varepsilon \to 0} |\log \varepsilon|^{-1} \mathcal{I}_1(C_\varepsilon) \le 2,$$

we use $\mu_\varepsilon := (1/(\pi\varepsilon^2))\chi_{C_\varepsilon}$ as a test measure in the definition of $\mathcal{I}_1(C_\varepsilon)$. Then, noting that for every $y \in C_\varepsilon$,

$$\int_{C_\varepsilon + y} \frac{dz}{|z|} \le \int_{[-\frac{1}{2},\frac{1}{2}] \times B_\varepsilon} \frac{dz}{|z|},$$

we obtain

$$\begin{aligned}
\mathcal{I}_1(C_\varepsilon) &\le \frac{1}{\pi^2\varepsilon^4} \int_{C_\varepsilon \times C_\varepsilon} \frac{dx\,dy}{|x-y|} = \frac{1}{\pi^2\varepsilon^4} \int_{C_\varepsilon} \left(\int_{C_\varepsilon + y} \frac{dz}{|z|} \right) dy \\
&\le \frac{1}{\pi\varepsilon^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{B_\varepsilon} \frac{1}{(z_1^2 + |(z_2,z_3)|^2)^{\frac{1}{2}}} = \frac{4}{\varepsilon^2} \int_0^{\frac{1}{2}} \int_0^\varepsilon \frac{r}{(z_1^2 + r^2)^{\frac{1}{2}}} \\
&= \frac{4}{\varepsilon^2} \int_0^{\frac{1}{2}} \sqrt{z_1^2 + \varepsilon^2} - z_1 \\
&= \frac{4}{\varepsilon^2} \left(\frac{1}{8}\sqrt{1 + 4\varepsilon^2} - \frac{1}{8} + \frac{\varepsilon^2}{2} \log\left(\frac{1}{2\varepsilon} + \sqrt{1 + \frac{1}{4\varepsilon^2}} \right) \right) \\
&= 2|\log \varepsilon| + o(|\log \varepsilon|).
\end{aligned}$$

In order to show the opposite inequality, we recall the following definition of capacity of a set E:

$$\operatorname{Cap}(E) := \min\left\{ \int_{\mathbb{R}^3} |\nabla\phi|^2 : \chi_E \le \phi,\ \phi \in H_0^1(\mathbb{R}^3) \right\}.$$

Then, if E is compact, we have [Landkof 1972; Goldman et al. 2015]

$$\mathcal{I}_1(E) = \frac{4\pi}{\operatorname{Cap}(E)}.$$

Thus (6-4) will be proved once we show that

$$\mathrm{Cap}(C_\varepsilon)|\log\varepsilon| \le 2\pi + o(1). \tag{6-6}$$

For this, let $\lambda > 0$ and $\mu > 0$ to be fixed later and let

$$f_\lambda(x') := \begin{cases} 1 & \text{for } |x'| \le \varepsilon, \\ 1 - \log(|x'|/\varepsilon)/\log(\lambda/\varepsilon) & \text{for } \varepsilon \le |x'| \le \lambda, \\ 0 & \text{for } |x'| \ge \lambda \end{cases}$$

and

$$\rho_\mu(z) := \begin{cases} 0 & \text{for } z \le -\mu, \\ (z+\mu)/\mu & \text{for } -\mu \le z \le 0, \\ 1 & \text{for } 0 \le z \le 1, \\ 1-(z-1)/\mu & \text{for } 1 \le z \le 1+\mu, \\ 0 & \text{for } z \ge 1+\mu. \end{cases}$$

We finally let $\phi(x', z) := f_\lambda(x')\rho_\mu(z)$. Since $\rho_\mu, f_\lambda \le 1$ and $|\rho'_\mu| \le \mu^{-1}$, by the definition of $\mathrm{Cap}(C_\varepsilon)$, we have

$$\begin{aligned} \mathrm{Cap}(C_\varepsilon) &\le \int_0^1 \frac{2\pi}{\log(\lambda/\varepsilon)^2} \int_\varepsilon^\lambda \frac{1}{r} + C\left(\frac{\mu}{\log(\lambda/\varepsilon)} + \frac{\lambda^2}{\mu}\right) \\ &\le \frac{2\pi}{\log(\lambda/\varepsilon)} + C\left(\frac{\mu}{\log(\lambda/\varepsilon)} + \frac{\lambda^2}{\mu}\right). \end{aligned}$$

We now choose $\lambda := |\log\varepsilon|^{-1} \gg \varepsilon$ and $\mu := |\log\lambda|^{-1} = (\log|\log\varepsilon|)^{-1}$ so that $\log(\lambda/\varepsilon) = |\log\varepsilon| + \log|\log\varepsilon|$, $\mu \to 0$ and $\mu \gg \lambda$; thus

$$\frac{\mu}{\log(\lambda/\varepsilon)} + \frac{\lambda^2}{\mu} = o(|\log\varepsilon|^{-1})$$

and we find (6-6).

The equality in (6-5) then follows by scaling. □

As a simple corollary we get the two-dimensional result

Corollary 6.7.
$$\lim_{\varepsilon\to 0} \frac{\mathcal{I}_1([0,1]\times[0,\varepsilon])}{|\log\varepsilon|} = 2. \tag{6-7}$$

Proof. The upper bound is obtained as above by testing with $\mu_\varepsilon := \varepsilon^{-1}\chi_{[0,1]\times[0,\varepsilon]}$. By identifying $[0,1]\times[0,\varepsilon]$ with $[0,1]\times[0,\varepsilon]\times\{0\} \subset C_\varepsilon$ we get that $\mathcal{I}_1([0,1]\times[0,\varepsilon]) \ge \mathcal{I}_1(C_\varepsilon)$. This gives, together with (6-4), the corresponding lower bound. □

We can now prove a compactness result analogous to Proposition 6.1.

Proposition 6.8. *Let $E_n \in \mathcal{A}_{n,1}$ be such that $\sup_n \widehat{\mathcal{F}}_{n,1}(E_n) < +\infty$. Then, up to extracting a subsequence and up to rigid motions, the sets E_n converge in the Hausdorff topology to a segment $[0, L]\times\{0\}^N$ for some $L \in (0, +\infty)$.*

Proof. We argue as in the proof of Proposition 6.1. Since the case $N = 2$ is easier, we focus on $N = 3$. Let $\mathcal{R}_n = \prod_{i=1}^3 [0, \lambda_{i,n}]$ be given by Lemma 2.2 and let us assume without loss of generality that $i \mapsto \lambda_{i,n}$ is decreasing. Then (2-1) applied with $V = Q_n^{-4}(\log Q_n)^{-2}$, directly yields an upper bound on $\lambda_{1,n}$ (and thus on $\operatorname{diam}(E_n)$).

We now show that the diameter of E_n is also uniformly bounded from below. Unfortunately, (2-2) does not give the right bound and we need to refine it using (6-4). As in Proposition 6.1, the energy bound $\mathcal{I}_1(E_n) \lesssim \log Q_n$, directly implies that

$$\lambda_{1,n} \gtrsim \frac{1}{\log Q_n},$$

from which, using (2-1) and $\prod_{i=1}^3 \lambda_{i,n} \sim Q_n^{-4}(\log Q_n)^{-2}$, we get

$$\lambda_{2,n} \lesssim Q_n^{-2}.$$

In particular, it follows that

$$\frac{\lambda_{2,n}}{\lambda_{1,n}} \lesssim \frac{\log Q_n}{Q_n^2}.$$

By Proposition 6.6, letting $\varepsilon_n := Q_n^{-2} \log Q_n$ we get

$$\begin{aligned}\lambda_{1,n} \log Q_n &\gtrsim \lambda_{1,n}\mathcal{I}_1(E_n) \sim \lambda_{1,n}\mathcal{I}_1(\mathcal{R}_n)\\ &= \mathcal{I}_1\left(\prod_{i=1}^3 \left[0, \frac{\lambda_{i,n}}{\lambda_{1,n}}\right]\right) \gtrsim \mathcal{I}_1(C_{\varepsilon_n})\\ &\sim |\log \varepsilon_n| \sim \log Q_n,\end{aligned}$$

which implies

$$\lambda_{1,n} \gtrsim 1,$$

and gives a lower bound on the diameter of E_n.

Arguing as in the proof of (2-2), we then get

$$\lambda_{3,n} \le \lambda_{2,n} \lesssim Q_n^{-2}(\log Q_n)^{-1}. \tag{6-8}$$

It follows that the sets E_n are compact in the Hausdorff topology, and any limit set is a segment of length $L \in (0, +\infty)$. □

Arguing as in Theorem 6.3, we obtain the following result.

Theorem 6.9. *The functionals $\widehat{\mathcal{F}}_{n,1}$ Γ-converge in the Hausdorff topology to the functional*

$$\widehat{\mathcal{F}}_1(E) := \begin{cases} C_N\, L^{\frac{1}{N-1}} + \frac{4}{L} & \text{if } E \simeq [0, L] \times \{0\}^{N-1},\\ +\infty & \text{otherwise},\end{cases}$$

where C_N is defined as in Theorem 6.3.

Proof. Since the case $N = 2$ is easier, we focus on $N = 3$. The compactness and lower bound for the perimeter are obtained exactly as in Theorem 6.3. For the upper bound, for $L > 0$ and $n \in \mathbb{N}$, we define E_n as in the proof of Theorem 6.3 by first letting $V_n := Q_n^{-4}(\log Q_n)^{-2}$ (recall that $N = 3$) and then for $t \in \left[0, \frac{L}{2}\right]$,

$$r(t) := \left(\frac{3V_n}{\pi L}\right)^{\frac{1}{2}} \left(1 - \frac{2t}{L}\right)$$

and

$$E_n \cap (\mathbb{R}^+ \times \mathbb{R}^2) := \bigcup_{t \in \left[0, \frac{L}{2}\right]} \{t\} \times B^2_{r(t)},$$

where $B^2_{r(t)}$ is the ball of radius $r(t)$ in $\mathbb{R}^2$.

As in the proof of Theorem 6.3, we have

$$\lim_{n \to +\infty} Q_n^2 \log Q_n \, P(E_n) = C_3 \, L^{\frac{1}{2}}.$$

Let μ_n be the optimal measure for $\mathcal{I}_1(E_n)$, and let

$$\varepsilon_n := \left(\frac{3V_n}{\pi L}\right)^{\frac{1}{2}}.$$

For $L > \delta > 0$, we have $\left[-\frac{L-\delta}{2}, \frac{L-\delta}{2}\right] \times B^2_{\varepsilon_n} \subset E_n$ so that by (6-5),

$$\mathcal{I}_1(E_n) \le \mathcal{I}_1\left(\left[-\frac{L-\delta}{2}, \frac{L-\delta}{2}\right] \times B^2_{\varepsilon_n}\right) = \frac{|\log V_n|}{(L-\delta)} + o(|\log V_n|).$$

Recalling that $|\log V_n| = 4|\log Q_n| + o(|\log Q_n|)$, we then get

$$\varlimsup_{n \to +\infty} \frac{\mathcal{I}_1(E_n)}{\log(Q_n)} \le \frac{4}{L-\delta}.$$

Letting $\delta \to 0^+$, we obtain the upper bound.

We are left to prove the lower bound for the nonlocal part of the energy. Let E_n be a sequence of convex sets such that $E_n \to [0, L] \times \{0\}^2$ and such that $|E_n| = Q_n^{-4}(\log Q_n)^{-2}$. We can assume that $\sup_n \widehat{\mathcal{F}}_{n,1}(E_n) < +\infty$, since otherwise there is nothing to prove. Let $\delta > 0$. Up to a rotation and a translation, we can assume that $[0, L-\delta] \times \{0\}^2 \subset E_n \subset [0, L+\delta] \times \mathbb{R}^2$ for n large enough. Let now $x^1 = (x_1^1, x_2^1, x_3^1)$ be such that

$$|(x_2^1, x_3^1)| = \max_{x \in E_n} |(x_2, x_3)|.$$

Up to a rotation of axis $\mathbb{R} \times \{0\}^2$, we can assume that $x^1 = (a, \ell_1^n, 0)$ for some $\ell_1^n \ge 0$. Let finally x^2 be such that

$$|x^2 \cdot e_3| = \max_{x \in E_n} |x \cdot e_3|$$

so that $x^2 = (b, c, \ell_2^n)$ with $\ell_2^n \le \ell_1^n$. Since by definition $E_n \subset [0, L+\delta] \times [-\ell_1^n, \ell_1^n] \times [-\ell_2^n, \ell_2^n]$, we have $Q_n^{-4}(\log Q_n)^{-2} = |E_n| \lesssim \ell_1^n \ell_2^n (L+\delta)$. On the other hand, by convexity, the tetrahedron T with vertices 0, x_1, x_2 and $(L-\delta, 0, 0)$ is contained in E_n. We thus have $|E_n| \ge |T|$. Since

$$|T| = \tfrac{1}{8} |\det(x^1, x^2, (L-\delta, 0, 0))| = \tfrac{1}{8}(L-\delta)\ell_1^n \ell_2^n,$$

we also have $Q_n^{-4}(\log Q_n)^{-2} \gtrsim \ell_1^n \ell_2^n (L-\delta)$. Arguing as in the proof of (2-2), we get from the energy bound, $(L-\delta)\ell_1^n \lesssim Q_n^{-2}(\log Q_n)^{-1}$, and thus

$$\ell_1^n \ell_2^n \gtrsim \frac{1}{(L-\delta) Q_n^4 (\log Q_n)^2}.$$

From this we get $\ell_1^n \sim \ell_2^n \sim Q_n^{-2}(\log Q_n)^{-1}$, where the constants involved might depend on L. We therefore have $E_n \subset [0, L+\delta] \times B_{CQ_n^{-2}(\log Q_n)^{-1}}$ for C large enough. From this we infer that

$$\begin{aligned}\liminf_{n\to+\infty} \frac{\mathcal{I}_1(E_n)}{\log Q_n} &\geq \liminf_{n\to+\infty} \frac{\mathcal{I}_1\big([0, L+\delta] \times B_{CQ_n^{-2}(\log Q_n)^{-1}}\big)}{\log Q_n} \\ &\geq 2 \liminf_{n\to+\infty} \frac{\mathcal{I}_1\big([0, L+\delta] \times B_{CQ_n^{-2}(\log Q_n)^{-1}}\big)}{\log(CQ_n^{-2}(\log Q_n)^{-1})} \geq 4(L+\delta)^{-1},\end{aligned}$$

where the last inequality follows from (6-5). Letting $\delta \to 0$, we conclude the proof. □

Remark 6.10. As before, optimizing $\widehat{\mathcal{F}}_1$ with respect to L, one easily obtains the values of $L_{N,1}$ given in Theorem 1.4.

Remark 6.11. By analogy with results obtained in the setting of minimal Riesz energy point configurations [Hardin and Saff 2005; Martínez-Finkelshtein et al. 2004], we believe that for every $N \geq 2$, $\alpha > 1$ and $L > 0$, (6-5) can be generalized to

$$\lim_{\varepsilon\to 0} \frac{\mathcal{I}_\alpha([0, L] \times [0, \varepsilon]^{N-1})}{\varepsilon^{1-\alpha}} = \frac{C_\alpha}{L^\alpha} \tag{6-9}$$

for some constant C_α depending only on α. This result would permit one to extend Theorem 6.9 beyond $\alpha = 1$. Let us point out that showing that the right-hand side of (6-9) is bigger than the left-hand side can be easily obtained by plugging in the uniform measure as a test measure. However, we are not able to prove the reverse inequality.

Acknowledgements

The authors wish to thank Guido De Philippis, Jimmy Lamboley, Antoine Lemenant and Cyrill Muratov for useful discussions on the subject of this paper. Novaga and Ruffini were partially supported by the Italian CNR-GNAMPA and by the University of Pisa via grant PRA-2015-0017.

References

[Alt and Caffarelli 1981] H. W. Alt and L. A. Caffarelli, "Existence and regularity for a minimum problem with free boundary", *J. Reine Angew. Math.* **1981**:325 (1981), 105–144. MR Zbl

[Ambrosio et al. 2000] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon, New York, 2000. MR Zbl

[Brunn 1887] H. Brunn, *Über Ovale und Eiflächen*, dissertation, Ludwig Maximilian University of Munich, 1887. JFM

[Crasta et al. 2005] G. Crasta, I. Fragalà, and F. Gazzola, "On a long-standing conjecture by Pólya–Szegö and related topics", *Z. Angew. Math. Phys.* **56**:5 (2005), 763–782. MR Zbl

[Dahlberg 1977] B. E. J. Dahlberg, "Estimates of harmonic measure", *Arch. Rational Mech. Anal.* **65**:3 (1977), 275–288. MR Zbl

[Esposito and Fusco 2011] L. Esposito and N. Fusco, "A remark on a free interface problem with volume constraint", *J. Convex Anal.* **18**:2 (2011), 417–426. MR Zbl

[Fontelos and Friedman 2004] M. A. Fontelos and A. Friedman, "Symmetry-breaking bifurcations of charged drops", *Arch. Ration. Mech. Anal.* **172**:2 (2004), 267–294. MR Zbl

[Fusco et al. 2008] N. Fusco, F. Maggi, and A. Pratelli, "The sharp quantitative isoperimetric inequality", *Ann. of Math.* (2) **168**:3 (2008), 941–980. MR Zbl

[Garnett and Marshall 2005] J. B. Garnett and D. E. Marshall, *Harmonic measure*, New Mathematical Monographs **2**, Cambridge University Press, 2005. MR Zbl

[Goldman and Novaga 2012] M. Goldman and M. Novaga, "Volume-constrained minimizers for the prescribed curvature problem in periodic media", *Calc. Var. Partial Differential Equations* **44**:3-4 (2012), 297–318. MR Zbl

[Goldman et al. 2015] M. Goldman, M. Novaga, and B. Ruffini, "Existence and stability for a non-local isoperimetric model of charged liquid drops", *Arch. Ration. Mech. Anal.* **217**:1 (2015), 1–36. MR Zbl

[Hardin and Saff 2005] D. P. Hardin and E. B. Saff, "Minimal Riesz energy point configurations for rectifiable d-dimensional manifolds", *Adv. Math.* **193**:1 (2005), 174–204. MR Zbl

[Jerison 1996] D. Jerison, "A Minkowski problem for electrostatic capacity", *Acta Math.* **176**:1 (1996), 1–47. MR Zbl

[Jerison and Kenig 1982] D. S. Jerison and C. E. Kenig, "Boundary behavior of harmonic functions in nontangentially accessible domains", *Adv. in Math.* **46**:1 (1982), 80–147. MR Zbl

[John 1948] F. John, "Extremum problems with inequalities as subsidiary conditions", pp. 187–204 in *Studies and essays presented to R. Courant on his 60th birthday, January 8, 1948*, Interscience, New York, 1948. MR Zbl

[Kenig and Toro 1997] C. E. Kenig and T. Toro, "Harmonic measure on locally flat domains", *Duke Math. J.* **87**:3 (1997), 509–551. MR Zbl

[Kenig and Toro 1999] C. E. Kenig and T. Toro, "Free boundary regularity for harmonic measures and Poisson kernels", *Ann. of Math.* (2) **150**:2 (1999), 369–454. MR Zbl

[Lamboley et al. 2012] J. Lamboley, A. Novruzi, and M. Pierre, "Regularity and singularities of optimal convex shapes in the plane", *Arch. Ration. Mech. Anal.* **205**:1 (2012), 311–343. MR Zbl

[Lamboley et al. 2016] J. Lamboley, A. Novruzi, and M. Pierre, "Estimates of first and second order shape derivatives in nonsmooth multidimensional domains and applications", *J. Funct. Anal.* **270**:7 (2016), 2616–2652. MR Zbl

[Landkof 1972] N. S. Landkof, *Foundations of modern potential theory*, Die Grundlehren der Mathematischen Wissenschaften **180**, Springer, 1972. MR Zbl

[Martínez-Finkelshtein et al. 2004] A. Martínez-Finkelshtein, V. Maymeskul, E. A. Rakhmanov, and E. B. Saff, "Asymptotics for minimal discrete Riesz energy on curves in $\mathbb{R}^d$", *Canad. J. Math.* **56**:3 (2004), 529–552. MR Zbl

[Maxwell 1877] C. Maxwell, "On the electrical capacity of a long narrow cylinder, and of a disk of sensible thickness", *Proc. Lond. Math. Soc.* **9** (1877), 94–99. MR Zbl

[Muratov and Novaga 2016] C. B. Muratov and M. Novaga, "On well-posedness of variational models of charged drops", *Proc. A.* **472**:2187 (2016), art. id. 20150808. MR Zbl

[Muratov et al. 2016] C. B. Muratov, M. Novaga, and B. Ruffini, "On equilibrium shapes of charged flat drops", preprint, 2016. To appear in *Comm. Pure Appl. Math.* arXiv

[Novaga and Ruffini 2015] M. Novaga and B. Ruffini, "Brunn–Minkowski inequality for the 1-Riesz capacity and level set convexity for the $1/2$-Laplacian", *J. Convex Anal.* **22**:4 (2015), 1125–1134. MR

[Pommerenke 1992] C. Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften **299**, Springer, 1992. MR Zbl

[Saff and Totik 1997] E. B. Saff and V. Totik, *Logarithmic potentials with external fields*, Grundlehren der Mathematischen Wissenschaften **316**, Springer, 1997. MR Zbl

[Strutt (Lord Rayleigh) 1882] J. W. Strutt (Lord Rayleigh), "On the equilibrium of liquid conducting masses charged with electricity", *Phil. Mag.* **14** (1882), 184–186.

[Taylor 1964] G. Taylor, "Disintegration of water drops in an electric field", *Proc. Roy. Soc. Lond. A* **280**:1382 (1964), 383–397. Zbl

[Warschawski and Schober 1966] S. E. Warschawski and G. E. Schober, "On conformal mapping of certain classes of Jordan domains", *Arch. Rational Mech. Anal.* **22** (1966), 201–209. MR Zbl

[Zeleny 1917] J. Zeleny, "Instability of electricfied liquid surfaces", *Phys. Rev.* **10**:1 (1917), 1–6.

Received 8 Nov 2016. Revised 6 Jul 2017. Accepted 2 Jan 2018.

MICHAEL GOLDMAN: goldman@math.univ-paris-diderot.fr
Université Paris-Diderot, Sorbonne Paris-Cité, Sorbonne Université, CNRS, Laboratoire Jacques-Louis Lions, Paris, France

MATTEO NOVAGA: matteo.novaga@unipi.it
Dipartimento di Matematica, Università di Pisa, Pisa, Italy

BERARDO RUFFINI: berardo.ruffini@umontpellier.fr
Institut Montpelliérain Alexander Grothendieck, University of Montpellier, CNRS, Montpellier, France

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2018 is US $275/year for the electronic version, and $480/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 11 No. 5 2018