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INEQUALITY FOR BURKHOLDER'S MARTINGALE TRANSFORM

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We find the sharp constant $C = C(\tau, p, \mathbb{E}G, \mathbb{E}F)$ of the inequality $\|(G^2 + \tau^2 F^2)^{1/2}\|_p \le C\|F\|_p$, where G is the transform of a martingale F under a predictable sequence ε with absolute value 1, $1 < p < 2$, and τ is any real number.

1. Introduction

Let I be an interval of the real line $\mathbb{R}$, and let $|I|$ be its Lebesgue length. We write $\mathcal{B}$ for the σ-algebra of Borel subsets of I. Let $\{F_n\}_{n=0}^{\infty}$ be a martingale on the probability space $(I, \mathcal{B}, dx/|I|)$ with a filtration $\{I, \varnothing\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$. Consider any sequence of functions $\{\varepsilon_n\}_{n=1}^{\infty}$ such that, for each $n \ge 1$, ε_n is $\mathcal{F}_{n-1}$ measurable and $|\varepsilon_n| \le 1$. Let G_0 be a constant function on I; for any $n \ge 1$, let G_n denote

$$G_0 + \sum_{k=1}^{n} \varepsilon_k (F_k - F_{k-1}).$$

The sequence $\{G_n\}_{n=0}^{\infty}$ is called the *martingale transform* of $\{F_n\}$. Obviously $\{G_n\}_{n=0}^{\infty}$ is a martingale with the same filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Note that, since $\{F_n\}$ and $\{G_n\}$ are martingales, we have $F_0 = \mathbb{E}F_n$ and $G_0 = \mathbb{E}G_n$ for any $n \ge 0$.

Burkholder [1984] proved that if $|G_0| \le |F_0|$, $1 < p < \infty$, then we have the sharp estimate

$$\|G_n\|_{L^p} \le (p^* - 1)\|F_n\|_{L^p} \quad \text{for all } n \ge 0, \tag{1}$$

where $p^* - 1 = \max\{p - 1, 1/(p-1)\}$. Burkholder showed that it is sufficient to prove inequality (1) for the sequences of numbers $\{\varepsilon_n\}$ such that $\varepsilon_n = \pm 1$ for all $n \ge 1$. It was also noted that such an estimate

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as (1) does not depend on the choice of filtration $\{\mathcal{F}_n\}$. For example, one can consider only the dyadic filtration. For more information on the estimate (1) we refer the reader to [Burkholder 1984; Choi 1992].

Vasyunin and Volberg [2010] slightly generalized the result by the Bellman function technique and Monge–Ampère equation, i.e., the estimate (1) holds if and only if

$$|G_0| \le (p^* - 1)|F_0|. \tag{2}$$

In what follows we assume that $\{\varepsilon_n\}$ is a predictable sequence of functions such that $|\varepsilon_n| = 1$.

In [Boros et al. 2012], a perturbation of the martingale transform was investigated. Namely, under the same assumptions as (2) it was proved that, for $2 \le p < \infty$, $\tau \in \mathbb{R}$, we have the sharp estimate

$$\|(G_n^2 + \tau^2 F_n^2)^{1/2}\|_{L^p} \le ((p^* - 1)^2 + \tau^2)^{1/2} \|F_n\|_{L^p} \quad \text{for all } n \ge 0. \tag{3}$$

It was also claimed to be proven that the same sharp estimate holds for $1 < p < 2$, $|\tau| \le 0.5$, and the case $1 < p < 2$, $|\tau| > 0.5$ was left open.

The inequality (3) stems from important questions concerning the L^p bounds for the perturbation of the Beurling–Ahlfors operator and hence it is of interest. We refer the reader to recent works regarding martingale inequalities and estimates of the Beurling–Ahlfors operator [Bañuelos and Janakiraman 2008; Bañuelos and Méndez-Hernández 2003; Bañuelos and Osękowski 2013; Bañuelos and Wang 1995; Boros et al. 2012] and references therein.

We should mention that Burkholder's [1984] method and the Bellman function approach [Vasyunin and Volberg 2010; Boros et al. 2012] have similar traces in the sense that both of them reduce the required estimate to finding a certain minimal diagonally concave function with prescribed boundary conditions. However, the methods of construction of such a function are different. Unlike Burkholder's method, in [Vasyunin and Volberg 2010; Boros et al. 2012] the construction of the function is based on the Monge–Ampère equation.

1.1. ***Our main results.*** Firstly, we should mention that the proof of (3) presented in [Boros et al. 2012] has a gap in the case $1 < p < 2$, $0 < |\tau| \le 0.5$ (the constructed function does not satisfy a necessary concavity condition).

In the present paper we obtain the sharp L^p estimate of the perturbed martingale transform for the remaining case $1 < p < 2$ and for all $\tau \in \mathbb{R}$. Moreover, we do not require condition (2).

We define

$$u(z) \stackrel{\text{def}}{=} \tau^p (p-1)(\tau^2 + z^2)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1.$$

Theorem 1. *Let $1 < p < 2$, and let $\{G_n\}_{n=0}^{\infty}$ be a martingale transform of $\{F_n\}_{n=0}^{\infty}$. Set $\beta = \dfrac{|G_0| - |F_0|}{|G_0| + |F_0|}$. The following estimates are sharp*:

(1) *If $u(1/(p-1)) \le 0$, then*

$$\|(\tau^2 F_n^2 + G_n^2)^{1/2}\|_{L^p} \le \left(\tau^2 + \max\left\{\left|\frac{G_0}{F_0}\right|, \frac{1}{p-1}\right\}^2\right)^{\frac{1}{2}} \|F_n\|_{L^p} \quad \textit{for all } n \ge 0.$$

(2) *If* $u(1/(p-1)) > 0$, *then*

$$\|(\tau^2 F_n^2 + G_n^2)^{1/2}\|_{L^p}^p \le C(\beta)\|F_n\|_{L^p}^p \quad \textit{for all } n \ge 0,$$

where $C(\beta)$ *is continuous, nondecreasing, and defined as follows*:

$$C(\beta) \stackrel{\text{def}}{=} \begin{cases} (\tau^2 + |G_0|^2/|F_0|^2)^{p/2} & \textit{if } \beta \ge s_0, \\ \tau^p \left(1 - \dfrac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}\right)^{-1} & \textit{if } \beta \le -1+2/p, \\ C(\beta) & \textit{if } \beta \in (-1+2/p, s_0), \end{cases}$$

where $s_0 \in (-1+2/p, 1)$ *is the solution of the equation* $u((1+s_0)/(1-s_0)) = 0$.

Explicit expression for the function $C(\beta)$ on the interval $(-1+2/p, s_0)$ was hard to present in a simple way. The reader can find the value of the function $C(\beta)$ in Theorem 39(ii).

Remark 2. The condition $u(1/(p-1)) \le 0$ holds when $|\tau| \le 0.822$. So we also obtain Burkholder's result in the limit case when $\tau = 0$. It is worth mentioning that although the proof of the estimate (3) has a gap in [Boros et al. 2012], the claimed result in the case $1 < p < 2$, $|\tau| < 0.5$ remains true as a result of Theorem 1.

One of the important results of the current paper is that we find the function (5), and the above estimates are corollaries of this result. The argument we exploit is different from [Vasyunin and Volberg 2010; Boros et al. 2012]. Instead of writing a lot of technical computations and checking which case is valid, we present some pure geometrical facts regarding minimal concave functions with prescribed boundary conditions, and in this way we avoid computations. Moreover, we explain to the reader how we construct our Bellman function (5) based on these geometrical facts, derived in Section 3.

1.2. ***Plan of the paper.*** In Section 2 we formulate results about how to reduce the estimate (3) to finding a certain function with required properties. These results are well known and can be found in [Boros et al. 2012]. A slightly different function was investigated in [Vasyunin and Volberg 2010]; however, it possesses almost the same properties and the proof works exactly in the same way. We only mention these results and the fact that we look for a minimal continuous diagonally concave function $H(x_1, x_2, x_3)$ (see Definition 7) in the domain $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1|^p \le x_3\}$ with the boundary condition $H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}$.

Section 3 is devoted to the investigation of the minimal concave functions in two variables. It is worth mentioning that the first crucial steps in this direction for some special cases were made in [Ivanishvili et al. 2012a] (see also [Ivanishvili et al. 2012b; ≥ 2015]). In Section 3 we develop this theory for a slightly more general case. We investigate a special foliation called the *cup* and another useful object, called *force functions*.

We should note that the theory of minimal concave functions in two variables does not include the minimal diagonally concave functions in three variables. Nevertheless, this knowledge allows us to construct the candidate for H in Section 4, but with some additional technical work not mentioned in Section 3.

In Section 5 we find the good estimates for the perturbed martingale transform. In Section 6 we prove that the candidate for H constructed in Section 4 coincides with H, and as a corollary we show the sharpness of the estimates found for the perturbed martingale transform in Section 5.

In conclusion, the reader can note that the hard technical part of the current paper lies in the construction of the minimal diagonally concave function in three variables with the given boundary condition.

2. Definitions and known results

Let $\mathbb{E}F \stackrel{\text{def}}{=} \langle F\rangle_I$, where

$$\langle F\rangle_J \stackrel{\text{def}}{=} \frac{1}{|J|}\int_J F(t)\,dt$$

for any interval J of the real line. Let F and G be real valued integrable functions. Let $G_n = \mathbb{E}(G|\mathcal{M}_n)$ and $F_n = \mathbb{E}(F|\mathcal{M}_n)$ for $n \geq 0$, where $\{\mathcal{M}_n\}$ is a dyadic filtration (see [Boros et al. 2012]).

Definition 3. If the martingale $\{G_n\}$ satisfies $|G_{n+1} - G_n| = |F_{n+1} - F_n|$ for each $n \geq 0$, then G is called the martingale transform of F.

Recall that we are interested in the estimate

$$\|(G^2 + \tau^2 F^2)^{1/2}\|_{L^p} \leq C\|F\|_{L^p}. \tag{4}$$

We introduce the Bellman function

$$H(\boldsymbol{x}) \stackrel{\text{def}}{=} \sup_{F,G}\bigl\{\mathbb{E}\boldsymbol{B}(\varphi(F,G)) : \mathbb{E}\varphi(F,G) = \boldsymbol{x},\ |G_{n+1} - G_n| = |F_{n+1} - F_n|,\ n \geq 0\bigr\}, \tag{5}$$

where $\varphi(x_1, x_2) = (x_1, x_2, |x_1|^p)$, $\boldsymbol{B}(\varphi(x_1, x_2)) = (x_2^2 + \tau^2 x_1^2)^{p/2}$ and $\boldsymbol{x} = (x_1, x_2, x_3)$.

Remark 4. In what follows, bold lowercase letters denote points in $\mathbb{R}^3$.

Then we see that the estimate (4) can be rewritten as follows:

$$H(x_1, x_2, x_3) \leq C^p x_3.$$

We mention that the Bellman function H does not depend on the choice of the interval I. Without loss of generality, we may assume that $I = [0, 1]$.

Definition 5. Given a point $\boldsymbol{x} \in \mathbb{R}^3$, a pair (F, G) is said to be admissible for $\boldsymbol{x}$ if G is the martingale transform of F and $\mathbb{E}(F, G, |F|^p) = \boldsymbol{x}$.

Proposition 6. *The domain of $H(\boldsymbol{x})$ is $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1|^p \leq x_3\}$, and H satisfies the boundary condition*

$$H(x_1, x_2, |x_1|^p) = (x_2^2 + \tau^2 x_1^2)^{p/2}. \tag{6}$$

Definition 7. A function U is said to be diagonally concave in Ω if it is concave in both

$$\Omega \cap \{(x_1, x_2, x_3) : x_1 + x_2 = A\} \quad\text{and}\quad \Omega \cap \{(x_1, x_2, x_3) : x_1 - x_2 = A\}$$

for every constant $A \in \mathbb{R}$.

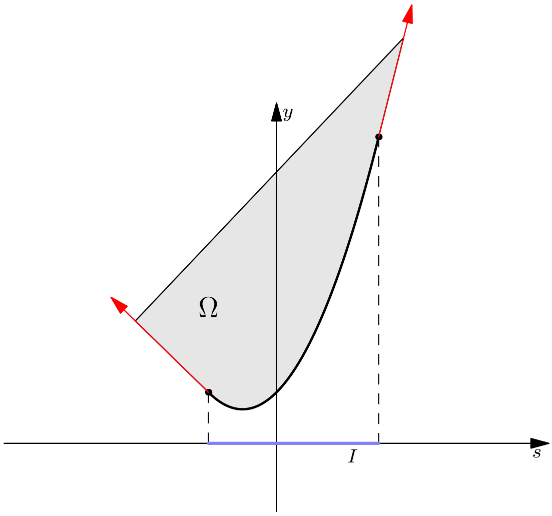

Figure 1. A domain Ω.

Proposition 8. *$H(\boldsymbol{x})$ is a diagonally concave function in Ω.*

Proposition 9. *If U is a continuous, diagonally concave function in Ω with the boundary condition $U(x_1, x_2, |x_1|^p) \geq (x_2^2 + \tau^2 x_1^2)^{p/2}$, then $U \geq H$ in Ω.*

We explain our strategy of finding the Bellman function H. We are going to find a minimal candidate $\boldsymbol{B}$ that is continuous and diagonally concave, with the fixed boundary condition $\boldsymbol{B}|_{\partial\Omega} = (y^2 + \tau^2 x^2)^{p/2}$. We warn the reader that the symbol $\boldsymbol{B}$ denoted boundary data previously, however, in Section 6 we are going to use the symbol $\boldsymbol{B}$ as the candidate for the minimal diagonally concave function. Obviously, $\boldsymbol{B} \geq H$ by Proposition 9. We will also see that, given $\boldsymbol{x} \in \Omega$ and any $\varepsilon > 0$, we can construct an admissible pair (F, G) such that $\boldsymbol{B}(\boldsymbol{x}) < \mathbb{E}(F^2 + \tau^2 G^2)^{p/2} + \varepsilon$. This will show that $\boldsymbol{B} \leq H$ and hence $\boldsymbol{B} = H$.

In order to construct the minimal candidate $\boldsymbol{B}$, we have to elaborate a few preliminary concepts from differential geometry. We introduce the notions of *foliation* and *force* functions.

3. Homogeneous Monge–Ampère equation and minimal concave functions

3.1. *Foliation.* Let $g(s) \in C^3(I)$ be such that $g'' > 0$, and let Ω be a convex domain which is bounded by the curve $(s, g(s))$ and the tangents that pass through the endpoints of the curve (see Figure 1). Fix some function $f(s) \in C^3(I)$. The first question we ask is the following: how the minimal concave function $B(x_1, x_2)$ with boundary data $B(s, g(s)) = f(s)$ looks *locally* in a subdomain of Ω. In other words, take a convex hull of the curve $(s, g(s), f(s))$, $s \in I$; then the question is how the boundary of this convex hull looks.

We recall that the concavity is equivalent to the following inequalities:

$$\det(\mathrm{d}^2 B) \geq 0, \tag{7}$$

$$B''_{x_1x_1} + B''_{x_2x_2} \leq 0. \tag{8}$$

The expression (7) is the Gaussian curvature of the surface $(x_1, x_2, B(x_1, x_2))$ up to a positive factor $(1+(B'_{x_1})^2+(B'_{x_2})^2)^2$. So, in order to minimize the function $B(x_1, x_2)$, it is reasonable to minimize the Gaussian curvature. Therefore, we will look for a surface with zero Gaussian curvature. Here the homogeneous Monge–Ampère equation arises. These surfaces are known as *developable surfaces*, that is, such a surface can be constructed by bending a plane region. The important property of such surfaces is that they consist of line segments, i.e., the function B satisfying the homogeneous Monge–Ampère equation $\det(\mathrm{d}^2 B)=0$ is linear along some *family of segments*. These considerations lead us to investigate such functions B. Firstly, we define a *foliation*. For any segment ℓ in the Euclidean space, by ℓ° we denote its open segment, ℓ without endpoints.

Fix any subinterval $J \subseteq I$. By $\Theta(J, g)$ we denote an arbitrary set of nontrivial segments (i.e., single points are excluded) in $\mathbb{R}^2$ with the following requirements:

(1) For any $\ell \in \Theta(J, g)$ we have $\ell^\circ \in \Omega$.

(2) For any $\ell_1, \ell_2 \in \Theta(J, g)$ we have $\ell_1 \cap \ell_2 = \varnothing$.

(3) For any $\ell \in \Theta(J, g)$ there exists only one point $s \in J$ such that $(s, g(s))$ is one of the endpoints of the segment ℓ and, vice versa, for any point $s \in J$ there exists $\ell \in \Theta(J, g)$ such that $(s, g(s))$ is one of the endpoints of the segment ℓ.

(4) There exists a C^1 smooth argument function $\theta(s)$.

We explain the meaning of the requirement (4). To each point $s \in J$ there corresponds only one segment $\ell \in \Theta(J, g)$ with an endpoint $(s, g(s))$. Take a nonzero vector with initial point $(s, g(s))$, parallel to the segment ℓ and having an endpoint in Ω. We define the value of $\theta(s)$ to be an argument of this vector. Since argument is defined up to addition by $2\pi k$, where $k \in \mathbb{Z}$, we take any representative from these angles. We do the same for all other points $s \in I$. In this way we get a family of functions $\theta(s)$. If there exists a $C^1(J)$ smooth function $\theta(s)$ from this family then requirement (4) is satisfied.

Remark 10. It is clear that, if $\theta(s)$ is a $C^1(J)$ smooth argument function, then, for any $k \in \mathbb{Z}$, $\theta(s)+2\pi k$ is also a $C^1(J)$ smooth argument function. Any two $C^1(J)$ smooth argument functions differ by a constant $2\pi n$ for some $n \in \mathbb{Z}$.

This remark is a consequence of the fact that the quantity $\theta'(s)$ is well defined regardless of the choices of $\theta(s)$. Next, we define $\Omega(\Theta(J, g)) = \bigcup_{\ell\in\Theta(J,g)} \ell^\circ$. Given a point $x \in \Omega(\Theta(J, g))$, we denote by $\ell(x)$ a segment in $\Theta(J, g)$ which passes through the point x. If $x = (s, g(s))$ then, instead of $\ell\big((s, g(s))\big)$, we just write $\ell(s)$. Surely such a segment exists, and it is unique. We denote by $s(x)$ a point $s(x) \in J$ such that $\big(s(x), g(s(x))\big)$ is one of the endpoints of the segment $\ell(x)$. Moreover, in a natural way we set $s(x) = s$ if $x = (s, g(s))$. It is clear that such $s(x)$ exists, and it is unique. We introduce a function

$$K(s) = g'(s)\cos\theta(s) - \sin\theta(s), \qquad s \in J. \tag{9}$$

Note that $K < 0$. This inequality becomes obvious if we rewrite

$$g'(s)\cos\theta(s) - \sin\theta(s) = \langle (1, g'), (-\sin\theta, \cos\theta)\rangle$$

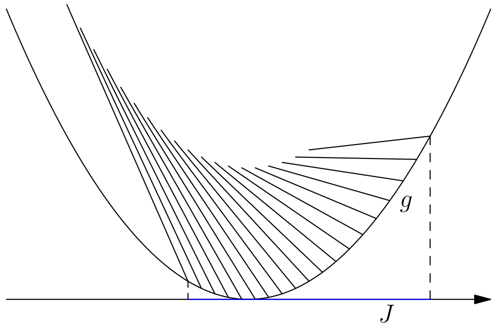

Figure 2. A foliation $\Theta(J, g)$.

and take into account requirement (1) of $\Theta(J, g)$. Note that $\langle \cdot, \cdot \rangle$ means scalar product in Euclidean space.

We need two more requirements on $\Theta(J, g)$.

(5) For any $x = (x_1, x_2) \in \Omega(\Theta(J, g))$, we have $K(s(x)) + \theta'(s(x)) \big\| \big(x_1 - s(x), x_2 - g(s(x))\big) \big\| < 0$.

(6) The function $s(x)$ is continuous in $\Omega(\Theta(J, g)) \cup \Gamma(J)$, where $\Gamma(J) = \{(s, g(s)) : s \in J\}$.

Note that if $\theta'(s) \le 0$ (which happens in most of the cases) then requirement (5) holds. If we know the endpoints of the segments $\Theta(J, g)$, then in order to verify (5) it is enough to check it at those points $x = (x_1, x_2)$, where x is an endpoint of the segment other than $(s, g(s))$. Roughly speaking, requirement (5) means the segments of $\Theta(J, g)$ do not rotate rapidly counterclockwise.

Definition 11. A set of segments $\Theta(J, g)$ with the requirements mentioned above is called a *foliation*. The set $\Omega(\Theta(J, g))$ is called the *domain of foliation*.

A typical example of a foliation is given in Figure 2.

Lemma 12. *The function $s(x)$ belongs to $C^1\big(\Omega(\Theta(J, g))\big)$. Moreover,*

$$(s'_{x_1}, s'_{x_2}) = \frac{(\sin\theta, -\cos\theta)}{-K(s) - \theta' \cdot \|(x_1 - s, x_2 - g(s))\|}. \tag{10}$$

Proof. The definition of the function $s(x)$ implies that

$$-(x_1 - s)\sin\theta(s) + (x_2 - g(s))\cos\theta(s) = 0.$$

Therefore the lemma is an immediate consequence of the implicit function theorem. □

Let $J = [s_1, s_2] \subseteq I$, and let $(s, g(s), f(s)) \in C^3(I)$ be such that $g'' > 0$ on I. Consider an arbitrary foliation $\Theta(J, g)$ with an arbitrary $C^1([s_1, s_2])$ smooth argument function $\theta(s)$. We need the following technical lemma:

Lemma 13. *The solutions of the system of equations*

$$t_1'(s)\cos\theta(s) + t_2'(s)\sin\theta(s) = 0, \tag{11}$$

$$t_1(s) + t_2(s)g'(s) = f'(s), \quad s \in J \tag{12}$$

are the functions

$$t_1(s) = \int_{s_1}^{s} \left(\frac{g''(r)}{K(r)} \sin\theta(r) \cdot t_2(r) - \frac{f''(r)}{K(r)} \sin\theta(r) \right) dr + f'(s_1) - t_2(s_1) g'(s_1),$$

$$t_2(s) = t_2(s_1) \exp\left(- \int_{s_1}^{s} \frac{g''(r)}{K(r)} \cos\theta(r)\, dr \right) + \int_{s_1}^{s} \frac{f''(y)}{K(y)} \exp\left(- \int_{y}^{s} \frac{g''(r)}{K(r)} \cos\theta(r)\, dr \right) \cos\theta(y)\, dy$$

for $s \in J$, where $t_2(s_1)$ is an arbitrary real number.

Proof. We differentiate (12) and combine it with (11) to obtain the system

$$\begin{pmatrix} \cos\theta & \sin\theta \\ 1 & g' \end{pmatrix} \begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -g'' \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f'' \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} t_1' \\ t_2' \end{pmatrix} = \frac{g''}{K} \begin{pmatrix} 0 & \sin\theta \\ 0 & -\cos\theta \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \frac{f''}{K} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{13}$$

By solving this system of differential equations and using the fact that $t_1(s_1) + g'(s_1)t_2(s_1) = f'(s_1)$, we get the desired result. □

Remark 14. Integration by parts allows us to rewrite the expression for $t_2(s)$ as

$$t_2(s) = \exp\left(- \int_{s_1}^{s} \frac{g''(r)}{K(r)} \cos\theta(r)\, dr \right) \left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)} \right) + \frac{f''(s)}{g''(s)} - \int_{s_1}^{s} \left[\frac{f''(y)}{g''(y)} \right]' \exp\left(- \int_{y}^{s} \frac{g''(r)}{K(r)} \cos\theta(r)\, dr \right) dy.$$

Definition 15. We say that a function B has a foliation $\Theta(J, g)$ if it is continuous on $\Omega(\Theta(J, g))$ and it is linear on each segment of $\Theta(J, g)$.

The following lemma describes how to construct a function B with a given foliation $\Theta(J, g)$ and boundary condition $B(s, g(s)) = f(s)$ such that B satisfies the homogeneous Monge–Ampère equation.

Consider the function B defined by

$$B(x) = f(s) + \langle t(s), x - (s, g(s)) \rangle, \quad x = (x_1, x_2) \in \Omega(\Theta(J, g)), \tag{14}$$

where $s = s(x)$, and $t(s) = (t_1(s), t_2(s))$ satisfies the system of equations (11), (12) with an arbitrary $t_2(s_1)$.

Lemma 16. *The function B defined by* (14) *satisfies the following properties*:

(1) $B \in C^2\big(\Omega(\Theta(J, g))\big) \cap C^1\big(\Omega(\Theta(J, g)) \cup \Gamma\big)$, *$B$ has the foliation $\Theta(J, g)$ and*

$$B(s, g(s)) = f(s) \quad \text{for all } s \in [s_1, s_2]. \tag{15}$$

(2) $\nabla B(x) = t(s)$, *where $s = s(x)$; moreover, B satisfies the homogeneous Monge–Ampère equation.*

Proof. The fact that B has the foliation $\Theta(J, g)$ and that it satisfies the equality (15) immediately follows from the definition of the function B. We check the condition of smoothness. By Lemma 12 and Lemma 13 we have $s(x) \in C^2\big(\Omega(\Theta(J, g))\big)$ and $t_1, t_2 \in C^1(J)$, therefore the right-hand side of (14) is differentiable with respect to x. So, after differentiation of (14), we get

$$\nabla B(x) = \big[f'(s) - \big\langle t(s), (1, g'(s))\big\rangle\big](s'_{x_1}, s'_{x_2}) + t(s) + \big\langle t'(s), x - (s, g(s))\big\rangle(s'_{x_1}, s'_{x_2}). \tag{16}$$

Using (11) and (12) we obtain $\nabla B(x) = t(s)$. Taking the derivative with respect to x a second time we get

$$\frac{\partial^2 B}{\partial x_1^2} = t_1'(s)s'_{x_1}, \quad \frac{\partial^2 B}{\partial x_2 \partial x_1} = t_1'(s)s'_{x_2}, \quad \frac{\partial^2 B}{\partial x_1 \partial x_2} = t_2'(s)s'_{x_1}, \quad \frac{\partial^2 B}{\partial x_2^2} = t_2'(s)s'_{x_2}.$$

Using (11) we get that $t_1'(s)s'_{x_2} = t_2'(s)s'_{x_1}$, therefore $B \in C^2\big(\Omega(\Theta(J, g))\big)$. Finally, we check that B satisfies the homogeneous Monge–Ampère equation. Indeed,

$$\det(\mathrm{d}^2 B) = \frac{\partial^2 B}{\partial x_1^2} \cdot \frac{\partial^2 B}{\partial x_2^2} - \frac{\partial^2 B}{\partial x_2 \partial x_1} \cdot \frac{\partial^2 B}{\partial x_1 \partial x_2} = t_1'(s)s'_{x_1} \cdot t_2'(s)s'_{x_1} - t_1'(s)s'_{x_2} \cdot t_2'(s)s'_{x_1} = 0. \qquad \square$$

Definition 17. The function $t(s) = (t_1(s), t_2(s)) = \nabla B(x)$, $s = s(x)$, is called the *gradient function* corresponding to B.

The following lemma investigates the concavity of the function B defined by (14). Let $\|\tilde{\ell}(x)\| = \big\|\big(s(x) - x_1, g(s(x)) - x_2\big)\big\|$, where $x = (x_1, x_2) \in \Omega(\Theta(J, g))$.

Lemma 18. *The following equalities hold*:

$$\begin{aligned}\frac{\partial^2 B}{\partial x_1^2} + \frac{\partial^2 B}{\partial x_2^2} &= \frac{g''}{K(K + \theta'\|\tilde{\ell}(x)\|)}\left(-t_2 + \frac{f''}{g''}\right)\\ &= \frac{g''}{K(K + \theta'\|\tilde{\ell}(x)\|)}\Bigg[-\exp\left(-\int_{s_1}^{s} \frac{g''(r)}{K(r)} \cos\theta(r)\, dr\right)\left(t_2(s_1) - \frac{f''(s_1)}{g''(s_1)}\right)\\ &\qquad + \int_{s_1}^{s} \left[\frac{f''(y)}{g''(y)}\right]' \exp\left(-\int_{y}^{s} \frac{g''(r)}{K(r)} \cos\theta(r)\, dr\right) dy\Bigg].\end{aligned}$$

Proof. Note that

$$\frac{\partial^2 B}{\partial x_1^2} + \frac{\partial^2 B}{\partial x_2^2} = t_1'(s)s_1' + t_2'(s)s_2'.$$

Therefore the lemma is a direct computation and application of (10), (11), (12) and Remark 14. $\square$

Finally, we get the following important statement:

Corollary 19. *The function B is concave in $\Omega(\Theta(J,g))$ if and only if $\mathcal{F}(s)\le 0$, where*

$$\begin{aligned}\mathcal{F}(s) &= -\exp\left(-\int_{s_1}^{s}\frac{g''(r)}{K(r)}\cos\theta(r)\,dr\right)\left(t_2(s_1)-\frac{f''(s_1)}{g''(s_1)}\right)\\ &\qquad\qquad + \int_{s_1}^{s}\left[\frac{f''(y)}{g''(y)}\right]'\exp\left(-\int_{y}^{s}\frac{g''(r)}{K(r)}\cos\theta(r)\,dr\right)dy\\ &= \frac{f''(s)}{g''(s)} - t_2(s).\end{aligned} \tag{17}$$

Proof. B satisfies the homogeneous Monge–Ampère equation. Therefore, B is concave if and only if

$$\frac{\partial^2 B}{\partial x_1^2}+\frac{\partial^2 B}{\partial x_2^2}\le 0. \tag{18}$$

Note that

$$\frac{g''}{K(K+\theta'\|\tilde{\ell}(x)\|)}>0.$$

Hence, according to Lemma 18, the inequality (18) holds if and only if $\mathcal{F}(s)\le 0$. □

Furthermore, the function $\mathcal{F}$ will be called a *force* function.

Remark 20. The fact that $t_2(s)=f''/g''-\mathcal{F}$ together with (13) implies that the force function $\mathcal{F}$ satisfies the differential equation

$$\begin{aligned}&\mathcal{F}'+\mathcal{F}\cdot\frac{\cos\theta}{K}g''-\left[\frac{f''}{g''}\right]'=0, \qquad s\in J,\\ &\mathcal{F}(s_1)=\frac{f''(s_1)}{g''(s_1)}-t_2(s_1).\end{aligned} \tag{19}$$

We remind the reader that, for an arbitrary smooth curve $\gamma=(s,g(s),f(s))$, the torsion has the expression

$$\frac{\det(\gamma',\gamma'',\gamma''')}{\|\gamma'\times\gamma''\|^2}=\frac{f'''g''-g'''f''}{\|\gamma'\times\gamma''\|^2}=\frac{(g'')^2}{\|\gamma'\times\gamma''\|^2}\cdot\left[\frac{f''}{g''}\right]'.$$

Corollary 21. *If $\mathcal{F}(s_1)\le 0$ and the torsion of a curve $(s,g(s),f(s))$, $s\in J$ is negative, then the function B defined by (14) is concave.*

Proof. The corollary is an immediate consequence of (17). □

Thus, we see that the torsion of the boundary data plays a crucial role in the concavity of a surface with zero Gaussian curvature. More detailed investigations about how we choose the constant $t_2(s_1)$ will be given in Section 3.2.

Let $\Theta(J,g)$ and $\tilde{\Theta}(J,g)$ be foliations with some argument functions $\theta(s)$ and $\tilde{\theta}(s)$, respectively. Let B and $\tilde{B}$ be the corresponding functions defined by (14), and let $\mathcal{F}$, $\tilde{\mathcal{F}}$ be the corresponding force functions. Note that $\mathcal{F}(s)=\tilde{\mathcal{F}}(s)$ is equivalent to the equality $t(s)=\tilde{t}(s)$, where $t(s)=(t_1(s),t_2(s))$ and $\tilde{t}(s)=(\tilde{t}_1(s),t_2(s))$ are the corresponding gradients of B and $\tilde{B}$ (see (12) and Corollary 19).

Assume that the functions B and $\tilde{B}$ are concave functions.

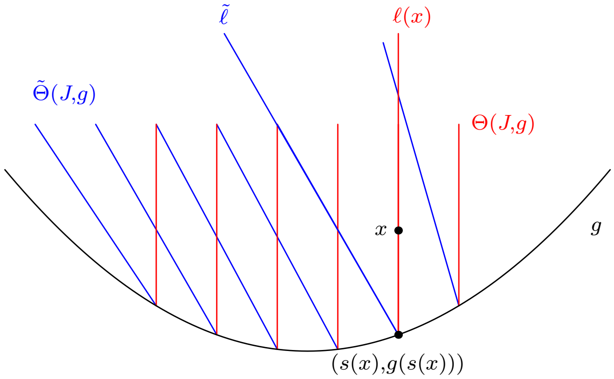

Figure 3. Foliations $\Theta(J, g)$ and $\tilde{\Theta}(J, g)$.

Lemma 22. *If* $\sin(\tilde{\theta} - \theta) \geq 0$ *for all* $s \in J$, *and* $\mathcal{F}(s_1) = \tilde{\mathcal{F}}(s_1)$, *then* $\tilde{B} \leq B$ *on* $\Omega(\Theta(J, g)) \cap \tilde{\Omega}(\Theta(J, g))$.

In other words, the lemma says that if, at the initial point $(s_1, g(s_1))$, gradients of the functions $\tilde{B}$ and B coincide and the foliation $\tilde{\Theta}(J, g)$ is "to the left of" the foliation $\Theta(J, g)$ (see Figure 3), then $\tilde{B} \leq B$ provided B and $\tilde{B}$ are concave.

Proof. Let K and $\tilde{K}$ be the corresponding functions of B and $\tilde{B}$ defined by (9). The condition $K, \tilde{K} < 0$ implies that the inequality $\sin(\tilde{\theta} - \theta) \geq 0$ is equivalent to the inequality

$$\frac{\cos\tilde{\theta}}{\tilde{K}} \geq \frac{\cos\theta}{K} \quad \text{for } s \in J. \tag{20}$$

Indeed, if we rewrite (20) as $K\cos\tilde{\theta} \geq \tilde{K}\cos\theta$ then this simplifies to $-\sin\theta\cos\tilde{\theta} \geq -\sin\tilde{\theta}\cos\theta$, so the result follows.

The force functions $\mathcal{F}$, $\tilde{\mathcal{F}}$ satisfy the differential equation (19) with the same boundary condition $\mathcal{F}(s_1) = \tilde{\mathcal{F}}(s_1)$. Then, by (20) and by comparison theorems, we get $\tilde{\mathcal{F}} \geq \mathcal{F}$ on J. This and (17) imply that $\tilde{t}_2 \leq t_2$ on J. Pick any point $x \in \Omega(\Theta(J, g)) \cap \tilde{\Omega}(\Theta(J, g))$. Then there exists a segment $\ell(x) \in \Theta(J, g)$. Let $\big(s(x), g(s(x))\big)$ be the corresponding endpoint of this segment. There exists a segment $\tilde{\ell} \in \tilde{\Theta}(J, g)$ which has $\big(s(x), g(s(x))\big)$ as an endpoint (see Figure 3).

Consider a tangent plane $L(x)$ to $(x_1, x_2, \tilde{B})$ at the point $\big(s(x), g(s(x))\big)$. The fact that the gradient of $\tilde{B}$ is constant on $\tilde{\ell}$ implies that L is tangent to $(x_1, x_2, \tilde{B})$ on $\tilde{\ell}$. Therefore,

$$L(x) = f(s) + \big\langle(\tilde{t}_1(s), \tilde{t}_2(s)), (x_1 - s, x_2 - g(s))\big\rangle,$$

where $x = (x_1, x_2)$ and $s = s(x)$. The concavity of $\tilde{B}$ implies that a value of the function $\tilde{B}$ at a point y seen from the point $(s(x), g(s(x)))$ is less than $L(y)$. In particular, $\tilde{B}(x) \leq L(x)$. Now it is enough to prove that $L(x) \leq B(x)$. By (14) we have

$$B(x) = f(s) + \big\langle(t_1(s), t_2(s)), (x_1 - s(x), x_2 - g(s))\big\rangle.$$

Therefore, using (12), the fact that $\big\langle(-g', 1), (x_1 - s, x_2 - g(s))\big\rangle \geq 0$ and $\tilde{t}_2 \leq t_2$, we get the desired result. □

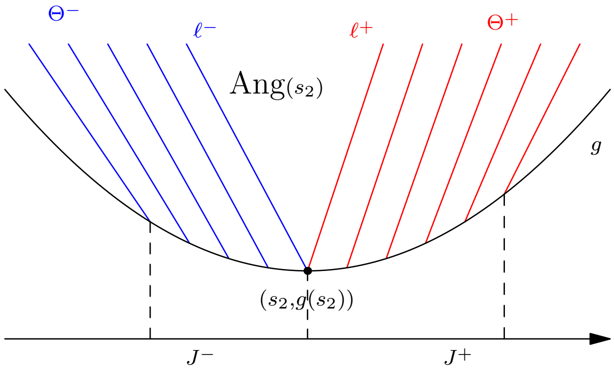

Figure 4. Gluing of B^- and B^+.

Let $J^- = [s_1, s_2]$ and $J^+ = [s_2, s_3]$, where $J^-, J^+ \subset I$. Consider arbitrary foliations $\Theta^- = \Theta^-(J^-, g)$ and $\Theta^+ = \Theta^+(J^+, g)$ such that $\Omega(\Theta^-) \cap \Omega(\Theta^+) = \varnothing$, and let θ^- and θ^+ be the corresponding argument functions. Let B^- and B^+ be the corresponding functions defined $\ell^+(s_2)$, where $\ell^-(s_2) \in \Theta^-$ by (14), and let $t^- = (t_1^-, t_2^-)$ and $t^+ = (t_1^+, t_2^+)$ be the corresponding gradient functions. Set $\mathrm{Ang}(s_2)$ to be a convex hull of $\ell^-(s_2)$ and and $\ell^+(s_2) \in \Theta^+$ are the segments with the endpoint $(s_2, g(s_2))$ (see Figure 4). We require that $\mathrm{Ang}(s_2) \cap \Omega(\Theta^-) = \ell^-$ and $\mathrm{Ang}(s_2) \cap \Omega(\Theta^+) = \ell^+$.

Let $\mathcal{F}^-$, $\mathcal{F}^+$ be the corresponding forces, and let B_{Ang} be the function defined linearly on $\mathrm{Ang}(s_2)$ via the values of B^- and B^+ on ℓ^-, ℓ^+ respectively.

Lemma 23. *If $t_2^-(s_2) = t_2^+(s_2)$, then the function B defined by*

$$B(x) = \begin{cases} B^-(x) & \text{if } x \in \Omega(\Theta(J^-, g)), \\ B_{\mathrm{Ang}}(x) & \text{if } x \in \mathrm{Ang}(s_2), \\ B^+(x) & \text{if } x \in \Omega(\Theta(J^+, g)), \end{cases}$$

belongs to the class $C^1(\Omega(\Theta^-) \cup \mathrm{Ang}(s_2) \cup \Omega(\Theta^+) \cup \Gamma(J^- \cup J^+))$.

Proof. By (12) the condition $t_2^-(s_2) = t_2^+(s_2)$ is equivalent to the condition $t^-(s_2) = t^+(s_2)$. We recall that the gradient of B^- is constant on $\ell^-(s_2)$, and the gradient of B^+ is constant on $\ell^+(s_2)$, therefore the lemma follows immediately from the fact that $B^-(s_2, g(s_2)) = B^+(s_2, g(s_2))$. □

Remark 24. The fact $B \in C^1$ implies that its gradient function $t(s) = \nabla B$ is well defined and is continuous. Unfortunately, it is not necessarily true that $t(s) \in C^1([s_1, s_3])$. However, it is clear that $t(s) \in C^1([s_1, s_2])$ and $t(s) \in C^1([s_2, s_3])$.

We finish this section with the following important corollary about *concave extension* of the functions with zero Gaussian curvature:

Let B^- and B^+ be defined as above (see Figure 4). Assume that $t_2^-(s_2) = t_2^+(s_2)$.

Corollary 25. *If B^- is concave in $\Omega(\Theta^-)$ and the torsion of the curve $(s, g(s), f(s))$ is nonnegative on $J^+ = [s_2, s_3]$ then the function B defined in Lemma 23 is concave in the domain $\Omega(\Theta^-) \cup \mathrm{Ang}(s_2) \cup \Omega(\Theta^+)$.*

In other words, the corollary tells us that, if we have constructed a concave function B^- which satisfies the homogeneous Monge–Ampère equation, and we glue B^- smoothly with B^+ (which also satisfies the homogeneous Monge–Ampère equation), then the result, B, is a concave function provided that the space curve $(s, g(s), f(s))$ has nonnegative torsion on the interval J^+.

Proof. By Corollary 19, concavity of B^- implies $\mathcal{F}^-(s_2) \le 0$. By (17) the condition $t_2^-(s_2) = t_2^+(s_2)$ is equivalent to $\mathcal{F}^-(s_2) = \mathcal{F}^+(s_2)$. By Corollary 21 we get that B^+ is concave. Thus, concavity of B follows from Lemma 23. □

3.2. *Cup.* In this subsection we are going to consider a special type of foliation, which is called a *cup*. Fix an interval I and consider an arbitrary curve $(s, g(s), f(s)) \in C^3(I)$. We suppose that $g'' > 0$ on I. Let $a(s) \in C^1(J)$ be a function such that $a'(s) < 0$ on J, where $J = [s_0, s_1]$ is a subinterval of I. Assume that $a(s_0) < s_0$ and $[a(s_1), a(s_0)] \subset I$. Consider a set of open segments $\Theta_{\text{cup}}(J, g)$ consisting of those segments $\ell(s, g(s))$, $s \in J$ such that $\ell(s, g(s))$ is a segment in the plane joining the points $(s, g(s))$ and $\big(a(s), g(a(s))\big)$ (see Figure 5).

Lemma 26. *The set of segments $\Theta_{\text{cup}}(J, g)$ described above forms a foliation.*

Proof. We need to check the six requirements for a set to be the foliation. Most of them are trivial except for (4) and (5). We know the endpoints of each segment, therefore we can consider the argument function

$$\theta(s) = \pi + \arctan\left(\frac{g(s) - g(a(s))}{s - a(s)}\right).$$

Surely $\theta(s) \in C^1(J)$, so requirement (4) is satisfied. We check requirement (5). It is clear that it is enough to check this requirement for $x = (a(s), g(a(s)))$. Let $s = s(x)$; then

$$\begin{aligned} &K(s) + \theta'(s)\|(a(s) - s, g(a(s)) - g(s))\| \\ &\qquad = \frac{\langle (1, g'), (g - g(a), a - s)\rangle}{\|(g(a) - g, s - a)\|} + \frac{(g' - a'g'(a))(s - a) - (1 - a')(g - g(a))}{\|(g(a) - g, s - a)\|} \\ &\qquad = \frac{a' \cdot \langle (1, g'(a)), (g - g(a), a - s)\rangle}{\|(g(a) - g, s - a)\|}, \end{aligned}$$

which is strictly negative. □

Let $\gamma(t) = (t, g(t), f(t)) \in C^3([a_0, b_0])$ be an arbitrary curve such that $g'' > 0$ on $[a_0, b_0]$. Assume that the torsion of γ is positive on $I^- = (a_0, c)$, and it is negative on $I^+ = (c, b_0)$ for some $c \in (a_0, b_0)$.

Lemma 27. *For all P such that $0 < P < \min\{c - a_0, b_0 - c\}$, there exist $a \in I^-$, $b \in I^+$ such that $b - a = P$ and*

$$\begin{vmatrix} 1 & 1 & a - b \\ g'(a) & g'(b) & g(a) - g(b) \\ f'(a) & f'(b) & f(a) - f(b) \end{vmatrix} = 0. \tag{21}$$

Proof. Pick a number $a \in (a_0, b_0)$ such that $b = a + P \in (a_0, b_0)$. We denote

$$\mathcal{M}(a, b) = (a - b)(g'(b) - g'(a))\left(\frac{g(a) - g(b)}{a - b} - g'(a)\right).$$

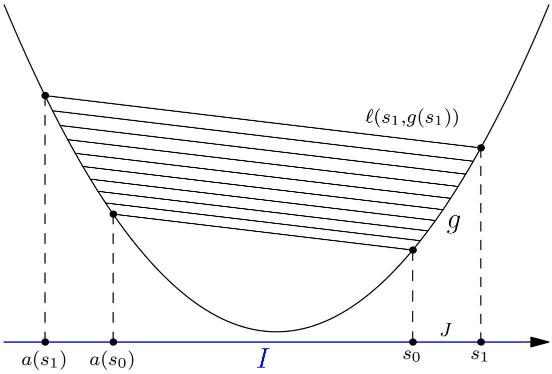

Figure 5. The foliation $\Theta_{\text{cup}}(J, g)$.

Note that the conditions $a \neq b$ and $g'' > 0$ imply $\mathcal{M}(a, b) \neq 0$. Then

$$\begin{vmatrix} 1 & 1 & a-b \\ g'(a) & g'(b) & g(a)-g(b) \\ f'(a) & f'(b) & f(a)-f(b) \end{vmatrix} = \mathcal{M}(a,b)\left[\frac{f(a)-f(b)-f'(a)(a-b)}{g(a)-g(b)-g'(a)(a-b)} - \frac{f'(b)-f'(a)}{g'(b)-g'(a)}\right].$$

Thus our equation (21) turns into

$$\frac{f(a)-f(b)-f'(a)(a-b)}{g(a)-g(b)-g'(a)(a-b)} - \frac{f'(b)-f'(a)}{g'(b)-g'(a)} = 0. \tag{22}$$

We consider the functions $V(x) = f(x) - f'(a)x$ and $U(x) = g(x) - g'(a)x$. Note that $U(a) \neq U(b)$ and $U' \neq 0$ on (a, b). Therefore, by Cauchy's mean value theorem there exists a point $\xi = \xi(a, b) \in (a, b)$ such that

$$\frac{f(a)-f(b)-f'(a)(a-b)}{g(a)-g(b)-g'(a)(a-b)} = \frac{V(a)-V(b)}{U(a)-U(b)} = \frac{V'(\xi)}{U'(\xi)} = \frac{f'(\xi)-f'(a)}{g'(\xi)-g'(a)}.$$

Now we define

$$W_a(z) \overset{\text{def}}{=} \frac{f'(z)-f'(a)}{g'(z)-g'(a)}, \quad z \in (a, b].$$

So the left-hand side of (22) takes the form $W_a(\xi) - W_a(b) = 0$ for some $\xi(a, P) \in (a, b)$. We consider the curve $v(s) = (g'(s), f'(s))$, which is a graph on $[a_0, b_0]$. The fact that the torsion of the curve $\gamma(s) = (s, g(s), f(s))$ changes sign from + to − at the point $c \in (a_0, b_0)$ means that the curve $v(s)$ is strictly convex on the interval (a_0, c) and it is strictly concave on the interval (c, b_0). We consider a function obtained from (22),

$$D(z) \overset{\text{def}}{=} \frac{f(z)-f(z+P)+f'(z)P}{g(z)-g(z+P)+g'(z)P} - \frac{f'(z+P)-f'(z)}{g'(z+P)-g'(z)}, \quad z \in [a_0, c]. \tag{23}$$

Note that $D(a_0) = W_{a_0}(\zeta) - W_{a_0}(a_0+P)$ for some $\zeta = \zeta(a_0, P) \in (a_0, a_0+P)$. We know that $v(s)$ is strictly convex on the interval $(a_0, a_0 + P)$. This implies that $W_{a_0}(z) - W_{a_0}(a_0 + P) < 0$ for all $z \in (a_0, a_0 + P)$. In particular, $D(a_0) < 0$. Similarly, concavity of $v(s)$ on $(c, c + P)$ implies that $D(c) > 0$. Hence, there exists $a \in (a_0, c)$ such that $D(a) = 0$. □

Let a_1 and b_1 be some solutions of (21) obtained by Lemma 27.

Lemma 28. *There exists a function* $a(s) \in C^1((c, b_1]) \cap C([c, b_1])$ *such that* $a(b_1) = a_1$, $a(c) = c$, $a'(s) < 0$, *and the pair* $(a(s), s)$ *solves* (21) *for all* $s \in [c, b_1]$.

Proof. The proof of the lemma is a consequence of the implicit function theorem. Let $a < b$, and consider the function

$$\Phi(a, b) \stackrel{\text{def}}{=} \begin{vmatrix} 1 & 1 & a-b \\ g'(a) & g'(b) & g(a)-g(b) \\ f'(a) & f'(b) & f(a)-f(b) \end{vmatrix}.$$

We are going to find the signs of the partial derivatives of $\Phi(a, b)$ at the point $(a, b) = (a_1, b_1)$. We present the calculation only for $\partial\Phi/\partial b$. The case for $\partial\Phi/\partial a$ is similar.

$$\begin{aligned} \frac{\partial\Phi(a, b)}{\partial b} &= \begin{vmatrix} 1 & 0 & a-b \\ g'(a) & g''(b) & g(a)-g(b) \\ f'(a) & f''(b) & f(a)-f(b) \end{vmatrix} \\ &= (a-b)g''(b)\left(\frac{g(a)-g(b)}{a-b} - g'(a)\right)\left[\frac{f(a)-f(b)-f'(a)(a-b)}{g(a)-g(b)-g'(a)(a-b)} - \frac{f''(b)}{g''(b)}\right]. \end{aligned}$$

Note that

$$(a-b)g''(b)\left(\frac{g(a)-g(b)}{a-b} - g'(a)\right) < 0,$$

therefore we see that the sign of $\partial\Phi/\partial b$ depends only on the sign of the expression

$$\frac{f(a)-f(b)-f'(a)(a-b)}{g(a)-g(b)-g'(a)(a-b)} - \frac{f''(b)}{g''(b)}. \tag{24}$$

We use the *cup equation* (22), and we obtain that the expression (24) at the point $(a, b) = (a_1, b_1)$ takes the form

$$\frac{f'(b)-f'(a)}{g'(b)-g'(a)} - \frac{f''(b)}{g''(b)}. \tag{25}$$

The above expression has the following geometric meaning. We consider the curve $v(s) = (g'(s), f'(s))$, and we draw a segment which connects the points $v(a)$ and $v(b)$. The above expression is the difference between the slope of the line which passes through the segment $[v(a), v(b)]$ and the slope of the tangent line of the curve $v(s)$ at the point b. In the case shown in Figure 6, this difference is positive. Recall that $v(s)$ is strictly convex on (a_1, c) and it is strictly concave on (c, b_1). Therefore, one can easily note that this expression (25) is always positive if the segment $[v(a), v(b)]$ also intersects the curve $v(s)$ at a point ξ such that $a < \xi < b$. This always happens in our case because (22) means that the points $v(a)$, $v(\xi)$, $v(b)$ lie on the same line, where ξ was determined from Cauchy's mean value theorem. Thus,

$$\frac{f'(b)-f'(a)}{g'(b)-g'(a)} - \frac{f''(b)}{g''(b)} > 0. \tag{26}$$

Similarly, we can obtain that $\partial\Phi/\partial a < 0$, because this is the same as to show that

$$\frac{f'(b)-f'(a)}{g'(b)-g'(a)} - \frac{f''(a)}{g''(a)} > 0. \tag{27}$$

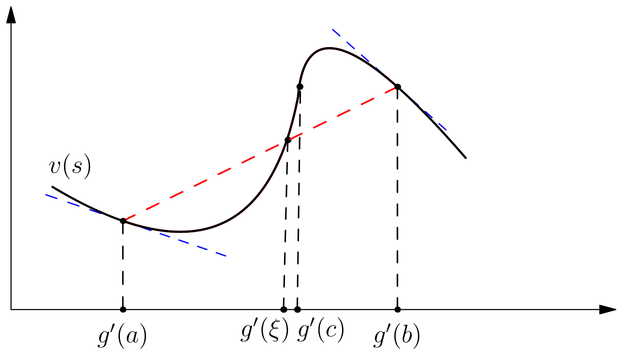

Figure 6. Graph of $v(s)$.

Thus, by the implicit function theorem there exists a C^1 function $a(s)$ in some neighborhood of b_1 such that $a'(s) = -\Phi'_b/\Phi'_a < 0$, and the pair $(a(s), s)$ solves (21).

Now we will explain that the function $a(s)$ can be defined on $(c, b_1]$ and, moreover, $\lim_{s\to c+0} a(s) = c$. Indeed, whenever $a(s) \in (a_1, c)$ and $s \in (c, b_1)$ we can use the implicit function theorem, and we can extend the function $a(s)$. It is clear that for each s we have $a(s) \in [a_1, c)$ and $s \in (c, b_1)$. Indeed, if $a(s), s \in (a_1, c]$, or $a(s), s \in [c, b_1)$, then (21) has a definite sign (see (23)). It follows that $\alpha(s) \in C^1((c, b_1])$, and the condition $a'(s) < 0$ implies $\lim_{s\to c+0} a(s) = c$. Hence $a(s) \in C([c, b_1])$. □

It is worth mentioning that we did not use the fact that the torsion of $(s, g(s), f(s))$ changes sign from $+$ to $-$. The only thing we needed was that the torsion changes sign.

Let a_1 and b_1 be any solutions of (21) from Lemma 27, and let $a(s)$ be any function from Lemma 28. Fix an arbitrary $s_1 \in (c, b_1)$ and consider the foliation $\Theta_{\text{cup}}([s_1, b_1], g)$ constructed by $a(s)$ (see Lemma 26). Let B be the function defined by (14), where

$$t_2(s_1) = \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}. \tag{28}$$

Set $\Omega_{\text{cup}} = \Omega(\Theta_{\text{cup}}([s_1, b_1], g))$, and let $\overline{\Omega}_{\text{cup}}$ be the closure of Ω_{cup}.

Lemma 29. *The function B satisfies the following properties*

(1) $B \in C^2(\Omega_{\text{cup}}) \cap C^1(\overline{\Omega}_{\text{cup}})$.

(2) $B\big(a(s), g(a(s))\big) = f(a(s))$ *for all* $s \in [s_1, b_1]$.

(3) B *is a concave function in* $\overline{\Omega}_{\text{cup}}$.

Proof. The first property follows from Lemma 16 and the fact that $\nabla B(x) = t(s)$ for $s = s(x)$, where $s(x)$ is a continuous function in $\overline{\Omega}_{\text{cup}}$.

We are going to check the second property. We recall (see (12)) that $t_1(s) = f'(s) - t_2(s)g'(s)$. Condition (28) implies that

$$t_1(s_1) + t_2(s_1)g'(a(s_1)) = f'(a(s_1)). \tag{29}$$

Let $B\big(a(s), g(a(s))\big) = \tilde{f}(a(s))$. After differentiation of this equality we get $t_1(s_1) + t_2(s_1)g'(a(s_1)) = \tilde{f}'(a(s_1))$. Hence, (29) implies that $f'(a(s_1)) = \tilde{f}'(a(s_1))$. It is clear that

$$\begin{aligned} t_1(s) + t_2(s)g'(s) &= f'(s), \\ t_1(s) + t_2(s)g'(a(s)) &= \tilde{f}'(a(s)), \\ t_1(s)(s - a(s)) + t_2(s)\big(g(s) - g(a(s))\big) &= f(s) - \tilde{f}(a(s)), \end{aligned}$$

which implies

$$\begin{vmatrix} 1 & 1 & s - a(s) \\ g'(s) & g'(a(s)) & g(s) - g(a(s)) \\ f'(s) & \tilde{f}'(a(s)) & f(s) - \tilde{f}(a(s)) \end{vmatrix} = 0.$$

This equality can be rewritten as follows:

$$f' \cdot \begin{vmatrix} 1 & s - a(s) \\ g'(a(s)) & g(s) - g(a(s)) \end{vmatrix} - \tilde{f}'(a) \begin{vmatrix} 1 & s - a(s) \\ g' & g(s) - g(a(s)) \end{vmatrix} + (f - \tilde{f}(a))(g'(a(s)) - g'(s)) = 0.$$

By virtue of Lemma 28 we have the same equality as above except $\tilde{f}$ is replaced by f. We subtract one from the other:

$$\big[f(a(s)) - \tilde{f}(a(s))\big] + \big[f'(a(s)) - \tilde{f}'(a(s))\big] \cdot \frac{1}{g'(a(s)) - g'(s)} \begin{vmatrix} 1 & s - a(s) \\ g' & g(s) - g(a(s)) \end{vmatrix} = 0.$$

Note that

$$\frac{1}{g'(a(s)) - g'(s)} \begin{vmatrix} 1 & s - a(s) \\ g' & g(s) - g(a(s)) \end{vmatrix} < 0$$

and $a(s)$ is invertible. Therefore, we get the differential equation $z(u)\boldsymbol{C}(u) + z'(u) = 0$, where $\boldsymbol{C}$ is in $C^1\big([a(b_1), a(s_1)]\big)$, $z(u) = f(u) - \tilde{f}(u)$ and $\boldsymbol{C} < 0$. The condition $z'(a(s_1)) = 0$ implies $z(a(s_1)) = 0$. Note that $z = 0$ is a trivial solution. Therefore, by uniqueness of solutions to ODEs, we get $z = 0$.

We are going to check the concavity of B. Let $\mathcal{F}$ be the force function corresponding to B. By Corollary 21 we only need to check that $\mathcal{F}(s_1) \le 0$. Note that (17) and (28) imply

$$\mathcal{F}(s_1) = \frac{f''(s_1)}{g''(s_1)} - t_2(s_1) = \frac{f''(s_1)}{g''(s_1)} - \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))},$$

which is negative by (26). □

Remark 30. The above lemma is true for all choices $s_1 \in (c, b_1)$. If we send s_1 to c then one can easily see that $\lim_{s_1 \to c+} t_2(s_1) = 0$, therefore the force function $\mathcal{F}$ takes the form

$$\mathcal{F}(s) = \int_c^s \left[\frac{f''(y)}{g''(y)}\right]' \exp\left(-\int_y^s \frac{g''(r)}{K(r)} \cos\theta(r)\, dr\right) dy.$$

This is another way to show that the force function is nonpositive.

The next lemma shows that, regardless of the choices of initial solution (a_1, b_1) of (21), the function $a(s)$ constructed by Lemma 28 is unique (i.e., it does not depend on the pair (a_1, b_1)).

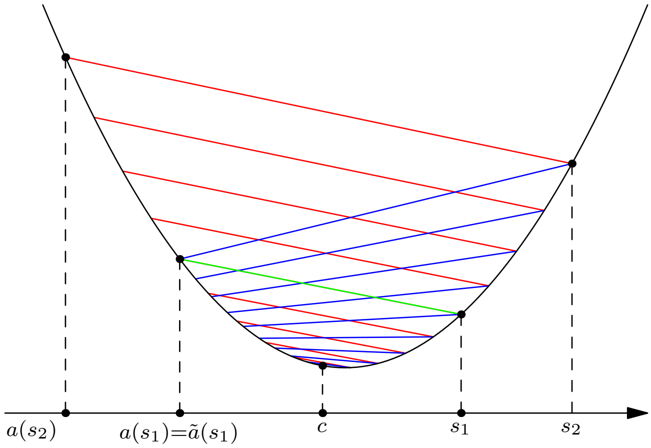

Figure 7. Uniqueness of the cup.

Lemma 31. *Let pairs* (a_1, b_1), $(\tilde{a}_1, \tilde{b}_1)$ *solve* (21), *and let* $a(s)$, $\tilde{a}(s)$ *be the corresponding functions obtained by* Lemma 28. *Then* $a(s) = \tilde{a}(s)$ *on* $[c, \min\{b_1, \tilde{b}_1\}]$.

Proof. By the uniqueness result of the implicit function theorem we only need to show existence of $s_1 \in (c, \min\{b_1, \tilde{b}_1\})$ such that $a(s_1) = \tilde{a}(s_1)$. Without loss of generality, assume that $\tilde{b}_1 = b_1 = s_2$. We can also assume that $\tilde{a}(s_2) > a(s_2)$, because other cases can be solved in a similar way.

Let $\Theta = \Theta_{\mathrm{cup}}([c, s_2], g)$ and $\tilde{\Theta} = \tilde{\Theta}_{\mathrm{cup}}([c, s_2], g)$ be the foliations corresponding to the functions $a(s)$ and $\tilde{a}(s)$. Let B and $\tilde{B}$ be the functions corresponding to these foliations from Lemma 29. We consider a chord T in $\mathbb{R}^3$ joining the points $\big(a(s_1), g(a(s_1)), f(a(s_1))\big)$ and $(s_1, g(s_1), f(s_1))$ (see Figure 7). We want to show that the chord T belongs to the graph of $\tilde{B}$. Indeed, concavity of $\tilde{B}$ (see Lemma 29) implies that the chord T lies below the graph of $\tilde{B}(x_1, x_2)$, where $(x_1, x_2) \in \Omega(\tilde{\Theta})$. Moreover, concavity of B, $\Omega(\tilde{\Theta}) \subset \Omega(\Theta)$ and the fact that the graph $\tilde{B}$ consists of chords joining the points of the curve $(t, g(t), f(t))$ imply that the graph B lies above the graph $\tilde{B}$. In particular, the chord T, belonging to the graph B, lies above the graph $\tilde{B}$. This can happen if and only if T belongs to the graph $\tilde{B}$. Now we show that, if $s_1 < s_2$, then the torsion of the curve $(s, g(s), f(s))$ is zero for $s \in [s_1, s_2]$. Indeed, let $\tilde{T}$ be a chord in $\mathbb{R}^3$ which joins the points $\big(a(s_1), g(a(s_1)), f(a(s_1))\big)$ and $(s_2, g(s_2), f(s_2))$. We consider the tangent plane $L(x)$ to the graph $\tilde{B}$ at the point $(x_1, x_2) = \big(a(s_1), g(a(s_1))\big)$. This tangent plane must contain both chords T and $\tilde{T}$, and it must be tangent to the surface at these chords. Concavity of $\tilde{B}$ implies that the tangent plane L coincides with $\tilde{B}$ at points belonging to the triangle, which is the convex hull of the points $\big(a(s_1), g(a(s_1))\big)$, $(s_1, g(s_1))$ and $(s_2, g(s_2))$. Therefore, it is clear that the tangent plane L coincides with $\tilde{B}$ on the segments $\ell \in \tilde{\Theta}$ with the endpoint at $(s, g(s))$ for $s \in [s_1, s_2]$. Thus $L\big((s, g(s))\big) = \tilde{B}\big((s, g(s))\big)$ for any $s \in [s_1, s_2]$. This means that the torsion of the curve $(s, g(s), f(s))$ is zero on $s \in [s_1, s_2]$, which contradicts our assumption about the torsion. Therefore $s_1 = s_2$. □

Corollary 32. *In the conditions of* Lemma 27, *for all* $0 < P < \min\{c - a_0, b_0 - c\}$ *there exists a unique pair* (a_1, b_1) *which solves* (21) *such that* $b_1 - a_1 = P$.

The above corollary implies that, if the pairs (a_1, b_1) and $(\tilde{a}_1, \tilde{b}_1)$ solve (21), then $a_1 \neq \tilde{a}_1$ and $b_1 \neq \tilde{b}_1$, and one of the following conditions holds: $(a_1, b_1) \subset (\tilde{a}_1, \tilde{b}_1)$ or $(\tilde{a}_1, \tilde{b}_1) \subset (a_1, b_1)$.

Remark 33. The function $a(s)$ is defined on the right of the point c. We extend naturally its definition on the left of the interval by $a(s) \stackrel{\text{def}}{=} a^{-1}(s)$.

4. Construction of the Bellman function

4.1. ***Reduction to the two-dimensional case.*** We are going to construct the Bellman function for the case $p < 2$. The case $p = 2$ is trivial, and the case $p > 2$ was solved in [Boros et al. 2012]. From the definition of H it follows that

$$H(x_1, x_2, x_3) = H(|x_1|, |x_2|, x_3) \quad \text{for all } (x_1, x_2, x_3) \in \Omega. \tag{30}$$

Also note the homogeneity condition

$$H(\lambda x_1, \lambda x_2, \lambda^p x_3) = \lambda^p H(x_1, x_2, x_3) \quad \text{for all } \lambda \geq 0. \tag{31}$$

These two conditions (30), (31), which follow from the nature of the boundary data $(x^2 + \tau^2 y^2)^{2/p}$, make the construction of H easier. However, in order to construct the function H, this information is not necessary. Further, we assume that H is $C^1(\Omega)$ smooth. Then, from the symmetry (30), it follows that

$$\frac{\partial H}{\partial x_j} = 0 \quad \text{on } x_j = 0 \text{ for } j = 1, 2. \tag{32}$$

For convenience, as in [Boros et al. 2012], we rotate the system of coordinates (x_1, x_2, x_3). Namely, let

$$y_1 \stackrel{\text{def}}{=} \frac{x_1 + x_2}{2}, \qquad y_2 \stackrel{\text{def}}{=} \frac{x_2 - x_1}{2}, \qquad y_3 \stackrel{\text{def}}{=} x_3. \tag{33}$$

We define

$$N(y_1, y_2, y_3) \stackrel{\text{def}}{=} H(y_1 - y_2, y_1 + y_2, y_3) \quad \text{on } \Omega_1,$$

where $\Omega_1 = \{(y_1, y_2, y_3) : y_3 \geq 0,\ |y_1 - y_2|^p \leq y_3\}$. It is clear that, for fixed y_1, the function N is concave in the variables y_2 and y_3; moreover, for fixed y_2, the function N is concave with respect to the other variables. The symmetry (30) for N turns into the condition

$$N(y_1, y_2, y_3) = N(y_2, y_1, y_3) = N(-y_1, -y_2, y_3). \tag{34}$$

Thus it is sufficient to construct the function N on the domain

$$\Omega_2 \stackrel{\text{def}}{=} \{(y_1, y_2, y_3) : y_1 \geq 0,\ -y_1 \leq y_2 \leq y_1,\ (y_1 - y_2)^p \leq y_3\}.$$

Condition (32) turns into

$$\frac{\partial N}{\partial y_1} = \frac{\partial N}{\partial y_2} \qquad \text{on the hyperplane } y_2 = y_1, \tag{35}$$

$$\frac{\partial N}{\partial y_1} = -\frac{\partial N}{\partial y_2} \qquad \text{on the hyperplane } y_2 = -y_1. \tag{36}$$

The boundary condition (6) becomes

$$N(y_1, y_2, |y_1 - y_2|^p) = ((y_1 + y_2)^2 + \tau^2(y_1 - y_2)^2)^{p/2}. \tag{37}$$

The homogeneity condition (31) implies that $N(\lambda y_1, \lambda y_2, \lambda^p y_3) = \lambda^p N(y_1, y_2, y_3)$ for $\lambda \geq 0$. We choose $\lambda = 1/y_1$, and we obtain that

$$N(y_1, y_2, y_3) = y_1^p N\left(1, \frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \tag{38}$$

Suppose we are able to construct the function $M(y_2, y_3) \stackrel{\text{def}}{=} N(1, y_2, y_3)$ on

$$\Omega_3 \stackrel{\text{def}}{=} \{(y_2, y_3) :\ -1 \leq y_2 \leq 1,\ (1 - y_2)^p \leq y_3\}$$

with the following conditions:

(1) M is concave in Ω_3.

(2) M satisfies (37) for $y_1 = 1$.

(3) The extension of M onto Ω_1 via formulas (38) and (34) is a function with the properties of N (see (35), (36), and concavity of N).

(4) M is minimal among those who satisfy the conditions (1)–(3).

Then the extended function M should be N. So we are going to construct M on Ω_3. We denote

$$g(t) \stackrel{\text{def}}{=} (1 - t)^p, \quad t \in [-1, 1], \tag{39}$$

$$f(t) \stackrel{\text{def}}{=} ((1 + t)^2 + \tau^2(1 - t)^2)^{p/2}, \quad t \in [-1, 1]. \tag{40}$$

Then we have the boundary condition

$$M(t, g(t)) = f(t), \quad t \in [-1, 1]. \tag{41}$$

We differentiate the condition (38) with respect to y_1 at the point $(y_1, y_2, y_3) = (1, -1, y_3)$ and we obtain that

$$\frac{\partial N}{\partial y_1}(1, -1, y_3) = pN(1, -1, y_3) + \frac{\partial N}{\partial y_2}(1, -1, y_3) - py_3\frac{\partial N}{\partial y_3}, \quad y_3 \geq 0.$$

Now we use (36), so we obtain another requirement for $M(y_2, y_3)$:

$$0 = pM(-1, y_3) + 2\frac{\partial M}{\partial y_2}(-1, y_3) - py_3\frac{\partial M}{\partial y_3}(-1, y_3) \quad \text{for } y_3 \geq 0. \tag{42}$$

Similarly, we differentiate (38) with respect to y_1 at the point $(y_1, y_2, y_3) = (1, 1, y_3)$ and use (35), so we obtain

$$0 = pM(1, y_3) - 2\frac{\partial M}{\partial y_2}(1, y_3) - py_3\frac{\partial M}{\partial y_3}(1, y_3) \quad \text{for } y_3 \geq 0. \tag{43}$$

So, in order to satisfy conditions (35) and (36), the requirements (42) and (43) are necessary. It is easy to see that these requirements are also sufficient in order to satisfy these conditions.

The minimum between two concave functions with fixed boundary data is a concave function with the same boundary data. Note also that the conditions (42) and (43) are still fulfilled after taking the minimum. Thus it is quite reasonable to construct a candidate for $M(y_2, y_3)$ as a minimal concave function on Ω_3 with the boundary conditions (41), (42) and (43). We recall that we should also have the concavity of the extended function $N(y_1, y_2, y_3)$ with respect to the variables y_1, y_3 for each fixed y_2. This condition can be verified after the construction of the function $M(y_2, y_3)$.

4.2. ***Construction of a candidate for M.*** We are going to construct a candidate B for M. Firstly, we show that, for $\tau > 0$, the torsion τ_γ of the boundary curve $\gamma(t) \overset{\text{def}}{=} (t, g(t), f(t))$ on $t \in (-1, 1)$, where f, g are defined by (39) and (40), changes sign once from + to −. We call this point the root of a cup. We construct the cup around this point. Note that $g' < 0$, $g'' > 0$ on $[-1, 1)$. Therefore,

$$\operatorname{sign} \tau_\gamma = \operatorname{sign}\left(f''' - \frac{g'''}{g''} f''\right) = \operatorname{sign}\left(f''' - \frac{2-p}{1-t} f''\right) = \operatorname{sign}(v(t)),$$

where

$$v(t) \overset{\text{def}}{=} -(1+\tau^2)^2(p-1)t^3 + (1+\tau^2)(3\tau^2+\tau^2 p+3-3p)t^2 \\ + (2\tau^2 p - 9\tau^4 + \tau^4 p + 3 - 3p - 6\tau^2)t - p + 5\tau^4 + 2\tau^2 p - \tau^4 p - 10\tau^2 + 1.$$

Note that $v(-1) = 16\tau^4 > 0$ and $v(1) = -8((p-1)+\tau^2) < 0$. So the function $v(t)$ changes sign from + to − at least once. Now, we show that $v(t)$ has only one root. For $\tau^2 < 3(p-1)/(3-p)$, note that the linear function $v''(t)$ is nonnegative, i.e., $v''(-1) = 8\tau^2 p(1+\tau^2) > 0$, $v''(1) = -4(1+\tau^2)(\tau^2 p - 3\tau^2 + 3p - 3) \geq 0$. Therefore, the convexity of $v(t)$ implies the uniqueness of the root $v(t)$ on $[-1, 1]$.

Suppose $\tau^2 < 3(p-1)/(3-p)$; we will show that $v' \leq 0$ on $[-1, 1]$. Indeed, the discriminant of the quadratic function $v'(x)$ has the expression

$$D = 16\tau^2(\tau^2+1)^2((3-p)^2\tau^2 - 9(p-1)),$$

which is negative for $0 < \tau^2 < 3(p-1)/(3-p)$. Moreover, $v'(-1) = -4\tau^2(\tau^2 p + 3\tau^2 + 3) < 0$. Thus we obtain that v' is negative.

We denote the root of v by c. It is an appropriate time to make the following remark:

Remark 34. Note that $v(-1+2/p) < 0$. Indeed,

$$v\left(-1+\frac{2}{p}\right) = \frac{(3p-2)(p^2-2p-4)\tau^4 + (16+5p^3-8p^2-16p)\tau^2 + 8(1-p)}{p^3},$$

which is negative because the coefficients of τ^4, τ^2, τ^0 are negative. Therefore, this inequality implies that $c < -1 + 2/p$.

Consider $a = -1$ and $b = 1$; the left side of (21) takes the positive value $-2^{2p-1}p(1-p)$. However, if we consider $a = -1$ and $b = c$, then the proof of Lemma 27 (see (23)) implies that the left side of (21) is negative. Therefore, there exists a unique $s_0 \in (c, 1)$ such that the pair $(-1, s_0)$ solves (21). Uniqueness follows from Corollary 32. The equation (21) for the pair $(-1, s_0)$ is equivalent to the

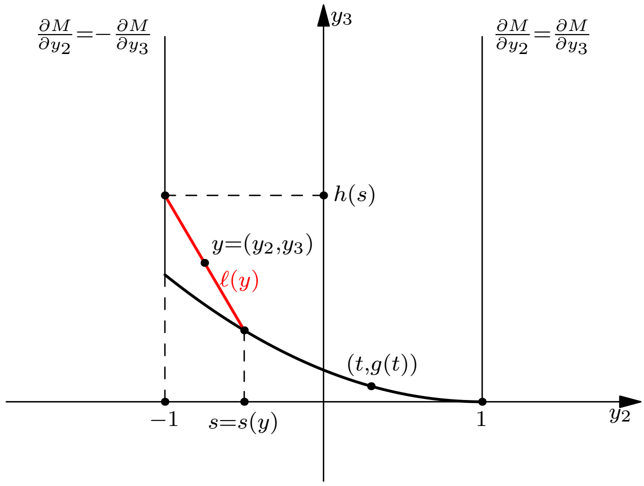

Figure 8. The segment $\ell(y)$.

equation $u((1+s_0)/(1-s_0))=0$, where

$$u(z) \stackrel{\text{def}}{=} \tau^p(p-1)(\tau^2+z^2)^{(2-p)/2} - \tau^2(p-1) + (1+z)^{2-p} - z(2-p) - 1. \tag{44}$$

Lemma 28 gives the function $a(s)$, and Lemma 29 gives the concave function $B(y_2, y_3)$ for $s_1 = c$ with the foliation $\Theta_{\text{cup}}((c, s_0], g)$ in the domain $\Omega\big(\Theta_{\text{cup}}((c, s_0], g)\big)$.

The above explanation implies the following corollary:

Corollary 35. *Pick any point $\tilde{y}_2 \in (-1, 1)$. The inequalities $s_0 < \tilde{y}_2$, $s_0 = \tilde{y}_2$ and $\tilde{y}_2 > s_0$ are equivalent to the following inequalities, respectively*: $u((1+\tilde{y}_2)/(1-\tilde{y}_2)) < 0$, $u((1+\tilde{y}_2)/(1-\tilde{y}_2)) = 0$ *and* $u((1+\tilde{y}_2)/(1-\tilde{y}_2)) > 0$.

Now we are going to extend C^1 smoothly the function B on the upper part of the cup. Recall that we are looking for a minimal concave function. If we construct a function with a foliation $\Theta([s_0, \tilde{y}_2], g)$, where $\tilde{y}_2 \in (s_0, 1)$, then the best thing we can do according to Lemma 23 and Lemma 22 is to minimize $\sin(\theta_{\text{cup}}(s_0) - \theta(s_0))$, where $\theta_{\text{cup}}(s)$ is an argument function of $\Theta_{\text{cup}}((c, s_0], g)$ and $\theta(s)$ is an argument function of $\Theta([s_0, \tilde{y}_2], g)$. In other words, we need to choose segments from $\Theta([s_0, \tilde{y}_2], g)$ close enough to the segments of $\Theta_{\text{cup}}((c, s_0], g)$.

Thus, we are going to construct the set of segments $\Theta([s_0, \tilde{y}_2])$ so that they start from $(s, g(s), f(s))$, $s \in [s_0, \tilde{y}_2]$, and they go to the boundary $y_2 = -1$ of Ω_3.

We explain how the conditions (42) and (43) allow us to construct such a type of foliation $\Theta([s_0, \tilde{y}_2], g)$ in a unique way. Let $\ell(y)$ be the segment with the endpoints $(s, g(s))$, where $s \in (s_0, \tilde{y}_2)$ and $(-1, h(s))$ (see Figure 8).

Let $t(s) = (t_1(s), t_2(s)) = \nabla B(y)$, where $s = s(y)$ is the corresponding gradient function. Then (42) takes the form

$$0 = pB(-1, h(s)) + 2t_1(s) - ph(s)t_2(s). \tag{45}$$

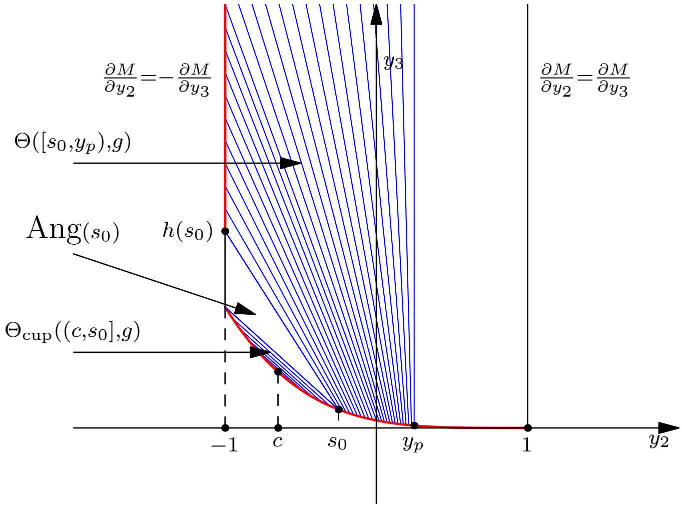

Figure 9. Foliations $\Theta_{\text{cup}}((c, s_0], g)$ and $\Theta([s_0, y_p), g)$.

We differentiate this expression with respect to s, and we obtain

$$2t_1'(s) - ph(s)t_2'(s) = 0. \tag{46}$$

Then, according to (11), we find the function $\tan\theta(s)$, and, hence, we find the quantity $h(s)$:

$$\tan\theta(s) = -\frac{ph(s)}{2} \iff \frac{h(s)-g(s)}{s+1} = \frac{ph(s)}{2}.$$

Therefore,

$$h(s) = \frac{2g(s)}{p}\left(\frac{1}{y_p - s}\right), \quad \text{where} \quad y_p \stackrel{\text{def}}{=} -1 + \frac{2}{p}. \tag{47}$$

We see that the function $h(s)$ is well defined, it increases, and it is differentiable on $-1 \le s < y_p$. So we conclude that if $s_0 < y_p$ then we are able to construct the set of segments $\Theta([s_0, y_p), g)$ that pass through the points $(s, g(s))$, where $s \in [s_0, y_p)$, and through the boundary $y_2 = -1$ (see Figure 9).

It is easy to check that $\Theta([s_0, y_p), g)$ is a foliation, so, taking the value $t_2(s_0)$ of B on $\Omega\big(\Theta([s_0, y_p), g)\big)$ according to Lemma 23, by Corollary 25 we have constructed a concave function B in the domain $\Omega\big(\Theta_{\text{cup}}((c, s_0], g)\big) \cup \text{Ang}(s_0) \cup \Omega\big(\Theta([s_0, y_p], g)\big)$.

It is clear that the foliation $\Theta([s_0, y_p), g)$ exists as long as $s_0 < y_p$. Note that $(1 + y_p)/(1 - y_p) = 1/(p-1)$. Therefore, Corollary 35 implies the following remark:

Remark 36. The inequalities $s_0 < y_p$, $s_0 = y_p$ and $s_0 > y_p$ are equivalent to the following inequalities respectively: $u(1/(p-1)) < 0$, $u(1/(p-1)) = 0$ and $u(1/(p-1)) > 0$.

At the point y_p, the segments from $\Theta([s_0, y_p), g)$ become vertical. After the point $(y_p, g(y_p))$, we should consider vertical segments $\Theta([y_p, 1], g)$ (see Figure 10), because by Lemma 22 this corresponds to the minimal function. Surely $\Theta([y_p, 1], g)$ is the foliation. Again, choosing the value $t_2(y_p)$ of B on $\Omega\big(\Theta([y_p, 1], g)\big)$ according to Lemma 23, by Corollary 25 we have constructed the concave function B

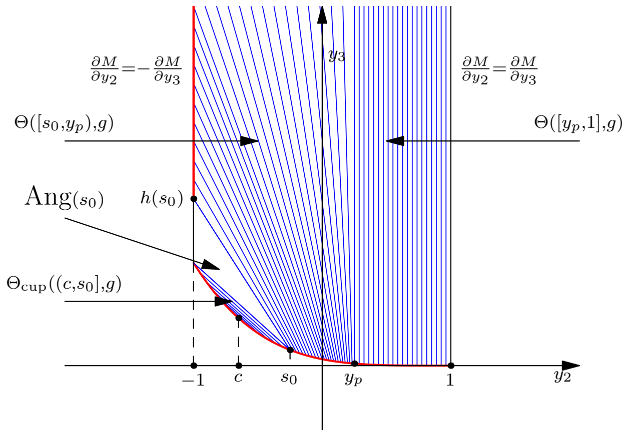

Figure 10. The case $u(1/(p-1)) < 0$.

on Ω_3. Note that if $s_0 \ge y_p$ (which corresponds to the inequality $u(1/(p-1)) > 0$) then we do not have the foliation $\Theta([s_0, y_p), g)$. In this case we consider only vertical segments $\Theta([s_0, 1], g)$ (see Figure 11), and again, choosing the value $t_2(s_0)$ of B on $\Omega(\Theta([s_0, 1], g))$ according to Lemma 23, by Corollary 25 we construct a concave function B on Ω_3. We believe that $B = M$.

We still have to check the requirements (42) and (43). A crucial role is played by symmetry of the boundary data of N. Further, the given proofs work for both of the cases $y_p < s_0$ and $y_p \ge s_0$, so we do not consider them separately.

The requirement (43) follows immediately. Indeed, the condition (14) at the point $y = (1, y_3)$ (note that in (14) instead of $x = (x_1, x_2)$ we consider $y = (y_2, y_3)$) implies that $B(1, y_3) = f(1) + t_2(1)(y_3 - g(1))$. Therefore, (43) takes the form $0 = pf(1) - 2t_1(1)$. Using (12), we obtain that $t_1(1) = f'(1)$. Therefore, we see that $pf(1) - 2t_1(1) = pf(1) - 2f'(1) = 0$.

Now, we are going to obtain the requirement (42) which is the same as (45). The quantities t_1, t_2 of B with the foliation $\Theta([s_0, y_p), g)$ satisfy the condition (46) which was obtained by differentiation of (45). So we only need to check the condition (45) at the initial point $s = s_0$. If we substitute the expression of B from (14) into (45), then (45) turns into the following equivalent condition:

$$t_1(s)(s - y_p) + t_2(s)g(s) = f(s). \tag{48}$$

Note that (12) allows us to rewrite (48) into the equivalent condition

$$t_2(s) = \frac{f(s) - (s - y_p)f'(s)}{g(s) - (s - y_p)g'(s)}. \tag{49}$$

And, as was mentioned above we only need to check condition (49) at the point $s = s_0$, i.e.,

$$t_2(s_0) = \frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)}. \tag{50}$$

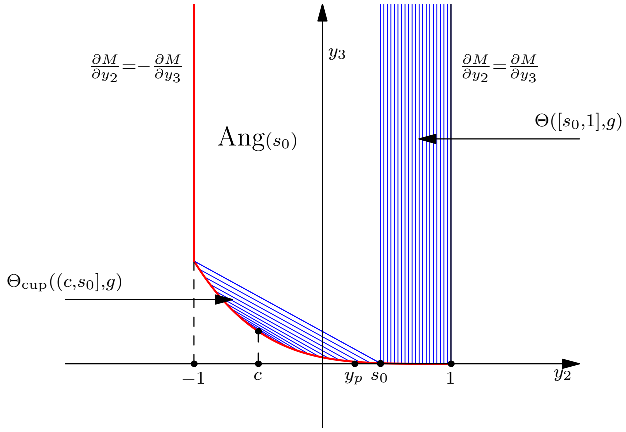

Figure 11. The case $u(1/(p-1)) \geq 0$.

On the other hand, if we differentiate the boundary condition $B(s, g(s)) = f(s)$ at the points $s = s_0, -1$, then we obtain

$$t_1(s_0) + t_2(s_0)g'(-1) = f'(-1),$$
$$t_1(s_0) + t_2(s_0)g'(s_0) = f'(s_0).$$

Thus we can find the value of $t_2(s_0)$:

$$t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}. \tag{51}$$

So these two values (51) and (50) must coincide. In other words, we need to show

$$\frac{f(s_0) - (s_0 - y_p)f'(s_0)}{g(s_0) - (s_0 - y_p)g'(s_0)} = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}. \tag{52}$$

It will be convenient for us to work with the following notations for the rest of the current subsection. We denote $g(-1) = g_-$, $g'(-1) = g'_-$, $f(-1) = f_-$, $f'(-1) = f'_-$ $g(s_0) = g$, $g'(s_0) = g'$, $f(s_0) = f$ and $f'(s_0) = f'$. The condition (52) is equivalent to

$$s_0 = \frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} + y_p = \left(\frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} - 1\right) + \frac{2}{p}. \tag{53}$$

On the other hand, from (21) for the pair $(-1, s_0)$, we obtain that

$$s_0 = \left(\frac{fg'_- + f'g - fg' - gf'_-}{f'g'_- - g'f'_-} - 1\right) + \frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-}.$$

So, from (53) we see that it suffices to show that

$$\frac{f'g_- + g'_-f_- - g'f_- - f'_-g_-}{g'f'_- - f'g'_-} = \frac{2}{p}.$$

We note that $g'_- = -(p/2)g_-$, $f'_- = -(p/2)f_-$, hence $g'_- f_- = f'_- g_-$. Therefore, we have

$$\frac{f'g_- + g'_- f_- - g'f_- - f'_- g_-}{g'f'_- - f'g'_-} = \frac{f'g_- - g'f_-}{g'f'_- - f'g'_-} = \frac{2}{p}.$$

4.3. ***Concavity in another direction.*** We are going to check the concavity of the extended function N via B in another direction. It is worth mentioning that both of the cases $y_p < s_0$, $y_p \geq s_0$ do not play any role in the following computations, therefore we consider them together. We define a candidate for N as

$$N(y_1, y_2, y_3) \stackrel{\text{def}}{=} y_1^p B\left(1, \frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \quad \text{for } \left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) \in \Omega_3, \tag{54}$$

and we extend N to Ω_1 by (34). Then, as was already discussed, $N \in C^1(\Omega_1)$. We need the following technical lemma:

Lemma 37. $$N''_{y_1y_1} N''_{y_3y_3} - (N''_{y_1y_3})^2 = -t'_2 s'_{y_3} p(p-1) y_1^{p-2}\left(st_1 + gt_2 - f + \frac{y_2}{y_1} t_1 \cdot \left(\frac{2}{p} - 1\right)\right),$$

where $s = s(y_2/y_1, y_3/y_1^p)$ *and* $(y_2/y_1, y_3/y_1^p) \in \operatorname{int}(\Omega_3) \setminus \operatorname{Ang}(s_0)$.

As was mentioned in Remark 24, the gradient function $t(s)$ is not necessarily differentiable at the point s_0; this is the reason for the requirement $(y_2/y_1, y_3/y_1^p) \in \operatorname{int}(\Omega_3) \setminus \operatorname{Ang}(s_0)$ in the lemma. However, from the proof of the lemma, the reader can easily see that $N''_{y_1y_1} N''_{y_3y_3} - (N''_{y_1y_3})^2 = 0$ whenever the points $(y_2/y_1, y_3/y_1^p)$ belong to the interior of the domain $\operatorname{Ang}(s_0)$.

Proof. The definition of the candidate N (see (54)) implies $N''_{y_3y_3} = t'_2(s)s'_{y_3}$, $N''_{y_3y_1} = t'_2 s'_{y_1}$ and

$$N'_{y_1} = y_1^{p-1}\left(pB\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) - t_1\frac{y_2}{y_1} - pt_2\frac{y_3}{y_1^p}\right). \tag{55}$$

Condition (14) implies

$$B\left(\frac{y_2}{y_1}, \frac{y_3}{y_1^p}\right) = f(s) + t_1 \cdot \left(\frac{y_2}{y_1} - s\right) + t_2 \cdot \left(\frac{y_3}{y_1^p} - g(s)\right).$$

We substitute this expression for $B(y_2/y_1, y_3/y_1^p)$ into (55), and we obtain

$$N'_{y_1} = y_1^{p-1}\left(pf + \frac{y_2}{y_1}t_1(p-1) - pst_1 - pgt_2\right). \tag{56}$$

The condition $(y_2/y_1, y_3/y_1^p) \in \operatorname{int}(\Omega_3) \setminus \operatorname{Ang}(s_0)$ implies the equality $N''_{y_1y_3} = N''_{y_3y_1}$, which in turn gives

$$t'_2 s'_{y_1} = y_1^{p-1}\left(pf' + \frac{y_2}{y_1}t'_1(p-1) - (pst_1 + pgt_2)'_s\right)s'_{y_3}.$$

Hence,

$$t'_2 \cdot (s'_{y_1})^2 = y_1^{p-1}\left(pf' + \frac{y_2}{y_1}t'_1(p-1) - (pst_1 + pgt_2)'_s\right)s'_{y_3}s'_{y_1}. \tag{57}$$

We keep in mind this identity, and continue our calculations:

$$N''_{y_1y_1} = (p-1)y_1^{p-2}\left(pf + \frac{y_2}{y_1}t_1(p-2) - pst_1 - pgt_2\right) + y_1^{p-1}\left(pf' + \frac{y_2}{y_1}t'_1(p-1) - (pst_1 + pgt_2)'_s\right)s'_{y_1}.$$

So, finally we obtain

$$N''_{y_1y_1}N''_{y_3y_3} - (N''_{y_1y_3})^2 = t_2'(N''_{y_1y_1}s'_{y_3} - t_2'(s'_{y_1})^2).$$

Now we use the identity (57), and we substitute the expression $t_2'(s'_{y_1})^2$:

$$\begin{aligned} N''_{y_1y_1}N''_{y_3y_3} - (N''_{y_1y_3})^2 &= t_2's'_{y_3}\left(N''_{y_1y_1} - y_1^{p-1}\left(pf' + \frac{y_2}{y_1}t_1'(p-1) - (pst_1 + pgt_2)'_s\right)s'_{y_1}\right) \\ &= t_2's'_{y_3}(p-1)y_1^{p-2}\left(pf + \frac{y_2}{y_1}t_1(p-2) - pst_1 - pgt_2\right) \\ &= -t_2's'_{y_3}p(p-1)y_1^{p-2}\left(st_1 + gt_2 - f + \frac{y_2}{y_1}t_1\cdot\left(\frac{2}{p}-1\right)\right). \qquad \square \end{aligned}$$

Now we are going to consider several cases, when the points $(y_2/y_1, y_3/y_1^p)$ belong to the different subdomains in Ω_3. Note that we always have $N''_{y_3y_3} \le 0$, because of the fact that B is concave in Ω_3 and (54). So we only have to check that the determinant of the Hessian of N is negative. If the determinant of the Hessian is zero, then it is sufficient to ensure that $N''_{y_3y_3}$ is strictly negative, and, if $N''_{y_3y_3}$ is also zero, then we need to ensure that N''_{y_1,y_1} is nonpositive.

The domain $\Omega(\Theta[s_0, y_p])$. In this case we can use the equality (48), and we obtain that

$$st_1 + gt_2 - f = y_pt_1.$$

Therefore,

$$N''_{y_1y_1}N''_{y_3y_3} - (N''_{y_1y_3})^2 = -t_2's'_{y_3}p(p-1)y_1^{p-2}t_1y_p\left(1 + \frac{y_2}{y_1}\right) \ge 0$$

because $t_1 \ge 0$. Indeed, $t_1(s)$ is continuous on $[c, 1]$, where c is the root of the cup and $B''_{y_2y_2} = t_1's'_{y_2} \le 0$; therefore, because of the fact $s'_{y_2} > 0$, it suffices to check that $t_1(1) \ge 0$, which follows from the inequality

$$t_1(1) = f'(1) - t_2(1)g'(1) = f'(1) > 0.$$

Domain of linearity $\mathrm{Ang}(s_0)$. This is the domain that consists of the triangle ABC with $A = (-1, g(-1))$, $B = (s_0, g(s_0))$ and $C = (-1, h(s_0))$ if $s_0 < y_p$, and the infinite domain of linearity, which is of rectangular type and which lies between the chords AB, BC', where $C' = (s_0, +\infty)$, and AC'', where $C'' = (-1, +\infty)$ (see Figure 11).

Suppose the points $(y_2/y_1, y_3/y_1^p)$ belong to the interior of $\mathrm{Ang}(s_0)$. Then the gradient function $t(s)$ of B is constant, and, moreover, $s(y_2/y_1, y_3/y_1^p)$ is constant. The fact that the determinant of the Hessian is zero in the domain of linearity (note that $s'_{y_3} = 0$) implies that we only need to check $N''_{y_1y_1} < 0$. The equality (56) implies

$$N''_{y_1y_1} = (p-1)y_1^{p-2}\left(pf + \frac{y_2}{y_1}t_1(p-2) - ps_0t_1 - pgt_2\right) \le (p-1)y_1^{p-2}(pf - ps_0t_1 - pgt_2 - t_1(p-2)) = 0.$$

The last equality follows from (48). The above inequality turns into an equality if and only if $y_2/y_1 = s_0$; this is the boundary point of $\mathrm{Ang}(s_0)$.

Domain of vertical segments. On the vertical segments, the determinant of the Hessian is zero (for example, because the vertical segment is a vertical segment in all directions) and $B''_{y_3y_3} = 0$; therefore, we must check that $N''_{y_1y_1} \le 0$. We note that $s(y_2, y_3) = y_2$, so

$$N''_{y_1y_1} = y_1^{p-2} \times \big[(p-1)(pf + st_1(p-2) - pst_1 - pgt_2) - s(pf' - t_1's - t_1 p - pg't_2)\big].$$

However, from (12) we have $pf' - t_1 p - pg't_2 = 0$; therefore,

$$N''_{y_1y_1} = y_1^{p-2} \times [(p-1)(pf - 2st_1 - pgt_2) + s^2 t_1'].$$

The condition $t_1' \le 0$ implies that it is sufficient to show $pf - 2st_1 - pgt_2 \le 0$. We use (12), and we find $t_1 = f' - g't_2$. Hence,

$$pf - 2st_1 - pgt_2 = pf - gpt_2 - 2s(f' - g't_2) = pf - 2sf' - t_2(gp - 2sg').$$

Note that $gp - 2sg' \ge 0$ (because $s \ge 0$ and $g' \le 0$). From (12) and the fact that on the vertical segments t_2 is constant (see the expression for t_2 in Lemma 13 and note that $\cos\theta(s) = 0$), it follows that $0 \ge t_1' = f'' - g''t_2$; therefore, we have $t_2 \ge f''/g''$. Therefore,

$$pf - 2sf' - t_2(gp - 2sg') \le pf - 2sf' - \frac{f''}{g''}(gp - 2sg').$$

Now we recall the values (41), (40), and after direct calculations we obtain

$$pf - 2sf' - \frac{f''}{g''}(gp - 2sg') = \frac{f(1-s^2)p(p-2)(\tau^2(1+s)^2 + (1-s)^2 + 2\tau^2(1-s^2))}{(p-1)((1+s)^2 + \tau^2(1-s)^2)^2} \le 0.$$

Domain of the cup $\Omega\big(\Theta_{\text{cup}}((c, s_0], g)\big)$. The condition that $N''_{y_3y_3}$ is strictly negative in the cup implies that we only need to show $st_2 + gt_3 - f + (y_2/y_1)t_1(2/p - 1) \ge 0$, where $s = s(y_2/y_1, y_3/y_1^p)$ and the points $y = (y_2/y_1, y_3/y_1^p)$ lie in the cup. Without loss of generality we can assume that $y_1 = 1$. Therefore it suffices to show that $st_2 + gt_3 - f + y_2t_1(2/p - 1) \ge 0$, where $y = (y_2, y_3) \in \Omega\big(\Theta_{\text{cup}}((c, s_0], g)\big)$. On a segment with the fixed endpoint $(s, g(s))$ the expressions s, t_1, $g(s)$, t_2 and $f(s)$ are constant, so the expression $st_1 + gt_2 - f + y_2t_1(2/p - 1)$ is linear with respect to y_2 on each segment of the cup. Therefore, the worst case appears when $y_2 = a(s)$ (it is the left end — an abscissa — of the given segment). This is true because $t_1 \ge 0$ (as was already shown) and $(2/p - 1) \ge 0$. So, as a result, we derive that it is sufficient to prove the inequality

$$st_1 + gt_2 - f + a(s)t_1 \cdot \left(\frac{2}{p} - 1\right) = t_1(s - a(s)) + gt_2 - f + \frac{2a(s)}{p}t_1 \ge 0. \tag{58}$$

We use the identity (14) at the point $y = \big(a(s), g(a(s))\big)$, and we find that

$$t_1(s - a(s)) + gt_2 - f = g(a(s))t_2 - f(a(s)).$$

We substitute this expression into (58), then we get that it suffices to prove the inequality

$$g(a(s))t_2 - f(a(s)) + \frac{2a(s)}{p}t_1 \ge 0. \tag{59}$$

We differentiate the condition $B\big(a(s), g(a(s))\big) = f(s)$ with respect to s. Then we find the expression for $t_1(s)$, namely $t_1(s) = f'(a(s)) - t_2(s)g'(a(s))$. After substituting this expression into (59) we obtain that

$$g(a(s))t_2 - f(a(s)) + \frac{2a(s)}{p}t_1 = \frac{1+z}{g'(z)}\left(\frac{(z-1)(\tau^2+1)f(z)}{((1+z)^2+\tau^2(1-z)^2)g'(z)} - t_2(s)\right),$$

where $z = a(s)$. So it suffices to show that

$$\frac{(z-1)(\tau^2+1)f(z)}{((1+z)^2+\tau^2(1-z)^2)g'(z)} - t_2(s) \le 0 \tag{60}$$

because g' is negative. We are going to show that it is sufficient to check the condition (60) at the point $z = -1$. Indeed, note that $(t_2)'_z \ge 0$ on $[-1, c]$, where c is the root of the cup, and also note that

$$\left(\frac{(z-1)(\tau^2+1)f}{((1+z)^2+\tau^2(1-z)^2)g'}\right)'_z = \frac{\tau^2+1}{p}(p-2)(1-z)^{-(p-1)}[(1+z)^2+\tau^2(1-z)^2]^{p/2-2}2(1+z) \le 0.$$

The condition (60) at the point $z = -1$ turns into the condition

$$t_2(s_0) - \frac{\tau^{p-2}(\tau^2+1)}{p} \ge 0.$$

Now we recall (27) and $t_2(s_0) = (f'(-1) - f'(s_0)/(g'(-1) - g'(s_0))$; therefore, we have

$$t_2(s_0) - \frac{\tau^{p-2}(\tau^2+1)}{p} \ge \frac{f''(-1)}{g''(-1)} - \frac{\tau^{p-2}(\tau^2+1)}{p} = \frac{\tau^p(p-1)^2+\tau^{p-2}}{p(p-1)} > 0.$$

Thus we finish this section with the following remark:

Remark 38. We still have to check the cases when the points $(y_2/y_1, y_3/y_1^p)$ belong to the boundary of $\mathrm{Ang}(s_0)$ and the vertical rays $y_2 = \pm 1$ in Ω_3. The reader can easily see that, in this case, the concavity of N follows from the observation that $N \in C^1(\Omega_1)$. Symmetry of N covers the rest of the cases when $(y_2/y_1, y_3/y_1^p) \notin \Omega_3$.

Thus we have constructed the candidate N.

5. Sharp constants via foliation

5.1. ***Main theorem.*** We remind the reader the definition of the functions $u(z)$, $g(s)$ and $f(s)$ (see (44), (39) and (40)), the value $y_p = -1 + 2/p$ and the definition of the function $a(s)$ (see Lemma 28, Lemma 31 and Remark 33).

Theorem 39. *Let $1 < p < 2$, let G be the martingale transform of F and let $|\mathbb{E}G| \le \beta|\mathbb{E}F|$. Set $\beta' = \dfrac{\beta-1}{\beta+1}$.*
(i) *If $u(1/(p-1)) \le 0$ then*

$$\mathbb{E}(\tau^2F^2+G^2)^{p/2} \le \left(\tau^2 + \max\left\{\beta, \frac{1}{p-1}\right\}^2\right)^{\frac{p}{2}} \mathbb{E}|F|^p. \tag{61}$$

(ii) *If $u(1/(p-1)) > 0$ then*

$$\mathbb{E}(\tau^2F^2+G^2)^{p/2} \le C(\beta')\mathbb{E}|F|^p,$$

where $C(\beta')$ is continuous, nondecreasing, and is defined by

$$C(\beta') \stackrel{\text{def}}{=} \begin{cases} (\tau^2+\beta^2)^{p/2} & \text{if } \beta' \geq s^*, \\ \tau^p\left(1-\dfrac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0)+2(2-p)}\right)^{-1} & \text{if } \beta' \leq -1+2/p, \\ \dfrac{f'(s_1)-f'(a(s_1))}{g'(s_1)-g'(a(s_1))} & \text{if } R(s_1,\beta')=0 \text{ for } s_1\in(\beta',s_0), \end{cases}$$

where $s_0 \in (-1+2/p, 1)$ is the solution of the equation $u((1+s_0)/(1-s_0)) = 0$, and the function $R(s,z)$ is defined as follows:

$$R(s,z) \stackrel{\text{def}}{=} -f(s) - \frac{f'(a(s))g'(s)-f'(s)g'(a(s))}{g'(s)-g'(a(s))}(z-s) + \frac{f'(s)-f'(a(s))}{g'(s)-g'(a(s))}g(s)$$

for $z \in [-1+2/p, s^*]$, $s \in [z, s_0]$. The value $s^* \in [-1+2/p, s_0]$ is the solution of the equation

$$\frac{f'(s^*)-f'(a(s^*))}{g'(s^*)-g'(a(s^*))} = \frac{f(s^*)}{g(s^*)}. \tag{62}$$

Proof. Before we investigate some of the cases mentioned in the theorem, we should make the following observation. The inequality (61) can be restated as follows:

$$H(x_1,x_2,x_3) \leq Cx_3, \tag{63}$$

where H is defined by (5) and $x_1 = \mathbb{E}F$, $x_2 = \mathbb{E}G$, $x_3 = \mathbb{E}|F|^p$. In order to derive the estimate (61), we have to find the sharp C in (63). Because of the property (30), we can assume that both of the values x_1, x_2 are nonnegative. So, the nonnegativity of x_1, x_2 and the condition $|\mathbb{E}G| \leq \beta|\mathbb{E}F|$ can be reformulated as

$$-\frac{x_1+x_2}{2} \leq \frac{x_2-x_1}{2} \leq \frac{\beta-1}{\beta+1}\cdot\frac{x_1+x_2}{2}. \tag{64}$$

The condition (64) with (63) in terms of the function N and the variables y_1, y_2, y_3 means that we have to find the sharp C such that

$$N(y_1,y_2,y_3) \leq Cy_3 \quad \text{for } -y_1 \leq y_2 \leq \frac{\beta-1}{\beta+1}y_1,\ \boldsymbol{y}\in\Omega_2.$$

Because of (38), the above condition can be reformulated as

$$B(y_2,y_3) \leq Cy_3 \quad \text{for } -1 \leq y_2 \leq \frac{\beta-1}{\beta+1},\ y_3 \geq g(y_2), \tag{65}$$

where $B(y_2,y_3) = N(1,y_2,y_3)$. So our aim is to find the sharp C, or in other words the value $\sup_{y_1,y_2} B/y_3$, where the supremum is taken from the domain mentioned in (65). Note that the quantity $B(y_2,y_3)/y_3$ increases with respect to the variable y_2. Indeed, $(B(y_2,y_3)/y_3)'_{y_2} = t_1(s(y))/y_3$, where the function $t_1(s)$ is nonnegative on $[c_0, 1]$ (see the end of the proof of the concavity condition in the domain $\Omega(\Theta[s_0, y_p])$). Note that, as we increase the value y_2, the range of y_3 also increases. This means that the supremum of the expression B/y_3 is attained on the subdomain where $y_2 = (\beta-1)/(\beta+1)$. It is worth mentioning that, since the quantity $(\beta-1)/(\beta+1)\in[-1,1]$ increases as β increases and because of the observation made above, we see that the value C increases as β' increases.

5.2. *The case $y_p \le s_0$.* We are going to investigate the simple case (i). Remark 36 implies that $s_0 \le y_p$; in other words, the foliation of vertical segments is $\Theta([y_p,1],g)$, where the value $\theta(s)$ on $[y_p,1]$ is equal to $\pi/2$. This means that $t_2(s)$ is constant on $[y_p,1]$ (see Lemma 13), and it is equal to $f(y_p)/g(y_p) = (\tau^2+1/(p-1)^2)^{p/2}$ (see (49)).

If $(\beta-1)/(\beta+1)\ge y_p$, or equivalently $\beta\ge 1/(p-1)$, then the function B on the vertical segment with the endpoint $(\beta', g(\beta'))$, where $(\beta-1)/(\beta+1)=\beta'\in[y_p,1)$, has the expression (see (14))

$$B(\beta',y_3)=f(\beta')+\frac{f(y_p)}{g(y_p)}(y_3-g(\beta')),\qquad y_3\ge g(\beta').$$

Therefore,

$$\frac{B(\beta',y_3)}{y_3}=\frac{f(y_p)}{g(y_p)}+\frac{g(\beta')}{y_3}\left(\frac{f(\beta')}{g(\beta')}-\frac{f(y_p)}{g(y_p)}\right),\qquad y_3\ge g(\beta'). \tag{66}$$

The expression $f(s)/g(s)$ is strictly increasing on $(-1,1)$; therefore, (66) attains its maximal value at the point $y_3=g(\beta')$. Thus, we have

$$\frac{B(y_2,y_3)}{y_3}\le\frac{B(\beta',y_3)}{y_3}\le\frac{B(\beta',g(\beta'))}{g(\beta')}=\frac{f(\beta')}{g(\beta')}=(\tau^2+\beta^2)^{p/2}\quad\text{for }\ -1\le y_2\le\beta',\ y_3\ge g(y_2).$$

If $(\beta-1)/(\beta+1)<y_p$, or equivalently $\beta<1/(p-1)$, then we can achieve the value for C which was achieved at the moment $\beta=1/(p-1)$, and, since the function $C=C(\beta')$ increases as β' increases, this value will be the best. Indeed, it suffices to look at the foliation (see Figure 10). For any fixed y_2 we send y_3 to $+\infty$, and we obtain that

$$\begin{aligned}\lim_{y_3\to\infty}\frac{B(y_2,y_3)}{y_3}&=\lim_{y_3\to\infty}\frac{f(s)+t_1(s)(y_2-s)+t_2(s)(y_3-g(s))}{y_3}\\&=\lim_{y_3\to\infty}t_2(s(y))=t_2(y_p)=\left(\tau^2+\frac{1}{(p-1)^2}\right)^{\frac{p}{2}}.\end{aligned}$$

5.3. *The case $y_p > s_0$.* As was already mentioned, the condition in case (ii) is equivalent to the inequality $s_0>y_p$ (see Remark 36). This means that the foliation of the vertical segments is $\Theta([s_0,1],g)$ (see Figure 11). We know that $C(\beta')$ is increasing. We recall that we are going to maximize the function $B(y_2,y_3)/y_3$ on the domain in (65). It was already mentioned that we can require $y_2=(\beta-1)/(\beta+1)=\beta'$. For such fixed $y_2=\beta'\in[-1,1]$, we are going to investigate the monotonicity of the function $B(\beta',y_3)/y_3$. We consider several cases. Let $\beta'\ge s_0$. We differentiate the function $B(\beta',y_3)/y_3$ with respect to y_3, and we use the expression (14) for B to obtain that

$$\frac{\partial}{\partial y_3}\left(\frac{B(\beta',y_3)}{y_3}\right)=\frac{t_2(\beta')y_3-B(\beta',y_3)}{y_3^2}=\frac{-f(\beta')+t_2(\beta')g(\beta')}{y_3^2}.$$

Recall that $t_2(s) = t_2(s_0)$ for $s \in [s_0, 1]$; therefore, direct calculations imply

$$t_2(\beta') = \frac{f(s_0) - (s_0 - y_p) f'(s_0)}{g(s_0) - (s_0 - y_p) g'(s_0)} < \frac{f(s_0)}{g(s_0)} \le \frac{f(\beta')}{g(\beta')}, \quad \beta' \ge s_0.$$

This implies that

$$C(\beta') = \sup_{y_3 \ge g(\beta')} \frac{B(\beta', y_3)}{y_3} = \frac{B(\beta', y_3)}{y_3}\bigg|_{y_3 = g(\beta')} = \frac{f(\beta')}{g(\beta')} = (\tau^2 + \beta^2)^{p/2}.$$

Now we consider the case $\beta' < s_0$. For each point $y = (\beta', y_3)$ that belongs to the line $y_2 = \beta'$, there exists a segment $\ell(y) \in \Theta((c, s_0], g)$ with the endpoint $(s, g(s))$, where $s \in [\max\{\beta', a(\beta')\}, s_0]$. If the point y belongs to the domain of linearity $\mathrm{Ang}(s_0)$, then we can choose the value s_0 and consider any segment with the endpoints y and $(s_0, g(s_0))$, which surely belongs to the domain of linearity. The reader can easily see that as we increase the value y_3 the value s increases as well. So,

$$\frac{\partial}{\partial y_3}\left(\frac{B(\beta', y_3)}{y_3}\right) = \frac{t_2(s) y_3 - B(\beta', y_3)}{y_3^2} = \frac{-f(s) - t_1(s)(\beta' - s) + t_2(s) g(s)}{y_3^2}.$$

Our aim is to investigate the sign of the expression $-f(s) - t_1(s)(\beta' - s) + t_2(s)g(s)$ as we vary the value $y_3 \in [g(\beta'), +\infty)$. Without loss of generality, we can forget about the variable y_3, and we can vary only the value s on the interval $[\max\{\alpha(\beta'), \beta'\}, s_0]$.

We consider the function $R(s, z) \stackrel{\text{def}}{=} -f(s) - t_1(s)(z - s) + t_2(s) g(s)$ with the domain $-1 \le z \le s_0$ and $s \in [\max\{\alpha(z), z\}, s_0]$ (see Figure 12). As we have already seen, $R(s_0, s_0) < 0$. Note that $R(s_0, -1) > 0$. Indeed, $R(s_0, -1) = t_2(s_0) g(-1) - f(-1)$. This equality follows from the fact that

$$f(s_0) - f(-1) = t_1(s_0)(s_0 + 1) + t_2(s_0)(g(s_0) - g(-1)),$$

which is a consequence of Lemma 29. So, (51) and (27) imply

$$t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} > \frac{f''(-1)}{g''(-1)} \ge \frac{f(-1)}{g(-1)}.$$

The function $R(z, s_0)$ is linear with respect to z. So, on the interval $[-1, s_0]$, it has the root $y_p = -1 + 2/p$. Indeed,

$$\frac{-f(s_0) + t_2(s_0) g(s_0) + t_1(s_0) s_0}{t_1(s_0)} = y_p.$$

The last equality follows from (51), (53) and (12). We need a few more properties of the function $R(s, z)$. For each fixed z, the function $R(s, z)$ is nonincreasing on $[\max\{\alpha(z), z\}, s_0]$. Indeed,

$$R'_s(s, z) = -f'(s) - t'_1(s)(z - s) + t_1(s) + t'_2(s) g(s) + t_2(s) g(s). \tag{67}$$

We take into account the condition (12), so the expression (67) simplifies to

$$R'_s(s, z) = t'_2(s) g(s) + t'_1(s)(s - z).$$

We remind the reader of the equality (11) and the fact that $t'_2(s) \le 0$. Therefore, we have $R'_s(s, z) = y_3 t'_2(s)$, where $y_3 = y_3(s) > 0$. Thus we see that $R(s, \beta') \ge 0$ for $\beta' \le y_p$.

So, if the function $R(\,\cdot\,, z)$ at the right end on its domain $[\max\{\alpha(z), z\}, s_0]$ is positive, this will mean that the function B/y_3 is increasing; hence, the constant $C(\beta')$ will be equal to

$$C(\beta') = \lim_{y_3\to\infty} \frac{B(z, y_3)}{y_3} = t_2(s_0) = \frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)}$$

(this follows from (51) and the structure of the foliation). Since $u((1+s_0)/(1-s_0)) = 0$ and given (52), direct computations show that

$$\frac{f'(-1) - f'(s_0)}{g'(-1) - g'(s_0)} = \tau^p \left(1 - \frac{2^{2-p}(1-s_0)^{p-1}}{(\tau^2+1)(p-1)(1-s_0) + 2(2-p)}\right)^{-1}. \tag{68}$$

So it follows that, if $\beta' \le y_p$, then (68) is the value of $C(\beta')$.

If the function $R(\,\cdot\,, z)$ on the left end of its domain is nonpositive, this will mean that the function B/y_3 is decreasing, so the sharp constant will be the value of the function $B(z, y_3)/y_3$ at the left end of its domain:

$$C(\beta') = \frac{B(z, y_3)}{y_3}\bigg|_{y_3=g(z)} = \frac{f(z)}{g(z)} = (\tau^2 + \beta^2)^{p/2}. \tag{69}$$

We recall that c is the root of the cup and $c < y_p$ (see Remark 34). We will show that the function $R(z, s)$ is decreasing on the boundary $s = z$ for $s \in (y_p, s_0]$. Indeed, (12) implies

$$(R(s, s))'_s = -f'(s) + t_2'(s)g(s) + t_2(s)g'(s) = -t_1(s) + t_2'(s)g(s) < 0.$$

The last inequality follows from the fact that $t_2'(s) \le 0$ and $t_1(s) > 0$ on $(c, 1]$. Surely $R(y_p, y_p) > R(s_0, y_p) = 0$, and we recall that $R(s_0, s_0) < 0$, so there exists a unique $s^* \in [y_p, s_0]$ such that $R(s^*, s^*) = 0$. This is equivalent to (62). So it is clear that $R(z, z) \le 0$ for $z \in [s^*, s_0]$. Therefore, $C(\beta')$ has the value (69) for $\beta' \ge s^*$.

The only case that remains is when $\beta' \in [y_p, s^*]$. We know that $R(z, z) \ge 0$ for $z \in [y_p, s^*]$ and $R(s_0, z) \le 0$ for $z \in [y_p, s^*]$. The fact that, for each fixed z, the function $R(s, z)$ is decreasing implies the following: for each $z \in [y_p, s^*]$, there exists a unique $s_1(z) \in [z, s_0]$ such that $R(z, s_1(z)) = 0$. Therefore, for $\beta' \in [y_p, s^*]$ we have

$$C(\beta') = \frac{B\big(\beta', y_3(s_1(\beta'))\big)}{y_3(s_1(\beta'))}, \tag{70}$$

where the value $s_1(\beta')$ is the root of the equation $R(s_1(\beta'), \beta') = 0$. Recall that

$$R(s_1(\beta'), \beta') = t_2(s_1)y_3(s_1) - B(\beta', y_3(s_1)) = -f(s_1) - t_1(s_1)(\beta' - s_1) + t_2(s_1)g(s_1). \tag{71}$$

So the expression (70) takes the form

$$C(\beta') = t_2(s_1) = \frac{f'(s_1) - f'(a(s_1))}{g'(s_1) - g'(a(s_1))}.$$

Finally, we remind the reader that

$$t_2(s) = \frac{f'(s) - f'(a(s))}{g'(s) - g'(a(s))},$$
$$t_1(s) = \frac{f'(a(s))g'(s) - f'(s)g'(a(s))}{g'(s) - g'(a(s))}$$

for $s \in (c, s_0]$, and we finish the proof of the theorem. □

6. Extremizers via foliation

We set $\Psi(F, G) = \mathbb{E}(G^2 + \tau^2 F^2)^{2/p}$. Let N be the candidate that we have constructed in Section 4 (see (54)). We define the candidate $\boldsymbol{B}$ for the Bellman function H (see (5)) as follows:

$$\boldsymbol{B}(x_1, x_2, x_3) = N\left(\frac{x_1 + x_2}{2}, \frac{x_2 - x_1}{2}, x_3\right), \quad (x_1, x_2, x_3) \in \Omega.$$

We want to show that $\boldsymbol{B} = H$. We already know that $\boldsymbol{B} \geq H$ (see Proposition 9). So, it remains to show that $\boldsymbol{B} \leq H$. We are going to do this as follows: for each point $\boldsymbol{x} \in \Omega$ and any $\varepsilon > 0$, we are going to find an admissible pair (F, G) such that

$$\Psi(F, G) > \boldsymbol{B}(\boldsymbol{x}) - \varepsilon. \tag{72}$$

Up to the end of the current section, we are going to work with the coordinates (y_1, y_2, y_3) (see (33)). It will be convenient for us to redefine the notion of admissibility of a pair.

Definition 40. We say that a pair (F, G) is admissible for the point $(y_1, y_2, y_3) \in \Omega_1$ if G is the martingale transform of F and $\mathbb{E}(F, G, |F|^p) = (y_1 - y_2, y_1 + y_2, y_3)$.

So, in this case, the condition (72) in terms of the function N takes the following form: for any point $\boldsymbol{y} \in \Omega_1$ and for any $\varepsilon > 0$, we are going to find an admissible pair (F, G) for the point $\boldsymbol{y}$ such that

$$\Psi(F, G) > N(\boldsymbol{y}) - \varepsilon. \tag{73}$$

We formulate the following obvious observations:

Lemma 41. (1) *A pair (F, G) is admissible for the point $\boldsymbol{y} = (y_1, y_2, y_3)$ if and only if $(\tilde{F}, \tilde{G}) = (\pm F, \mp G)$ is admissible for the point $\tilde{\boldsymbol{y}} = (\mp y_2, \mp y_1, y_3)$; moreover,* $\Psi(\tilde{F}, \tilde{G}) = \Psi(F, G)$.

(2) *A pair (F, G) is admissible for the point $\boldsymbol{y} = (y_1, y_2, y_3)$ if and only if $(\tilde{F}, \tilde{G}) = (\lambda F, \lambda G)$ (where $\lambda \neq 0$) is admissible for the point $\tilde{\boldsymbol{y}} = (\lambda y_1, \lambda y_2, |\lambda|^p y_3)$; moreover,* $\Psi(\tilde{F}, \tilde{G}) = |\lambda|^p \Psi(F, G)$.

Definition 42. The pair of functions (F, G) is called an *ε-extremizer* for the point $\boldsymbol{y} \in \Omega_1$ if (F, G) is admissible for the point $\boldsymbol{y}$ and $\Psi(F, G) > N(\boldsymbol{y}) - \varepsilon$.

Lemma 41, homogeneity, and the symmetry of N imply that we only need to check (73) for the points $\boldsymbol{y} \in \Omega_1$ where $y_1 = 1$ and $(y_2, y_3) \in \Omega_3$. In other words, we show that $\Psi(F, G) > \boldsymbol{B}(y_2, y_3) - \varepsilon$ for some admissible pair (F, G) for the point $(1, y_2, y_3)$, where $(y_2, y_3) \in \Omega_3$. Further, instead of saying that (F, G) is an admissible pair (or ε-extremizer) for the point $(1, y_2, y_3)$ we just say that it is an

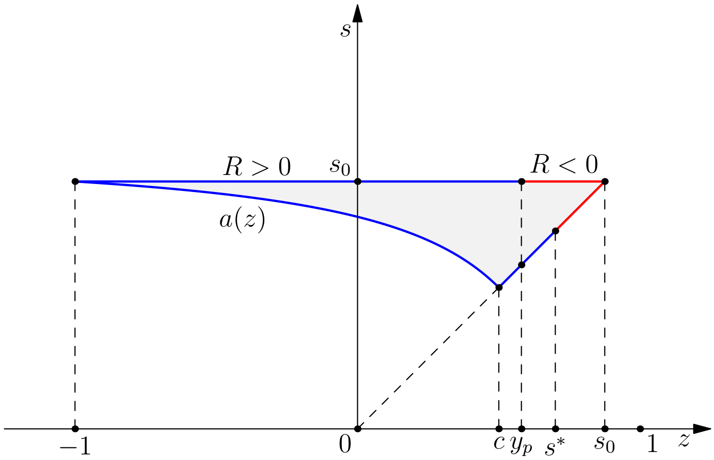

Figure 12. The domain of $R(s, z)$.

admissible pair (or an ε-extremizer) for the point (y_2, y_3). So we only have to construct ε-extremizers in the domain Ω_3.

It is worth mentioning that we construct ε-extremizers (F, G) such that G will be the martingale transform of F with respect to some filtration other than dyadic. The reader can find a detailed explanation on how to pass from one filtration to another in [Slavin and Vasyunin 2011].

We need a few more observations. For $\alpha \in (0, 1)$, we define the *α-concatenation* of the pairs (F, G) and $(\tilde{F}, \tilde{G})$ as follows:

$$(F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha(x) = \begin{cases} (F, G)(x/\alpha) & \text{if } x \in [0, \alpha], \\ (\tilde{F}, \tilde{G})((x-\alpha)/(1-\alpha)) & \text{if } x \in [\alpha, 1]. \end{cases}$$

Clearly, $\Psi((F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha(x)) = \alpha\Psi(F, G) + (1-\alpha)\Psi(\tilde{F}, \tilde{G})$.

Definition 43. Any domain of the type $\Omega_1 \cap \{y_1 = A\}$, where A is some real number, is said to be a *positive domain*. Any domain of the type $\Omega_1 \cap \{y_2 = B\}$, where B is some real number, is said to be a *negative domain*.

The following lemma is obvious:

Lemma 44. *If (F, G) is an admissible pair for a point $\boldsymbol{y}$ and $(\tilde{F}, \tilde{G})$ is an admissible pair for a point $\tilde{\boldsymbol{y}}$ such that either of the following is true: $\boldsymbol{y}$, $\tilde{\boldsymbol{y}}$ belong to a positive domain, or $\boldsymbol{y}$, $\tilde{\boldsymbol{y}}$ belong to a negative domain, then $(F \bullet \tilde{F}, G \bullet \tilde{G})_\alpha$ is an admissible pair for the point $\alpha\boldsymbol{y} + (1-\alpha)\tilde{\boldsymbol{y}}$.*

Let (F, G) be an admissible pair for a point $\boldsymbol{y}$, and let $(\tilde{F}, \tilde{G})$ be an admissible pair for a point $\tilde{\boldsymbol{y}}$. Let $\hat{\boldsymbol{y}}$ be a point which belongs to the chord joining the points $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$.

Remark 45. It is clear that, if (F^+, G^+) is admissible for a point (y_2^+, y_3^+) and (F^-, G^-) is admissible for a point (y_2^-, y_3^-), then an α-concatenation of these pairs is admissible for the point $(y_2, y_3) = \alpha \cdot (y_2^+, y_3^+) + (1-\alpha) \cdot (y_2^-, y_3^-)$.

Now we are ready to construct ε-extremizers in Ω_3. The main idea is that these functions Ψ and $\boldsymbol{B}$ are very similar: they obey almost the same properties. Moreover, foliation plays a crucial role in the contraction of ε-extremizers.

6.1. ***The case $s_0 \le y_p$.*** We want to find ε-extremizers for the points in Ω_3.

Extremizers in the domain $\Omega\big(\Theta_{\text{cup}}((c, s_0], g)\big)$. Pick any $y = (y_2, y_3) \in \Omega(\Theta_{\text{cup}}((c, s_0], g))$. Then there exists a segment $\ell(y) \in \Theta_{\text{cup}}((c, s_0], g)$. Let $y^+ = (s, g(s))$ and $y^- = (a(s), g(a(s))$ be the endpoints of $\ell(y)$ in Ω_3. We know ε-extremizers at these points y^+, y^-. Indeed, we can take the ε-extremizers $(F^+, G^+) = (1-s, 1+s)$ and $(F^-, G^-) = (1-a(s), 1+a(s))$ (i.e., constant functions). Consider an α-concatenation $(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha$, where α is chosen so that $y = \alpha y^+ + (1-\alpha) y^-$. We have

$$\Psi[(F^+\bullet F^-, G^+\bullet G^-)_\alpha]=\alpha\Psi(F^+, G^+)+(1-\alpha)\Psi(F^{-1}, G^-) > \alpha\boldsymbol{B}(y^+)+(1-\alpha)\boldsymbol{B}(y^-)-\varepsilon=\boldsymbol{B}(y)-\varepsilon.$$

The last equality follows from the linearity of $\boldsymbol{B}$ on $\ell(y)$.

Extremizers on the vertical line $(-1, y_3)$, $y_3 \ge h(s_0)$. Now we are going to find ε-extremizers for the points $(-1, y_3)$, where $y_3 \ge h(s_0)$. We use a similar idea to one mentioned in [Vasyunin and Volberg 2010] (see the proof of Lemma 3). We define the functions (F, G) recursively:

$$G(t) = \begin{cases} -w & \text{if } 0 \le t < \varepsilon, \\ \gamma \cdot g\left(\dfrac{t-\varepsilon}{1-2\varepsilon}\right) & \text{if } \varepsilon \le t \le 1-\varepsilon, \\ w & \text{if } 1-\varepsilon < t \le 1, \end{cases}$$

$$F(t) = \begin{cases} d_- & \text{if } 0 \le t < \varepsilon, \\ \gamma \cdot f\left(\dfrac{t-\varepsilon}{1-2\varepsilon}\right) & \text{if } \varepsilon \le t \le 1-\varepsilon, \\ d_+ & \text{if } 1-\varepsilon < t \le 1, \end{cases}$$

where the nonnegative constants w, d_-, d_+ and γ will be obtained from the requirement $\mathbb{E}(F, G, |F|^p) = (2, 0, y_3)$ and the fact that G is the martingale transform of F. Surely $\langle G\rangle_{[0,1]} = 0$. The condition $\langle F\rangle_{[0,1]} = 2$ means that

$$(d_- + d_+)\varepsilon + 2\gamma(1-2\varepsilon) = 2. \tag{74}$$

The condition $\langle |F|^p\rangle_{[0,1]} = y_3$ implies that

$$y_3 = \frac{\varepsilon(d_+^p + d_-^p)}{1-(1-2\varepsilon)\gamma^p}. \tag{75}$$

Now we use the condition $|F_0 - F_1| = |G_0 - G_1|$. In the first step we split the interval $[0, 1]$ at the point ε with the requirement $F_0 - F_1 = G_0 - G_1$, from which we obtain $w = 2 - d_-$. In the second step we split at the point $1-\varepsilon$ with the requirement $F_1 - F_2 = G_2 - G_1$, obtaining $w = 2\gamma - d_+$. From these two conditions we obtain $d_- + d_+ = 2(1+\gamma) - 2w$, and by substituting in (74) we find

$$\gamma = 1 + \frac{\varepsilon w}{1-\varepsilon}.$$

Now we investigate what happens as ε tends to zero. Our aim will be to focus on the limit value $\lim_{\varepsilon\to 0} w = w_0$. We have $1-(1-2\varepsilon)\gamma^p \approx \varepsilon(2-wp)$. So (75) becomes

$$y_3 = \frac{\varepsilon(d_+^p + d_-^p)}{1-(1-2\varepsilon)\gamma^p} \to \frac{2(2-w_0)^p}{2-w_0 p}. \tag{76}$$

Note that, for $w_0 = 1+s$, equation (76) is the same as (47). By direct calculations we see that as $\varepsilon \to 0$ we have

$$\langle (G^2+\tau^2F^2)^{p/2}\rangle_{[0,1]} = \frac{\varepsilon[(w^2+\tau^2 d_-^2)^{p/2} + (w^2+\tau^2 d_+^2)^{p/2}]}{1-(1-2\varepsilon)\gamma^p} \to \frac{2f(w_0-1)}{2-w_0 p}.$$

Now we are going to calculate the value $\boldsymbol{B}(-1,h(s))$, where $h(s)=y_3$. From (45) we have

$$\boldsymbol{B}(-1,h(s)) = h(s)t_2(s) - \frac{2}{p}t_1(s).$$

By using (12) we express t_1 via t_2; also because of (47) and (50), we have

$$\begin{aligned}\boldsymbol{B}(-1,h(s)) &= h(s)t_2(s) - \frac{2}{p}t_1(s) = h(s)t_2 - \frac{2}{p}(f'-t_2g') = t_2\Big(h(s)+\frac{2}{p}g'\Big) - f'\frac{2}{p}\\ &= \frac{f(s)-(s-y_p)f'(s)}{g(s)-(s-y_p)g'(s)}\Big(\frac{2g}{p(y_p-s)}+\frac{2}{p}g'\Big) - f'\frac{2}{p}\\ &= \frac{2}{p}\Big[\frac{f(s)}{y_p-s}\Big] = \frac{2(2-w_0)^p}{2-w_0p}.\end{aligned}$$

Thus we obtain the desired result

$$\langle (G^2+\tau^2F^2)^{p/2}\rangle_{[0,1]} \to \boldsymbol{B}(-1,y_3) \quad \text{as } \varepsilon \to 0.$$

Extremizers in the domain $\Omega\big(\Theta([s_0,y_p),g)\big)$. Pick any point $y=(y_2,y_3)\in\Omega\big(\Theta([s_0,y_p],g)\big)$. Then there exists a segment $\ell(y)\in\Theta([s_0,y_p],g)$. Let y^+ and y^- be the endpoints of this segment, so that $y^+=(-1,\tilde{y}_3)$ for some $\tilde{y}_3\ge h(s_0)$ and $y^-=(\tilde{s},g(\tilde{s}))$ for some $\tilde{s}\in[y_p,s_0)$. We remind the reader that we know ε-extremizers for the points $(s,g(s))$, where $s\in[s_0,1]$, and we know ε-extremizers on the vertical line $(-1,y_3)$, where $y_3\ge h(s_0)$. Therefore, as in the case of a cup, taking the appropriate α-concatenation of these ε-extremizers and using the fact that $\boldsymbol{B}$ is linear on $\ell(y)$, we find an ε-extremizer at the point y.

Extremizers in the domain $\mathrm{Ang}(s_0)$. Pick any $y=(y_1,y_2)\in\mathrm{Ang}(s_0)$. There exist points $y^+\in\ell^+$, $y^-\in\ell^-$, where $\ell^+=\ell^+(s_0,g(s_0))\in\Theta([s_0,y_p),g)$ and $\ell^-=\ell^-(s_0,g(s_0))\in\Theta((c,s_0],g)$, such that $y=\alpha y^+ + (1-\alpha)y^-$ for some $\alpha\in[0,1]$. We know ε-extremizers at the points y^+ and y^-. Then by taking an α-concatenation of these extremizers and using the linearity of $\boldsymbol{B}$ on $\mathrm{Ang}(s_0)$ we can obtain an ε-extremizer at the point y.

Extremizers in the domain $\Omega\big(\Theta([y_p,1],g)\big)$. Finally, we consider the domain of vertical segments $\Omega(\Theta[y_p,1],g)$. Pick any point $y=(y_2,y_3)\in\Omega(\Theta[y_p,1])$. Take an arbitrary point $y^+=(-1,y_3^+)$, where y_3^+ is sufficiently large such that $y=\alpha y^+ + (1-\alpha)y^-$ for some $\alpha\in(0,1)$ and some $y^-=(y_2^-,y_3^-)$

with $(1, y_2^-, y_3^-) \in \partial\Omega_1$. Surely, y^+ and y^- belong to a positive domain. The condition $(1, y_2^-, y_3^-) \in \partial\Omega_1$ implies that we know an ε-extremizer (F^-, G^-) at the point y^- (these are constant functions). We also know an ε-extremizer at the point y^+. Let $(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha$ be an α-concatenation of these extremizers. Then

$$\Psi[(F^+ \bullet F^-, G^+ \bullet G^-)_\alpha] > \alpha \boldsymbol{B}(y^+) + (1-\alpha)\boldsymbol{B}(y^-) - \varepsilon.$$

Note that the condition $y = \alpha y^+ + (1-\alpha)y^-$ implies that

$$\alpha = \frac{y_3 - (y_2/y_2^-)y_3^-}{y_3^+ + y_3^-/y_2^-}.$$

Recall that $\boldsymbol{B}(y_2, g(y_2)) = f(y_2)$ and $\boldsymbol{B}(y^+) = f(s) + t_1(s)(-1-s) + t_2(s)(y_3^+ - g(s))$, where $s \in [s_0, y_p]$ is such that a segment $\ell(s, g(s)) \in \Theta([s_0, y_p), g)$ has an endpoint y^+.

Note that as $y_3^+ \to \infty$ all terms remain bounded; moreover, $\alpha \to 0$, $y^- \to (y_2, g(y_2))$ and $s \to y_p$. This means that

$$\begin{aligned}\lim_{y_3^+ \to \infty} \alpha \boldsymbol{B}(y^+) + (1-\alpha)\boldsymbol{B}(y^-) - \varepsilon &= \lim_{y_3^+ \to \infty} t_2(s) y_3^+ \left(\frac{y_3 - (y_2/y_2^-)y_3^-}{y_3^+ + y_3^-/y_2^-} \right) + f(y_2) - \varepsilon \\ &= t_2(y_p)(y_3 - g(y_2)) + f(y_2) - \varepsilon.\end{aligned}$$

We recall that $t_2(s) = t_2(y_p)$ for $s \in [y_p, 1]$. Then

$$\boldsymbol{B}(y) = f(y_2) + t_2(s(y))(y_3 - g(y_2)) = f(y_2) + t_2(y_p)(y_3 - g(y_2)).$$

Thus, if we choose y_3^+ sufficiently large then we can obtain a 2ε-extremizer for the point y.

6.2. ***The case $s_0 > y_p$.*** In this case we have $s_0 \geq y_p$ (see Figure 11). This case is a little bit more complicated than the previous one. The construction of ε-extremizers (F, G) will be similar to the one presented in [Reznikov et al. 2013].

We need a few more definitions.

Definition 46. Let (F, G) be an arbitrary pair of functions. Let $(y_2, g(y_2)) \in \Omega_3$ and let J be a subinterval of $[0, 1]$. We define a new pair $(\tilde{F}, \tilde{G})$ as follows:

$$(\tilde{F}, \tilde{G})(x) = \begin{cases} (F, G)(x) & \text{if } x \in [0,1] \setminus J \\ (1 - y_2, 1 + y_2) & \text{if } x \in J. \end{cases}$$

We will refer to the new pair $(\tilde{F}, \tilde{G})$ as *putting the constant* $(y_2, g(y_2))$ *on the interval* J for the pair (F, G).

Sometimes we will denote the new pair $(\tilde{F}, \tilde{G})$ by the same symbol (F, G).

Definition 47. We say that the pairs (F_α, G_α), $(F_{1-\alpha}, G_{1-\alpha})$ are obtained from the pair (F, G) by splitting at the point $\alpha \in (0, 1)$ if

$$\begin{aligned}(F_\alpha, G_\alpha) &= (F, G)(x \cdot \alpha), & x &\in [0, 1], \\ (F_{1-\alpha}, G_{1-\alpha}) &= (F, G)(x \cdot (1-\alpha) + \alpha), & x &\in [0, 1].\end{aligned}$$

It is clear that $\Psi(F,G)=\alpha\Psi(F_\alpha,G_\alpha)+(1-\alpha)\Psi(F_{1-\alpha},G_{1-\alpha})$. Also note that, if (F_α,G_α) and $(F_{1-\alpha},G_{1-\alpha})$ are obtained from the pair (F,G) by splitting at the point $\alpha\in(0,1)$, then (F,G) is an α-concatenation of the pairs (F_α,G_α) and $(F_{1-\alpha},G_{1-\alpha})$. Thus, splitting and concatenation are opposite operations.

Instead of explicitly presenting an admissible pair (F,G) and showing that it is an ε-extremizer, we present an algorithm which constructs the admissible pair, and we show that the result is an ε-extremizer.

By the same explanations as in the case $s_0\le y_p$, it is enough to construct an ε-extremizer (F,G) on the vertical line $y_2=-1$ of the domain Ω_3. Moreover, linearity of $\boldsymbol{B}$ implies that, for any $A>0$, it is enough to construct ε-extremizers for the points $(-1,y_3)$, where $y_3\ge A$. Pick any point $(-1,y_3)$, where $y_3=y_3^{(0)}>g(-1)$. Linearity of $\boldsymbol{B}$ on $\mathrm{Ang}(s_0)$ and direct calculations (see (14), (51)) show that

$$\boldsymbol{B}(-1,y_3)=f(-1)+t_3(s_0)(y_3-g(-1))=f(-1)+(y_3-g(-1))\frac{f'(-1)-f'(s_0)}{g'(-1)-g'(s_0)}. \tag{77}$$

We describe the first iteration. Let (F,G) be an admissible pair for the point $(-1,y_3)$, whose explicit expression will be described during the algorithm. For a pair (F,G), we put a constant $(s_0,g(s_0))$ on an interval $[0,\varepsilon]$, where the value $\varepsilon\in(0,1)$ will be given later. Thus we obtain a new pair (F,G), which we denote by the same symbol. We want (F,G) to be an admissible pair for the point $(-1,y_3)$. Let the pairs $(F_\varepsilon,G_\varepsilon)$ and $(F_{1-\varepsilon},G_{1-\varepsilon})$ be obtained from the pair (F,G) by splitting at the point ε. It is clear that $(F_\varepsilon,G_\varepsilon)$ is an admissible pair for the point $(s_0,g(s_0))$. We want $(F_{1-\varepsilon},G_{1-\varepsilon})$ to be an admissible pair for the point $P=(\tilde y_2,\tilde y_3)$, so that

$$(-1,y_3)=\varepsilon(s_0,g(s_0))+(1-\varepsilon)P. \tag{78}$$

Therefore we require

$$P=\left(\frac{-1-\varepsilon s_0}{1-\varepsilon},\frac{y_3-\varepsilon g(s_0)}{1-\varepsilon}\right). \tag{79}$$

So we make the following simple observation: if $(F_{1-\varepsilon},G_{1-\varepsilon})$ were an admissible pair for the point P, then (F,G) (which is an ε-concatenation of the pairs $(1-s_0,1+s_0)$ and $(F_{1-\varepsilon},G_{1-\varepsilon})$) would be an admissible pair for the point $(-1,y_3)$. The explanation of this observation is simple: note that the pairs $(F_{1-\varepsilon},G_{1-\varepsilon})$ and $(1-s_0,1+s_0)$ are admissible pairs for the points P and $(s_0,g(s_0))$, which belong to a positive domain (see (78)); therefore, the rest immediately follows from Lemma 44. So we want to construct the admissible pair $(F_{1-\varepsilon},G_{1-\varepsilon})$ for the point (79).

We recall Lemma 41, which implies that the pair $(F_{1-\varepsilon},G_{1-\varepsilon})$ is admissible for the point

$$\left(1,\frac{-1-\varepsilon s_0}{1-\varepsilon},\frac{y_3-\varepsilon g(s_0)}{1-\varepsilon}\right)$$

if and only if the pair $(\tilde F,\tilde G)$, where $(F_{1-\varepsilon},-G_{1-\varepsilon})=(1+\varepsilon s_0)/(1-\varepsilon)(\tilde F,\tilde G)$, is admissible for a point

$$W=\left(1,\frac{\varepsilon-1}{1+\varepsilon s_0},\frac{y_3-\varepsilon g(s_0)}{(1+\varepsilon s_0)^p}\cdot(1-\varepsilon)^{p-1}\right).$$

So, if we find the admissible pair $(\tilde F,\tilde G)$ then we automatically find the admissible pair (F,G).

Choose ε small enough so that $\big((\varepsilon-1)/(1+\varepsilon s_0), (y_3-\varepsilon g(s_0))/(1+\varepsilon s_0)^p\cdot(1-\varepsilon)^{p-1}\big)\in\Omega_3$ and

$$\left(\frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{y_3-\varepsilon g(s_0)}{(1+\varepsilon s_0)^p}\cdot(1-\varepsilon)^{p-1}\right)=\delta(s_0, g(s_0))+(1-\delta)(-1, y_3^{(1)})$$

for some $\delta\in(0,1)$ and $y_3^{(1)}\ge g(-1)$. Then

$$\begin{aligned}\delta&=\frac{\varepsilon}{1+\varepsilon s_0}=\varepsilon+O(\varepsilon^2),\\ y_3^{(1)}&=\frac{\big((y_3-\varepsilon g(s_0))/(1+\varepsilon s_0)^p\big)\cdot(1-\varepsilon)^{p-1}-(\varepsilon/(1+\varepsilon s_0))g(s_0)}{1-\varepsilon/(1+\varepsilon s_0)}\\ &=y_3(1-\varepsilon(p+ps_0-2))-2\varepsilon g(s_0)+O(\varepsilon^2).\end{aligned}\tag{80}$$

For the pair $(\tilde F, \tilde G)$, we put a constant $(s_0, g(s_0))$ on the interval $[0,\delta]$. We split the new pair $(\tilde F, \tilde G)$ at δ, so we get the pairs $(\tilde F_\delta, \tilde G_\delta)$ and $(\tilde F_{1-\delta}, \tilde G_{1-\delta})$. We make a similar observation as above. It is clear that if we know the admissible pair $(\tilde F_{1-\delta}, \tilde G_{1-\delta})$ for the point $(-1, y_3^{(1)})$, then we can obtain an admissible pair $(\tilde F, \tilde G)$ for the point

$$\left(\frac{\varepsilon-1}{1+\varepsilon s_0}, \frac{y_3-\varepsilon g(s_0)}{(1+\varepsilon s_0)^p}\cdot(1-\varepsilon)^{p-1}\right).$$

Surely $(\tilde F, \tilde G)$ is a δ-concatenation of the pairs $(1-s_0, 1+s_0)$ and $(\tilde F_{1-\delta}, \tilde G_{1-\delta})$.

We summarize the first iteration. We took $\varepsilon\in(0,1)$, and we started from the pair $(F^{(0)}, G^{(0)})=(F,G)$, and after one iteration we came to the pair $(F^{(1)}, G^{(1)})=(\tilde F_{1-\delta}, \tilde G_{1-\delta})$. We showed that, if $(F^{(1)}, G^{(1)})$ is an admissible pair for the point $(1, y_3^{(1)})$, then the pair $(F^{(0)}, G^{(0)})$ can be obtained from the pair $(F^{(1)}, G^{(1)})$; moreover, it is admissible for the point $(1, y_3^{(0)})$.

Continuing these iterations, we obtain the sequence of numbers $\{y_3^{(j)}\}_{j=0}^N$ and the sequence of pairs $\{(F^{(j)}, G^{(j)})\}_{j=0}^N$. Let N be such that $y_3^{(N)}\ge g(-1)$. It is clear that, if $(F^{(N)}, G^{(N)})$ is an admissible pair for the point $(-1, y_3^{(N)})$, then the pairs $\{(F^{(j)}, G^{(j)})\}_{j=0}^{N-1}$ can be determined uniquely, and, moreover, $(F^{(j)}, G^{(j)})$ is an admissible pair for the point $(-1, y_3^{(j)})$ for all $j=0,\dots,N-1$.

Note that we can choose sufficiently small $\varepsilon\in(0,1)$, and we can find $N=N(\varepsilon)$ such that $y_3^{(N)}=g(-1)$ (see (80), and recall that $s_0>y_p$). In this case the admissible pair $(F^{(N)}, G^{(N)})$ for the point $(-1, y_3^{(N)})=(-1, g(-1))$ is a constant function, namely, $(F^{(N)}, G^{(N)})=(2,0)$. Now we try to find N in terms of ε, and we try to find the value of $\Psi(F^{(0)}, G^{(0)})$.

Condition (80) implies that $y_3^{(1)}=y_3^{(0)}(1-\varepsilon(p+ps_0-2))-2\varepsilon g(s_0)+O(\varepsilon^2)$. We denote $\delta_0=p+ps_0-2>0$. Therefore, after the N-th iteration we obtain

$$y_3^{(N)}=(1-\varepsilon\delta_0)^N\left(y_3^{(0)}+\frac{2g(s_0)}{\delta_0}\right)-\frac{2g(s_0)}{\delta_0}+O(\varepsilon).$$

The requirement $y_3^{(N)}=g(-1)$ implies that

$$(1-\varepsilon\delta_0)^{-N}=\frac{y_3^{(0)}+2g(s_0)/\delta_0}{g(-1)+2g(s_0)/\delta_0}+O(\varepsilon).$$

This implies that $\limsup_{\varepsilon\to 0} \varepsilon\cdot N = \limsup_{\varepsilon\to 0} \varepsilon \cdot N(\varepsilon) < \infty$. Therefore, we get

$$e^{\varepsilon\delta_0 N} = \frac{y_3^{(0)} + 2g(s_0)/\delta_0}{g(-1)+2g(s_0)/\delta_0} + O(\varepsilon). \tag{81}$$

Also note that

$$\begin{aligned}\Psi(F^{(0)}, G^{(0)}) = \Psi(F, G) &= \varepsilon\Psi(F_\varepsilon, G_\varepsilon) + (1-\varepsilon)\Psi(F_{1-\varepsilon}, G_{1-\varepsilon})\\ &= \varepsilon f(s_0) + (1-\varepsilon)\Psi(F_{1-\varepsilon}, G_{1-\varepsilon}) = \varepsilon f(s_0) + (1-\varepsilon)\left(\frac{1+\varepsilon s_0}{1-\varepsilon}\right)^p \Psi(\tilde F, \tilde G)\\ &= \varepsilon f(s_0) + (1-\varepsilon)(1-\varepsilon)\left(\frac{1+\varepsilon s_0}{1-\varepsilon}\right)^p [\delta f(s_0) + (1-\delta)\Psi(\tilde F_{1-\delta}, \tilde G_{1-\delta})]\\ &= 2\varepsilon f(s_0) + (1+\varepsilon\delta_0)\Psi(F^{(1)}, G^{(1)}) + O(\varepsilon^2).\end{aligned}$$

Therefore, after the N-th iteration (and using the fact that $\Psi(F^{(N)}, G^{(N)}) = f(-1)$), we obtain

$$\begin{aligned}\Psi(F^{(0)}, G^{(0)}) &= (1+\varepsilon\delta_0)^N\left(f(-1) + \frac{2f(s_0)}{\delta_0}\right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon)\\ &= e^{\varepsilon\delta_0 N}\left(f(-1) + \frac{2f(s_0)}{\delta_0}\right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon).\end{aligned} \tag{82}$$

The last equality follows from the fact that $\limsup_{\varepsilon\to 0}\varepsilon\cdot N(\varepsilon) < \infty$.

Therefore (81) and (82) imply that

$$\begin{aligned}\Psi(F^{(0)}, G^{(0)}) &= \left(\frac{y_3^{(0)} + 2g(s_0)/\delta_0}{g(-1)+2g(s_0)/\delta_0}\right)\left(f(-1) + \frac{2f(s_0)}{\delta_0}\right) - \frac{2f(s_0)}{\delta_0} + O(\varepsilon)\\ &= f(-1) + (y_3^{(0)} - g(-1))\left(\frac{f(-1)+2f(s_0)/\delta_0}{g(-1)+2g(s_0)/\delta_0}\right) + O(\varepsilon).\end{aligned}$$

Now we recall (77). So, if we show that

$$\frac{f(-1)+2f(s_0)/\delta_0}{g(-1)+2g(s_0)/\delta_0} = \frac{f'(-1)-f'(s_0)}{g'(-1)-g'(s_0)}, \tag{83}$$

then (83) will imply that $\Psi(F^{(0)}, G^{(0)}) = \boldsymbol{B}(-1, y_3^{(0)}) + O(\varepsilon)$. So, choosing ε sufficiently small, we can obtain the extremizer $(F^{(0)}, G^{(0)})$ for the point $(-1, y_3)$. Therefore, we need only to prove equality (83). It will be convenient to use the following notations: set $f_- = f(-1)$, $f'_- = f'(-1)$, $f = f(s_0)$, $f' = f'(s_0)$, $g_- = g(-1)$, $g'_- = g'(-1)$, $g = g(s_0)$ and $g' = g(s_0)$. Then (83) turns into

$$\frac{\delta_0}{2} = \frac{fg'_- - fg' - f'_- g + f'g}{g'f_- - f'g_-}. \tag{84}$$

This simplifies into

$$s_0 - y_p = \frac{2}{p}\cdot\frac{fg'_- - fg' - f'_- g + f'g}{g'f_- - f'g_-} = \frac{fg'_- - fg' - f'_- g + f'g}{-g'f'_- + f'g'_-},$$

which is true by (53).

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