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**A GLUING FORMULA FOR THE ANALYTIC TORSION
ON SINGULAR SPACES**

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To my family

We prove a gluing formula for the analytic torsion on noncompact (i.e., singular) Riemannian manifolds. Let $M = U \cup_{\partial M_1} M_1$, where M_1 is a compact manifold with boundary and U represents a model of the singularity. For general elliptic operators we formulate a criterion, which can be checked solely on U , for the existence of a global heat expansion, in particular for the existence of the analytic torsion in the case of the Laplace operator. The main result then is the gluing formula for the analytic torsion. Here, decompositions $M = M_1 \cup_Y M_2$ along any compact closed hypersurface Y with M_1, M_2 both noncompact are allowed; however a product structure near Y is assumed. We work with the de Rham complex coupled to an arbitrary flat bundle F ; the metric on F is not assumed to be flat. In an appendix the corresponding algebraic gluing formula is proved. As a consequence we obtain a framework for proving a Cheeger–Müller-type theorem for singular manifolds; the latter has been the main motivation for this work.

The main tool is Vishik’s theory of moving boundary value problems for the de Rham complex which has also been successfully applied to Dirac-type operators and the eta invariant by J. Brüning and the author. The paper also serves as a new, self-contained, and brief approach to Vishik’s important work.

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1. Introduction

The Cheeger–Müller theorem [Cheeger 1979a; Müller 1978; 1993] on the equality of the analytic and combinatorial torsion is one of the cornerstones of modern global analysis. To extend the theorem to certain singular manifolds is an intriguing open challenge.

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In his seminal work, Cheeger [1979b; 1983] initiated the program of “extending the theory of the Laplace operator to certain Riemannian spaces with singularities”. Since then a lot of work on this program has been done. It is impossible to give a proper account here, but let us mention [Brüning and Seeley 1988; 1987], Melrose [1993] and collaborators, and Schulze [1991] and collaborators. While the basic spectral theory (index theory, heat kernel analysis) for several types of singularities (cones [Lesch 1997], cylinders [Melrose 1993], cusps [Müller 1983], edges [Mazzeo 1991]) is fairly well understood, an analogue of the Cheeger–Müller theorem has not yet been established for any type of singular manifold, except compact manifolds with boundary.

We will not solve this problem in this paper. However, we will provide a framework for attacking the problem.

To describe this we must go back a little. Let M be a Riemannian manifold (boundaryless but not necessarily compact; also the *interior* of a manifold with boundary is allowed) and let P^0 be an elliptic differential operator acting on the sections $\Gamma^\infty(E)$ of the Hermitian vector bundle E . We consider P^0 as an unbounded operator in the Hilbert space $L^2(M, E)$ of L^2 -sections of E . Moreover, we assume P^0 to be bounded below; for example, $P^0 = D^t D$ for an elliptic operator D . Fix a bounded below self-adjoint extension $P \geq -C > -\infty$.

We know that e^{-tP} is an integral operator with a smooth kernel $k_t(x, y)$ which on the diagonal has a pointwise asymptotic expansion

$$k_t(x, x) \sim_{t \searrow 0} \sum_{j=0}^{\infty} a_j(x) t^{\frac{j - \dim M}{\text{ord } P}}. \tag{1-1}$$

This asymptotic expansion is *uniform on compact subsets of M* and hence if, e.g., M is compact, it may be integrated over the manifold to obtain an asymptotic expansion for the trace of e^{-tP} . For general noncompact M one cannot expect the operator e^{-tP} to be of trace class. Even if it is of trace class and even if the coefficients $a_j(x)$ in (1-1) are integrable, integration of (1-1) does not necessarily lead to an asymptotic expansion of $\text{Tr}(e^{-tP})$. It is therefore a fundamental problem to give criteria which ensure that e^{-tP} is of trace class and such that there is an asymptotic expansion

$$\text{Tr}(e^{-tP}) \sim_{t \searrow 0} \sum_{\substack{\Re \alpha \rightarrow \infty \\ 0 \leq k \leq k(\alpha)}} a_{\alpha k} t^\alpha \log^k t. \tag{1-2}$$

It is not realistic to find such criteria for arbitrary open manifolds. Instead one looks at geometric differential operators on manifolds with singular exits which occur in geometry. A rather generic description of this situation can be given as follows: suppose that there is a compact manifold $M_1 \subset M$ and a “well understood” model manifold U such that

$$M = U \cup_{\partial M_1} M_1. \tag{1-3}$$

We list a couple of examples for U which are reasonably well understood and which are of geometrical significance:

1. *Smooth boundary.* $U = (0, \epsilon) \times Y$ is a cylinder with metric $dx^2 + g_Y$ over a smooth compact boundaryless manifold Y . Then M is just the interior of a compact manifold with boundary. To this situation the theory of elliptic boundary value problems applies. Heat trace expansions are established, for example, for all well-posed elliptic boundary value problems associated to Laplace-type operators [Grubb 1999].
2. *Isolated asymptotically conical singularities.* $U = (0, \epsilon) \times Y$ with metric $dx^2 + x^2 g_Y(x)$. Then M is a manifold with an isolated (asymptotically) conical singularity. This is the best understood case of a singular manifold; it is impossible here to do justice to all the scientists who contributed. So we just reiterate that its study was initiated by Cheeger [1979b; 1983].
3. *Simple edge singularities.* In the hierarchy of singularities of stratified spaces, which are in general of iterated cone type, this is the next simple class after isolated conical ones: simplifying a little, U is of the form $(0, \epsilon) \times F \times B$ with metric $dx^2 + x^2 g_F(x) + g_B(x)$. The heat trace expansion and the existence of the analytic torsion for this class of singularities has been established recently by Mazzeo and Vertman [2012].
4. *Complete cylindrical ends.* This case is at the heart of Melrose’s celebrated b-calculus [1993]. An exact b-metric on $(0, \epsilon) \times Y$ is of the form $dx^2/x^2 + g_Y$. Making the change of variables $x = e^{-y}$ we obtain a metric cylinder $(-\log \epsilon, \infty) \times Y$ with metric $dy^2 + g_Y$. M is then a complete manifold. Therefore, the Laplacian, for example, is essentially self-adjoint. However, it is not a discrete operator and hence its heat operator is not of trace class.
5. *Cusps.* Cusps occur naturally as singularities of Riemann surfaces of constant negative curvature. A cusp is given by $U = (0, \infty) \times Y$ with metric $dx^2 + e^{-2x} g_Y$. Then M has finite volume. As in the previous case, however, the Laplacian is not a discrete operator. In this situation (and also in the previous one) one employs methods from scattering theory. There has been seminal work on this by Werner Müller [1992].

The results of this paper apply to situations where the operator P is discrete (has compact resolvent). This is the case in the examples 1–3 above, but *not* in 4 and 5. Nevertheless we are confident that our method can be extended to relative heat traces and relative determinants, for example, for surfaces of finite area.

To explain our results without becoming too technical, suppose that for $P_U = P \upharpoonright U$ and $P_1 = P \upharpoonright M_1$ (of course suitable extensions have to be chosen for P_U and P_1) we have proved expansions (1-2). Then in terms of a suitable cut-off function φ which is 1 in a neighborhood of M_1 one expects to hold:

Principle 1.1 (Duhamel’s principle for heat asymptotics; informal version). If P_U and P_1 are discrete with trace-class heat kernels then so is P and

$$\text{Tr}(e^{-tP}) = \text{Tr}(\varphi e^{-tP_1}) + \text{Tr}((1 - \varphi)e^{-tP_U}) + O(t^N) \quad \text{as } t \rightarrow 0+, \quad \text{for all } N. \quad (1-4)$$

We reiterate that the heat operator is a *global operator*. On a *closed* manifold its short-time asymptotic expansion is local in the sense that the heat trace coefficients are integrals over local densities as described

above. This kind of local behavior cannot be expected on noncompact manifolds. However, [Principle 1.1](#) shows that the heat trace coefficients localize near the singularity; they may still be global *on* the singularity as is the case, for example, for Atiyah–Patodi–Singer boundary conditions [[Atiyah et al. 1975](#)].

[Principle 1.1](#) is a folklore theorem which appears in various versions in the literature. In [Section 3](#) below we will prove a fairly general rigorous version of it ([Corollary 3.7](#)).

Once the asymptotic expansion (1-2) is in place one obtains, via the Mellin transform, the meromorphic continuation of the ζ -function

$$\zeta(P; s) := \sum_{\lambda \in \text{spec}(P) \setminus \{0\}} \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}((I - \Pi_{\ker P})e^{-tP}) dt. \tag{1-5}$$

Let us specialize to the de Rham complex. Suppose that we have chosen an ideal boundary condition (essentially this means that we have chosen closed extensions for the exterior derivative) (\mathcal{D}, D) for the de Rham complex such that the corresponding extensions $\Delta_j = D_j^* D_j + D_{j-1} D_{j-1}^*$ of the Laplace operators satisfy (1-2). Then we can form the analytic torsion of (\mathcal{D}, D) :

$$\log T(\mathcal{D}, D) := \frac{1}{2} \sum_{j \geq 0} (-1)^j j \frac{d}{ds} \Big|_{s=0} \zeta(\Delta_j; s). \tag{1-6}$$

For a closed manifold the celebrated Cheeger–Müller theorem [[Cheeger 1979a](#); [Müller 1978](#)] relates the analytic torsion to the combinatorial torsion (Reidemeister torsion).

In terms of the decomposition (1-3) the problem of proving a CM-type theorem for the singular manifold M decomposes into the following steps.

- (1) Prove that the analytic torsion exists for the model manifold U .
- (2) Compare the analytic torsion with a suitable combinatorial torsion for U .
- (3) Prove a gluing formula for the analytic and combinatorial torsion and apply the known Cheeger–Müller theorem for the manifold with boundary M_1 .

A gluing formula for the combinatorial torsion is more or less an algebraic fact due to Milnor; see also the [Appendix](#). The following theorem, which follows from our gluing formula, solves (3) under a product structure assumption:

Theorem 1.2. *Let M be a singular manifold expressed as (1-3) and assume that near ∂M_1 all structures are product. Then for establishing a Cheeger–Müller theorem for M it suffices to prove it for the model space U of the singularity.*

The theorem basically says that, under product assumptions, one gets step (3) for free. Otherwise the specific form of U is completely irrelevant. We conjecture that the product assumption in [Theorem 1.2](#) can be dispensed with. This would follow once the anomaly formula of Brüning and Ma [[2006](#)] were established for the model U of the singularity; this would allow us to compare the analytic torsion for (U, g) to the torsion of (U, g_1) , where g_1 is product near ∂M_1 and coincides with g outside a relatively compact collar.

The theorem is less obvious than it sounds since torsion invariants are global in nature. However, we will show here that under minimal technical assumptions the analytic torsion satisfies a gluing formula. That the combinatorial torsion satisfies a gluing formula is a purely algebraic fact (see [Appendix](#)). The blueprint for our proof is a technique of moving boundary conditions due to Vishik [\[1995\]](#) who applied it to prove the Cheeger–Müller theorem for compact manifolds with smooth boundary. Brüning and the author [\[Brüning and Lesch 1999\]](#) applied Vishik’s moving boundary conditions to generalized Atiyah–Patodi–Singer nonlocal boundary conditions and to give an alternative proof of the gluing formula for the eta-invariant. We emphasize, however, that the technical part of the present paper is completely independent of (and in our slightly biased view simpler than) [\[Vishik 1995\]](#). Also we work with the de Rham complex coupled to an arbitrary flat bundle F . Besides the product structure assumption we do not impose any restrictions on the metric h^F on F ; in particular h^F is not assumed to be flat.

We note here that in the context of *closed* manifolds gluing formulas for the analytic torsion have been proved in [\[Vishik 1995; Burghelea et al. 1999\]](#), and recently [\[Brüning and Ma 2013\]](#). In contrast our method applies to a wide class of singular manifolds.

Some more comments on conic singularities, the most basic singularities, are in order: let (N, g) be a compact closed Riemannian manifold and let $CN = (0, 1) \times N$ with metric $dx^2 + x^2g$ be the cone over N . We emphasize that sadly near $\partial CN = \{1\} \times N$ we do not have product structure. Let g_1 be a metric on CN that is product near $\{1\} \times N$ and that coincides with g near the cone tip.

Vertman [\[2009\]](#) gave formulas for the torsion of the cone (CN, g) in terms of spectral data of the cone base. What is still not yet understood is how these formulas for the analytic torsion can be related to a combinatorial torsion of the cone, at least not in the interesting odd-dimensional case. For CN even-dimensional, Hartmann and Spreafico [\[2010\]](#) express the torsion of (CN, g) in terms of the intersection torsion introduced by A. Dar [\[1987\]](#) and the anomaly term of Brüning and Ma [\[2006\]](#). If it were also possible to apply the latter to the singular manifold CN to compare the torsion of the metric cone (CN, g) to that of the cone (CN, g_1) , where the metric near $\{1\} \times N$ is modified to a product metric, then one would obtain a (very sophisticated) new proof of Dar’s theorem that for an even-dimensional manifold with conical singularities the analytic and the intersection torsions both vanish.¹ It would be more interesting, of course, to have this program worked out in the odd-dimensional case.

The paper is organized as follows. [Section 2](#) serves to introduce some terminology and notation. In a purely functional analytic context we discuss self-adjoint operators with *discrete dimension spectrum*; this terminology is borrowed from Connes and Moscovici’s celebrated paper [\[1995\]](#) on the local index theorem in noncommutative geometry. For *Hilbert complexes* [\[Brüning and Lesch 1992\]](#) whose Laplacians have discrete dimension spectrum one can introduce the analytic torsion. We state a formula for the torsion of a product complex ([Proposition 2.3](#)) and in [Section 2B](#) we collect some algebraic facts about determinants and the torsion of a finite-dimensional Hilbert complex. The main result of the section is [Proposition 2.4](#) which, under appropriate assumptions, provides a variation formula for the analytic torsion of a one-parameter family of Hilbert complexes.

¹ For this to hold one needs to assume that the metric on the twisting bundle F is also flat.

In [Section 3](#) we discuss the gluing of operators in a fairly general setting: we assume that we have two pairs (M_j, P_j^0) , $j = 1, 2$, consisting of Riemannian manifolds M_j^m and elliptic operators P_j^0 such that each M_j is the interior of a manifold \bar{M}_j with compact boundary Y (\bar{M}_j is not necessarily compact). Let $W = Y \times (-c, c)$ be a common collar of Y in M_1 and in M_2 such that $\partial M_1 = Y \times \{1\}$ and $\partial M_2 = Y \times \{-1\}$ and such that P_1^0 coincides with P_2^0 over W . Then P_j^0 give rise naturally to a differential operator $P^0 = P_1^0 \cup P_2^0$ on $M := (M_1 \setminus (Y \times (0, c))) \cup_{Y \times \{0\}} (M_2 \setminus (Y \times (-c, 0)))$. Without becoming too technical here we will show in [Proposition 3.5](#) that certain semibounded symmetric extensions P_j , $j = 1, 2$, of P_j^0 satisfying a noninteraction condition [\(3-18\)](#) give rise naturally to a semibounded self-adjoint extension of P^0 . Furthermore, if the P_j have discrete dimension spectrum outside W (compare the paragraph before [Corollary 3.7](#)), then the operator P has discrete dimension spectrum and up to an error of order $O(t^\infty)$ the short-time heat trace expansion of P can be calculated easily from the corresponding expansions of P_j .

We also prove similar results for perturbed operators of the form $P_j + V_j$, where V_j is a certain non-pseudodifferential operator; such operators will occur naturally in [Section 5](#), our main technical section.

In [Section 4](#) we describe the details of the gluing situation, review Vishik's moving boundary conditions for the de Rham complex in this context, and introduce various one-parameter families of de Rham complexes. The main technical result of the paper is [Theorem 4.1](#) which analyzes the variation of the torsions of these various families of de Rham complexes. The proof of [Theorem 4.1](#) occupies the whole [Section 5](#). The proof is completely independent of Vishik's original approach. The main feature of our proof is a gauge transformation à la Witten of the de Rham complex which transforms the de Rham operator, originally a family of operators with varying domains, onto a family of operators with constant domain; this family can then easily be differentiated by the parameter.

[Theorem 6.1](#) in [Section 6](#) then finally is the main result of the paper, whose proof, thanks to [Theorem 4.1](#), is now more or less an exercise in diagram chasing.

The [Appendix](#) contains the analogues of our main results for finite-dimensional Hilbert complexes.

The paper has a somewhat lengthy history. The material of [Sections 4](#) and [5](#), in the context of smooth manifolds only, was developed in the summer of 1999, while I was on a Heisenberg fellowship in Bonn. In light of the negative feedback received at conferences I felt that the subject was dying and abandoned it.

In recent years there has been revived interest in generalizing the Cheeger–Müller theorem to manifolds with singularities [[Mazzeo and Vertman 2012](#); [Vertman 2009](#); [Müller and Vertman 2011](#); [Hartmann and Spreafico 2010](#)]. I noticed that my techniques (an adaption of Vishik's work [[1995](#)] plus simple observations based on Duhamel's principle) do not require the manifold to be closed. The bare minimal assumptions required for the analytic torsion to exist ("discrete dimension spectrum"; see [Section 2](#)) and a mild but obvious noninteraction restriction on the choice of the ideal boundary conditions ([Definition 3.4](#)) for the de Rham complex actually suffice to prove a gluing formula for the analytic torsion. Since a more concise and more accessible account of the long and important paper [[Vishik 1995](#)] is overdue anyway, I eventually made a final effort to write up this paper, in part because Werner Müller and Boris Vertman had been pushing me to do so.

2. Operators with meromorphic ζ -function

Let \mathcal{H} be a separable complex Hilbert space, T a nonnegative self-adjoint operator in \mathcal{H} with p -summable resolvent for some $1 \leq p < \infty$. The summability condition implies that T is a *discrete* operator; that is, the spectrum of T consists of eigenvalues of finite multiplicity with $+\infty$ being the only accumulation point. Moreover,

$$\text{Tr}(e^{-tT}) = \sum_{\lambda \in \text{spec } T} e^{-t\lambda} = \dim \ker T + O(e^{-t\lambda_1}) \quad \text{as } t \rightarrow \infty \tag{2-1}$$

and

$$\text{Tr}(e^{-tT}) = O(t^{-p}) \quad \text{as } t \rightarrow 0+. \tag{2-2}$$

Here $\lambda_1 := \min(\text{spec } T \setminus \{0\})$ denotes the smallest nonzero eigenvalue of T .

As a consequence, the ζ -function

$$\zeta(T; s) := \sum_{\lambda \in \text{spec}(T) \setminus \{0\}} \lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}((I - \Pi_{\ker T})e^{-tT}) dt \tag{2-3}$$

is a holomorphic function in the half plane $\Re s > p$; $\Pi_{\ker T}$ denotes the orthogonal projection onto $\ker T$.

Definition 2.1. Following [Connes and Moscovici 1995] we say that T has *discrete dimension spectrum* if $\zeta(T; s)$ extends meromorphically to the complex plane \mathbb{C} such that on finite vertical strips $|\Gamma(s)\zeta(T; s)| = O(|s|^{-N})$, $|\Im s| \rightarrow \infty$, for each N . Denote by $\Sigma(T)$ the set of poles of the function $\Gamma(s)\zeta(T; s)$.

It then follows that for fixed real numbers $a < b$ there are only finitely many poles in the strip $a < \Re s < b$. Moreover, as explained, e.g., in [Brüning and Lesch 1999, Section 2], the discrete dimension spectrum condition is equivalent to the existence of an asymptotic expansion

$$\text{Tr}(e^{-tT}) \sim_{t \rightarrow 0+} \sum_{\substack{\alpha \in -\Sigma \\ 0 \leq k \leq k(\alpha)}} a_{\alpha k} t^\alpha \log^k t. \tag{2-4}$$

Furthermore, there is the following simple relation between the coefficients of the asymptotic expansion and the principal parts of the Laurent expansion at the poles of $\Gamma(s)\zeta(T; s)$:

$$\Gamma(s)\zeta(T; s) \sim \sum_{\substack{\alpha \in -\Sigma \\ 0 \leq k \leq k(\alpha)}} \frac{a_{\alpha k} (-1)^k k!}{(s + \alpha)^{k+1}} - \frac{\dim \ker T}{s}. \tag{2-5}$$

2A. Hilbert complexes and the analytic torsion. We use the convenient language of Hilbert complexes as outlined in [Brüning and Lesch 1992]. Recall that a Hilbert complex (\mathcal{D}, D) consists of a sequence of Hilbert spaces H_j , $0 \leq j \leq N$, together with closed operators D_j mapping a dense linear subspace $\mathcal{D}_j \subset H_j$ into H_{j+1} . The complex property means that actually $\text{ran } D_j \subset \mathcal{D}_{j+1}$ and $D_{j+1} \circ D_j = 0$. We say that a Hilbert complex has discrete dimension spectrum if all its Laplace operators $\Delta_j = D_j^* D_j + D_{j-1} D_{j-1}^*$ do have discrete dimension spectrum in the sense of Definition 2.1. Note that since Δ_j has compact resolvent, (\mathcal{D}, D) is automatically a Fredholm complex, by [loc. cit., Theorem 2.4]. For a Hilbert complex (\mathcal{D}, D) which is Fredholm, the finite-dimensional cohomology group $H^j(\mathcal{D}, D) = \ker D_j / \text{ran } D_{j-1}$ is

the quotient space of the Hilbert space $\ker D_j$ by the closed subspace $\text{ran } D_{j-1}$ and therefore is naturally equipped with a Hilbert space structure. From the Hodge decomposition [Brüning and Lesch 1992, Corollary 2.5]

$$H_j = \ker D_j \cap \ker D_{j-1}^* \oplus \text{ran } D_{j-1} \oplus \text{ran } D_j^* = \ker \Delta_j \oplus \text{ran } D_{j-1} \oplus \text{ran } D_j^*, \quad (2-6)$$

one then sees that the natural isomorphism $\hat{H}^j(\mathcal{D}, D) := \ker \Delta_j = \ker D_j \cap \ker D_{j-1}^* \rightarrow H^j(\mathcal{D}, D)$ is an isometric isomorphism. We will always tacitly assume that the cohomology groups are equipped with this natural Hilbert space structure.

Recall the *Euler characteristic*

$$\chi(\mathcal{D}, D) := \sum_{j \geq 0} (-1)^j \dim H^j(\mathcal{D}, D) = \sum_{j \geq 0} (-1)^j \dim \ker \Delta_j. \quad (2-7)$$

The discrete dimension spectrum assumption implies the validity of the *McKean–Singer formula*

$$\chi(\mathcal{D}, D) = \sum_{j \geq 0} (-1)^j \text{Tr}(e^{-t\Delta_j}) \quad \text{for } t > 0. \quad (2-8)$$

Definition 2.2. Let (\mathcal{D}, D) be a Hilbert complex with discrete dimension spectrum. The *analytic torsion* of (\mathcal{D}, D) is defined by

$$\log T(\mathcal{D}, D) := \frac{1}{2} \sum_{j \geq 0} (-1)^j j \left. \frac{d}{ds} \right|_{s=0} \zeta(\Delta_j; s).$$

If $\zeta(\Delta_j; s)$ has a pole at $s = 0$ then by $\left. \frac{d}{ds} \right|_{s=0} \zeta(\Delta_j; s)$ we understand the coefficient of s in the Laurent expansion at 0.

Obviously $\log T(\mathcal{D}, D)$ can be defined under the weaker assumption that the function

$$F(\mathcal{D}, D; s) := \frac{1}{2} \sum_{j \geq 0} (-1)^j j \zeta(\Delta_j; s) \quad (2-9)$$

extends meromorphically to \mathbb{C} .

The analytic torsion can also be expressed in terms of the *closed* and *coclosed* Laplacians: put

$$\Delta_{j,\text{cl}} := \Delta_j \upharpoonright \text{ran } D_{j-1} = D_{j-1} D_{j-1}^* \upharpoonright \text{ran } D_{j-1}, \quad (2-10)$$

$$\Delta_{j,\text{ccl}} := \Delta_j \upharpoonright \text{ran } D_j^* = D_j^* D_j \upharpoonright \text{ran } D_j^*. \quad (2-11)$$

Note that by definition $\Delta_{0,\text{ccl}} = 0$ and $\Delta_{N,\text{cl}} = 0$ act on the trivial Hilbert space $\{0\}$; recall that N is the length of the Hilbert complex. By the Hodge decomposition (2-6) the operators $\Delta_{j,\text{cl}}$ and $\Delta_{j,\text{ccl}}$ are invertible. Moreover,

$$\Delta_{j+1,\text{cl}} D_j \upharpoonright \text{ran } D_j^* = D_j \Delta_{j,\text{ccl}}. \quad (2-12)$$

Hence the eigenvalues of $\Delta_{j,\text{ccl}}$ and $\Delta_{j+1,\text{cl}}$ coincide including multiplicities. Putting for the moment $A_j := \text{Tr}(e^{-t\Delta_{j,\text{cl}}}) = \text{Tr}(e^{-t\Delta_{j-1,\text{ccl}}})$ for $j \geq 1$ and $A_0 := 0$ we therefore have

$$\text{Tr}(e^{-t\Delta_j}) - \dim H^j(\mathcal{D}, D) = \text{Tr}(e^{-t\Delta_{j,\text{cl}}}) + \text{Tr}(e^{-t\Delta_{j,\text{ccl}}}) = A_j + A_{j+1}, \quad (2-13)$$

and hence

$$\begin{aligned} \sum_{j \geq 0} (-1)^j j (\text{Tr}(e^{-t\Delta_j}) - \dim H^j(\mathcal{D}, D)) \\ = \sum_{j \geq 0} (-1)^j j (A_j + A_{j+1}) = \sum_{j \geq 0} (-1)^j j A_j - \sum_{j \geq 0} (-1)^j (j-1) A_j \\ = \sum_{j \geq 0} (-1)^j \text{Tr}(e^{-t\Delta_{j,\text{cl}}}) = - \sum_{j \geq 0} (-1)^j \text{Tr}(e^{-t\Delta_{j,\text{ccl}}}). \end{aligned} \tag{2-14}$$

To avoid cumbersome distinction of cases we understand that $\text{Tr}(e^{-t\Delta_{0,\text{ccl}}}) = 0$.

Proposition 2.3. *Let (\mathcal{D}', D') , (\mathcal{D}'', D'') be two Hilbert complexes with discrete dimension spectrum. Let $(\mathcal{D}, D) := (\mathcal{D}', D') \hat{\otimes} (\mathcal{D}'', D'')$ be their tensor product. Denote by $\Delta', \Delta'', \Delta$ the Laplacians of (\mathcal{D}', D') , (\mathcal{D}'', D'') , (\mathcal{D}, D) , respectively.*

Then the function $F(\mathcal{D}, D; s) := \frac{1}{2} \sum_{j \geq 0} (-1)^j j \zeta(\Delta_j; s)$ extends meromorphically to \mathbb{C} . More precisely, in terms of the corresponding function for the complexes (\mathcal{D}', D') , (\mathcal{D}'', D'') we have the equations

$$\chi(\mathcal{D}, D) = \chi(\mathcal{D}', D') \cdot \chi(\mathcal{D}'', D''), \tag{2-15}$$

$$F(\mathcal{D}, D; s) = \chi(\mathcal{D}', D') \cdot F(\mathcal{D}'', D''; s) + \chi(\mathcal{D}'', D'') \cdot F(\mathcal{D}', D'; s); \tag{2-16}$$

in particular

$$\log T(\mathcal{D}, D) = \chi(\mathcal{D}', D') \cdot \log T(\mathcal{D}'', D'') + \chi(\mathcal{D}'', D'') \cdot \log T(\mathcal{D}', D'). \tag{2-17}$$

Proof. This is an elementary calculation; compare [Vishik 1995, Proposition 2.1] and [Ray and Singer 1971, Theorem 2.5]. Since

$$\Delta_k = \bigoplus_{i+j=k} \Delta'_i \otimes I + I \otimes \Delta''_j,$$

we have

$$\ker(\Delta_k - \lambda) = \bigoplus_{\lambda' + \lambda'' = \lambda} \bigoplus_{i+j=k} \ker(\Delta'_i - \lambda') \otimes \ker(\Delta''_j - \lambda''). \tag{2-18}$$

This proves (2-15), which follows also from the Künneth theorem for Hilbert complexes [Brüning and Lesch 1992, Corollary 2.15]. Furthermore,

$$\begin{aligned} \sum_{k \geq 0} (-1)^k k \text{Tr} e^{-t\Delta_k} \\ = \sum_{k \geq 0} (-1)^k k \sum_{i+j=k} \sum_{\substack{\lambda' \in \text{spec} \Delta'_i \\ \lambda'' \in \text{spec} \Delta''_j}} e^{-t\lambda'} e^{-t\lambda''} = \sum_{i,j \geq 0} (-1)^{i+j} (i+j) \sum_{\substack{\lambda' \in \text{spec} \Delta'_i \\ \lambda'' \in \text{spec} \Delta''_j}} e^{-t\lambda'} e^{-t\lambda''} \\ = \left(\sum_{i \geq 0} (-1)^i \text{Tr} e^{-t\Delta'_i} \right) \left(\sum_{j \geq 0} (-1)^j j \text{Tr} e^{-t\Delta''_j} \right) + \left(\sum_{j \geq 0} (-1)^j \text{Tr} e^{-t\Delta''_j} \right) \left(\sum_{i \geq 0} (-1)^i i \text{Tr} e^{-t\Delta'_i} \right). \end{aligned} \tag{2-19}$$

The claim now follows from (2-3) and the McKean–Singer formula (2-8) applied to Δ'_i, Δ''_j . □

Next we state an abstract differentiability result (compare [Dai and Freed 1994, Appendix; Bohn 2009, Appendix D]).

Proposition 2.4. *Let $(\mathcal{D}^\theta, D^\theta)$, where $\theta \in J \subset \mathbb{R}$, be a one-parameter family of Hilbert complexes with discrete dimension spectrum. Let $\Delta_j^\theta = (D_j^\theta)^* D_j^\theta + D_{j-1}^\theta (D_{j-1}^\theta)^*$ be the corresponding Laplacians. Assume that*

(1) $H_T(\mathcal{D}^\theta, D^\theta)(t) = \sum_{j \geq 0} (-1)^j j \operatorname{Tr}(e^{-t\Delta_j^\theta})$ is differentiable in $(t, \theta) \in (0, \infty) \times J$ and

$$\frac{d}{d\theta} H_T(\mathcal{D}^\theta, D^\theta)(t) = t \frac{d}{dt} \operatorname{Tr}(P e^{-t\Delta^\theta}) \quad (2-20)$$

with some operator P in $H = \bigoplus_{j \geq 0} H_j$ with $P(I + \Delta^\theta)^{-N}$ bounded for some N ;

(2) Δ^θ is a graph smooth family of self-adjoint operators with constant domain and $\dim \ker \Delta^\theta$ independent of θ ;

(3) there is an asymptotic expansion

$$\operatorname{Tr}(P e^{-t\Delta^\theta}) \sim_{t \rightarrow 0^+} \sum_{\substack{\alpha \in -\Sigma \\ 0 \leq k \leq k(\alpha)}} a_{\alpha k}^\theta t^\alpha \log^k t, \quad (2-21)$$

which is locally uniformly in θ and with $a_{\alpha k}^\theta$ depending smoothly on θ ;

(4) $a_{0k}^\theta = 0$ for $k > 0$; that is, in the asymptotic expansion (2-21) there are no terms of the form $t^0 \log^k t$ for $k > 0$.

Then $\theta \mapsto \log T(\mathcal{D}^\theta, D^\theta)$ is differentiable and

$$\frac{d}{d\theta} \log T(\mathcal{D}^\theta, D^\theta) = -\frac{1}{2} \operatorname{LIM}_{t \rightarrow 0^+} \operatorname{Tr}(P e^{-t\Delta^\theta}) + \frac{1}{2} \operatorname{LIM}_{t \rightarrow \infty} \operatorname{Tr}(P e^{-t\Delta^\theta}) = -\frac{1}{2} a_{00}^\theta + \frac{1}{2} \operatorname{Tr}(P \upharpoonright \ker \Delta^\theta).$$

Here $\operatorname{LIM}_{t \rightarrow a}$ stands, as usual, for the constant term in the asymptotic expansion as $t \rightarrow a$. In (1) we have used the abbreviation $\Delta^\theta := \bigoplus_{j \geq 0} \Delta_j^\theta$.

Proof. Assumptions (2) and (3) of Proposition 2.4 guarantee that in the following we may interchange differentiation by s and by θ :

$$\begin{aligned} 2 \frac{d}{d\theta} \log T(\mathcal{D}^\theta, D^\theta) &= \frac{d}{d\theta} \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{j \geq 0} (-1)^j j \operatorname{Tr}(e^{-t\Delta_j^\theta} - \Pi_{\ker \Delta_j^\theta}) dt \\ &= \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \operatorname{Tr}(P e^{-t\Delta^\theta}) dt \\ &= -\frac{d}{ds} \Big|_{s=0} \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(P e^{-t\Delta^\theta}) dt \\ &= -\frac{d}{ds} \Big|_{s=0} \frac{s}{\Gamma(s)} \left[\left(\frac{a_{00}^\theta}{s} + c_0^\theta + c_1^\theta s + \dots \right) - \frac{\operatorname{Tr}(P \upharpoonright \ker \Delta^\theta)}{s} \right] \\ &= -a_{00}^\theta + \operatorname{Tr}(P \upharpoonright \ker \Delta^\theta). \end{aligned} \quad (2-22)$$

Assumption (1) was used in the second equality and assumptions (3), (4) were used in the penultimate equality. Without assumption (4) the higher derivatives of the function $1/\Gamma(s)$ at $s = 0$ would cause additional terms. Assumption (2) guarantees in particular that $\text{Tr}(\Pi_{\ker \Delta_j^\theta})$ is independent of θ . \square

2B. Torsion of a finite-dimensional Hilbert complex. This subsection mainly serves the purpose of fixing some notation. Let H_1, H_2 be finite-dimensional Hilbert spaces. For a linear map $T : H_1 \rightarrow H_2$ we put

$$\text{Det}(T) := \det(T^*T)^{1/2}. \tag{2-23}$$

If $T : H_1 \rightarrow H_2, S : H_2 \rightarrow H_3$ are linear maps then obviously $\text{Det}(TS) = \text{Det}(T) \text{Det}(S)$. Furthermore, given orthogonal decompositions $H_j = H_j^{(1)} \oplus H_j^{(2)}, j = 1, 2$, such that with respect to these decompositions we have

$$T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}, \tag{2-24}$$

then $\text{Det}(T) = \text{Det}(T_1) \text{Det}(T_2)$.

Let $0 \rightarrow C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} C^n \rightarrow 0$ be a finite-dimensional Hilbert complex. Then the torsion of this complex satisfies

$$\log T(C^*, d) = \sum_{p \geq 0} (-1)^p \log \text{Det}(d_p : \ker d_p^\perp \rightarrow \text{im } d_p) =: \log \tau(C^*, d). \tag{2-25}$$

Needless to say each finite-dimensional Hilbert complex is automatically a Hilbert complex with discrete dimension spectrum. In fact, since the zeta function is entire in this case, for the Laplacian of the complex the set $\Sigma(\Delta)$ defined in Definition 2.1 then equals the set of poles of the Γ -function, $\{0, -1, -2, \dots\}$.

The following two standard results about the torsion and the determinant will be needed at several places. The first one is elementary; the second one is due to Milnor [1966].

Lemma 2.5. *Let $(C_k^*, d^k), k = 1, 2$, be finite-dimensional Hilbert complexes and $\alpha : (C_1^*, d^1) \rightarrow (C_2^*, d^2)$ be a chain isomorphism. Then*

$$\begin{aligned} \log \tau(C_1^*, d^1) &= \log \tau(C_2^*, d^2) + \sum_{j \geq 0} (-1)^j \log \text{Det}(\alpha_j : C_1^j \rightarrow C_2^j) \\ &\quad - \sum_{j \geq 0} (-1)^j \log \text{Det}(\alpha_{j,*} : H^j(C_1^*, d^1) \rightarrow H^j(C_2^*, d^2)). \end{aligned} \tag{2-26}$$

Proof. For complexes of length 2 the formula follows directly from (2-24). Then one proceeds by induction on the length of the complexes C_1, C_2 . We omit the elementary but a little tedious details. \square

Proposition 2.6 [Milnor 1966, Theorem 3.1/3.2]. *Let $0 \rightarrow C_1 \xrightarrow{\alpha} C \xrightarrow{\beta} C_2 \rightarrow 0$ be an exact sequence of finite-dimensional Hilbert complexes and let*

$$\mathcal{H} : 0 \rightarrow H^0(C_1) \xrightarrow{\alpha_*} H^0(C) \xrightarrow{\beta_*} H^0(C_2) \xrightarrow{\delta} H^1(C_1) \rightarrow \dots \tag{2-27}$$

be their long exact cohomology sequence. Then

$$\log \tau(C^*, d) = \log \tau(C_1^*, d^1) + \log \tau(C_2^*, d^2) + \log \tau(\mathcal{H}) - \sum_{j \geq 0} (-1)^j \log \tau(0 \rightarrow C_1^j \xrightarrow{\alpha} C^j \xrightarrow{\beta} C_2^j \rightarrow 0). \quad (2-28)$$

In fact the proposition as stated is a combination of [Milnor 1966, Theorem 3.2] and Lemma 2.5. The last term in (2-28) does not appear in [Milnor 1966, Theorem 3.2] since there one is given preferred bases of C_1, C, C_2 which are compatible. In our Hilbert complex setting the preferred bases are the orthonormal ones. The last term in (2-28) makes up for the fact that in general it is not possible to choose orthonormal bases of C_1, C, C_2 which are compatible in the sense of [loc. cit.]. For a proof in the more general von Neumann setting see [Burghilea et al. 1999, Theorem 1.14].

For future reference we note that for the acyclic complex $(0 \rightarrow C_1^j \xrightarrow{\alpha} C^j \xrightarrow{\beta} C_2^j \rightarrow 0)$ of length 2 on the right of (2-28) it follows from the definition (2-25) that

$$\log \tau(0 \rightarrow C_1^j \xrightarrow{\alpha} C^j \xrightarrow{\beta} C_2^j \rightarrow 0) = \frac{1}{2} \log \text{Det}(C_1^j \xrightarrow{\alpha^* \alpha} C_1^j) - \frac{1}{2} \log \text{Det}(C_2^j \xrightarrow{\beta \beta^*} C_2^j). \quad (2-29)$$

Finally, we remind the reader of the (trivial) fact that if in Proposition 2.6 the complex C equals $C_1 \oplus C_2$, α the inclusion and β the projection onto the second summand, then $\log \tau(\mathcal{H}) = 0$ and $\log \tau(C^*, d) = \log \tau(C_1^*, d^1) + \log \tau(C_2^*, d^2)$.

3. Elementary operator gluing and heat kernel estimates on noncompact manifolds

3A. Standing assumptions. Let M^m be a Riemannian manifold of dimension m ; it is essential to note that M^m is not necessarily complete; see Figure 1. Furthermore, let $P_0 : \Gamma_c^\infty(M, E) \rightarrow \Gamma_c^\infty(M, E)$ be a second-order formally self-adjoint elliptic differential operator acting on the compactly supported sections, $\Gamma_c^\infty(M, E)$, of the Hermitian vector bundle E . We assume that P_0 is bounded below and we fix

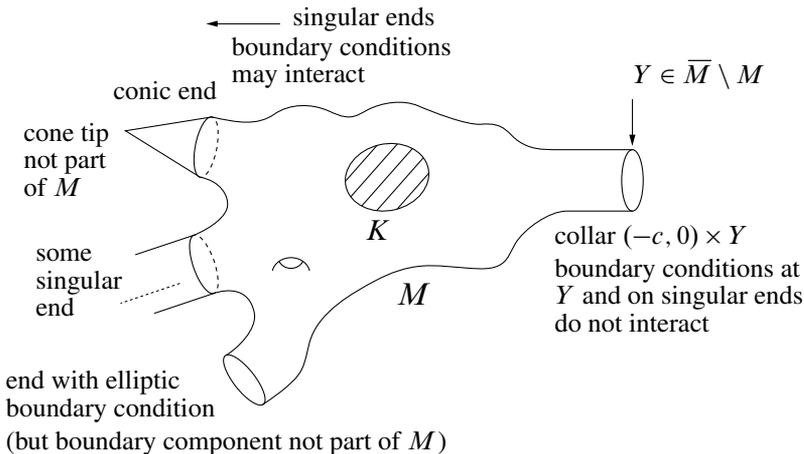


Figure 1. Example of a singular manifold.

once and for all a bounded below self-adjoint extension P of P_0 in the Hilbert space of square-integrable sections $L^2(M, E)$, for example, the Friedrichs extension.

Later on we will need a class of operators which is slightly more general than (pseudo)differential operators. For our purposes it will suffice to consider an auxiliary operator

$$V : H_{\text{loc}}^s(M, E) \rightarrow H_{\text{comp}}^{s-1}(M, E), \tag{3-1}$$

which for each real s maps the space $H_{\text{loc}}^s(M, E)$ of sections, which are locally of Sobolev class s , continuously into the space of compactly supported sections of Sobolev class $s - 1$; see [Shubin 2001, Section I.7]. We assume that V is symmetric with respect to the L^2 -scalar product on E ; that is, $\langle Vf, g \rangle = \langle f, Vg \rangle$ for $f \in H_{\text{loc}}^1(M, E)$, $g \in L_{\text{loc}}^2(M, E)$.

Finally, we assume that V is confined to a compact subset $\mathcal{H} \subset M$ in the sense that

$$M_\varphi V = VM_\varphi = 0, \tag{3-2}$$

for any smooth function vanishing in a neighborhood of \mathcal{H} . Equation (3-2) implies that V commutes with M_φ for any smooth function which is constant in a neighborhood of \mathcal{H} . Our main example is the operator $\tilde{\Delta}^\theta$ defined after (5-8) below.

In view of (3-1) and the ellipticity of P_0 , the operator V is P -bounded with arbitrarily small bound; thus $P + V$ is self-adjoint and bounded below as well.

With regard to the mapping property (3-1) of V we introduce the space $\text{Op}_c^a(M, E)$ of linear operators A mapping H_{loc}^s continuously into H_{comp}^{s-a} and whose Schwartz kernel K_A is compactly supported. Obvious examples are pseudodifferential operators with compactly supported Schwartz kernel, but also certain Fourier integral operators. The point is that elements in Op_c are not necessarily pseudolocal. Note that V is in $\text{Op}_c^1(M, E)$.

The set-up outlined in this Section 3A will be in effect during the remainder of Section 3.

3B. Heat kernel estimates for $P + V$.

Lemma 3.1. *For all $s \geq 0$ we have $\mathcal{D}(P + V)^s = \mathcal{D}(P^s)$. Furthermore, the operator $e^{-t(P+V)}$, $t > 0$, has a smooth integral kernel.*

Proof. By complex interpolation [Taylor 1996, Section 4.2] it suffices to prove the first claim for $s = k \in \mathbb{N}$ where it follows easily by induction exploiting the elliptic regularity for P and (3-1).

Consequently, $e^{-t(P+V)}$ is a self-adjoint operator which maps $L^2(M, E)$ into

$$\bigcap_{k \geq 0} \mathcal{D}((P + V)^k) = \bigcap_{k \geq 0} \mathcal{D}(P^k), \tag{3-3}$$

and the latter is contained in $\Gamma^\infty(M, E)$ by elliptic regularity. This implies smoothness of the kernel of $e^{-t(P+V)}$. □

Proposition 3.2. *Let $A \in \text{Op}_c^a(M, E)$, $B \in \text{Op}_c^b(M, E)$ with compactly supported Schwartz kernels K_A, K_B . Denote by $\pi_j : M \times M \rightarrow M$, $j = 1, 2$, the projections onto the first and second factor and suppose that $\pi_2(\text{supp } K_A) \cap \pi_1(\text{supp } K_B) = \emptyset$ and $\pi_2(\text{supp } K_A) \cap \mathcal{H} = \emptyset$. (For \mathcal{H} , see Section 3A.)*

Then $Ae^{-t(P+V)}B$ is a trace class operator and

$$\|Ae^{-t(P+V)}B\|_{\text{tr}} = O(t^\infty), \quad t \rightarrow 0+. \tag{3-4}$$

Here $O(t^\infty)$ is an abbreviation for $O(t^N)$ for any N ; the O -constant may depend on N . Furthermore, $\|\cdot\|_{\text{tr}}$ denotes the trace norm on the Schatten ideal of trace class operators.

Proof. (Compare [Lesch 1997, Section I.4].) Since the Schwartz kernels are compactly supported it suffices to prove that for all real α, β and all $N > 0$ we have

$$\|Ae^{-t(P+V)}B\|_{\alpha,\beta} = O(t^N), \quad t \rightarrow 0+. \tag{3-5}$$

Here, $\|\cdot\|_{\alpha,\beta}$ stands for the mapping norm between the Sobolev spaces

$$H^\alpha(\pi_2(\text{supp } K_B), E) \quad \text{and} \quad H^\beta(\pi_1(\text{supp } K_A), E).$$

The O -constant may depend on A, B, α, β, N .

Equation (3-5) follows from Duhamel’s formula by a standard bootstrapping argument as follows: note first that the mapping properties of A, B and $P + V$ imply that, for real α ,

$$\|Ae^{-t(P+V)}B\|_{\alpha,\alpha-a-b} = O(1), \quad t \rightarrow 0+. \tag{3-6}$$

Assume by induction that, for fixed l, N , for all A, B satisfying our assumptions and for all real α ,

$$\|Ae^{-t(P+V)}B\|_{\alpha,\alpha-a-b+l} = O(t^N), \quad t \rightarrow 0+. \tag{3-7}$$

Fix plateau functions $\chi, \varphi, \psi \in C_c^\infty(M)$ with the following properties:

- (1) $\varphi \equiv 1$ in a neighborhood of $\pi_2(\text{supp } K_A)$ and $\text{supp } \varphi \cap \mathcal{K} = \emptyset$.
- (2) $\psi \equiv 1$ in a neighborhood of $\pi_1(\text{supp } K_B)$.
- (3) $\chi \equiv 1$ in a neighborhood of $\text{supp } \varphi$ and $\text{supp } \chi \cap \mathcal{K} = \emptyset$.
- (4) $\text{supp } \chi \cap \text{supp } \psi = \emptyset$.

Then

$$\begin{aligned} \|Ae^{-t(P+V)}B\|_{\alpha,\alpha-a-b+l+1/2} &= \|A\varphi e^{-t(P+V)}\psi B\|_{\alpha,\alpha-a-b+l+1/2} \\ &\leq C_1 \|\varphi e^{-t(P+V)}\psi\|_{\alpha-b,\alpha-b+l+1/2}. \end{aligned} \tag{3-8}$$

From

$$(\partial_t + P + V)\varphi e^{-t(P+V)}\psi = \chi[P_0, \varphi]e^{-t(P+V)}\psi, \tag{3-9}$$

where $[P_0, \varphi]$ denotes the commutator between the differential expression P_0 and multiplication by φ , we infer

$$\varphi e^{-t(P+V)}\psi = \int_0^t \chi e^{-(t-s)(P+V)} \chi[P_0, \varphi] e^{-s(P+V)} \psi \, ds; \tag{3-10}$$

here we have used the assumptions on the supports of χ, ψ, φ and (3-2). In the displayed formulas we wrote, to save some space, χ, ψ, φ for the multiplication operators $M_\chi, M_\psi, M_\varphi$.

For $\tilde{\alpha} = \alpha - b$ we now find

$$\|\varphi e^{-t(P+V)}\psi\|_{\tilde{\alpha}, \tilde{\alpha}+l+\frac{1}{2}} \leq \int_0^t \|\chi e^{-(t-s)(P+V)}\chi\|_{\tilde{\alpha}-1+l, \tilde{\alpha}+l+\frac{1}{2}} \| [P_0, \varphi] e^{-s(P+V)}\psi\|_{\tilde{\alpha}, \tilde{\alpha}-1+l} ds. \tag{3-11}$$

Since $[P_0, \varphi]$ is in Op_c^1 we find, using (3-7),

$$\|[P_0, \varphi]e^{-s(P+V)}\psi\|_{\tilde{\alpha}, \tilde{\alpha}-1+l} = O(s^N) \quad \text{as } s \rightarrow 0+. \tag{3-12}$$

Furthermore, denoting by C a constant such that $P \geq -C + 1$,

$$\begin{aligned} &\|\chi e^{-u(P+V)}\chi\|_{\tilde{\alpha}-1+l, \tilde{\alpha}+l+\frac{1}{2}} \\ &\leq \|(P+V+C)^{\frac{\tilde{\alpha}+l-1}{2}}\chi\|_{\tilde{\alpha}-1+l, 0} \|(P+V+C)^{\frac{3}{4}}e^{-u(P+V)}\|_{0,0} \|\chi(P+V+C)^{-\frac{\tilde{\alpha}+l+1/2}{2}}\|_{0, \tilde{\alpha}+l+\frac{1}{2}}. \end{aligned} \tag{3-13}$$

The first and the third factors on the right are bounded while for the second factor we have, by the spectral theorem,

$$\|(P + V + C)^{3/4}e^{-u(P+V)}\|_{0,0} = O(u^{-3/4}) \quad \text{as } u \rightarrow 0+. \tag{3-14}$$

Thus

$$\|\varphi e^{-t(P+V)}\psi\|_{\tilde{\alpha}, \tilde{\alpha}+l+1/2} \leq C_1 \int_0^t (t-s)^{-3/4} s^N ds = O(t^{N+1/4}), \quad t \rightarrow 0+. \tag{3-15}$$

Thus we have improved the parameters l and N in (3-7) by $\frac{1}{2}$ and $\frac{1}{4}$, respectively, and therefore the result follows by induction. □

Proposition 3.3. *Under the standing assumptions of Section 3A, let $\varphi, \psi \in C^\infty(M)$ with $\text{supp } \varphi \cap \text{supp } \psi$ being compact (the individual supports of φ or ψ may be noncompact!) such that $d\varphi, d\psi$ are compactly supported and that $\text{supp } d\varphi \cap \mathcal{K} = \emptyset = \text{supp } d\psi \cap \mathcal{K}$. Furthermore, assume that multiplication by φ and by ψ preserves $\mathcal{D}(P + V) = \mathcal{D}(P)$.*

Then for $t > 0$ the operator $\varphi e^{-t(P+V)}\psi$ is trace class and

$$\|\varphi e^{-t(P+V)}\psi\|_{\text{tr}} = O(t^{-m/2-0}), \quad t \rightarrow 0+. \tag{3-16}$$

If $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$ then the right-hand side can be improved to $O(t^\infty)$, $t \rightarrow 0+$.

Here $O(t^{-m/2-0})$ is an abbreviation for $O(t^{-m/2-\epsilon})$ for any $\epsilon > 0$; the O -constant may depend on ϵ .

Proof. Assume first that additionally ψ is compactly supported. Again applying Duhamel we find

$$\varphi e^{-t(P+V)}\psi = \int_0^t e^{-(t-s)(P+V)} [P_0, \varphi] e^{-s(P+V)}\psi ds. \tag{3-17}$$

Now apply Lemma 3.1 and Proposition 3.2 to the operator $[P_0, \varphi]e^{-s(P+V)}\psi$. If $\text{supp } \varphi \cap \text{supp } \psi \neq \emptyset$ then the trace norm estimate is a simple consequence of Sobolev embedding and the established mapping

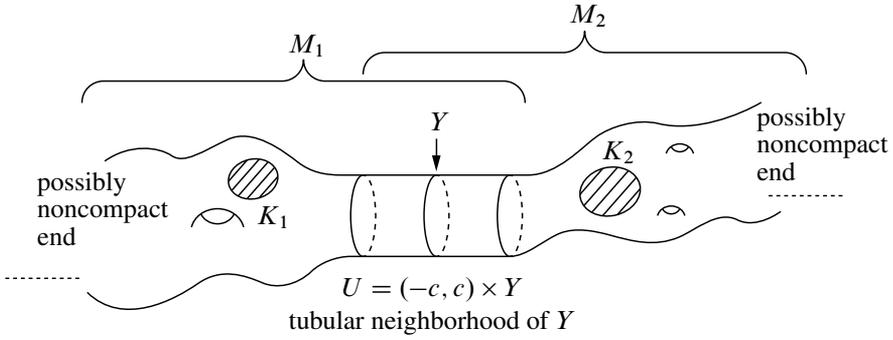


Figure 2. The gluing situation.

properties. If $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$ then Proposition 3.2 implies $\| [P_0, \varphi] e^{-s(P+V)} \psi \|_{\text{tr}} = O(t^\infty)$ and the claim follows in this case.

Since $e^{-t(P+V)}$ is self-adjoint the roles of φ, ψ may be interchanged by taking adjoints and hence the proposition is proved if φ or ψ is compactly supported. The general case now follows from (3-17) since the compactness of $\text{supp } d\varphi$ implies the compactness of the support of the Schwartz kernel of $[P_0, \varphi]$. \square

3C. Operator gluing. Now we assume that we have two triples $(M_j, P_j^0, V_j), j = 1, 2$, consisting of Riemannian manifolds M_j^m and operators P_j^0, V_j satisfying the standing assumptions of Section 3A.

Furthermore, we assume that each M_j is the interior of a manifold \bar{M}_j with compact boundary Y (it is essential that \bar{M}_j is not necessarily compact). Let $U = Y \times (-c, c)$ be a common collar of Y in M_1 and in M_2 , such that $\partial M_1 = Y \times \{1\}$ and $\partial M_2 = Y \times \{-1\}$.

We assume that the sets \mathcal{K}_j corresponding to V_j (cf. (3-2)) lie in $M_j \setminus U$ and that P_1^0 coincides with P_2^0 over U . Then the P_j^0 gives rise naturally to a differential operator $P^0 = P_1^0 \cup P_2^0$ on

$$M := (M_1 \setminus (Y \times (0, c))) \cup_{Y \times \{0\}} (M_2 \setminus (Y \times (-c, 0))),$$

and the V_j to an operator $V = V_1 + V_2 \in \text{Op}_c^1(M, E)$, where E is the bundle obtained by gluing the bundles E_1 and E_2 in the obvious way (due to (3-2) the operators V_1, V_2 extend to M in a natural way).

Definition 3.4. By $C_U^\infty(M_j)$ we denote the space of those smooth functions $\varphi \in C^\infty(M_j)$ such that φ is constant in a neighborhood of $M_j \setminus U$ and $\varphi \equiv 0$ in a neighborhood of $\partial \bar{M}_j$; see Figure 3.

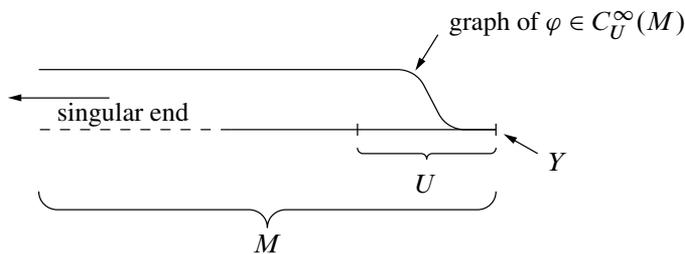


Figure 3. Schematic sketch of a function in $C_U^\infty(M)$. The line indicates the manifold M ; to the left are the possible noncomplete ends. On the right there is the collar U .

A function $\varphi \in C_U^\infty(M_j)$ extends by 0 to a smooth function on M .

Proposition 3.5. *Let $P_j, j = 1, 2$, be closed symmetric extensions of P_j^0 which are bounded below and for which*

$$\varphi \mathcal{D}(P_j^*) \subset \mathcal{D}(P_j) \quad \text{for all } \varphi \in C_U^\infty(M_j). \tag{3-18}$$

For a fixed pair of functions $\varphi_j \in C_U^\infty(M_j), j = 1, 2$, put

$$\mathcal{D}(P) := \{f \in \mathcal{D}(P_{\max}^0) \mid \varphi_j f \in \mathcal{D}(P_j), j = 1, 2\} = H_{\text{comp}}^2(U, E) + \varphi_1 \mathcal{D}(P_1) + \varphi_2 \mathcal{D}(P_2). \tag{3-19}$$

$\mathcal{D}(P)$ is indeed independent of the particular choice of φ_j and the operator P which is defined by restricting $P_{\max}^0 = (P^0)^*$ to $\mathcal{D}(P)$ is self-adjoint and bounded below. V is P -bounded with arbitrarily small bound and hence $P + V$ is self-adjoint and bounded below as well.

Furthermore, if for fixed $j \in \{1, 2\}$ we have $\varphi, \psi \in C_U^\infty(M_j)$ satisfying (3-18), then the operator $\varphi e^{-t(P_j+V)}\psi - \varphi e^{-t(P+V)}\psi$ is trace class and its trace norm is $O(t^\infty)$ as $t \rightarrow 0+$.

Remark 3.6. 1. It is not assumed that $\varphi e^{-t(P_j+V)}\psi$ or $\varphi e^{-t(P+V)}\psi$ is of trace class individually!

2. Equation (3-18) says that the “boundary conditions” at the exits of M_1 and M_2 are separated. We illustrate this with an example: let $M_1 = (-1, \frac{1}{2}), M_2 = (-\frac{1}{2}, 1), U = (-\frac{1}{2}, \frac{1}{2}), M = (-1, 1)$, and let $P_1^0 = P_2^0 = -d^2/dx^2 = \Delta$ be the Laplacian on functions. Let P_1^{per} be the Laplacian Δ on M_1 with periodic boundary conditions. These boundary conditions are not separated and indeed for $\varphi \in C^\infty(-1, \frac{1}{2})$ with $\varphi(x) = 1$ for $x \leq -\frac{1}{4}$ and $\varphi(x) = 0$ for $x \geq \frac{1}{4}$ the space $\varphi \mathcal{D}(P_1^{\text{per}})$ equals $\varphi H^2([-1, \frac{1}{2}])$ and this is not contained in $\mathcal{D}(P_1^{\text{per}})$.

However, for any pair of self-adjoint extensions P_j of $P_j^0, j = 1, 2$, with separated boundary conditions at the ends of the intervals M_j one has $\varphi \mathcal{D}(P_j) \subset \mathcal{D}(P_j)$; that is, the condition (3-18) is satisfied and Proposition 3.5 applies to this pair.

Proof. Since $H_{\text{comp}}^2(U, E) \subset \mathcal{D}(P_{j,\text{min}}^0)$ the second equality in (3-19), the symmetry of P and the independence of $\mathcal{D}(P)$ of the particular choice of φ_j are easy consequences of (3-18).

To prove self-adjointness let $f \in \mathcal{D}(P^*)$. We claim that for $\varphi_1 \in C_U^\infty(M_1)$ we have $\varphi_1 f \in \mathcal{D}(P_1^*)$. Indeed for $g \in \mathcal{D}(P_1)$ we have

$$\langle \varphi_1 f, P_1 g \rangle = \langle f, \bar{\varphi}_1 P_1 g \rangle = \langle f, [\bar{\varphi}_1, P_1^0]g \rangle + \langle f, P \bar{\varphi}_1 g \rangle. \tag{3-20}$$

Since $\text{supp } d\varphi_1 \subset U$ is compact and since $[\bar{\varphi}_1, P_1^0]$ is a compactly supported first-order differential operator on U we find

$$\dots = \langle [P_1^0, \varphi_1]f + \varphi_1 P^* f, g \rangle, \tag{3-21}$$

proving $\varphi_1 f \in \mathcal{D}(P_1^*)$. In view of (3-18) we see, by choosing another plateau function $\psi \in C_U^\infty(M_1)$ with $\psi \varphi_1 = \varphi_1$, that $\varphi_1 f \in \mathcal{D}(P_1)$. In the same way we conclude $\varphi_2 f \in \mathcal{D}(P_2)$ for $\varphi_2 \in C_U^\infty(M_2)$ and thus $f \in \mathcal{D}(P)$.

To prove the trace class property and the trace estimate we choose another plateau function $\chi \in C_U^\infty(M_j)$ such that $\chi \equiv 1$ in a neighborhood of $\text{supp } \psi$ with $\chi - \psi \in C_c^\infty(M_j)$; hence χ also satisfies (3-18).

Consider first $K_t := \chi e^{-t(P_j+V_j)}\psi - \chi e^{-t(P+V)}\psi$. $K_{t=0} = 0$ and

$$(\partial_t + P + V)K_t = [P_j^0, \chi]e^{-t(P_j+V_j)}\psi - [P^0, \chi]e^{-t(P+V)}\psi. \tag{3-22}$$

Here we have used that multiplication by χ commutes with V and V_j ; see (3-2). Propositions 3.2 and 3.3 now imply that K_t is trace class for $t > 0$ and that $\|K_t\|_{\text{tr}} = O(t^\infty)$ as $t \rightarrow 0+$. Consequently

$$\|\chi\varphi e^{-t(P_j+V)}\psi - \chi\varphi e^{-t(P+V)}\psi\|_{\text{tr}} \leq \|\varphi\|_\infty \|K_t\|_{\text{tr}} = O(t^\infty).$$

To $(1-\chi)\varphi e^{-t(P_j+V)}\psi - (1-\chi)\varphi e^{-t(P+V)}\psi$ we can apply Proposition 3.3 since $(\text{supp } \psi) \cap \text{supp}(1-\chi) = \emptyset$ and the proof is complete. \square

Finally, we discuss heat expansions. Under the assumptions of Proposition 3.5 assume that $P_j + V_j$ has *discrete dimension spectrum outside U* . By this we understand that for $\varphi \in C_U^\infty(M_j)$ the operator $\varphi e^{-t(P_j+V_j)}$ is trace class and that there is an asymptotic expansion of the form (2-4) with $a_{\alpha k} = a_{\alpha k}(\varphi)$. Then:

Corollary 3.7. *Under the additional assumption of discrete dimension spectrum for $P_j + V_j$ outside U the operator $P + V$ has discrete dimension spectrum and for any $\varphi \in C_U^\infty(M_1)$ we have*

$$\text{Tr}(e^{-t(P+V)}) = \text{Tr}(\varphi e^{-t(P_1+V_1)}) + \text{Tr}((1-\varphi)e^{-t(P_2+V_2)}) + O(t^\infty) \quad \text{as } t \rightarrow 0+. \tag{3-23}$$

Proof. This is immediate from Proposition 3.5 and the discrete dimension spectrum assumption. \square

We add, however, a little more explanation since the term “discrete dimension spectrum outside U ” might lead to some confusion: since $\mathcal{H} \cap U = \emptyset$ (cf. (3-2) and the second paragraph of this section, on page 236) for $f \in \Gamma_c^\infty(U, E)$ we have $(P + V)f = Pf$. The classical interior parametric elliptic calculus (see [Shubin 2001], for example) then implies that for $\varphi \in C_c^\infty(U)$ there is an asymptotic expansion

$$\text{Tr}(\varphi e^{-t(P+V)}) \sim_{t \searrow 0} \sum_{j \geq 0} a_j(P, \varphi) t^{j-m/2}, \tag{3-24}$$

where $a_j(P, \varphi) = \int_M \tilde{a}_j(x, P)\varphi(x) dx$ and $\tilde{a}_j(x, P)$ are the local heat invariants of P . Thus over any compact subset in the *interior* of $M \setminus \mathcal{H}$ the discrete dimension spectrum assumption follows from standard elliptic theory and hence is a nonissue. Rather it is a condition on the behavior of P on noncompact “ends” and a condition on V over \mathcal{H} .

3D. Ideal boundary conditions with discrete dimension spectrum. The remarks of Section 3C extend to ideal boundary conditions of elliptic complexes in a straightforward fashion. Let X be a Riemannian manifold which is the interior of a Riemannian manifold \bar{X} with compact boundary Y , and let $U = (-c, 0) \times Y$ be a collar of the boundary. Since \bar{X} is allowed to be noncompact it is not excluded that away from U there are “ends” of \bar{X} which can be completed by adding another boundary component; see Figure 1.

As an example which illustrates what can happen consider a compact manifold Z with boundary, where $\partial Z = Y_1 \cup Y_2 \cup Y_3$ consists of the disjoint union of three compact closed manifolds Y_j , $j = 1, 2, 3$. Attach a cone $C(Y_3) = Y_3 \times (0, 1)$ with metric $dr^2 + r^2 g_{Y_3}$ to Y_3 (and smooth it out near $Y_3 \times \{1\}$).

Then put $X := (Z \setminus (Y_1 \cup Y_2)) \cup_{Y_3} C(Y_3)$ and $\bar{X} := (Z \setminus Y_2) \cup_{Y_3} C(Y_3)$. Then Y_1 plays the role of Y above, but \bar{X} is not compact. Compare [Figure 1](#) on page [232](#).

When introducing closed extensions (that is to say, boundary conditions) for elliptic operators on X it is important that the boundary conditions at Y_1 and Y_2 on the one hand, and on the cone on the other, do *not* interact in order to ensure [\(3-18\)](#) holds.

Leaving this example behind, let $(\Gamma_c^\infty(E), d)$ be an elliptic complex and let (\mathcal{D}, D) be an ideal boundary condition for $(\Gamma_c^\infty(E), d)$ — that is, a Hilbert complex such that $D_j, j = 1, 2$, is a closed extension of d_j .

We say that the ideal boundary condition (\mathcal{D}, D) has discrete dimension spectrum outside U if the Laplacians $\Delta_j = D_j^* D_j + D_{j-1} D_{j-1}^*$ have discrete dimension spectrum outside U ; cf. the paragraph before [Corollary 3.7](#). Then [Proposition 3.5](#) and [Corollary 3.7](#) hold for the Laplacians.

More concretely, let X, Y be as before and let (F, ∇) be a flat bundle over \bar{X} . Assume that we are given an ideal boundary condition (\mathcal{D}, D) of the de Rham complex $(\Omega^\bullet(X; F), d)$ with values in the flat bundle F with discrete dimension spectrum over the open set $X \setminus U, U = (-c, 0) \times Y$. Fix a smooth function $\varphi \in C^\infty(-c, 0)$ which is 1 near $-c$ and 0 near 0 and extend it to a smooth function on X in the obvious way.

We then define the absolute and relative boundary conditions at Y as follows:

$$\mathcal{D}^j(X; F) := \varphi \mathcal{D}(D_j) + (1 - \varphi) \mathcal{D}(d_{j, \max}), \quad \mathcal{D}^j(X, Y; F) := \varphi \mathcal{D}(D_j) + (1 - \varphi) \mathcal{D}(d_{j, \min}). \quad (3-25)$$

The Laplacians of the maximal and minimal ideal boundary conditions are, near Y , realizations of local elliptic boundary conditions (see, e.g., [\[Gilkey 1995, Section 2.7\]](#)). This, together with [Proposition 3.5](#) and [Corollary 3.7](#) applied to $M_1 = X, M_2 = Y \times (-c, 0), U = Y \times (-c, -c/2)$, implies that the Hilbert complexes $(\mathcal{D}(X; F), d)$ and $(\mathcal{D}(X, Y; F), d)$ are Hilbert complexes with discrete dimension spectrum.

4. Vishik’s moving boundary conditions

4A. Standing assumptions. We discuss here Vishik’s [\[1995\]](#) moving boundary conditions for the de Rham complex in our slightly more general setting. Let X be a (not necessarily compact or complete!) Riemannian manifold; see [Figure 1](#). Furthermore, let (F, ∇) be a flat bundle with a (not necessarily flat) Hermitian metric h^F . We assume furthermore that X contains a compact separating hypersurface $Y \subset X$ such that in a collar neighborhood $W = (-c, c) \times Y$ all structures are product. In particular we assume that ∇^F is in temporal gauge on W ; that is, $\nabla^F \upharpoonright W = \pi^* \tilde{\nabla}^F$ for a flat connection $\tilde{\nabla}^F$ on $F \upharpoonright Y$, where π denotes the natural projection map $W \rightarrow Y$. In other words X is obtained by gluing two manifolds with boundary X^\pm along their common boundary Y where all structures are product near Y ; compare [Figure 2](#).

We make the fundamental assumption that

$$\begin{aligned} &\text{we are given ideal boundary conditions } (\mathcal{D}^\pm, D^\pm) \text{ of the twisted de Rham} \\ &\text{complexes } (\Omega^\bullet(X^{\circ, \pm}; F), d) \text{ which have discrete dimension spectrum over} \\ &U^\pm := X^\pm \setminus W. \text{ We put } X^{\text{cut}} := X^- \amalg X^+. \end{aligned} \quad (4-1)$$

4B. Some exact sequences and the main deformation result. As explained in [Section 3D](#) we therefore have the following Hilbert complexes with discrete dimension spectrum: $\mathcal{D}^\bullet(X^\pm; F)$ (absolute boundary condition at Y), $\mathcal{D}^\bullet(X^\pm, Y; F)$ (relative boundary condition at Y), $\mathcal{D}^\bullet(X; F)$ (continuous transmission condition at Y). By construction we have the exact sequences of Hilbert complexes

$$0 \longrightarrow \mathcal{D}^\bullet(X^-, Y; F) \xrightarrow{\alpha_-} \mathcal{D}^\bullet(X; F) \xrightarrow{\beta} \mathcal{D}^\bullet(X^+, F) \longrightarrow 0, \quad (4-2)$$

$$0 \longrightarrow \mathcal{D}^\bullet(X^\pm, Y; F) \xrightarrow{\gamma_\pm} \mathcal{D}^\bullet(X^\pm; F) \xrightarrow{i_\pm^*} \mathcal{D}^\bullet(Y; F) \longrightarrow 0, \quad (4-3)$$

$$0 \longrightarrow \mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+, Y; F) \xrightarrow{\alpha_+ + \alpha_-} \mathcal{D}^\bullet(X; F) \xrightarrow{r} \mathcal{D}^\bullet(Y; F) \longrightarrow 0. \quad (4-4)$$

Here α_\pm are extensions by 0, β is the pullback (i.e., restriction) to X^+ , γ_\pm is the natural inclusion of the complex $\mathcal{D}^\bullet(X^\pm, Y; F)$ with relative boundary condition at Y into the complex $\mathcal{D}^\bullet(X^\pm; F)$ with absolute boundary condition, and $i_\pm : Y \hookrightarrow X^\pm$ is the inclusion map. Finally $r\omega = \frac{\sqrt{2}}{2}(i_+^*\omega + i_-^*\omega) = \sqrt{2}i_\pm^*\omega$ for $\omega \in \mathcal{D}^\bullet(X; F)$.

It is a consequence of standard trace theorems for Sobolev spaces that $i_\pm^* : \mathcal{D}^\bullet(X^\pm; F) \rightarrow \mathcal{D}^\bullet(Y; F)$ is well-defined; see, for example, [\[Paquet 1982; Lions and Magenes 1972; Brüning and Lesch 2001, Section 1\]](#). To save some space we have omitted the operator D from the notation in the complexes in [\(4-2\)](#), [\(4-3\)](#), and [\(4-4\)](#). Clearly, the complex differential is always the exterior derivative on the indicated domains.

Each of the complexes [\(4-2\)](#)–[\(4-4\)](#) induces a long exact sequence in cohomology. We abbreviate these long exact cohomology sequences by

$$\mathcal{H}((X^-, Y), X, X^+; F), \quad \mathcal{H}((X^\pm, Y), X^\pm, Y; F), \quad \mathcal{H}((X^-, Y) \cup (X^+, Y), X, Y; F),$$

respectively. The long exact cohomology sequences of the complexes [\(4-2\)](#), [\(4-3\)](#), [\(4-4\)](#) are exact sequences of finite-dimensional *Hilbert spaces* and therefore their torsion $\tau(\mathcal{H}(\dots))$ is defined; cf. [\(2-25\)](#). The Euler characteristics — see [\(2-7\)](#) — of the complexes in [\(4-2\)](#)–[\(4-4\)](#) are denoted by $\chi(X^\pm, Y; F)$, $\chi(X^\pm; F)$, $\chi(X; F)$, $\chi(Y; F)$, etc.

Next we introduce parametrized versions of the exact sequences [\(4-2\)](#) and [\(4-4\)](#). The idea is due to Vishik [\[1995\]](#) who applied it to give a new proof of the Ray–Singer conjecture for compact smooth manifolds with boundary. Namely, for $\theta \in \mathbb{R}$ consider the following ideal boundary condition of the twisted de Rham complex on the disjoint union $X^{\text{cut}} = X^- \amalg X^+$:

$$\mathcal{D}_\theta^j(X; F) := \{(\omega_1, \omega_2) \in \mathcal{D}^j(X^-; F) \oplus \mathcal{D}^j(X^+; F) \mid \cos \theta \cdot i_-^*\omega_1 = \sin \theta \cdot i_+^*\omega_2\}. \quad (4-5)$$

We will see that for each real θ the complex $(\mathcal{D}_\theta^\bullet(X; F), d)$ is indeed a Hilbert complex with discrete dimension spectrum. In fact near Y it is a realization of a local elliptic boundary value problem for the de Rham complex on the manifold X^{cut} , and away from Y we may apply [Corollary 3.7](#) and our assumption [\(4-1\)](#) that the Hilbert complexes (\mathcal{D}^\pm, D^\pm) have discrete dimension spectrum over $X^\pm \setminus W$.

For $\theta = 0$ we have $\mathcal{D}_0^\bullet(X; F) = \mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+, F)$, and for $\theta = \pi/4$ we see that the total Gauss–Bonnet operators $d + d^*$ of the complexes $\mathcal{D}_{\pi/4}(X; F)$ and $\mathcal{D}(X; F)$ coincide (see [\[Vishik 1995,](#)

Proposition 1.1, p. 16]). Hence the family of complexes $(\mathcal{D}_\theta^\bullet(X; F), d^\theta)$ interpolates in a sense between the direct sum $\mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+; F)$ and the complex $\mathcal{D}^\bullet(X; F)$ on the manifold X .

The parametrized versions of (4-2), (4-4) are then

$$0 \longrightarrow \mathcal{D}^\bullet(X^-, Y; F) \xleftarrow{\alpha_\theta} \mathcal{D}_\theta^\bullet(X; F) \xrightarrow{\beta_\theta} \mathcal{D}^\bullet(X^+; F) \longrightarrow 0, \tag{4-6}$$

$$0 \longrightarrow \mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+, Y; F) \xrightarrow{\gamma_+ + \gamma_-} \mathcal{D}_\theta^\bullet(X; F) \xrightarrow{r_\theta} \mathcal{D}^\bullet(Y; F) \longrightarrow 0, \tag{4-7}$$

where $\alpha_\theta \omega = (\omega, 0)$ is extension by 0, $\beta_\theta(\omega_1, \omega_2) = \omega_2$ is restriction to X^+ , $\gamma_+ \oplus \gamma_-(\omega_1, \omega_2) = (\omega_1, \omega_2)$ is inclusion and $r_\theta(\omega_1, \omega_2) = \sin \theta \cdot i_-^* \omega_1 + \cos \theta \cdot i_+^* \omega_2$. We denote by $\mathcal{H}_\theta((X^-, Y), X, X^+; F)$ and $\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F)$ be the corresponding long exact cohomology sequences.

We denote the cohomology groups of the complex $\mathcal{D}_\theta^\bullet(X; F)$ by $H_\theta^j(X; F)$; the corresponding space of harmonic forms will be denoted by $\hat{H}_\theta^j(X; F)$. For the next result we need some more notation. Let \mathcal{H} be a Hilbert space and let $T : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator. For a finite-dimensional subspace $V \subset \mathcal{H}$ we write $\text{Tr}(T \upharpoonright V)$ for $\text{Tr}(P_V T P_V)$ where P_V is the orthogonal projection onto V . If e_j, j, \dots, n , is an orthonormal basis of V then

$$\text{Tr}(T \upharpoonright V) = \sum_{j=1}^n \langle T e_j, e_j \rangle. \tag{4-8}$$

We will apply this to β_θ on the space $H_\theta^j(X; F)$. If e_j, j, \dots, n , is an orthonormal basis of $\hat{H}_\theta^j(X; F)$ then

$$\text{Tr}(\beta_\theta \upharpoonright H_\theta^j(X; F)) = \sum_{j=1}^n \|e_j \upharpoonright X^+\|_{X^+}^2 = \sum_{j=1}^n \int_{X^+} e_j \wedge *e_j. \tag{4-9}$$

After these preparations we are able to state our main technical result. It is inspired by Lemma 2.2 and Section 2.6 in [Vishik 1995].

Theorem 4.1. *The functions*

$$\theta \mapsto \log T(\mathcal{D}_\theta^\bullet(X; F)), \quad \log \tau(\mathcal{H}_\theta((X^-, Y), X, X^+; F)), \quad \log \tau(\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F))$$

are differentiable for $0 < \theta < \pi/2$. Moreover, for $0 < \theta < \pi/2$,

$$\frac{d}{d\theta} \log T(\mathcal{D}_\theta^\bullet(X; F)) = \frac{2}{\sin 2\theta} \left[- \sum_{j \geq 0} (-1)^j \text{Tr}(\beta_\theta \upharpoonright H_\theta^j(X; F)) + \chi(X^+; F) \right] - \tan \theta \cdot \chi(Y; F), \tag{4-10}$$

$$\frac{d}{d\theta} \log \tau(\mathcal{H}_\theta((X^-, Y), X, X^+; F)) = \frac{2}{\sin 2\theta} \left[- \sum_{j \geq 0} (-1)^j \text{Tr}(\beta_\theta \upharpoonright H_\theta^j(X; F)) + \chi(X^+; F) \right], \tag{4-11}$$

$$\frac{d}{d\theta} \log \tau(\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F)) = \frac{d}{d\theta} \log T(\mathcal{D}_\theta^\bullet(X; F)). \tag{4-12}$$

Furthermore, with \mathcal{H}_θ standing for either $\mathcal{H}_\theta((X^-, Y), X, X^+; F)$ or $\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F)$, the map

$$\theta \mapsto \log T(\mathcal{D}_\theta^\bullet(X; F)) - \log \tau(\mathcal{H}_\theta) \quad (4-13)$$

is differentiable for $0 \leq \theta < \pi/2$.

The proof of [Theorem 4.1](#) will occupy [Section 5](#).

5. Gauge transforming the parametrized de Rham complex à la Witten

Consider the manifold X as described in [Section 4](#). Recall that in the collar $W := (-c, c) \times Y$ of Y all structures are assumed to be product. We introduce $W^{\text{cut}} := (-c, 0] \times Y \sqcup [0, c) \times Y$. Furthermore, let $S : W^{\text{cut}} \rightarrow W^{\text{cut}}$, $(t, p) \mapsto (-t, p)$ be the reflection map at Y . Finally, we introduce the map

$$T : \Omega^\bullet(W^{\text{cut}}; F) \rightarrow \Omega^\bullet(W^{\text{cut}}; F), \quad T(\omega_1, \omega_2) := (S^* \omega_2, -S^* \omega_1). \quad (5-1)$$

T is a skew-adjoint operator in $L^2(W^{\text{cut}}, \Lambda^\bullet T^* W^{\text{cut}} \otimes F)$ with $T^2 = -I$. Note furthermore that T commutes with the exterior derivative d . We denote by D^θ (on X^{cut} or W^{cut}) the closed extension of the exterior derivative with boundary conditions as in [\(4-5\)](#) along Y . More precisely, D^θ acts on the domain

$$\mathcal{D}_\theta^j(W; F) := \{(\omega_1, \omega_2) \in \mathcal{D}(d_{j, \max}) \mid \cos \theta \cdot i_-^* \omega_1 = \sin \theta \cdot i_+^* \omega_2\}. \quad (5-2)$$

The operator family has varying domain. In order to obtain variation formulas for functions of D^θ we will apply the method of gauge-transforming D^θ onto a family with constant domain; compare, for instance, [\[Douglas and Wojciechowski 1991\]](#) and [\[Lesch and Wojciechowski 1996\]](#).

We choose a cut-off function $\varphi \in C_c^\infty((-c, c) \times Y)$ with $\varphi \equiv 1$ in a neighborhood of $\{0\} \times Y$ and which satisfies $\varphi(-t, p) = \varphi(t, p)$, $(t, p) \in (-c, c) \times Y$. Then we introduce the gauge transformation

$$\Phi_\theta := e^{\theta\varphi T} = \cos(\theta\varphi)I + \sin(\theta\varphi)T : \Omega^\bullet(W^{\text{cut}}; F) \rightarrow \Omega^\bullet(W^{\text{cut}}; F). \quad (5-3)$$

Since $e^{\theta\varphi(t,p)T} = 1$ for $|t|$ sufficiently close to c , Φ_θ extends in an obvious way to a unitary transformation of $L^2(\Lambda^* T^* X^{\text{cut}}; F)$ which maps smooth forms to smooth forms.

Lemma 5.1. *For $\theta, \theta' \in \mathbb{R}$ the operator Φ_θ maps $\mathcal{D}_{\theta'}^j(X; F)$ onto $\mathcal{D}_{\theta+\theta'}^j(X; F)$, and accordingly $\mathcal{D}_{\theta'}^j(W^{\text{cut}}; F)$ onto $\mathcal{D}_{\theta+\theta'}^j(W^{\text{cut}}; F)$. Furthermore,*

$$\Phi_\theta^* D^{\theta+\theta'} \Phi_\theta = D^{\theta'} + \theta \text{ext}(d\varphi)T. \quad (5-4)$$

Proof. It obviously suffices to prove the lemma for W^{cut} . Consider $(\omega_1, \omega_2) \in \mathcal{D}_{\theta'}^j(W^{\text{cut}}; F)$. Then

$$i_-^* \Phi_\theta(\omega_1, \omega_2) = \cos \theta \cdot i_-^* \omega_1 + \sin \theta \cdot i_+^* \omega_2, \quad (5-5)$$

$$i_+^* \Phi_\theta(\omega_1, \omega_2) = \cos \theta \cdot i_-^* \omega_2 - \sin \theta \cdot i_+^* \omega_1. \quad (5-6)$$

A direct calculation now shows

$$\cos(\theta + \theta') i_-^* \Phi_\theta(\omega_1, \omega_2) = \sin(\theta + \theta') i_+^* \Phi_\theta(\omega_1, \omega_2), \quad (5-7)$$

proving the first claim. The formula [\(5-4\)](#) follows since T commutes with exterior differentiation. \square

Note that $D^{\pi/4} + \theta \operatorname{ext}(d\varphi)T$ is a deformed de Rham operator acting on smooth differential forms on the smooth manifold X (respectively, W). T is not a differential operator. However, the reflection map S allows us to identify $(-c, 0) \times Y$ with $(0, c) \times Y$ and hence sections in a vector bundle E over $(-c, 0) \times Y \amalg (0, c) \times Y$ may be viewed as sections in the vector bundle $E \oplus S^*E$ over $(0, c) \times Y$. Therefore, since $\operatorname{supp}(d\varphi)$ is compact in $(-c, 0) \times Y \amalg (0, c) \times Y$, T may be viewed as a bundle endomorphism acting on the bundle $(\Lambda^*T^*(0, c) \times Y) \otimes (F \oplus F)$. In particular employing the classical interior parametric elliptic calculus, as, for example, in [Shubin 2001], we infer that the Laplacian corresponding to $D^{\pi/4} + \theta \operatorname{ext}(d\varphi)T$ has discrete dimension spectrum over any such compact neighborhood of $\operatorname{supp}(d\varphi)$ which does have positive distance from $\pm c \times Y$.

From now on let

$$\tilde{D}^\theta := D^{\pi/4} + \theta \operatorname{ext}(d\varphi)T, \tag{5-8}$$

with domain $\mathcal{D}_{\theta=\pi/4}^\bullet(X; F)$ and $\tilde{\Delta}^\theta = (\tilde{D}^\theta)^* \tilde{D}^\theta + \tilde{D}^\theta (\tilde{D}^\theta)^*$ the corresponding Laplacian. On the collar $(-c, c) \times Y$ the operator $\tilde{\Delta}^\theta$ is of the form $P + V$ as discussed in Section 3A, where P is the form Laplacian and $V = \tilde{\Delta}^\theta - \Delta$ is induced by $\theta \operatorname{ext}(d\varphi)T$. The subset \mathcal{H} of (3-2) is the support of $d\varphi$. The operator $\tilde{\Delta}^\theta$ is now obtained as in (3-19) by gluing the domains of the form Laplacians of the given de Rham complexes on X^\pm . Proposition 3.5 and Corollary 3.7 now give:

Theorem 5.2. *The Hilbert complexes $\mathcal{D}_\theta^\bullet(X; F)$ defined in (4-5) are Hilbert complexes with discrete dimension spectrum.*

Theorem 5.3. *For $0 < \theta < \pi/2$ the Hilbert complexes $\mathcal{D}_\theta^\bullet(X; F)$ satisfy (1)–(4) of Proposition 2.4. More precisely,*

$$\frac{d}{d\theta} H_T(\mathcal{D}_\theta^\bullet(X; F)) = -t \frac{d}{dt} \frac{4}{\sin 2\theta} \sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta e^{-t\Delta_j^\theta}) \tag{5-9}$$

and

$$\sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta e^{-t\Delta_j^\theta}) = \chi(X^+; F) - \sin^2 \theta \cdot \chi(Y; F) + O(t^\infty), \tag{5-10}$$

as $t \rightarrow 0+$.

5A. Proof of Theorem 5.3. Note that

$$\frac{d}{d\theta} D^{\pi/4} = \operatorname{ext}(d\varphi)T = [d, \varphi T]. \tag{5-11}$$

Let us reiterate that although $[d, \varphi T]$ is strictly speaking not a 0th-order differential operator it may be viewed as one over $(\Lambda^*T^*(0, c) \times Y) \otimes (F \oplus F)$, which implies that it lies in $\operatorname{Op}_c^0(W^{\text{cut}}) \subset \operatorname{Op}_c^0(X^{\text{cut}})$.

We remind the reader of the definition of the closed and coclosed Laplacians in (2-10), (2-11). We find

$$\begin{aligned} \frac{d}{d\theta} \operatorname{Tr}(e^{-t\Delta_{p,\text{ccl}}^\theta}) &= \frac{d}{d\theta} \operatorname{Tr}(e^{-t\tilde{\Delta}_{p,\text{ccl}}^\theta}) = -t \operatorname{Tr}(((\tilde{D}_p^\theta)^t \operatorname{ext}(d\varphi)T + (\operatorname{ext}(d\varphi)T)^t \tilde{D}_p^\theta) e^{-t\tilde{\Delta}_{p,\text{ccl}}^\theta}) \\ &= -t \operatorname{Tr}(((D_p^\theta)^t \operatorname{ext}(d\varphi)T + (\operatorname{ext}(d\varphi)T)^t D_p^\theta) e^{-t\Delta_{p,\text{ccl}}^\theta}), \end{aligned} \tag{5-12}$$

where in the last line we have used that Φ_θ commutes with $\operatorname{ext}(d\varphi)T$.

Next let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of $\text{ran}(D_p^\theta)^*$ consisting of eigenvectors of $\Delta_{p,\text{ccl}}^\theta$ to eigenvalues $\lambda_n > 0$. Then $(\tilde{e}_n = \lambda_n^{-1/2} e_n)_n$ is an orthonormal basis of $\text{ran } D_p^\theta$ consisting of eigenvectors of $\Delta_{p+1,\text{cl}}^\theta$ (see (2-10), (2-11) and thereafter). Equation (5-12) gives

$$\begin{aligned} \frac{d}{d\theta} \text{Tr}(e^{-t\Delta_{p,\text{ccl}}^\theta}) &= -t \sum_n \langle (d_p^\theta)^t \text{ext}(d\varphi) T e^{-t\Delta_{p,\text{ccl}}^\theta} e_n, e_n \rangle - t \sum_n \langle e^{-t\Delta_{p,\text{ccl}}^\theta} e_n, (d_p^\theta)^t \text{ext}(d\varphi) T e_n \rangle \\ &= -2t \Re \left(\sum_n \langle (d_p^\theta)^t \text{ext}(d\varphi) T e^{-t\Delta_{p,\text{ccl}}^\theta} e_n, e_n \rangle \right). \end{aligned} \quad (5-13)$$

Stokes' theorem and the boundary conditions will allow us to rewrite the individual summands of the last sum. To this end let $\omega, \eta \in \mathcal{D}(\Delta_p^\theta)$. Then since $d\varphi$ is compactly supported in the interior of W^{cut} we have

$$\begin{aligned} \langle (d_p^\theta)^t \text{ext}(d\varphi) T \omega, \eta \rangle &= \langle \text{ext}(d\varphi) T \omega, d\eta \rangle = \langle d\varphi \wedge T \omega, d\eta \rangle = \langle d(\varphi T \omega), d\eta \rangle - \langle \varphi T d\omega, d\eta \rangle \\ &= \int_{\partial X^{\text{cut}}} T \omega \wedge \tilde{*} d\eta + \langle \varphi T \omega, d^t d\eta \rangle - \langle \varphi T d\omega, d\eta \rangle. \end{aligned} \quad (5-14)$$

Here, $\tilde{*}$ denotes the natural isometry $\wedge^p T^* M \otimes F \rightarrow \wedge^{m-p} T^* M \otimes F^\dagger$. In the last equality we have applied Stokes' theorem on the manifold with boundary X^{cut} . Note that $\varphi T \omega$ is a compactly supported (locally of Sobolev class at least 2) form on X^{cut} .

The boundary of X^{cut} consists of two copies of Y with opposite orientations. To calculate the integral in the last equation we orient Y as the boundary of X^+ . Then using that ω and η satisfy the boundary conditions (4-5) at Y we find

$$\begin{aligned} \int_{\partial X^{\text{cut}}} T \omega \wedge \tilde{*} d\eta &= \int_Y i_+^*(T \omega \wedge \tilde{*} d\eta) - i_-^*(T \omega \wedge \tilde{*} d\eta) = - \int_Y i_-^* \omega \wedge i_+^* \tilde{*} d\eta + i_+^* \omega \wedge i_-^* \tilde{*} d\eta \\ &= -(\tan \theta + \cot \theta) \int_Y i_+^*(\omega \wedge \tilde{*} d\eta) = -\frac{2}{\sin 2\theta} \left(\int_{X^+} d\omega \wedge \tilde{*} d\eta + (-1)^{|\omega|} \omega \wedge d\tilde{*} d\eta \right) \\ &= -\frac{2}{\sin 2\theta} (\langle d\omega, d\eta \rangle_{X^+} - \langle \omega, d^t d\eta \rangle_{X^+}). \end{aligned} \quad (5-15)$$

Here $\langle \cdot, \cdot \rangle_{X^+}$ denotes the L^2 -scalar product of forms over X^+ .

Plugging into (5-14) gives

$$\langle (d_p^\theta)^t \text{ext}(d\varphi) T \omega, \eta \rangle = \langle \varphi T \omega, d^t d\eta \rangle - \langle \varphi T d\omega, d\eta \rangle - \frac{2}{\sin 2\theta} (\langle d\omega, d\eta \rangle_{X^+} - \langle \omega, d^t d\eta \rangle_{X^+}). \quad (5-16)$$

Similarly,

$$\langle (\text{ext}(d\varphi) T)^t D_p^\theta \omega, \eta \rangle = \langle d^t d\omega, \varphi T \eta \rangle - \langle d\omega, \varphi T d\eta \rangle - \frac{2}{\sin 2\theta} (\langle d\omega, d\eta \rangle_{X^+} - \langle \omega, d^t d\eta \rangle_{X^+}). \quad (5-17)$$

We now apply (5-16) to the summands on the right of (5-13) and find using (2-12)

$$\begin{aligned} \langle (d_p^\theta)^t \text{ext}(d\varphi) T e^{-t\Delta_{p,\text{ccl}}^\theta} e_n, e_n \rangle &= \langle \varphi T e^{-t\Delta_{p,\text{ccl}}^\theta} \Delta_{p,\text{ccl}}^\theta e_n, e_n \rangle - \langle \varphi T e^{-t\Delta_{p+1,\text{cl}}^\theta} \Delta_{p+1,\text{cl}}^\theta \tilde{e}_n, \tilde{e}_n \rangle \\ &\quad - \frac{2}{\sin 2\theta} (\langle \beta_\theta e^{-t\Delta_{p+1,\text{cl}}^\theta} \Delta_{p+1,\text{cl}}^\theta \tilde{e}_n, \tilde{e}_n \rangle - \langle \beta_\theta e^{-t\Delta_{p,\text{ccl}}^\theta} \Delta_{p,\text{ccl}}^\theta e_n, e_n \rangle), \end{aligned} \quad (5-18)$$

and summing over n gives

$$\begin{aligned} \frac{d}{d\theta} \operatorname{Tr}(e^{-t\Delta_{p,\text{ccl}}^\theta}) &= -2t \Im(\operatorname{Tr}(\varphi T e^{-t\Delta_{p,\text{ccl}}^\theta} \Delta_{p,\text{ccl}}^\theta) - \operatorname{Tr}(\varphi T e^{-t\Delta_{p+1,\text{cl}}^\theta} \Delta_{p+1,\text{cl}}^\theta)) \\ &\quad + \frac{4t}{\sin 2\theta} \Im(\operatorname{Tr}(\beta_\theta \Delta_{p+1,\text{cl}}^\theta e^{-t\Delta_{p+1,\text{cl}}^\theta} - \operatorname{Tr}(\beta_\theta \Delta_{p,\text{ccl}}^\theta e^{-t\Delta_{p,\text{ccl}}^\theta})) \\ &= 2t \frac{d}{dt} \frac{2}{\sin 2\theta} (\operatorname{Tr}(\beta_\theta e^{-t\Delta_{p,\text{ccl}}^\theta}) - \operatorname{Tr}(\beta_\theta e^{-t\Delta_{p+1,\text{cl}}^\theta})). \end{aligned} \tag{5-19}$$

Here we have used that, since φT is skew-adjoint, $\operatorname{Tr}(\varphi T A)$ is purely imaginary for every self-adjoint trace class operator A and similarly that since β_θ is self-adjoint that $\operatorname{Tr}(\beta_\theta A)$ is real. Consequently, using (2-14),

$$\frac{d}{d\theta} H_T(\mathcal{D}_\theta^\bullet(X; F)) = \frac{d}{d\theta} \sum_{j \geq 0} (-1)^{j+1} \operatorname{Tr}(e^{-t\Delta_{j,\text{ccl}}^\theta}) = -2t \frac{d}{dt} \frac{2}{\sin 2\theta} \sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta e^{-t\Delta_j^\theta}). \tag{5-20}$$

Finally, for calculating the asymptotic expansion (5-10) as $t \rightarrow 0+$ we may again invoke Corollary 3.7. The asymptotic expansion (5-10) on X^{cut} differs from the corresponding expansion for the double $-X^+ \amalg X^+$ by an error term $O(t^\infty)$; here $-X^+$ stands for X^+ with the opposite orientation. However, on the double $-X^+ \amalg X^+$ we may write down the heat kernel for Δ_p^θ explicitly in terms of the heat kernels for Δ_p with relative and absolute boundary conditions at Y [Vishik 1995, (2.118), p. 60]. Namely, let Δ_p^r, Δ_p^a be the Laplacians of the relative and absolute de Rham complexes on X^+ as in (3-25) and denote by $E_t^{p,r/a}$ their corresponding heat kernels. Let S be the reflection map which interchanges the two copies of X^+ in $-X^+ \amalg X^+$. Its restriction to W is the reflection map S defined before (5-1) and hence denoting it by the same letter is justified.

Finally, let $E_t^p(x, y)$ be the heat kernel of $\Delta_p^{\pi/4}$ on $-X^+ \amalg X^+$, that is, the Laplacian with continuous transmission boundary conditions at Y . The absolute/relative heat kernels are given in terms of E_t^p by

$$E_t^{p,a} = (E_t^p + S^* \circ E_t^p) \upharpoonright X^+, \quad E_t^{p,r} = (E_t^p - S^* \circ E_t^p) \upharpoonright X^+. \tag{5-21}$$

More generally, we put for $x, y \in -X^+ \amalg X^+$:

$$E_t^{p,\theta}(x, y) := \begin{cases} E_t^p(x, y) + \cos(2\theta)(S^* \circ E_t^p)(x, y) & \text{if } x, y \in X^+, \\ \sin(2\theta)E_t^p(x, y) & \text{if } x \in (-X^+), y \in X^+. \end{cases} \tag{5-22}$$

One immediately checks that $E_t^{p,\theta}$ is the heat kernel of Δ_p^θ on $-X^+ \amalg X^+$. Consequently

$$\operatorname{Tr}(\beta_\theta e^{-t\Delta_p^\theta}) = \operatorname{Tr}(\beta_\theta E_t^p) + \cos(2\theta) \operatorname{Tr}(S^* \circ E_t^p) = \cos^2(\theta) \operatorname{Tr}(E_t^{p,a}) + \sin^2(\theta) \operatorname{Tr}(E_t^{p,r}). \tag{5-23}$$

In view of our standing assumptions (Section 4A) the complexes $\mathcal{D}^\bullet(X^+, Y; F)$ and $\mathcal{D}^\bullet(X^+; F)$ are Fredholm complexes, so the McKean–Singer formula (2-8) holds and hence taking alternating sums yields

$$\sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta e^{-t\Delta_j^\theta}) = \cos^2 \theta \cdot \chi(X^+; F) + \sin^2 \theta \cdot \chi(X^+, Y; F) = \chi(X^+; F) - \sin^2 \theta \cdot \chi(Y; F), \tag{5-24}$$

and the proof of (5-10) is complete. In the last equality we used that $\chi(X^+; F) = \chi(X^+, Y; F) + \chi(Y; F)$; this formula follows from the exact sequence (4-4). □

5B. Proof of Theorem 4.1.

Proof of (4-10). Combining Proposition 2.4 and Theorem 5.3 we find

$$\begin{aligned} \frac{d}{d\theta} \log T(\mathcal{D}_\theta^\bullet(X; F)) &= -\frac{1}{2} \frac{-4}{\sin 2\theta} (\chi(X^+; F) - \sin^2 \theta \chi(Y; F)) + \frac{1}{2} \frac{-4}{\sin 2\theta} \operatorname{Tr} \left(\sum_{j \geq 0} (-1)^j \beta_\theta \upharpoonright H_\theta^j(X; F) \right) \\ &= \frac{2}{\sin 2\theta} \left(-\sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta \upharpoonright H_\theta^j(X; F)) + \chi(X^+; F) \right) - \tan \theta \cdot \chi(Y; F), \end{aligned} \quad (5-25)$$

which is the right-hand side of (4-10). □

Proof of (4-11) and (4-12). Let $0 < \theta, \theta' < \pi/2$ and consider the following commutative diagram (cf. (4-6)):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{D}^\bullet(X^-, Y; F) & \xrightarrow{\alpha_\theta} & \mathcal{D}_\theta^\bullet(X; F) & \xrightarrow{\beta_\theta} & \mathcal{D}^\bullet(X^+; F) & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \phi_{\theta, \theta'} & & \downarrow \phi_{\theta, \theta'}^+ & & \\ 0 & \longrightarrow & \mathcal{D}^\bullet(X^-, Y; F) & \xrightarrow{\alpha_{\theta'}} & \mathcal{D}_{\theta'}^\bullet(X; F) & \xrightarrow{\beta_{\theta'}} & \mathcal{D}^\bullet(X^+; F) & \longrightarrow & 0, \end{array} \quad (5-26)$$

where $\phi_{\theta, \theta'}$ and $\phi_{\theta, \theta'}^+$ are Hilbert complex isomorphisms, defined by

$$\phi_{\theta, \theta'}(\omega_1, \omega_2) = \left(\omega_1, \frac{\tan \theta}{\tan \theta'} \omega_2 \right), \quad \phi_{\theta, \theta'}^+(\omega_2) = \frac{\tan \theta}{\tan \theta'} \omega_2.$$

Hence we obtain a cochain isomorphism between the long exact cohomology sequences of the upper and lower horizontal exact sequences (F omitted to save horizontal space):

$$\begin{array}{ccccccccc} \dots & H^k(X^-, Y) & \xrightarrow{\alpha_{\theta, *}} & H_\theta^k(X) & \xrightarrow{\beta_{\theta, *}} & H^k(X^-) & \xrightarrow{\delta_\theta} & H^{k+1}(X^-, Y) & \dots \\ & \downarrow \text{id} & & \downarrow \phi_{\theta, \theta', * } & & \downarrow \phi_{\theta, \theta', * }^+ & & \downarrow \text{id} & \\ \dots & H^k(X^-, Y) & \xrightarrow{\alpha_{\theta', *}} & H_{\theta'}^k(X) & \xrightarrow{\beta_{\theta', *}} & H^k(X^-) & \xrightarrow{\delta_{\theta'}} & H^{k+1}(X^-, Y) & \dots \end{array} \quad (5-27)$$

Let e_1, \dots, e_r be an orthonormal basis of $H_\theta^k(X; F)$. Then

$$\operatorname{Det}(\phi_{\theta, \theta', * }^k)^2 = \operatorname{Det}(\langle \phi_{\theta, \theta', * } e_i, \phi_{\theta, \theta', * } e_j \rangle_{i, j=1}^r); \quad (5-28)$$

hence

$$\begin{aligned} \frac{d}{d\theta'} \Big|_{\theta'=\theta} \log \operatorname{Det}(\phi_{\theta, \theta', * }^k)^2 &= \operatorname{Tr} \left(\left(\left\langle \frac{d}{d\theta'} \Big|_{\theta'=\theta} \phi_{\theta, \theta', * } e_i, e_j \right\rangle + \left\langle e_i, \frac{d}{d\theta'} \Big|_{\theta'=\theta} \phi_{\theta, \theta', * } e_j \right\rangle \right)_{i, j=1}^r \right) \\ &= -2 \frac{2}{\sin 2\theta} \sum_{j=0}^r \langle \beta_\theta e_j, e_j \rangle = -2 \frac{2}{\sin 2\theta} \operatorname{Tr}(\beta_\theta \upharpoonright H_\theta^k(X; F)); \end{aligned} \quad (5-29)$$

see (4-8). Furthermore, since $\phi_{\theta, \theta'}^+$ is multiplication by $\tan \theta / \tan \theta'$ we have

$$\text{Det}(\phi_{\theta, \theta', *})^+ = \left(\frac{\tan \theta}{\tan \theta'} \right)^{\chi(X^+; F)}, \quad (5-30)$$

and hence

$$\frac{d}{d\theta'} \Big|_{\theta'=\theta} \log \text{Det}(\phi_{\theta, \theta', *})^+ = -2 \frac{2}{\sin 2\theta} \chi(X^+; F). \quad (5-31)$$

By Lemma 2.5 we have

$$\begin{aligned} \log \tau(\mathcal{H}_{\theta'}((X^-, Y), X, X^+; F)) - \log \tau(\mathcal{H}_{\theta}((X^-, Y), X, X^+; F)) \\ = \frac{1}{2} \log \text{Det}(\phi_{\theta, \theta', *})^+ - \frac{1}{2} \log \text{Det}(\phi_{\theta, \theta', *})^+; \end{aligned} \quad (5-32)$$

combined with (5-29) and (5-31) we therefore find (4-11).

That the left-hand side of (4-12) equals the right-hand side of (4-10) is proved analogously. One just has to replace the commutative diagram (5-26) by

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+, Y; F) & \xrightarrow{\gamma^+ + \gamma^-} & \mathcal{D}_\theta^\bullet(X; F) & \xrightarrow{r_\theta} & \mathcal{D}^\bullet(Y; F) \longrightarrow 0 \\ & & \downarrow \text{id} \oplus \phi_{\theta, \theta'}^+ & & \downarrow \phi_{\theta, \theta'} & & \downarrow \psi_{\theta, \theta'} \\ 0 & \longrightarrow & \mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+, Y; F) & \xrightarrow{\gamma^+ + \gamma^-} & \mathcal{D}_{\theta'}^\bullet(X; F) & \xrightarrow{r_{\theta'}} & \mathcal{D}^\bullet(Y; F) \longrightarrow 0, \end{array} \quad (5-33)$$

where $\psi_{\theta, \theta'}(\omega) = \frac{\sin \theta}{\sin \theta'} \omega$. See also (A-20) and thereafter. □

Proof of the differentiability of (4-13) at 0. The problem is that the dimensions of the cohomology groups $H_\theta^j(X; F)$ may jump at 0; note that the isomorphism $\phi_{\theta, \theta'}$ defined after (5-27) between $\mathcal{D}_\theta^\bullet(X; F)$ and $\mathcal{D}_{\theta'}^\bullet(X; F)$ is defined only for $0 < \theta, \theta' < \pi/2$. By our standing assumptions of Section 4A (see also Section 3D), $\mathcal{D}^\bullet(X^-, Y; F)$ and $\mathcal{D}^\bullet(X^+, Y; F)$ are Hilbert complexes with discrete dimension spectrum. Hence we may choose $a > 0$ such that a is smaller than the smallest nonzero eigenvalues of the Laplacians of $\mathcal{D}^\bullet(X^-, Y; F)$ and $\mathcal{D}^\bullet(X^+, Y; F)$. Furthermore, we denote by Π_θ^P the orthogonal projection onto

$$H_{\theta, a}^P(X; F) := \bigoplus_{0 \leq \lambda < a} \ker(\Delta_P^\theta - \lambda). \quad (5-34)$$

Since for $\theta = 0$ the complex $\mathcal{D}_\theta^\bullet(X; F)$ is canonically isomorphic to the direct sum $\mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+, Y; F)$ and since the gauge-transformed Laplacian $\tilde{\Delta}^\theta$ of $\mathcal{D}_\theta^\bullet(X; F)$ in view of (5-8) certainly depends smoothly on θ there exists a $\theta_0 > 0$ such that the projection Π_θ^P depends smoothly on θ for $0 \leq \theta < \theta_0$. In particular,

$$\text{rank } \Pi_\theta^P = \dim H_{\theta=0}^P(X; F)$$

is constant for $0 \leq \theta < \theta_0$.

$(H_{\theta, a}^\bullet(X; F), d)$ is a finite-dimensional Hilbert complex and the orthogonal projections Π_θ^P give rise to a natural orthogonal decomposition of Hilbert complexes

$$\mathcal{D}_\theta^\bullet(X; F) =: (H_{\theta, a}^\bullet(X; F), d) \oplus \mathcal{D}_{\theta, a}^\bullet(X; F). \quad (5-35)$$

By construction of Π_a^p we have

$$\log T(\mathcal{D}_\theta^\bullet(X; F)) = \log \tau(H_{\theta,a}^\bullet(X; F), d) + \log T(\mathcal{D}_{\theta,a}^\bullet(X; F)), \tag{5-36}$$

and $\theta \mapsto \log T(\mathcal{D}_{\theta,a}^\bullet(X; F))$ is differentiable for $0 \leq \theta < \theta_0$.

Since surjectivity is an open condition we conclude that the sequence

$$0 \longrightarrow H^*(X^-, Y; F) \xrightarrow{\alpha_\theta} H_{\theta,a}^*(X; F) \xrightarrow{\beta_\theta} H^*(X^+; F) \longrightarrow 0 \tag{5-37}$$

is exact for $0 \leq \theta < \theta_1 \leq \theta_0$. Here, α_θ is defined in the obvious way while

$$\beta_\theta := \text{orthogonal projection onto } H^*(X^+; F) \text{ of } \omega \upharpoonright X^+. \tag{5-38}$$

Note that the differentials of the left and right complexes vanish and hence so do their torsions. The space of harmonics of the middle complex equals the space of harmonics of the complex $\mathcal{D}_\theta^\bullet(X; F)$ and hence the cohomology of the middle complex is (isometrically) isomorphic to the cohomology of $\mathcal{D}_\theta^\bullet(X; F)$. One immediately checks that the long exact cohomology sequence of (5-37) is exactly the exact cohomology sequence $\mathcal{H}((X^-, Y), X, X^+; F)$. Hence Proposition 2.6 yields

$$\begin{aligned} \log \tau(H_{\theta,a}^*(X; F)) &= \log \tau(\mathcal{H}((X^-, Y), X, X^+; F)) \\ &\quad - \sum_{p \geq 0} \log \tau(0 \rightarrow H^p(X^-, Y; F) \xrightarrow{\alpha_\theta} H_{\theta,a}^p(X; F) \xrightarrow{\beta_\theta} H^p(X^+; F) \rightarrow 0). \end{aligned} \tag{5-39}$$

This shows the differentiability of the difference $\log \tau(H_{\theta,a}^*(X; F)) - \log \tau(\mathcal{H}((X^-, Y), X, X^+; F))$ at $\theta = 0$. In view of (5-36) the claim is proved. \square

6. The gluing formula

We can now state and prove the main result of this paper. The standing assumptions (Section 4A) are still in effect. Furthermore, we will use freely the notation introduced in Section 4B.

Theorem 6.1. *For the analytic torsions of the Hilbert complexes $\mathcal{D}^\bullet(X^\pm, Y; F)$, $\mathcal{D}^\bullet(X^\pm; F)$, $\mathcal{D}^\bullet(X; F)$ we have the following formulas:*

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(X^+; F)) \\ &\quad + \log \tau(\mathcal{H}((X^-, Y), X, X^+; F)) - \frac{1}{2} \log 2 \cdot \chi(Y; F), \end{aligned} \tag{6-1}$$

$$\log T(\mathcal{D}^\bullet(X^-; F)) = \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(Y; F)) + \log \tau(\mathcal{H}((X^-, Y), X^-, Y; F)), \tag{6-2}$$

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(X^+, Y; F)) \\ &\quad + \log \tau(\mathcal{H}((X^-, Y) \cup (X^+, Y), X, Y; F)) + \log T(\mathcal{D}^\bullet(Y; F)). \end{aligned} \tag{6-3}$$

6A. Proof of Theorem 6.1. In the course of the proof we will make heavy use of Theorem 4.1.

Proof of (6-1). As noted after (4-5) we have for $\theta = 0$ that $\mathcal{D}_{\theta=0}^\bullet(X; F) = \mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+; F)$ and that for $\theta = \pi/4$ the complexes $\mathcal{D}_{\theta=\pi/4}^\bullet(X; F)$ and $\mathcal{D}^\bullet(X; F)$ are isometric. Hence we have

$$\begin{aligned} & \log T(\mathcal{D}^\bullet(X; F)) - \log T(\mathcal{D}^\bullet(X^-, Y; F)) - \log T(\mathcal{D}^\bullet(X^+; F)) \\ &= \log T(\mathcal{D}_{\pi/4}^\bullet(X; F)) - \log T(\mathcal{D}_{\theta=0}^\bullet(X; F)) \\ &= \log T(\mathcal{D}_{\pi/4}^\bullet(X; F)) - \log \tau(\mathcal{H}_{\pi/4}((X^-, Y), X, X^+; F)) - \log T(\mathcal{D}_{\theta=0}^\bullet(X; F)) \\ & \quad + \log \tau(\mathcal{H}_{\theta=0}((X^-, Y), X, X^+; F)) + \log \tau(\mathcal{H}_{\pi/4}((X^-, Y), X, X^+; F)). \end{aligned} \quad (6-4)$$

Recall that for $\theta = 0$ the complex $\mathcal{D}_{\theta=0}^\bullet(X; F)$ is just the direct sum complex $\mathcal{D}^\bullet(X^-, Y; F) \oplus \mathcal{D}^\bullet(X^+; F)$ and hence $\log \tau(\mathcal{H}_{\theta=0}((X^-, Y), X, X^+; F)) = 0$ (see also the sentence after (2-29)). Furthermore, $\log \tau(\mathcal{H}_{\pi/4}((X^-, Y), X, X^+; F)) = \log \tau(\mathcal{H}((X^-, Y), X, X^+; F))$; hence by Theorem 4.1

$$\begin{aligned} \dots &= \int_0^{\pi/4} -\tan \theta d\theta \chi(Y; F) + \log \tau(\mathcal{H}((X^-, Y), X, X^+; F)) \\ &= -\frac{1}{2} \log 2 \chi(Y; F) + \log \tau(\mathcal{H}((X^-, Y), X, X^+; F)), \end{aligned} \quad (6-5)$$

and we arrive at (6-1). □

Proof of (6-2). Consider $\epsilon > 0$ and apply the proved equation (6-1) to the manifold $X_\epsilon^- := X^- \cup_Y [0, \epsilon] \times Y$. Then

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X_\epsilon^-; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet([0, \epsilon] \times Y; F)) \\ & \quad - \frac{1}{2} \log 2 \chi(Y; F) + \log \tau(\mathcal{H}((X_\epsilon^-, Y), X_\epsilon^-, [0, \epsilon] \times Y; F)). \end{aligned} \quad (6-6)$$

For the cylinder $[0, \epsilon] \times Y$ it is well-known (it also follows easily from Proposition 2.3) that

$$\chi([0, \epsilon] \times Y; F) = \chi(Y; F) = \chi(Y) \text{ rank } F, \quad (6-7)$$

$$\begin{aligned} \log T(\mathcal{D}^\bullet([0, \epsilon] \times Y; F)) &= \log T(\mathcal{D}^\bullet(Y; F)) \chi([0, \epsilon]) + \chi(Y; F) \log T(\mathcal{D}^\bullet([0, \epsilon])) \\ &= \log T(\mathcal{D}^\bullet(Y; F)) + \frac{1}{2} \log(2\epsilon) \chi(Y; F). \end{aligned} \quad (6-8)$$

Hence

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X_\epsilon^-; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(Y; F)) \\ & \quad + \frac{1}{2} \log \epsilon \chi(Y; F) + \log \tau(\mathcal{H}((X_\epsilon^-, Y), X_\epsilon^-, [0, \epsilon] \times Y; F)). \end{aligned} \quad (6-9)$$

In the sequel we will, to save some space, omit the bundle F from the notation in commutative diagrams. Our first commutative diagram is

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^k(X^-, Y) & \xrightarrow{\alpha_{-,*}} & H^k(X_\epsilon^-) & \xrightarrow{\beta_*} & H^k([0, \epsilon] \times Y) & \longrightarrow & \dots \\ & & \downarrow \psi_\epsilon^* & & \downarrow \psi_\epsilon^* & & \downarrow \chi_\epsilon^* & & \\ \dots & \longrightarrow & H^k(X^-, Y) & \longrightarrow & H^k(X^-) & \longrightarrow & H^k(Y) & \longrightarrow & \dots \end{array} \quad (6-10)$$

The first row is the long exact cohomology sequence of (4-2) for $X_\epsilon^- = X^- \cup_Y [0, \epsilon] \times Y$ instead of X ;

the second row is the long exact cohomology sequence of (4-2) for $X = X^- \cup_Y X^+$. The map ψ_ϵ is a diffeomorphism $X^- \rightarrow X_\epsilon^-$ obtained as follows: choose a diffeomorphism $f : [-c, 0] \rightarrow [-c, \epsilon]$ such that $f(x) = x$ for x near $-c$ and $f(x) = x + \epsilon$ for x near 0 . Then ψ_ϵ is obtained by patching the identity on $X^- \setminus [-c, 0] \times Y$ and $f \times \text{id}_Y$. Furthermore $\chi_\epsilon : Y \rightarrow [0, \epsilon] \times Y$, $p \mapsto (\epsilon, p)$.

For a harmonic form $\omega \in \mathcal{D}(d_{k,\max}) \cap \mathcal{D}((d_{k-1,\max})^*) \subset \Omega^k([0, \epsilon] \times Y; F)$ one has $\omega = \pi^* \chi_\epsilon^*(\omega)$ ($\pi : [0, \epsilon] \times Y \rightarrow Y$ the projection) and thus

$$\int_{[0,\epsilon] \times Y} \omega \wedge \tilde{*}\omega = \epsilon \int_Y \chi_\epsilon^* \omega \wedge \tilde{*}\chi_\epsilon^* \omega. \quad (6-11)$$

Therefore the determinant (in the sense of (2-23)) of χ_ϵ^* on the cohomology is given by $\epsilon^{-\frac{1}{2}\chi(Y;F)}$. Consequently by Lemma 2.5

$$\begin{aligned} \log \tau(\mathcal{H}((X_\epsilon^-, Y), X_\epsilon^-, [0, \epsilon] \times Y; F)) &= \log \tau(\mathcal{H}((X^-, Y), X^-, Y; F)) - \frac{1}{2} \log \epsilon \chi(Y; F) \\ &\quad + \log \text{Det}(\psi_\epsilon^* : H^*(X^-, Y; F) \rightarrow H^*(X^-, Y; F)) \\ &\quad - \log \text{Det}(\psi_\epsilon^* : H^*(X_\epsilon^-; F) \rightarrow H^*(X^-; F)). \end{aligned} \quad (6-12)$$

Summing up (6-9), (6-12)

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X_\epsilon^-; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(Y; F)) + \log \tau(\mathcal{H}((X^-, Y), X^-, Y; F)) \\ &\quad + \log \text{Det}(\psi_\epsilon^* : H^*(X^-, Y; F) \rightarrow H^*(X^-, Y; F)) \\ &\quad - \log \text{Det}(\psi_\epsilon^* : H^*(X_\epsilon^-; F) \rightarrow H^*(X^-; F)). \end{aligned} \quad (6-13)$$

As $\epsilon \rightarrow 0$ the determinants of

$$\psi_\epsilon^* : H^*(X^-, Y; F) \rightarrow H^*(X^-, Y; F), \quad \text{resp.}, \quad \psi_\epsilon^* : H^*(X_\epsilon^-; F) \rightarrow H^*(X^-; F)$$

tend to 1 and we obtain (6-2). □

Proof of (6-3). We note that $\tau(\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F))|_{\theta=0} = \tau(\mathcal{H}((X^+, Y), X^+, Y; F))$; hence by (4-12) and (4-13)

$$\begin{aligned} \log T(\mathcal{D}_{\theta=\pi/4}^\bullet(X; F)) - \log \tau(\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F))|_{\theta=\pi/4} \\ &= \log T(\mathcal{D}_{\theta=0}^\bullet(X; F)) - \log \tau(\mathcal{H}_\theta((X^-, Y) \cup (X^+, Y), X, Y; F))|_{\theta=0} \\ &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(X^+, Y; F)) - \log \tau(\mathcal{H}((X^+, Y), X^+, Y; F)) \\ &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(X^+, Y; F)) + \log T(\mathcal{D}^\bullet(Y; F)), \end{aligned} \quad (6-14)$$

where in the last equality we have used the proved identity (6-2). □

Appendix: The homological algebra gluing formula

We present here the analogues of Theorems 6.1 and 4.1 for finite-dimensional Hilbert complexes. This applies, for example, to the cochain complexes of a triangulation twisted by a unitary representation of the fundamental group; see, e.g., [Müller 1993, Section 1].

Let (C_j^*, d^j) , $j = 1, 2$, be finite-dimensional Hilbert complexes. Let (B^*, d) be another such Hilbert complex and assume that we are given *surjective* homomorphisms of cochain complexes

$$r_j : (C_j, d^j) \rightarrow (B, d), \quad j = 1, 2. \tag{A-1}$$

We denote by $C_{j,r} \subset C_j$ the kernel of r_j , by $\alpha : C_1 \rightarrow C_1 \oplus C_2$ the inclusion and by $\beta : C_1 \oplus C_2 \rightarrow C_2$ the projection onto the second factor.

For $\theta \in \mathbb{R}$ we define the following homological algebra analogue of the complex $\mathcal{D}_\theta^*(X; F)$ of (4-5) by putting

$$(C_1 \oplus_\theta C_2)^j := \{(\xi_1, \xi_2) \in C_1^j \oplus C_2^j \mid \cos \theta \cdot r_1 \xi_1 = \sin \theta \cdot r_2 \xi_2\}. \tag{A-2}$$

$(C_1 \oplus_\theta C_2, d = d^1 \oplus d^2)$ is a subcomplex of $(C_1 \oplus C_2, d^1 \oplus d^2)$. For $\theta = 0$ we have $C_1 \oplus_\theta C_2 = C_{1,r} \oplus C_2$ and for $\theta = \pi/4$ we have a homological algebra analogue of the complex $\mathcal{D}_\theta^*(X; F)$.

Furthermore, we have the following analogues of the exact sequences (4-3), (4-6), (4-7) (note that the exact sequences (4-2), (4-4) are special cases of the exact sequences (4-6), (4-7)):

$$0 \rightarrow C_{j,r} \xrightarrow{\gamma_j} C_j \xrightarrow{r_j} B \rightarrow 0, \tag{A-3}$$

$$0 \rightarrow C_{1,r} \xrightarrow{\alpha_\theta} C_1 \oplus_\theta C_2 \xrightarrow{\beta_\theta} C_2 \rightarrow 0, \tag{A-4}$$

$$0 \rightarrow C_{1,r} \oplus C_{2,r} \xrightarrow{\gamma_1 + \gamma_2} C_1 \oplus_\theta C_2 \xrightarrow{r_\theta} B \rightarrow 0. \tag{A-5}$$

Here, γ_j is the natural inclusion, $\beta_\theta = \beta \upharpoonright C_1 \oplus_\theta C_2$, $\alpha_\theta(\xi) = (\xi, 0)$, and $r_\theta(\xi_1, \xi_2) = \sin \theta \cdot r_1 \xi_1 + \cos \theta \cdot r_2 \xi_2$. Denote by $\mathcal{H}(C_{j,r}, C_j, B)$, $\mathcal{H}(C_{1,r}, C_1 \oplus_\theta C_2, C_2)$, $\mathcal{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_\theta C_2, B)$ the long exact cohomology sequences of (A-3), (A-4), (A-5), respectively.

Since all complexes are finite-dimensional we have Lemma 2.5 and Proposition 2.6 at our disposal. The latter applied to (A-3) immediately gives the analogue of (6-2):

$$\log \tau(C_1) = \log \tau(C_{1,r}) + \log \tau(B) + \log \tau(\mathcal{H}(C_{1,r}, C_1, B)). \tag{A-6}$$

The other claims of Theorems 6.1 and 4.1 have exact counterparts in this context as summarized in the following:

Theorem A.2. (1) *The functions*

$$\theta \mapsto \log \tau(C_1 \oplus_\theta C_2), \quad \log \tau(\mathcal{H}(C_{1,r}, C_1 \oplus_\theta C_2, C_2)), \quad \log \tau(\mathcal{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_\theta C_2, B))$$

are differentiable for $0 < \theta < \pi/2$. Moreover, for $0 < \theta < \pi/2$,

$$\frac{d}{d\theta} \log \tau(C_1 \oplus_\theta C_2) = \frac{2}{\sin 2\theta} \left(- \sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta \upharpoonright H^j(C_1 \oplus_\theta C_2)) + \sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta \upharpoonright (C_1 \oplus_\theta C_2)^j) \right), \tag{A-7}$$

$$\frac{d}{d\theta} \log \tau(\mathcal{H}(C_{1,r}, C_1 \oplus_\theta C_2, C_2)) = \frac{2}{\sin 2\theta} \left(- \sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta \upharpoonright H^j(C_1 \oplus_\theta C_2)) + \chi(C_2) \right), \tag{A-8}$$

and

$$\begin{aligned} \frac{d}{d\theta} \log \tau(\mathfrak{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_\theta C_2, B)) \\ = \frac{2}{\sin 2\theta} \left(-\sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_\theta \upharpoonright H^j(C_1 \oplus_\theta C_2)) + \chi(C_2) \right) - \tan \theta \chi(B). \end{aligned} \quad (\text{A-9})$$

Furthermore,

$$\theta \mapsto \log T(C_1 \oplus_\theta C_2) - \log \tau(\mathfrak{H}_\theta) \quad (\text{A-10})$$

is differentiable for $0 \leq \theta < \pi/2$. Here, \mathfrak{H}_θ stands for either

$$\mathfrak{H}(C_{1,r}, C_1 \oplus_\theta C_2, C_2) \quad \text{or} \quad \mathfrak{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_\theta C_2, B).$$

(2) Under the additional assumption that the r_j are partial isometries we have

$$\frac{d}{d\theta} \log \tau(C_1 \oplus_\theta C_2) = \frac{d}{d\theta} \log \tau(\mathfrak{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_\theta C_2, B)) \quad (\text{A-11})$$

and

$$\log \tau(C_1 \oplus_\theta C_2) = \log \tau(C_{1,r}) + \log \tau(C_{2,r}) + \log \tau(\mathfrak{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_\theta C_2, B)) \quad (\text{A-12})$$

$$= \log \tau(C_{1,r}) + \log \tau(C_2) + \log \tau(\mathfrak{H}(C_{1,r}, C_1 \oplus_\theta C_2, C_2)) + \log \cos \theta \chi(B). \quad (\text{A-13})$$

When comparing the last formula with [Theorem 6.1](#) one should note that for $\theta = \pi/4$ we have

$$\log \cos \theta = \log \frac{1}{\sqrt{2}} = -\frac{1}{2} \log 2.$$

Proof. For $0 < \theta, \theta' < \pi/2$ we have the cochain isomorphism (cf. [\(5-26\)](#))

$$\phi_{\theta, \theta'} : C_1 \oplus_\theta C_2 \rightarrow C_1 \oplus_{\theta'} C_2, \quad (\xi_1, \xi_2) \mapsto \left(\xi_1, \frac{\tan \theta}{\tan \theta'} \xi_2 \right); \quad (\text{A-14})$$

hence by [Lemma 2.5](#)

$$\begin{aligned} \log \tau(C_1 \oplus_\theta C_2) = \log \tau(C_1 \oplus_{\theta'} C_2) - \sum_{j \geq 0} (-1)^j \log \operatorname{Det}(\phi_{\theta, \theta'} \upharpoonright H^j(C_1 \oplus_\theta C_2)) \\ + \sum_{j \geq 0} (-1)^j \log \operatorname{Det}(\phi_{\theta, \theta'} \upharpoonright (C_1 \oplus_\theta C_2)^j). \end{aligned} \quad (\text{A-15})$$

Taking $d/d\theta'$ at $\theta' = \theta$ yields [\(A-7\)](#).

Next we look at the analogues of [\(5-26\)](#) and [\(5-27\)](#):

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{1,r} & \xrightarrow{\alpha_\theta} & C_1 \oplus_\theta C_2 & \xrightarrow{\beta_\theta} & C_2 \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \phi_{\theta, \theta'} & & \downarrow \tilde{\phi}_{\theta, \theta'} \\ 0 & \longrightarrow & C_{1,r} & \xrightarrow{\alpha_{\theta'}} & C_1 \oplus_{\theta'} C_2 & \xrightarrow{\beta_{\theta'}} & C_2 \longrightarrow 0, \end{array} \quad (\text{A-16})$$

where $\tilde{\phi}_{\theta, \theta'}(\xi) = \frac{\tan \theta}{\tan \theta'} \xi$, and at the corresponding isomorphism between the long exact cohomology

sequences

$$\begin{array}{ccccccc}
 \cdots & H^k(C_{1,r}) & \xrightarrow{\alpha_{\theta,*}} & H^k(C_1 \oplus_{\theta} C_2) & \xrightarrow{\beta_{\theta,*}} & H^k(C_2) & \xrightarrow{\delta_{\theta}} & H^{k+1}(C_{1,r}) \cdots \\
 & \downarrow \text{id} & & \downarrow \phi_{\theta,\theta',*} & & \downarrow \tilde{\phi}_{\theta,\theta',*} & & \downarrow \text{id} \\
 \cdots & H^k(C_{1,r}) & \xrightarrow{\alpha_{\theta',*}} & H^k(C_1 \oplus_{\theta'} C_2) & \xrightarrow{\beta_{\theta',*}} & H^k(C_2) & \xrightarrow{\delta_{\theta'}} & H^{k+1}(C_{1,r}) \cdots
 \end{array} \tag{A-17}$$

Following the argument after (5-27) we find that

$$\frac{d}{d\theta'} \Big|_{\theta'=\theta} \log \text{Det}(\phi_{\theta,\theta',*}^j)^2 = -2 \frac{2}{\sin 2\theta} \text{Tr}(\beta_{\theta} \upharpoonright H^j(C_1 \oplus_{\theta} C_2)), \tag{A-18}$$

$$\frac{d}{d\theta'} \Big|_{\theta'=\theta} \log \text{Det}(\tilde{\phi}_{\theta,\theta',*}^j) = -2 \frac{2}{\sin 2\theta} \dim C_2^j, \tag{A-19}$$

and hence with Lemma 2.5 applied to (A-17) we arrive at (A-8).

The analogue of (5-33) is

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_{1,r} \oplus C_{2,r} & \xrightarrow{\gamma_1 \oplus \gamma_2} & C_1 \oplus_{\theta} C_2 & \xrightarrow{r_{\theta}} & B & \longrightarrow & 0 \\
 & & \downarrow \text{id} \oplus \tilde{\phi}_{\theta,\theta'} & & \downarrow \phi_{\theta,\theta'} & & \downarrow \psi_{\theta,\theta'} = \frac{\tan \theta}{\tan \theta'} \text{id} & & \\
 0 & \longrightarrow & C_{1,r} \oplus C_{2,r} & \xrightarrow{\alpha_{\theta'}} & C_1 \oplus_{\theta'} C_2 & \xrightarrow{r_{\theta'}} & B & \longrightarrow & 0.
 \end{array} \tag{A-20}$$

We apply Lemma 2.5 to the induced isomorphism of the long exact cohomology sequences and find

$$\begin{aligned}
 & \log \tau(\mathcal{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_{\theta} C_2, B)) - \log \tau(\mathcal{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_{\theta'} C_2, B)) \\
 &= - \sum_{j \geq 0} (-1)^j \log \text{Det}(\phi_{\theta,\theta',*} : H^j \rightarrow H^j) + \sum_{j \geq 0} (-1)^j \log \text{Det}(\tilde{\phi}_{\theta,\theta',*} : H^j \rightarrow H^j) \\
 & \quad + \sum_{j \geq 0} (-1)^j \log \text{Det}(\Psi_{\theta,\theta',*} : H^j \rightarrow H^j), \tag{A-21}
 \end{aligned}$$

where H^j is shorthand for the respective cohomology groups. Since $\tilde{\phi}_{\theta,\theta'}$ and $\phi_{\theta,\theta'}$ are multiplication operators we have

$$\sum_{j \geq 0} (-1)^j \log \text{Det}(\tilde{\phi}_{\theta,\theta',*} : H^j \rightarrow H^j) = \chi(C_{2,r}) \log \frac{\tan \theta}{\tan \theta'}, \tag{A-22}$$

$$\sum_{j \geq 0} (-1)^j \log \text{Det}(\Psi_{\theta,\theta',*} : H^j \rightarrow H^j) = \chi(B) \log \frac{\sin \theta}{\sin \theta'}, \tag{A-23}$$

and together with (A-18) we obtain

$$\begin{aligned} \frac{d}{d\theta} \log \tau(\mathcal{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_{\theta} C_2, B)) \\ = \frac{2}{\sin 2\theta} \left(-\sum_{j \geq 0} (-1)^j \operatorname{Tr}(\beta_{\theta} \upharpoonright H^j(C_1 \oplus_{\theta} C_2)) + \chi(C_{2,r}) \right) - \frac{\cos \theta}{\sin \theta} \chi(B). \end{aligned} \quad (\text{A-24})$$

Taking into account that $\chi(C_{2,r}) = \chi(C_2) - \chi(B)$ (see (A-3)) and that

$$\frac{\cos \theta}{\sin \theta} - \frac{2}{\sin 2\theta} = -\tan \theta,$$

we find (A-9).

Next we apply Proposition 2.6 to the exact sequence (A-4) and get

$$\begin{aligned} \log \tau(C_1 \oplus_{\theta} C_2) = \log \tau(C_{1,r}) + \log \tau(C_2) + \log \tau(\mathcal{H}(C_{1,r}, C_1 \oplus_{\theta} C_2, C_2)) \\ + \frac{1}{2} \sum_{j \geq 0} (-1)^j \log \operatorname{Det}(\beta\beta^* : C_2^j \rightarrow C_2^j). \end{aligned} \quad (\text{A-25})$$

Here we have used (2-29) and that α is a partial isometry and thus $\alpha^*\alpha = \operatorname{id}$. Analogously, we infer from (A-5) that

$$\begin{aligned} \log \tau(C_1 \oplus_{\theta} C_2) = \log \tau(C_{1,r}) + \log \tau(C_{2,r}) + \log \tau(\mathcal{H}(C_{1,r} \oplus C_{2,r}, C_1 \oplus_{\theta} C_2, B)) \\ + \frac{1}{2} \sum_{j \geq 0} (-1)^j \log \operatorname{Det}(r_{\theta}r_{\theta}^* : B^j \rightarrow B^j). \end{aligned} \quad (\text{A-26})$$

From (A-25) and (A-26) one deduces the differentiability statement (A-10).

Finally we discuss the case that the maps $r_j, j = 1, 2$ are partial isometries. Then for $(\xi_1, \xi_2) \in C_1 \oplus_{\theta} C_2, \eta \in B$ we calculate

$$\langle r_{\theta}(\xi_1, \xi_2), b \rangle = \sin \theta \cdot \langle r_1 \xi_1, b \rangle + \cos \theta \cdot \langle r_2 \xi_2, b \rangle = \langle (\xi_1, \xi_2), (\sin \theta \cdot r_1^* b, \cos \theta \cdot r_2^* b) \rangle. \quad (\text{A-27})$$

If r_1 and r_2 are partial isometries then $(\sin \theta \cdot r_1^* b, \cos \theta \cdot r_2^* b) \in C_1 \oplus_{\theta} C_2$ and hence it equals $r_{\theta}^*(b)$. Consequently $r_{\theta}r_{\theta}^*b = (\sin^2 \theta + \cos^2 \theta)b = b$ and thus $\operatorname{Det}(r_{\theta}r_{\theta}^* : B^j \rightarrow B^j) = 1$. Therefore (A-26) reduces to (A-12).

Similarly, one calculates

$$\operatorname{Det}(\beta\beta^* : C_2^j \rightarrow C_2^j) = (1 + \tan^2)^{-\dim B^j}; \quad (\text{A-28})$$

then (A-13) follows from (A-25). □

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