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APPLICATIONS OF SMALL-SCALE QUANTUM ERGODICITY IN NODAL SETS

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The goal of this article is to draw new applications of small-scale quantum ergodicity in nodal sets of eigenfunctions. We show that if quantum ergodicity holds on balls of shrinking radius $r(\lambda) \to 0$ then one can achieve improvements on the recent upper bounds of Logunov (2016) and Logunov and Malinnikova (2016) on the size of nodal sets, according to a certain power of $r(\lambda)$. We also show that the doubling estimates and the order-of-vanishing results of Donnelly and Fefferman (1988, 1990) can be improved. Due to results of Han (2015) and Hezari and Rivière (2016), small-scale QE holds on negatively curved manifolds at logarithmically shrinking rates, and thus we get logarithmic improvements on such manifolds for the above measurements of eigenfunctions. We also get $o(1)$ improvements for manifolds with ergodic geodesic flows. Our results work for a full density subsequence of any given orthonormal basis of eigenfunctions.

1. Introduction

Let (X, g) be a smooth compact connected boundaryless Riemannian manifold of dimension n. Suppose Δ_g is the positive Laplace–Beltrami operator on (X, g) and ψ_λ is a sequence of L^2 normalized eigenfunctions of Δ_g with eigenvalues λ. It was shown in [Hezari and Rivière 2016] that if for some shrinking radius $r = r(\lambda) \to 0$ and for all geodesic balls $B_r(x)$ one has $K_1 r^n \le \|\psi_\lambda\|^2_{B_r(x)} \le K_2 r^n$, then one gets improved upper bounds[1] of the form $(r^2\lambda)^{\delta(p)}$ on the L^p norms of ψ_λ, where $\delta(p)$ is Sogge's exponent. The purpose of this article is to prove more applications of small-scale L^2 equidistribution of eigenfunctions. We will show that upper bounds on the size of nodal sets, as well as the order of vanishing of eigenfunctions, can be improved by certain powers of r. Since by [Hezari and Rivière 2016][2] such equidistribution properties hold on negatively curved manifolds[3] with $r = (\log\lambda)^{-\kappa}$ for any $\kappa \in \left(0, \frac{1}{2n}\right)$, we obtain improvements of the results of [Logunov 2016a; Logunov and Malinnikova 2016; Donnelly and Fefferman 1988; 1990a; Dong 1992]. We also get slight improvements for quantum ergodic eigenfunctions because roughly speaking they equidistribute on balls of radius $r = o(1)$.

In the following $\mathcal{H}^{n-1}(Z_{\psi_\lambda})$ means the $(n-1)$-dimensional Hausdorff measure of the nodal set of ψ_λ, denoted by Z_{ψ_λ}, and $\nu_x(\psi_\lambda)$ means the order of vanishing of ψ_λ at a point x in X.

We recall that for $n \ge 3$, a recent result of [Logunov 2016a] gives a polynomial upper bound for $\mathcal{H}^{n-1}(Z_{\psi_\lambda})$ of the form λ^α for some $\alpha > \frac{1}{2}$ depending only on n, and for $n = 2$ another recent result

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[1] It was shown by Sogge [2016] that $\|\psi_\lambda\|^2_{B_r(x)} \le K_2 r^n$ suffices.

[2] In [Han 2015], this is proved for $\kappa \in \left(0, \frac{1}{3n}\right)$.

[3] For a full density subsequence of any given orthonormal basis of eigenfunctions.

of [Logunov and Malinnikova 2016] shows upper bounds of the form $\lambda^{\frac{3}{4}-\beta}$ for some small universal $\beta \in \left(0, \frac{1}{4}\right)$. Our first result is the following refinement of the results of the above-mentioned papers and also the order-of-vanishing results of [Donnelly and Fefferman 1988; 1990a; Dong 1992].

Theorem 1.1. *Let (X, g) be a boundaryless compact Riemannian manifold of dimension n with volume measure dv_g, and ψ_λ be an eigenfunction of Δ_g of eigenvalue $\lambda > 0$. Then there exists $r_0(g) > 0$ such that if $\lambda^{-\frac{1}{2}} < r_0(g)$, and if for some $r \in [\lambda^{-\frac{1}{2}}, r_0(g)]$ and for all geodesic balls $\{B_r(x)\}_{x \in X}$ we have*

$$K_1 r^n \leq \int_{B_r(x)} |\psi_\lambda|^2 \, dv_g \leq K_2 r^n \tag{1-1}$$

for some positive constants K_1 and K_2 independent of x, then:
For $n \geq 3$,

$$\mathcal{H}^{n-1}(Z_{\psi_\lambda}) \leq c_1 r^{2\alpha - 1} \lambda^\alpha, \tag{1-2}$$

$$\nu_x(\psi_\lambda) \leq c_2 r \sqrt{\lambda}. \tag{1-3}$$

For $n = 2$,

$$\mathcal{H}^1(Z_{\psi_\lambda}) \leq c_3 r^{\frac{1}{2} - 2\beta} \lambda^{\frac{3}{4} - \beta}, \tag{1-4}$$

$$\sum_{z \in Z_{\psi_\lambda} \cap B_{r^{1/2}\lambda^{-1/4}}(x)} (\nu_z(\psi_\lambda) - 1) \leq c_4 r \sqrt{\lambda}. \tag{1-5}$$

Here, $\alpha = \alpha(n) > \frac{1}{2}$ and $\beta \in \left(0, \frac{1}{4}\right)$ are the universal exponents from [Logunov 2016a; Logunov and Malinnikova 2016], and the constants c_1, c_2, c_3, c_4 are positive and depend only on (X, g), K_1, and K_2, and are independent of λ, r, and x. Note that the quantity on the left-hand side of (1-5) counts the number of singular points

$$\mathcal{S} = \{\psi_\lambda = |\nabla \psi_\lambda| = 0\}$$

in geodesic balls of radius $r^{\frac{1}{2}} \lambda^{-\frac{1}{4}}$.

Combining this with our result [Hezari and Rivière 2016], which states that on negatively curved manifolds (1-1) holds with $r = (\log \lambda)^{-\kappa}$ for any $\kappa \in \left(0, \frac{1}{2n}\right)$, the following unconditional results on such manifolds are immediate.

Theorem 1.2. *Let (X, g) be a boundaryless compact connected smooth Riemannian manifold of dimension n, with negative sectional curvatures. Let $\{\psi_{\lambda_j}\}_{j \in \mathbb{N}}$ be any orthonormal basis of $L^2(X)$ consisting of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$. Let $\epsilon > 0$ be arbitrary. Then there exists $S \subset \mathbb{N}$ of* full density[4] *such that for $j \in S$,*

$$\text{if } n \geq 3, \quad \mathcal{H}^{n-1}(Z_{\psi_{\lambda_j}}) \leq c_1 (\log \lambda_j)^{\frac{1-2\alpha}{2n} + \epsilon} \lambda_j^\alpha,$$

$$\text{if } n = 2, \quad \mathcal{H}^1(Z_{\psi_{\lambda_j}}) \leq c_3 (\log \lambda_j)^{-\frac{1}{8} + \frac{\beta}{2} + \epsilon} \lambda_j^{\frac{3}{4} - \beta}.$$

In addition, for all dimensions

$$\nu_x(\psi_{\lambda_j}) \leq c_2 (\log \lambda_j)^{-\frac{1}{2n} + \epsilon} \sqrt{\lambda_j}.$$

[4] It means that $\lim_{N \to \infty} \frac{1}{N} \operatorname{card}(S \cap [1, N]) = 1$.

We repeat that here $\alpha = \alpha(n) > \frac{1}{2}$ and $\beta \in \left(0, \frac{1}{4}\right)$ are the universal exponents from [Logunov 2016a; Logunov and Malinnikova 2016], and c_1, c_2, c_3 depend only on (X, g) and ϵ.

We will also prove the following $o(1)$ improvements for quantum ergodic sequences of eigenfunctions. In fact equidistribution on X (instead of the phase space S^*X) suffices.

Theorem 1.3. *Let (X, g) be a boundaryless compact connected smooth Riemannian manifold of dimension n. Let $\{\psi_{\lambda_j}\}_{j\in S}$ be a sequence of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j\in S}$ such that for all $r \in \left(0, \frac{1}{2}\operatorname{inj}(g)\right)$ and all $x \in X$*

$$\int_{B_r(x)} |\psi_{\lambda_j}|^2 \to \frac{\operatorname{Vol}_g(B_r(x))}{\operatorname{Vol}_g(X)}, \qquad \lambda_j \xrightarrow{j\in S} \infty. \tag{1-6}$$

Then, along this sequence, for $n \geq 3$

$$\mathcal{H}^{n-1}(Z_{\psi_{\lambda_j}}) = o(\lambda_j^{\alpha}),$$

and for $n = 2$

$$\mathcal{H}^{1}(Z_{\psi_{\lambda_j}}) = o(\lambda_j^{\frac{3}{4}-\beta}).$$

Also in all dimensions

$$\nu_x(\psi_{\lambda_j}) = o(\sqrt{\lambda_j}) \quad (\textit{uniformly in } x).$$

In particular the above theorem holds for manifolds with ergodic geodesic flows by the quantum ergodicity theorem of Shnirel'man [1974], Colin de Verdière [1985] and Zelditch [1987]. Hence given any orthonormal basis of eigenfunctions, on such a manifold one can pass to a full density subsequence where (1-6), whence Theorem 1.3 holds.

Remark 1.4. We point out that the equidistribution property (1-6), which is weaker than quantum ergodicity, holds for some nonergodic manifolds such as the flat torus and the rational polygons; see [Marklof and Rudnick 2012; Rivière 2013; Taylor 2015].

Main idea. The major idea in proving our upper bounds is to lower the doubling index

$$N(B_s(x)) := \log\left(\frac{\sup_{B_{2s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2}\right)$$

under the assumption

$$K_1 r^n \leq \int_{B_r(x)} |\psi_j|^2 \leq K_2 r^n.$$

We recall that Donnelly and Fefferman [1988] showed that an eigenfunction ψ_λ of Δ_g with eigenvalue λ satisfies

$$N(B_s(x)) \leq c\sqrt{\lambda}$$

for all $s < s_0$, where s_0 and c depend only on (X, g). We will prove in Lemma 2.1 that

$$N(B_s(x)) \leq c\, r\sqrt{\lambda} \tag{1-7}$$

for all $s < 10r$, where c depends only on (X, g). We then apply this modified growth estimate to the proofs of [Logunov 2016a; Logunov and Malinnikova 2016; Donnelly and Fefferman 1988; 1990a; Dong 1992] to obtain our improvements.

Remark 1.5. It is worth mentioning that in order to prove (1-2) of Theorem 1.1, we will need the improved doubling estimates (1-7) to hold for all $0 < s < 10r$ and not just s comparable to r. This is because the doubling exponent of a ball B (or a cube Q) as defined in [Logunov 2016a], see definition (2-12), is, roughly speaking, the supremum of $N(B_s(x))$ over all balls $B_s(x)$ contained in $2B$ (or $2Q$ respectively). The main result of that paper (see Theorem 2.5) gives an upper bound on the nodal sets in terms of this maximal doubling index. For the estimates (1-4) and (1-5) we need the validity of (1-7) for $0 < s < Cr^{\frac{1}{2}}\lambda^{-\frac{1}{4}}$.

Background on the size of nodal sets. For any smooth compact connected Riemannian manifold (X, g) of dimension n, Yau's conjecture states that there exist constants $c > 0$ and $C > 0$ independent of λ such that

$$c\sqrt{\lambda} \le \mathcal{H}^{n-1}(Z_{\psi_\lambda}) \le C\sqrt{\lambda}.$$

The conjecture was proved by Donnelly and Fefferman [1988] in the real analytic case. In dimension 2 and the C^∞ case, Brüning [1978] and Yau proved the lower bound $c\sqrt{\lambda}$. Until the recent result of Logunov and Malinnikova [2016] the best upper bound in dimension 2 was $C\lambda^{\frac{3}{4}}$, which was proved independently by Donnelly and Fefferman [1990a] and Dong [1992]. The result of Logunov and Malinnikova [2016] gives $C\lambda^{\frac{3}{4}-\beta}$ for some small universal constant $\beta < \frac{1}{4}$. In dimensions $n \ge 3$ until very recently, the best lower bound was $c\lambda^{\frac{3-n}{4}}$, proved[5] by Colding and Minicozzi [2011]. However, a recent breakthrough result of Logunov [2016b] proves the lower bound $c\sqrt{\lambda}$ for all $n \ge 3$. Also another result of Logunov [2016a] shows a polynomial upper bound $C\lambda^\alpha$ for some $\alpha > \frac{1}{2}$ which depends only on n. The best upper bound before this was the exponential bound $e^{c\sqrt{\lambda}\log\lambda}$ of Hardt and Simon [1989].

Background on small-scale quantum ergodicity. First, we recall that the quantum ergodicity result of Shnirel'man [1974], Colin de Verdière [1985] and Zelditch [1987] implies in particular that if the geodesic flow of a smooth compact Riemannian manifold without boundary is ergodic then for any orthonormal basis $\{\psi_{\lambda_j}\}_{j=1}^{\infty}$ consisting of the eigenfunctions of Δ_g, there exists a full density subset $S \subset \mathbb{N}$ such that for any $r < \mathrm{inj}(g)$, independent of λ_j, one has

$$\|\psi_{\lambda_j}\|^2_{L^2(B_r(x))} \sim \frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)}, \quad \text{as } \lambda_j \to \infty,\ j \in S. \tag{1-8}$$

The analogous result on manifolds with piecewise smooth boundary and with ergodic billiard flows was proved by Zelditch and Zworski [1996].

The small-scale equidistribution problem asks whether (1-8) holds for r dependent on λ_j. A quantitative QE result of Luo and Sarnak [1995] shows that the Hecke eigenfunctions on the modular surface satisfy

[5] Different proofs were given later by [Hezari and Wang 2012; Hezari and Sogge 2012; Sogge and Zelditch 2012] based on the earlier work [Sogge and Zelditch 2011], and by [Steinerberger 2014] using heat equation techniques. Also logarithmic improvements of the form $\lambda^{\frac{3-n}{4}}(\log\lambda)^\alpha$ were given in [Hezari and Rivière 2016] on negatively curved manifolds and in [Blair and Sogge 2015] on nonpositively curved manifolds.

this property along a density one subsequence for $r = \lambda^{-\kappa}$ for some small $\kappa > 0$. Also, under the Generalized Riemann Hypothesis, Young [2016] proved that small-scale equidistribution holds for Hecke eigenfunctions for $r = \lambda^{-\frac{1}{4}+\epsilon}$.

This problem was studied in [Han 2015; Hezari and Rivière 2016] for the eigenfunctions of negatively curved manifolds. To be precise, it was proved that on compact negatively curved manifolds without boundary, for any $\epsilon > 0$ and any orthonormal basis $\{\psi_{\lambda_j}\}_{j=1}^{\infty}$ of $L^2(X)$ consisting of the eigenfunctions of Δ_g, there exists a subset $S \subset \mathbb{N}$ of full density such that for all $x \in X$ and $j \in S$,

$$K_1 r^n \leq \|\psi_{\lambda_j}\|^2_{L^2(B_r(x))} \leq K_2 r^n, \quad \text{with } r = (\log \lambda_j)^{-\frac{1}{2n}+\epsilon}, \tag{1-9}$$

for some positive constants K_1, K_2 which depend only on (X, g) and ϵ. The same result was proved in [Han 2015] for $r = (\log \lambda_j)^{-\frac{1}{3n}+\epsilon}$.

We also point out that although eigenfunctions on the flat torus $\mathbb{R}^n/\mathbb{Z}^n$ are not quantum ergodic, they equidistribute on the configuration space $\mathbb{R}^n/\mathbb{Z}^n$; see [Marklof and Rudnick 2012], and also [Rivière 2013; Taylor 2015] for later proofs. So one can investigate the small-scale equidistribution property for toral eigenfunctions. It was proved in [Hezari and Rivière 2017] that a commensurability of L^2 masses such as (1-9) is valid for a full density subsequence with $r = \lambda^{-\frac{1}{7n+4}}$. Lester and Rudnick [2017] improved this rate of shrinking to $r = \lambda^{-\frac{1}{2n-2}+\epsilon}$, and in fact they proved that the stronger statement (1-8) holds. They also showed that their results are almost[6] sharp. The case of interest is $n = 2$, which gives $r = \lambda^{-\frac{1}{2}+\epsilon}$. A natural conjecture is that this should be the optimal rate of shrinking on negatively curved manifolds. A recent result of [Han 2017] proves that random eigenbases on the torus enjoy small-scale QE for $r = \lambda^{-\frac{n-2}{4n}+\epsilon}$, which is better than [Lester and Rudnick 2017] for $n \geq 5$.

Some remarks.

Remark 1.6. In our proof we have used both *local and global harmonic analysis*; see [Zelditch 2008] for background. The local analysis is used in [Logunov 2016a; Logunov and Malinnikova 2016], and the global analysis is used in [Hezari and Rivière 2016] to obtain equidistribution on small balls. We emphasize that our improvements of [Logunov 2016a; Logunov and Malinnikova 2016] are robust, in the sense that any upper bounds of the form λ^α for $\alpha > \frac{1}{2}$ that result from a purely local analysis of eigenfunctions can be improved using our combined method.

Remark 1.7. The most important assumption of Theorem 1.1 is the lower bound $K_1 r^n \leq \int_{B_r(x)} |\psi_\lambda|^2$ and the upper bound in (1-1) can be discarded at the expense of messy estimates in Theorem 1.1. In fact using Sogge's "trivial local L^2 estimates" [2016], which assert that one always has $\int_{B_r(x)} |\psi_\lambda|^2 \leq K_2 r$, we can still prove modified doubling estimates of the form

$$\sup_{B_{2s}(x)} |\psi_\lambda|^2 \leq r^{-b} e^{cr\sqrt{\lambda}} \sup_{B_s(x)} |\psi_\lambda|^2 \quad \text{for some } b = b(n) > 0 \text{ and all } s < 10r.$$

We can use this inequality and obtain estimates similar to those in Theorem 1.1; however we have not done so for the sake of more polished estimates. Another reason that we have not discarded the assumption

[6]They showed that the equidistribution property fails for $r = \lambda^{-\frac{1}{2n-2}-\epsilon}$ for a positive density subset of some orthonormal basis.

$\int_{B_r(x)} |\psi_\lambda|^2 \le K_2 r^n$ is that all the examples (such as QE eigenfunctions) for which we know the lower bounds are satisfied, also satisfy the upper bounds in (1-1).

Remark 1.8. As we discussed in the previous section, a result of [Luo and Sarnak 1995] implies that small-scale QE holds for a full density subsequence of Hecke eigenfunctions on the modular surface for balls of radius $r = \lambda^{-\kappa}$ for some explicitly calculable $\kappa > 0$. Hence using (1-3), we get upper bounds of the form $\lambda^{\frac{1}{2}-\kappa}$ on the order of vanishing of these eigenfunctions. We could not find any arithmetic results in the literature discussing improvements on the upper bound $\sqrt{\lambda}$ of Donnelly and Fefferman. Of course a natural conjecture to impose is that for Hecke eigenfunctions $\nu_x(\psi_\lambda) \le c\lambda^{\epsilon}$. Although the available graphs of nodal lines of Hecke eigenfunctions with high energy do not show any singular points, i.e., places where nodal lines intersect each other, there are many almost-intersecting nodal lines.

Remark 1.9. By our discussion in the previous section on the work of [Lester and Rudnick 2017], and using (1-3), we get that for a full density subsequence of toral eigenfunctions on the 2-torus, we have $\nu_x(\psi_\lambda) \le c\lambda^{\epsilon}$. However, it is proved in [Bourgain and Rudnick 2011] that $\nu_x(\psi_\lambda) \le c\lambda^{\frac{c}{\log\log\lambda}}$ for all eigenfunctions on $\mathbb{T}^2$.

Remark 1.10. Theorem 1.1 is local in nature, meaning that if the eigenfunctions satisfy (1-1) for balls centered on an open set, then we get the upper bounds in this theorem on that open set. In particular we get all the upper bounds in Theorem 1.3 for eigenfunctions on ergodic billiards (and also rational polygons) as long as we stay a positive distance away from the boundary. One would expect that the results of [Logunov 2016a; Logunov and Malinnikova 2016] can be extended to the eigenfunctions of the Laplacian on manifolds with boundary (with Dirichlet or Neumann boundary conditions) using the method of [Donnelly and Fefferman 1990b].

2. Proofs of upper bounds for nodal sets and order of vanishing

The following lemma is the main ingredient of the proofs. It gives improved growth estimates for eigenfunctions under our L^2 assumption on small balls.

Lemma 2.1. *Let (X, g) be a smooth Riemannian manifold, $p \in X$ a fixed point, and $R > 0$ a fixed radius so that the geodesic ball $B_{2R}(p)$ is embedded. Then there exists $r_0(g)$ such that the following statement holds*:

Suppose $\lambda^{-\frac{1}{2}} \le r_0(g)$ and ψ_λ is a smooth function such that $\Delta_g \psi_\lambda = \lambda\psi_\lambda$ on $B_{2R}(p)$. If for some $r \in [\lambda^{-\frac{1}{2}}, r_0(g)]$ and all $x \in B_R(p)$

$$K_1 r^n \le \int_{B_r(x)} |\psi_\lambda|^2 \le K_2 r^n \tag{2-1}$$

holds for some positive constants K_1 and K_2 independent of x, then one has the refined doubling estimates

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \quad \int_{B_{2\delta}(x)} |\psi_\lambda|^2 \le e^{cr\sqrt{\lambda}} \int_{B_\delta(x)} |\psi_\lambda|^2, \tag{2-2}$$

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \quad \sup_{B_{2\delta}(x)} |\psi_\lambda|^2 \le e^{cr\sqrt{\lambda}} \sup_{B_\delta(x)} |\psi_\lambda|^2. \tag{2-3}$$

We also have

$$\text{for } \delta\in\left(0,\tfrac{1}{2}r\right),\ x\in B_{\frac{R}{2}}(p),\qquad \frac{1}{\delta^n}\int_{B_\delta(x)}|\psi_\lambda|^2\geq\left(\frac{r}{\delta}\right)^{-cr\sqrt{\lambda}}, \tag{2-4}$$

$$\text{for } \delta\in\left(0,\tfrac{1}{2}r\right),\ x\in B_{\frac{R}{2}}(p),\qquad \sup_{B_\delta(x)}|\psi_\lambda|^2\geq\left(\frac{r}{\delta}\right)^{-cr\sqrt{\lambda}}. \tag{2-5}$$

Here c is positive and is uniform in x, r, δ, and λ, but depends on K_1, K_2, and $(B_{2R}(p),g)$.

Proof. We will give two proofs for (2-3). All other statements will follow from this as we will show. The first proof of (2-3) follows from a rescaling argument applied to the following theorem of Donnelly and Fefferman, which is a purely local result based on Carleman estimates. The second proof relies on a theorem of [Mangoubi 2013].

Theorem 2.2 [Donnelly and Fefferman 1988, Proposition 3.10(ii)]. *Let $(\widetilde{X},\tilde{g})$ be a smooth Riemannian manifold, $p\in\widetilde{X}$ a fixed point, and $\widetilde{R}>0$ a fixed radius such that the $\tilde{g}$-geodesic ball $\widetilde{B}_{2\widetilde{R}}(p)$ is embedded. Let $\psi_{\tilde{\lambda}}$ be a smooth function such that for some $\tilde{\lambda}\geq 1$ we have $\Delta_{\tilde{g}}\psi_{\tilde{\lambda}}=\tilde{\lambda}\psi_{\tilde{\lambda}}$ on $\widetilde{B}_{2\widetilde{R}}(p)$. Then there exists a suitably small $h_0(\tilde{g})>0$ such that for all $h\leq h_0(\tilde{g})$, $\delta<\frac{1}{2}h$, and $x\in\widetilde{B}_{\frac{\widetilde{R}}{2}}(p)$,*

$$\sup_{\widetilde{B}_{2\delta}(x)}|\psi_{\tilde{\lambda}}|^2\leq e^{\kappa_1\sqrt{\tilde{\lambda}}}\left(\frac{\sup_{\widetilde{B}_h(x)}|\psi_{\tilde{\lambda}}|^2}{\sup_{\widetilde{B}_{h/5}(x)\setminus\widetilde{B}_{h/10}(x)}|\psi_{\tilde{\lambda}}|^2}\right)^{\kappa_2}\sup_{\widetilde{B}_\delta(x)}|\psi_{\tilde{\lambda}}|^2. \tag{2-6}$$

The constant $h_0(\tilde{g})$ is controlled by $\widetilde{R}$ and the reciprocal of the square root of $\sup_{\widetilde{B}_{2\widetilde{R}}(p)}|\operatorname{Sec}(\tilde{g})|$, and the constants κ_1 and κ_2 are controlled by $\sup_{\widetilde{B}_{2\widetilde{R}}(p)}|\operatorname{Sec}(\tilde{g})|$.

To prove our lemma, we define $(\widetilde{X},\tilde{g})=(X,\frac{1}{r^2}g)$, and $\widetilde{R}=\frac{1}{r}R$. Then the equation

$$\Delta_g\psi_\lambda=\lambda\psi_\lambda\quad\text{on } B_{2R}(p)$$

becomes

$$\Delta_{\tilde{g}}\psi_{\tilde{\lambda}}=\tilde{\lambda}\psi_{\tilde{\lambda}}\quad\text{on } \widetilde{B}_{2\widetilde{R}}(p),$$

with

$$\tilde{\lambda}=r^2\lambda\quad\text{and}\quad\psi_{\tilde{\lambda}}=\psi_\lambda.$$

We then note that by [Donnelly and Fefferman 1988], although not explicitly stated, we have

$$h_0(\tilde{g})=C\min\left(\tfrac{1}{2}\widetilde{R},\left(\sup_{\widetilde{B}_{2R}(p)}|\operatorname{Sec}(\tilde{g})|\right)^{-\frac{1}{2}}\right)=\frac{C}{r}\min\left(\tfrac{1}{2}R,\left(\sup_{B_{2R}(p)}|\operatorname{Sec}(g)|\right)^{-\frac{1}{2}}\right)$$

for some suitably small C that is uniform in r. Hence if we set

$$r_0(g)\leq\tfrac{1}{20}C\min\left(\tfrac{1}{2}R,\left(\sup_{B_{2R}(p)}|\operatorname{Sec}(g)|\right)^{-\frac{1}{2}}\right)$$

then for all $r\leq r_0(g)$ we have $h_0(\tilde{g})\geq 20$, and therefore we can choose $h=20$. As a result, by (2-6)

$$\text{for } \delta\in(0,10),\ x\in\widetilde{B}_{\frac{\widetilde{R}}{2}}(p),\qquad \sup_{\widetilde{B}_{2\delta}(x)}|\psi_{\tilde{\lambda}}|^2\leq e^{\kappa_1\sqrt{\tilde{\lambda}}}\left(\frac{\sup_{\widetilde{B}_{20}(x)}|\psi_{\tilde{\lambda}}|^2}{\sup_{\widetilde{B}_4(x)\setminus\widetilde{B}_2(x)}|\psi_{\tilde{\lambda}}|^2}\right)^{\kappa_2}\sup_{\widetilde{B}_\delta(x)}|\psi_{\tilde{\lambda}}|^2.$$

Writing this inequality with respect to the metric g we get

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2\delta}(x)} |\psi_\lambda|^2 \le e^{\kappa_1 r\sqrt{\lambda}} \left(\frac{\sup_{B_{20r}(x)} |\psi_\lambda|^2}{\sup_{B_{4r}(x)\setminus B_{2r}(x)} |\psi_\lambda|^2} \right)^{\kappa_2} \sup_{B_\delta(x)} |\psi_\lambda|^2. \tag{2-7}$$

Remark 2.3. We emphasize that since $|\operatorname{Sec}(\tilde{g})| = r^2 |\operatorname{Sec}(g)|$, and since r is bounded by $r_0(g)$, the constants κ_1 and κ_2 can be chosen independently from r.

We now bound the expression in parenthesis using our local L^2 assumptions (2-1). First we find y such that

$$B_r(y) \subset B_{4r}(x)\setminus B_{2r}(x).$$

Since by assumption $\int_{B_r(y)} |\psi_j|^2 \ge K_1 r^n$, we must have

$$\sup_{B_{4r}(x)\setminus B_{2r}(x)} |\psi_\lambda|^2 \ge \sup_{B_r(y)} |\psi_\lambda|^2 \ge \frac{r^n}{\operatorname{Vol}(B_r(y))} K_1.$$

By making $r_0(g)$ sufficiently smaller, we obtain that for any $r \le r_0(g)$ which satisfies (2-1), we have

$$\sup_{B_{4r}(x)\setminus B_{2r}(x)} |\psi_\lambda|^2 \ge a K_1 \tag{2-8}$$

for some constant a which is uniform in $x \in B_{\frac{R}{2}}(p)$, $r \in (0, r_0(g))$, and λ. For the numerator in the parenthesis we claim that[7]

$$\sup_{B_{20r}(x)} |\psi_\lambda|^2 \le b K_2 (r\sqrt{\lambda})^n \tag{2-9}$$

for some constant b which is uniform in $x \in B_{\frac{R}{2}}(p)$, $r \in (0, r_0(g))$ and λ. To prove (2-9) we cover $B_{20r}(x)$ using balls of radius $\frac{r}{2}$. It is therefore enough to show that

$$\sup_{B_{r/2}(y)} |\psi_\lambda|^2 \le b\lambda^{\frac{n}{2}} \sup_{z\in B_{r/2}(y)} \int_{B_r(z)} |\psi_\lambda|^2 \tag{2-10}$$

for some b that is uniform in y, r, and λ. This estimate, however, follows from standard elliptic estimates, see for example [Gilbarg and Trudinger 1998, Theorem 8.17 and Corollary 9.21], which assert that there exists $a_0 < 1$ suitably small such that for $z \in B_R(p)$ we have

$$\text{for all } s \in (0, a_0\lambda^{-\frac{1}{2}}], \qquad \sup_{B_{s/2}(z)} |\psi_\lambda|^2 \le b_0 s^{-n} \int_{B_s(z)} |\psi_\lambda|^2 \tag{2-11}$$

for some b_0 which is uniform in λ, z, and s. Since $\lambda^{-\frac{1}{2}} \le r$, we have $B_{a_0\lambda^{-1/2}}(z) \subset B_r(z)$ and hence to get (2-10) we just need to observe that

$$\sup_{B_{r/2}(y)} |\psi_\lambda|^2 \le \sup_{z\in B_{r/2}(y)} \ \sup_{B_{(a_0/2)\lambda^{-1/2}}(z)} |\psi_\lambda|^2 \le b\lambda^{\frac{n}{2}} \sup_{z\in B_{r/2}(y)} \int_{B_r(z)} |\psi_\lambda|^2,$$

[7] In fact when (X, g) is a *closed manifold* the better estimate $bK_2(r\sqrt{\lambda})^{n-1}$ holds using Sogge's local L^∞ estimates [2016], but we do not need this better estimate.

with $b = b_0 a_0^{-n}$. Now we apply (2-8) and (2-9) to (2-7) to achieve

$$\text{for } \delta \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2\delta}(x)} |\psi_\lambda|^2 \le d e^{\kappa_1 r\sqrt{\lambda}} (r\sqrt{\lambda})^{n\kappa_2} \sup_{B_\delta(x)} |\psi_\lambda|^2$$

for some uniform constant d which depends on K_1 and K_2. We note since $r\sqrt{\lambda} \ge 1$, if we choose M to be an integer larger than κ_1 and $n\kappa_2$ then

$$(r\sqrt{\lambda})^{n\kappa_2} e^{\kappa_1 r\sqrt{\lambda}} \le M!\, e^{2Mr\sqrt{\lambda}}.$$

Finally by choosing

$$c \ge \max(\log d, M \log M, 2M),$$

we get (2-3).

To prove (2-2) we use (2-3). It is enough to show that

$$\frac{\int_{B_{2\delta}(x)} |\psi_\lambda|^2}{\int_{B_\delta(x)} |\psi_\lambda|^2} \le K(\delta\sqrt{\lambda})^n \frac{\sup_{B_{2\delta}(x)} |\psi_\lambda|^2}{\sup_{B_{\delta/2}(x)} |\psi_\lambda|^2},$$

because $(\delta\sqrt{\lambda})^n \le (10r\sqrt{\lambda})^n \le e^{cr\sqrt{\lambda}}$ for some appropriate c, as we found in the above argument. The above comparison of ratios follows from the trivial estimate

$$\int_{B_{2\delta}(x)} |\psi_\lambda|^2 \le \frac{1}{a} (2\delta)^n \sup_{B_{2\delta}(x)} |\psi_\lambda|^2$$

applied to the numerator, and the estimate

$$\int_{B_\delta(x)} |\psi_\lambda|^2 \ge \frac{1}{b_0} \left(\min\left(\lambda^{-\frac{1}{2}}, \tfrac{1}{4}\delta\right)\right)^n \sup_{B_{\delta/2}(x)} |\psi_\lambda|^2$$

applied to the denominator. The last estimate follows from the elliptic estimate (2-11) by setting $s = \min\left(a_0\lambda^{-\frac{1}{2}}, \frac{1}{4}\delta\right)$ and writing

$$\sup_{B_{\delta/2}(x)} |\psi_\lambda|^2 \le \sup_{z \in B_{\delta/2}(x)} \sup_{B_{s/2}(z)} |\psi_\lambda|^2 \le b_0 s^{-n} \sup_{z \in B_{\delta/2}(x)} \int_{B_s(z)} |\psi_\lambda|^2 \le b_0 s^{-n} \int_{B_\delta(x)} |\psi_\lambda|^2.$$

The proofs of (2-4) and (2-5) are obtained by iterations of inequalities (2-2) and (2-3). Since they are very similar we only give the proof of (2-5). Fix $\delta \le \frac{r}{2}$ and let m be the greatest integer such that $2^{m-1}\delta \le r$. Then if we write inequalities (2-3) for $\delta, 2\delta, 4\delta, \dots, 2^{m-1}\delta$ and multiply them all we get

$$\sup_{B_\delta(x)} |\psi_\lambda|^2 \ge e^{-mcr\sqrt{\lambda}} \sup_{B_{2^m\delta}(x)} |\psi_\lambda|^2.$$

Because of the choice of m, we have $2^m\delta > r$. Hence

$$\sup_{B_\delta(x)} |\psi_\lambda|^2 \ge e^{-mcr\sqrt{\lambda}} \sup_{B_r(x)} |\psi_\lambda|^2 \ge \frac{e^{-mcr\sqrt{\lambda}}}{\operatorname{Vol}(B_r(x))} \int_{B_r(x)} |\psi_\lambda|^2 \ge aK_2 e^{-mcr\sqrt{\lambda}}.$$

Since $m \ge \log\left(\frac{r}{\delta}\right)$ and $r\sqrt{\lambda} \ge 1$, by selecting c slightly larger the lower bound (2-5) follows. □

***Second proof of improved L^∞-growth estimates* (2-3).** We recall the following result of [Mangoubi 2013], which is similar to estimate (2-7).

Theorem 2.4 [Mangoubi 2013, Theorem 3.2]. *Let (X, g) be a smooth Riemannian manifold, $p \in X$, and $R > 0$ so that the geodesic ball $B_{2R}(p)$ is embedded, and set $S = \sup_{B_{2R(p)}} |\mathrm{Sec}(g)|$. Suppose ψ_λ is a smooth function such that $\Delta_g \psi_\lambda = \lambda \psi_\lambda$ on $B_{2R}(p)$ for some $\lambda \geq 0$. Then for all $\delta \leq s \leq \min\big(CS^{-\frac{1}{2}}, \frac{1}{6}R\big)$, and all $x \in B_{\frac{R}{2}}(p)$*

$$\sup_{B_{3\delta}(x)} |\psi_\lambda|^2 \leq c_0 e^{c_1 s\sqrt{\lambda}} \left(\frac{\sup_{B_{3s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \right)^{1+c_2\delta^2 S} \sup_{B_{2\delta}(x)} |\psi_\lambda|^2,$$

where C, c_1 and c_2 are positive constants which depend only on R, and c_0 depends on bounds on $(g^{-1})_{ij}$, its derivatives and its ellipticity constant on the ball $B_{2R}(p)$.

Using this theorem twice, we get for $\delta \leq s \leq \min\big(CS^{-\frac{1}{2}}, \frac{1}{6}R\big)$

$$\frac{\sup_{B_{2\delta}(x)} |\psi_\lambda|^2}{\sup_{B_\delta(x)} |\psi_\lambda|^2} \leq \frac{\sup_{B_{(9/4)\delta}(x)} |\psi_\lambda|^2}{\sup_{B_{(3/2)\delta}(x)} |\psi_\lambda|^2} \frac{\sup_{B_{(3/2)\delta}(x)} |\psi_\lambda|^2}{\sup_{B_\delta(x)} |\psi_\lambda|^2} \leq c_0^2 e^{2c_1 s\sqrt{\lambda}} \left(\frac{\sup_{B_{3s}(x)} |\psi_\lambda|^2}{\sup_{B_s(x)} |\psi_\lambda|^2} \right)^{2+c_2'\delta^2 S}$$

for a new constant c_2'. Now we choose $r_0(g) \leq \frac{1}{10} \min\big(CS^{-\frac{1}{2}}, \frac{1}{6}R\big)$, we put $s = 10r$, and argue as we did following inequality (2-7).

Proof of* (1-3)*: upper bound on the order of vanishing. Let us show that the upper bound (1-3) on the order of vanishing $\nu_x(\psi_\lambda)$ follows from the lower bound (2-5). Suppose ψ_λ vanishes at x to order M. Then there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$

$$C_{\psi_\lambda, \delta_0} \delta^M \geq \sup_{B_\delta(x)} |\psi_\lambda|^2.$$

Therefore using (2-5), for all $0 < \delta < \min\big(\delta_0, \frac{1}{2}r\big)$

$$C_{\psi_\lambda, \delta_0} \delta^M \geq \left(\frac{\delta}{r}\right)^{cr\sqrt{\lambda}}.$$

Dividing by δ^M and letting $\delta \to 0$ we see that we must have $M \leq cr\sqrt{\lambda}$.

Proof of* (1-2)*: upper bounds on the size of nodal sets for $n \geq 3$. The main tool is the following result.

Theorem 2.5 [Logunov 2016a, Theorem 6.1]. *Let $(\widetilde{X}, \tilde{g})$ be a smooth Riemannian manifold of dimension d, $\tilde{p} \in \widetilde{X}$, and $\widetilde{R} > 0$ so that the geodesic ball $B_{2\widetilde{R}}(\tilde{p})$ is embedded. Suppose H is a harmonic function on $B_{2\widetilde{R}}(\tilde{p})$; that is, $\Delta_{\tilde{g}} H = 0$ on $B_{2\widetilde{R}}(\tilde{p})$. Then there exists $R_0 = R_0(B_{2\widetilde{R}}(\tilde{p}), g) < \widetilde{R}$ such that for any Euclidean[8] cube $\widetilde{Q} \subset B_{R_0}(\tilde{p})$ one has*

$$\mathcal{H}^{d-1}\big(\{H = 0\} \cap \widetilde{Q}\big) \leq \kappa \operatorname{diam}(\widetilde{Q})^{d-1} N(H, \widetilde{Q})^{2\alpha}$$

[8]It means that $\widetilde{Q}$ is a cube in the chart associated to the geodesic normal coordinates at $\tilde{p}$.

for some $\alpha > \frac{1}{2}$ that is only dependent on d, and some κ that depends only on $(B_{2\widetilde{R}}(\tilde{p}), g)$. Here,

$$N(H, \widetilde{Q}) = \sup_{B_s^e(x) \subset 2\widetilde{Q}} \log\left(\frac{\sup_{B_{2s}^e(x)} |H|^2}{\sup_{B_s^e(x)} |H|^2}\right), \tag{2-12}$$

where $B_s^e(x)$ stands for the Euclidean ball of radius s centered at x in the normal chart of $B_{\widetilde{R}}(\tilde{p})$.

To prove (1-2), we use our modified growth estimates (2-2) and the above theorem. We first cover (X, g) using geodesic balls $\{B_r(x_i)\}_{x_i \in \mathcal{I}}$ such that each point in X is contained in $C(X, g)$ many of the double balls $\{B_{2r}(x_i)\}_{x_i \in \mathcal{I}}$, where $C(X, g)$ is independent of r and depends only on the injectivity radius of (X, g) and a bound on the Ricci curvature of (X, g). Such a thing is possible by the Bishop–Gromov volume comparison theorem. For a proof see, for example, [Colding and Minicozzi 2011, Lemma 2]. It is then easy to see that such a covering has at most $C_0 r^{-n}$ open balls for some uniform constant $C_0 = C_0(X, g)$. Next we estimate $\mathcal{H}^{n-1}(Z_{\psi_\lambda} \cap B_r(p))$ for each $p \in \mathcal{I}$. To do this we define

$$\widetilde{X} = X \times \mathbb{R}, \quad d = n + 1, \quad \tilde{g} = \text{product metric}.$$

We shall also use $\tilde{x} = (x, t)$. We then put

$$H(\tilde{x}) = \psi_\lambda(x) e^{t\sqrt{\lambda}}.$$

Then clearly $\Delta_{\tilde{g}} H = 0$. We now cover the compact manifold $X \times [-1, 1]$ by finitely many balls $\{\widetilde{B}_j\}_{1 \le j \le M}$ each of which satisfies the property of the ball $\widetilde{B}_{R_0}$ in Theorem 2.5. Let L_0 be the Lebesgue number of this finite cover and assume $r \le \frac{1}{2} L_0$. Also for each $p \in X$, let $Q_r(p)$ be the Euclidean cube in X of side lengths $2r$ centered at p. Then we observe that for some $1 \le j \le M$ we have

$$\widetilde{Q}_r(\tilde{p}) := Q_r(p) \times [-r, r] \subset \widetilde{B}_{2r}(\tilde{p}) \subset \widetilde{B}_{L_0}(\tilde{p}) \subset \widetilde{B}_j,$$

where $\tilde{p} = (p, 0)$. By applying Theorem 2.5 for the cube $\widetilde{Q}(\tilde{p})$ in the ball $\widetilde{B}_j$, we get that

$$\begin{aligned} \mathcal{H}^{n-1}\big(\{\psi_\lambda = 0\} \cap B_r(p)\big) &\le \mathcal{H}^{n-1}\big(\{\psi_\lambda = 0\} \cap Q_r(p)\big) \\ &= \frac{1}{2r} \mathcal{H}^n\big(\{H = 0\} \cap \widetilde{Q}_r(\tilde{p})\big) \\ &\le \frac{\kappa}{2r} (2r)^n N(H, \widetilde{Q}_r(\tilde{p}))^{2\alpha} \\ &= \kappa' r^{n-1} N(H, \widetilde{Q}_r(\tilde{p}))^{2\alpha}. \end{aligned}$$

Now we use our doubling estimates to show that $N(H, \widetilde{Q}_r(\tilde{p})) \le c' r \sqrt{\lambda}$ for some c' that is uniform in r, λ, and p. We emphasize that our doubling estimates involve geodesic balls, but the definition of the doubling index N in [Logunov 2016a] uses Euclidean balls $B_s^e(\tilde{x})$ in a fixed normal chart of $B_{2\widetilde{R}}(\tilde{p})$. However, by choosing R_0 sufficiently small we can make sure that

$$B_{\frac{s}{2}}(\tilde{x}) \subset B_s^e(\tilde{x}) \subset B_{\frac{3s}{2}}(\tilde{x})$$

for all $\tilde{x} \in B_{R_0}(\tilde{p})$ and all $s < R_0$. As a result of this if we assume $r < \frac{1}{20} R_0$, then using (2-3) four times we get

$$N(H, \widetilde{Q}_r(\tilde{p})) = \sup_{B^e_s(\tilde{x}) \subset \widetilde{Q}_{2r}(\tilde{p})} \log\left(\frac{\sup_{B^e_{2s}(\tilde{x})} |H(\tilde{x})|^2}{\sup_{B^e_s(\tilde{x})} |H(\tilde{x})|^2} \right)$$
$$\leq \sup_{B_{s/2}(\tilde{x}) \subset \widetilde{Q}_{2r}(\tilde{p})} \log\left(\frac{\sup_{B_{3s}(\tilde{x})} |H(\tilde{x})|^2}{\sup_{B_{s/2}(\tilde{x})} |H(\tilde{x})|^2} \right)$$
$$\leq \sup_{B_{s/2}(x) \subset Q_{2r}(p)} \log\left(e^{5s\sqrt{\lambda}} \frac{\sup_{B_{3s}(x)} |\psi_\lambda(x)|^2}{\sup_{B_{s/4}(x)} |\psi_\lambda(x)|^2} \right) \leq c' r \sqrt{\lambda}.$$

Finally

$$\mathcal{H}^{n-1}(Z_{\psi_\lambda}) \leq \sum_{x_i \in \mathcal{I}} \mathcal{H}^{n-1}(Z_{\psi_\lambda} \cap B_r(x_i)) \leq C_0 r^{-n} \kappa' r^{n-1} (c'^2 r^2 \lambda)^\alpha \leq c_1 r^{2\alpha - 1} \lambda^\alpha$$

for some c_1 that is uniform in r and λ.

Proof of* (1-4)*: upper bounds on the size of nodal sets for surfaces. The main tool is the following local result.

Theorem 2.6 [Logunov and Malinnikova 2016]. *Let $(\widetilde{X}, \tilde{g})$ be a smooth Riemannian manifold of dimension $n = 2$, $p \in \widetilde{X}$ a point, and $\widetilde{R} > 0$ a radius such that the $\tilde{g}$-geodesic ball $\widetilde{B}_{2\widetilde{R}}(p)$ is embedded. Let $\psi_{\tilde{\lambda}}$ be a smooth function such that for some $\tilde{\lambda} \geq 1$ we have $\Delta_{\tilde{g}} \psi_{\tilde{\lambda}} = \tilde{\lambda} \psi_{\tilde{\lambda}}$ on $\widetilde{B}_{2\widetilde{R}}(p)$. Suppose we also know that there exists some $s_0 \leq \frac{1}{10} R$ such that for all $s < s_0$ we have*

$$\frac{\sup_{\widetilde{B}_{2s}(x)} |\psi_{\tilde{\lambda}}|^2}{\sup_{\widetilde{B}_s(x)} |\psi_{\tilde{\lambda}}|^2} \leq C_1 e^{c\sqrt{\tilde{\lambda}}}$$

for some constants c and C_1 that are uniform for $x \in \widetilde{B}_{\widetilde{R}}(p)$. Then

$$\mathcal{H}^1_{\tilde{g}}\big(\{\psi_{\tilde{\lambda}} = 0\} \cap \widetilde{B}_{\frac{\widetilde{R}}{2}}(p)\big) \leq C_2 \tilde{\lambda}^{\frac{3}{4} - \beta}, \tag{2-13}$$

where $\beta \in \big(0, \frac{1}{4}\big)$ is a small universal constant and C_2 is controlled by c, C_1, and the $\mathcal{C}^k$ norm of $(\tilde{g}^{-1})_{ij}$ on $\widetilde{B}_{2\widetilde{R}}(p)$ for some universal k.

To prove (1-4), suppose ψ_λ is an eigenfunction of Δ_g on (X, g). We cover X by geodesic balls $\{B_{\frac{r}{2}}(x_i)\}_{x_i \in \mathcal{I}}$ of radius $\frac{1}{2} r$ in such a way that the number of them is at most $C_0 r^{-n}$. As we saw earlier, this is always possible. We then estimate the size of the nodal set of ψ_λ in each $B_{\frac{r}{2}}(x)$ using Theorem 2.6. To do this, we first define $(\widetilde{X}, \tilde{g}) = \big(X, \frac{1}{r^2} g\big)$. Under such a rescaling, a ball of radius r scales to a ball of radius 1. Hence we put $\widetilde{R} = 1$. Then the equation

$$-\Delta_g \psi_\lambda = \lambda \psi_\lambda \quad \text{on } B_{2r}(p),$$

becomes

$$-\Delta_{\tilde{g}} \psi_\lambda = \tilde{\lambda} \psi_\lambda \quad \text{on } \widetilde{B}_2(p),$$

with

$$\tilde{\lambda} = r^2\lambda \quad \text{and} \quad \psi_{\tilde{\lambda}} = \psi_\lambda.$$

We can see that the doubling condition of Theorem 2.6 is valid because for all $s \leq \frac{1}{10}$, using (2-3)

$$\frac{\sup_{\widetilde{B}_{2s}(x)} |\psi_\lambda|^2}{\sup_{\widetilde{B}_s(x)} |\psi_\lambda|^2} = \frac{\sup_{B_{2sr}(x)} |\psi_\lambda|^2}{\sup_{B_{sr}(x)} |\psi_\lambda|^2} \leq e^{cr\sqrt{\lambda}} = e^{c\sqrt{\tilde{\lambda}}}$$

for some c that is uniform in $\tilde{\lambda}$, s, and x, and is controlled by K_1, K_2, and the $\mathcal{C}^k$ norm of $(\tilde{g})^{ij}$ on $\widetilde{B}_2(p)$ for some universal k. Therefore, by Theorem 2.6

$$\mathcal{H}^1_g\big(\{\psi_\lambda{=}0\} \cap B_{\frac{r}{2}}(p)\big) = r^{n-1}\mathcal{H}^1_{\tilde{g}}\big(\{\psi_{\tilde{\lambda}}{=}0\} \cap \widetilde{B}_{\frac{1}{2}}(p)\big) \leq C_2 r^{n-1}\tilde{\lambda}^{\frac{3}{4}-\beta}.$$

We emphasize that since $(\tilde{g})^{ij} = r^2 g^{ij}$, for small enough $r_0(g)$ and all $r < r_0(h)$, the $\mathcal{C}^k$ norm of $(\tilde{g})^{ij}$ on $\widetilde{B}_2(p)$ is bounded by the $\mathcal{C}^k$ norm of $(g)^{ij}$ on $B_{2r}(p)$. Hence C_2 is independent of r, λ, and p, and is controlled only by K_1 and K_2 and (X, g). Adding these up, we get

$$\mathcal{H}^1_g\big(\{\psi_\lambda{=}0\}\big) \leq \sum_{x_i \in \mathcal{I}} \mathcal{H}^1_g\big(\{\psi_\lambda{=}0\} \cap B_{\frac{r}{2}}(x_i)\big) \leq (C_0 r^{-n}) C_2 r^{n-1}\tilde{\lambda}^{\frac{3}{4}-\beta} = c_3 r^{1-2\beta}\lambda^{\frac{3}{4}-\beta}.$$

Proof of* (1-5): *number of singular points for surfaces. We shall use the results of [Dong 1992] instead of [Donnelly and Fefferman 1990a], although both methods would work. Another goal is to simplify a less detailed part of the argument of [Dong 1992]. Let us first recall some statements from that paper.

Theorem 2.7 [Dong 1992, Theorems 2.2 and 3.4]. *Let (X, g) be a smooth Riemannian manifold of dimension 2, $p \in X$, and $R > 0$ so that the geodesic ball $B_{2R}(p)$ is embedded. Suppose ψ_λ is a smooth function such that $\Delta_g \psi_\lambda = \lambda\psi_\lambda$ on $B_{2R}(p)$ for some $\lambda \geq 1$. Then for all $x \in B_{\frac{R}{2}}(p)$ and all $s < \frac{1}{8}R$*

$$\sum_{z \in Z_{\psi_\lambda} \cap B_s(x)} (\nu_z(\psi_\lambda) - 1) \leq \alpha_1\sqrt{\lambda} + \alpha_2 s^2\lambda. \tag{2-14}$$

The constants α_1, α_2 are uniform in x, s, and λ, and depend only on $(B_{2R}(p), g)$.

In fact by a glance at the proof of (2-14), see [Dong 1992, Theorem 3.4, pp. 502–503], one sees that the following statement holds:

$$\sum_{z \in Z_{\psi_\lambda} \cap B_s(x)} (\nu_z(\psi_\lambda) - 1) \leq \alpha_1' \log\left(\frac{\sup_{B_{4s}(x)} q_\lambda}{\sup_{B_s(x)} q_\lambda}\right) + \alpha_2 s^2\lambda, \tag{2-15}$$

where

$$q_\lambda = |\nabla\psi_\lambda|^2 + \frac{\lambda}{2}|\psi_\lambda|^2$$

and α_1' and α_2 are some uniform constants.

The estimate (2-14) follows quickly from (2-15) if one knows that

$$\text{for } s \in \big(0, \tfrac{1}{8}R\big),\ x \in B_{\frac{R}{2}}(p), \quad \frac{\sup_{B_{4s}(x)} q_\lambda}{\sup_{B_s(x)} q_\lambda} \leq \alpha_3 e^{c_2\sqrt{\lambda}}.$$

The above growth estimate is proved in [Dong 1992] using the theory of frequency functions and monotonicity formulas; see [Garofalo and Lin 1986; Han and Lin 2007; Lin 1991] for background. However the proof of the monotonicity formula associated to q_λ, see [Dong 1992, pp. 498–499], is carried out only for the Euclidean metric and the proof of the upper bound $\sqrt{\lambda}$ on the frequency function uses the methods of [Lin 1991]. Here we give a simpler proof of this growth estimate which is based on gradient estimates for solutions of elliptic equations. More precisely, we show that if doubling estimates (2-3)

$$\text{for } s \in (0, 10r),\ x \in B_{\frac{R}{2}}(p), \qquad \sup_{B_{2s}(x)} |\psi_\lambda|^2 \le e^{cr\sqrt{\lambda}} \sup_{B_s(x)} |\psi_\lambda|^2$$

hold, then

$$\text{for } s \in (\lambda^{-\frac{1}{2}}, 2r),\ x \in B_{\frac{R}{2}}(p), \qquad \frac{\sup_{B_{4s}(x)} q_\lambda}{\sup_{B_s(x)} q_\lambda} \le \alpha_3 e^{c_2 r\sqrt{\lambda}} \tag{2-16}$$

for uniform constants α_3 and c_2. For the proof we use an application of standard elliptic estimates to the gradient of eigenfunctions, as performed in [Shi and Xu 2010].

Theorem 2.8 [Shi and Xu 2010, Theorem 1]. *Let (X, g) be a smooth connected compact Riemannian manifold without boundary. Suppose ψ_λ is an eigenfunction of Δ_g with eigenvalue λ. Then*

$$\beta_1\sqrt{\lambda}\sup_X |\psi_\lambda| \le \sup_X |\nabla\psi_\lambda| \le \beta_2\sqrt{\lambda}\sup_X |\psi_\lambda|$$

for some positive constants β_1 and β_2 independent of λ.

In fact by looking at the proof of this theorem we notice that a stronger statement holds. More precisely, one can see that, see [Shi and Xu 2010, p. 23, Fact (1) and equation (6)], for all $s < \frac{1}{4}\operatorname{inj}(g)$

$$\beta_1\sqrt{\lambda} \sup_{B_s(x)} |\psi_\lambda| \le \sup_{B_{s+(\gamma_0/\lambda^{1/2})}(x)} |\nabla\psi_\lambda|$$

$$\sup_{B_s(x)} |\nabla\psi_\lambda| \le \beta_2\sqrt{\lambda} \sup_{B_{s+(1/\lambda^{1/2})}(x)} |\psi_\lambda|, \tag{2-17}$$

where γ_0 is a positive constant that depends only on the Riemannian manifold (X, g). In fact it is the Brüning constant that guarantees that in every ball of radius $\gamma_0/\lambda^{\frac{1}{2}}$ there is a zero of ψ_λ. However, to prove (2-16) we only need the upper bound (2-17) for the gradient.[9] Let $s \in (\lambda^{-\frac{1}{2}}, 2r)$. Then since $4s + \lambda^{-\frac{1}{2}} < 10r$, using our doubling estimate (2-3) three times, we get

$$\begin{aligned}
\sup_{B_{4s}(x)} q_\lambda &= \sup_{B_{4s}(x)} \left(|\nabla\psi_\lambda|^2 + \tfrac{1}{2}\lambda|\psi_\lambda|^2\right) \\
&\le \beta_2'\lambda \sup_{B_{4s+(1/\lambda^{1/2})}(x)} |\psi_\lambda|^2 \\
&\le \beta_2'\lambda e^{3cr\sqrt{\lambda}} \sup_{B_{s/2+(1/8\lambda^{1/2})}(x)} |\psi_\lambda|^2 \\
&\le 2\beta_2' e^{3cr\sqrt{\lambda}} \sup_{B_s(x)} q_\lambda.
\end{aligned}$$

This proves (2-16) with $\alpha_3 = 2\beta_2'$ and $c_2 = 3c$.

[9]This is proved easily by a rescaling argument and elliptic estimates such as Theorem 8.32 in [Gilbarg and Trudinger 1998].

To finish the proof of our upper bounds for the number of singular points for surfaces, we apply (2-16) to the inequality (2-15) and obtain

$$\sum_{z\in Z_{\psi_\lambda}\cap B_s(x)} (\nu_z(\psi_\lambda)-1) \le \alpha_3'' r\sqrt{\lambda}+\alpha_2 s^2\lambda.$$

We now put $s=r^{\frac{1}{2}}\lambda^{-\frac{1}{4}}$. We underline that this choice of s is in fact in the allowable range $(\lambda^{-\frac{1}{2}},2r)$ because $r\ge\lambda^{-\frac{1}{2}}$. From this, (1-5) follows immediately.

Proof of Theorem 1.3*: *upper bounds for QE eigenfunctions. This theorem follows quickly from the lemma below combined with Theorem 1.1.

Lemma 2.9. *Let $\{\psi_j\}_{j\in S}$ be a sequence of eigenfunctions of Δ_g with eigenvalues $\{\lambda_j\}_{j\in S}$ such that for all $r\in\big(0,\frac{1}{2}\operatorname{inj}(g)\big)$ and all $x\in X$*

$$\int_{B_r(x)}|\psi_j|^2 \to \frac{\mathrm{Vol}_g(B_r(x))}{\mathrm{Vol}_g(X)},\quad \lambda_j\xrightarrow{j\in S}\infty. \tag{2-18}$$

Then there exists $r_0(g)$ such that for each $r\in(0,r_0(g))$ there exists Λ_r such that for $\lambda_j\ge\Lambda_r$ we have

$$K_1r^n\le\int_{B_r(x)}|\psi_j|^2\le K_2r^n$$

uniformly for all $x\in X$. Here, K_1 and K_2 are independent of r, j, and x.

We point out that this lemma is obvious when x is fixed; however to obtain uniform L^2 estimates we need to use a covering argument as follows.

Proof. First we choose $r_0(g)<\frac{1}{4}\operatorname{inj}(g)$ small enough so that for all $r<r_0(g)$

$$a_1r^n\le\mathrm{Vol}(B_{\frac{r}{2}}(x))<\mathrm{Vol}(B_{2r}(x))\le a_2r^n$$

for some positive a_1 and a_2 that are independent of r and x. Next, we cover (X,g) using geodesic balls $\{B_{\frac{r}{2}}(x_i)\}_{x_i\in\mathcal{I}}$ such that card $(\mathcal{I})$ is at most C_0r^{-n}, where C_0 depends only on (X,g). The existence of such a covering was discussed in the proof of (1-2). For each $x_i\in\mathcal{I}$, by using (2-18) twice, we can find $\Lambda_{i,r}$ large enough so that for $\lambda_j\ge\Lambda_{i,r}$

$$K_1r^n\le\int_{B_{r/2}(x_i)}|\psi_j|^2\le\int_{B_{2r}(x_i)}|\psi_j|^2\le K_2r^n,$$

with $K_1=a_1/(2\,\mathrm{Vol}(X))$ and $K_2=2a_2/\mathrm{Vol}(X)$. We claim that $\Lambda_r=\max_{i\in\mathcal{I}}\{\Lambda_{i,r}\}$ would do the job for all x in X. So let x be in X and r be as above. Then $x\in B_{\frac{r}{2}}(x_i)$ for some $i\in\mathcal{I}$ and clearly one has $B_{\frac{r}{2}}(x_i)\subset B_r(x)\subset B_{2r}(x_i)$. This and the above inequalities prove the lemma. □

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