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REGULAR SPECTRUM OF ELEMENTS IN TOPOLOGICAL ALGEBRAS

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Abstract. Main properties of the regular (or extended) spectrum of elements in topological algebras (introduced by L. Waelbroeck and G. R. Allan for unital locally convex algebras) are presented. Descriptions of the relationship between the usual spectrum and the regular spectrum of elements in topological algebras with jointly continuous multiplication are given. It is shown that the usual spectrum and the regular spectrum of elements coincide for Hausdorff locally convex Waelbroeck algebras. Main properties of the disolvent map of elements in topological algebras are studied.

1. Preliminaries

Let A be a *topological algebra* over $\mathbb{C}$ with separately continuous multiplication (in short, a topological algebra). In particular, when the multiplication as a map $A \times A \to A$ is continuous, we speak about a *topological algebra with jointly continuous multiplication*. A topological algebra A is *locally convex*, if A has a base of neighborhoods of zero, consisting of absolutely convex neighborhoods.

Let $\mathrm{Inv}A$ denote the set of all invertible elements in A, and $\mathrm{Qinv}A$ is the set of all quasi-invertible elements in A (that is, of elements $a \in A$ for which there is an element a_q^{-1} (the *quasi-inverse* of a) such that $a + a_q^{-1} = aa_q^{-1} = a_q^{-1}a$). A topological algebra A is called a *Q-algebra* if the set $\mathrm{Qinv}A$ (for unital algebras $\mathrm{Inv}A$) is open in A, and a Q-algebra is called a *Waelbroeck algebra*, if the quasi-inversion $a \to a_q^{-1}$ (in the case of unital algebra, the inversion $a \to a^{-1}$) is continuous.

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In 1954 the notion of Waelbroeck algebra is introduced by Waelbroeck in [15] under the name "*continuous inverse algebra*" for unital locally convex Hausdorff algebras, but in 1965 the name "*Waelbroeck algebra*" is used first by Ouzulou in [10]. Moreover, a topological algebra A is *locally complete* (in locally convex case Allan used in [4, p. 401], the term *pseudo-complete algebra*) if every subalgebra of A, generated by a closed, bounded, idempotent, and absolutely pseudoconvex subset U, is complete (remember, U is *idempotent*, if $UU \subset U$; *absolutely k-convex*, if

$$U = \Gamma_k(U) = \Big\{ \sum_{v=1}^{n} \alpha_v u_v : n \in \mathbb{N}, u_1, \ldots, u_n \in U, \alpha_1, \ldots, \alpha_n \in \mathbb{C}, \sum_{v=1}^{n} |\alpha_v|^k \leqslant 1 \Big\},$$

and *absolutely pseudoconvex*, if U is absolutely k-convex for some $k \in (0,1]$). In addition, A is a *topological algebra with idempotently pseudoconvex von Neumann bornology* if, for every idempotent and bounded subset U of A, there is a number $k \in (0,1]$ such that $\Gamma_k(U)$ is bounded in A.

2. Introduction

1. **Bounded elements in topological algebras**. Let A be a unital locally convex algebra over the field $\mathbb{C}$ of complex numbers, and let e_A denote the unit element of A. In 1954 Waelbroeck introduced in [14] (see also [13]) the notion of regular element of algebra. He said that $a \in A$ is *regular* if there is a neighborhood O of ∞ such that *the resolvent map* R_a of a, defined by

$$R_a(\lambda) = (a - \lambda e_A)^{-1}$$

for each $\lambda \notin \mathrm{sp}_A(a)$, has a bounded limit in O.

After that, in 1956, Warner said in [18] that an element a is *idempotently bounded* in a locally m-convex algebra A, if there is a $\lambda > 0$ such that

$$I(\{\lambda a\}) = \bigcup_{n \in \mathbb{N}} \{\lambda a\}^n$$

is bounded in A.

Next, in 1965, Allan in [4] said that an element $a \in A$ is *bounded*, if the set

$$S(a, \lambda) = \Big\{ \Big(\frac{a}{\lambda}\Big)^n : n \in \mathbb{N} \Big\} \tag{2.1}$$

is bounded in A for some $\lambda > 0$. Hence, every idempotently bounded element of a locally m-convex algebra is bounded and vice versa. Moreover (see, for example, [17, Proposition 11]), these three notions, given by Waelbroeck, Warner, and Allan, are the same in case of a unital *b-algebra* (that is, of a unital locally convex algebra, the von Neumann bornology of which has a base of completant, idempotent, and absolutely convex sets). In case, when (2.1) holds for an element of arbitrary topological (that is, not necessarily unital and locally convex) algebra with separately continuous multiplication, we use the term "bounded element" and the set of all bounded elements in A is denoted, as it was used in [4], by A_0.

2. **Regular spectrum of elements in topological algebras**. Let A be a unital algebra over $\mathbb{C}$. The *spectrum* $\mathrm{sp}_A(a)$ of $a \in A$ is defined by

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\},$$

and in case, when A is not necessarily a unital algebra, then by

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \notin \mathrm{Qinv}A\} \cup \{0\}.$$

In 1954, Waelbroeck introduced in [13] the notion "regular spectrum of an element" in a unital commutative locally convex algebra A. He said that the set $\mathrm{sp}^r_A(a)$, defined by

$$\mathrm{sp}^r_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \text{ is not a regular in } A\} \cup S$$

(where $S = \{\infty\}$ if and only if a is not regular and $S = \{\emptyset\}$ otherwise), is a *regular spectrum* of $a \in A$.

Allan defined in [4] the *extended spectrum* of a in locally convex algebras similarly, using only instead of the term "regular element of a" the name "bounded element of a", that is in case of unital locally convex algebra A

$$sp^r_A(a) = \mathrm{sp}_A(a) \cup \{\lambda \in \mathbb{C} : a - \lambda e_A \in \mathrm{Inv}A \text{ but } (a - \lambda e_A)^{-1} \notin A_0\} \cup S, \quad (2.2)$$

where $S = \{\infty\}$ if and only if $a \notin A_0$ and $S = \{\emptyset\}$ otherwise. Since W. Żelazko used in [19, p. 130], the name "extended spectrum of an element" in the other sense, we shall use later on throughout this paper the term "regular spectrum of an element" of a unital topological algebra to refer to the set given by (2.2).

Next, we generalize that notion to the case of a (not necessarily unital) topological algebra A with separately continuous multiplication.

We say that the set

$$sp^r_A(a) = \mathrm{sp}_A(a) \cup \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \in \mathrm{Qinv}A \text{ but } \left(\frac{a}{\lambda}\right)^{-1}_q \notin A_0\} \cup S, \quad (2.3)$$

where $S = \{\infty\}$ if and only if $a \notin A_0$ and $S = \{\emptyset\}$ otherwise, is the *regular specrum* of $a \in A$. That is, $\mathrm{sp}^r_A(a)$ of $a \in A$ consists of all $\lambda \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ such that either $\frac{a}{\lambda}$ has not quasi-inverse in A or $\frac{a}{\lambda}$ has in A the quasi-inverse, but it does not belong to A_0.

We show first that the sets in the right side of (2.3) and of (2.2) coincide if A is a unital topological algebra. For that, we need the following result.

Proposition 2.1. *Let A be a unital topological algebra over $\mathbb{C}$. Then*

(a) $a \in \mathrm{Qinv}A$ *if and only if* $e_A - a \in \mathrm{Inv}A$
and

(b) $a \in \mathrm{Qinv}A$ *and* $a^{-1}_q \in A_0$ *if and only if* $e_A - a \in \mathrm{Inv}A$ *and* $(e_A - a)^{-1} \in A_0$.

Proof. a) Since $(e_A - a)^{-1} = e_A - a^{-1}_q$ for each $a \in \mathrm{Qinv}A$, then the statement (a) holds.

b) Let now $a \in \mathrm{Qinv}A$, $a^{-1}_q \in A_0$, $n \in \mathbb{N}$, and let O be a neighborhood of zero in A, and let O_1 be a balanced neighborhood of zero in A such that

$$\underbrace{O_1 + \cdots + O_1}_{n+1 \text{ summands}} \subset O.$$

Then there is a $\lambda_0 > 1$ such that the set $S(a_q^{-1}, \lambda_0)$ is bounded in A. Therefore, there is a $\mu > 0$ such that

$$\left(\frac{a_q^{-1}}{\lambda_0}\right)^n \in \mu O_1$$

for each $n \in \mathbb{N}$. Since

$$\begin{aligned}(e_A - a_q^{-1})^n &= \sum_{v=0}^{n} \binom{n}{k} (-1)^{n-v} (a_q^{-1})^{n-v} \\ &= \sum_{v=0}^{n} \binom{n}{v} (-1)^{n-v} \lambda_0^{n-v} \left(\frac{a_q^{-1}}{\lambda_0}\right)^{n-v} \in \mu(1+\lambda_0)^n \sum_{v=0}^{n} \alpha_v O_1,\end{aligned}$$

where

$$\alpha_v = \frac{\binom{n}{v}(-1)^{n-v}\lambda_0^{n-v}}{(1+\lambda_0)^n}$$

for each $1 \leqslant v \leqslant n$,

$$|\alpha_v| = \frac{\binom{n}{v}(\lambda_0)^{n-v}}{(1+\lambda_0)^n} \leqslant \frac{\sum_{v=0}^{n}\binom{n}{v}(\lambda_0)^{n-v}}{(1+\lambda_0)^n} = 1,$$

and O_1 is balanced, then

$$(e_A - a_q^{-1})^n \in \mu(1+\lambda_0)^n (\underbrace{O_1 + \cdots + O_1}_{n+1 \text{ summands}}) \subset \mu(1+\lambda_0)^n O.$$

Hence,

$$\left(\frac{(e_A - a)^{-1}}{1+\lambda_0}\right)^n = \left(\frac{e_A - a_q^{-1}}{1+\lambda_0}\right)^n \in \mu O,$$

because $(e_A - a)^{-1} = e_A - a_q^{-1}$. As n was arbitrary, then $e_A - a \in \operatorname{Inv} A$ and $(e_A - a)^{-1} \in A_0$.

The converse part of the proof is similar, because $a_q^{-1} = e_A - (e_A - a)^{-1}$ if $e_A - a \in \operatorname{Inv} A$. □

Corollary 2.2. *For any topological unital algebra A the regular spectrum $\operatorname{sp}_A^r(a)$ of $a \in A$, defined by (2.3), coincides with the regular spectrum $\operatorname{sp}_A^r(a)$, defined by (2.2).*

Proof. Because $(e_A - a)^{-1} = e_A - a_q^{-1}$ for each $a \in \operatorname{Qinv} A = e_A - \operatorname{Inv} A$ and $a - \lambda e_A = -\lambda(e_A - \lambda^{-1} a)$ for every $\lambda \in \mathbb{C}_0 = \mathbb{C} \setminus \{0\}$, then

$$\{\lambda \in \mathbb{C}_0 : a - \lambda e_A \notin \operatorname{Inv} A\} = \{\lambda \in \mathbb{C}_0 : \frac{a}{\lambda} \notin \operatorname{Qinv} A\}$$

and

$$\{\lambda \in \mathbb{C}_0 : a - \lambda e_A \in \operatorname{Inv} A, (a - \lambda e_A)^{-1} \notin A_0\} = \{\lambda \in \mathbb{C}_0 : \frac{a}{\lambda} \in \operatorname{Qinv} A, \left(\frac{a}{\lambda}\right)_q^{-1} \notin A_0\},$$

by Proposition 2.1. Hence the regular spectrum $\operatorname{sp}_A^r(a)$ of $a \in A$, defined by (2.3), coincides with the regular spectrum $\operatorname{sp}_A^r(a)$, defined by (2.2). □

In case, when A has not the unit element, instead of A, we shall use the unitization $A_1 = A \times \mathbb{C}$ of A.

Proposition 2.3. *Let A be a topological algebra without unit element; then*

$$sp^r_A(a) = \mathrm{sp}^r_{A_1}((a,0)) \tag{2.4}$$

for each $a \in A$.

Proof. Let $a \in A$. For proving the equality (2.4), we can consider only nonzero elements of these spectra, because $0 \in \mathrm{sp}^r_A(a)$, for each $a \in A$, by the definition of the regular spectrum of elements and $0 \in \mathrm{sp}^r_{A_1}((a,0))$ for each $a \in A$ because $A \times \{0\}$ is a two-sided ideal in A_1. Therefore, let first, $\lambda \in \mathrm{sp}^r_A(a) \setminus \{0\}$. Then $\frac{a}{\lambda} \notin \mathrm{Qinv}A$ or $\frac{a}{\lambda} \in \mathrm{Qinv}A$ but $\left(\frac{a}{\lambda}\right)_q^{-1} \notin A_0$. Let first $\frac{a}{\lambda} \notin \mathrm{Qinv}A$. Suppose that $(a, -\lambda) = (a,0) - \lambda(\theta_A, 1) \in \mathrm{Inv}A_1$. Then there exists an element $(b, \mu) \in A_1$ such that

$$(ab - \lambda b + \mu a, -\lambda\mu) = (a, -\lambda)(b, \mu) = (\theta_A, 1),$$

from what follows that $\mu = -\frac{1}{\lambda}$ and $\frac{a}{\lambda} \circ \lambda b = \theta_A$. Hence, $\frac{a}{\lambda} \in \mathrm{Qinv}A$. As this is impossible, then $(a, -\lambda) \notin \mathrm{Inv}A_1$ or $\lambda \in \mathrm{sp}_{A_1}((a,0)$.

Let now $\frac{a}{\lambda} \in \mathrm{Qinv}A$ and $\left(\frac{a}{\lambda}\right)_q^{-1} \notin A_0$. Then there exists

$$(a, -\lambda)^{-1} = \left(\frac{1}{\lambda}\left(\frac{a}{\lambda}\right)_q^{-1}, -\frac{1}{\lambda}\right) = -\frac{1}{\lambda}\left(-\left(\frac{a}{\lambda}\right)_q^{-1}, 1\right)$$

in A_1. Suppose that $(a, -\lambda)^{-1} \in (A_1)_0$. Then there exists a number $M > 0$ such that the set $S((a, -\lambda)^{-1}, M)$ is bounded in A_1. Since

$$(\theta_A, 1) - \left(\frac{a}{\lambda}, 0\right) = \left(-\frac{a}{\lambda}, 1\right) = -\frac{1}{\lambda}(a, -\lambda),$$

then

$$\left(-\frac{a}{\lambda}, 1\right) \in \mathrm{Inv}A_1.$$

Moreover, $S((-\frac{a}{\lambda}, 1)^{-1}, |\lambda| M) = S(-\frac{\lambda}{|\lambda|}(a, -\lambda)^{-1}, M)$. Hence, $\left(-\frac{a}{\lambda}, 1\right)^{-1} \in (A_1)_0$. Therefore, $(\frac{a}{\lambda}, 0) \in \mathrm{Qinv}A_1$ and $(\frac{a}{\lambda}, 0)_q^{-1} \in (A_1)_0$ by Proposition 2.1 (b). Hence, $\left(\frac{a}{\lambda}\right)_q^{-1} \in A_0$, because

$$\left(\frac{1}{M^n}\left(\left(\frac{a}{\lambda}\right)_q^{-1}\right)^n, 0\right) = \frac{1}{M^n}\left(\left(\frac{a}{\lambda}, 0\right)_q^{-1}\right)^n$$

for each $n \in \mathbb{N}$ and $M > 0$. As this is impossible, then $(a, -\lambda)^{-1} \notin (A_1)_0$ or $\lambda \in \mathrm{sp}^r_{A_1}((a,0))$. Consequently, $\mathrm{sp}^r_A(a) \subseteq \mathrm{sp}^r_{A_1}((a,0))$ for each $a \in A$.

Let now $\lambda \in \mathrm{sp}^r_{A_1}((a,0)) \setminus \{0\}$. Then $(a, -\lambda) = (a,0) - \lambda(\theta_A, 1) \notin \mathrm{Inv}A_1$ or $(a, -\lambda) \in \mathrm{Inv}A_1$ but $(a, -\lambda)^{-1} \notin (A_1)_0$. First we consider the case, when

$$(a, -\lambda) \notin \mathrm{Inv}A_1.$$

If $\frac{a}{\lambda} \in \mathrm{Qinv}A$, then

$$(a, -\lambda)^{-1} = \left(\frac{1}{\lambda}\left(\frac{a}{\lambda}\right)_q^{-1}, -\frac{1}{\lambda}\right)$$

but this is not possible by assumption. Hence, $\frac{a}{\lambda} \notin \mathrm{Qinv}A$. It means that $\lambda \in \mathrm{sp}_A(a)$.

Let next $(a, -\lambda) \in \mathrm{Inv}A_1$ and $(a, -\lambda)^{-1} \notin (A_1)_0$. Then there are $b \in A$ and $\mu \in \mathbb{K}$ such that $(a, -\lambda)(b, \mu) = (\theta_A, 1)$. Hence $\frac{a}{\lambda} \in \mathrm{Qinv}A$ and $\left(\frac{a}{\lambda}\right)_q^{-1} = \lambda b$. Since

$\left(\frac{a}{\lambda},0\right)_q^{-1} = \left(\left(\frac{a}{\lambda}\right)_q^{-1},0\right)$, then $\left(\frac{a}{\lambda},0\right) \in \mathrm{Qinv}A_1$. Suppose that $\left(\frac{a}{\lambda},0\right)_q^{-1} \in (A_1)_0$. Then $\left(-\frac{a}{\lambda},1\right) \in \mathrm{Inv}A_1$ and $\left(-\frac{a}{\lambda},1\right)^{-1} \in (A_1)_0$ by Proposition 2.1 (b). Hence, there exists a number $M > 0$ such that $S\big(\left(-\frac{a}{\lambda},1\right)^{-1}, M\big)$ is bounded in A_1. Since

$$\left(-\frac{a}{\lambda},1\right)^{-1} = -\lambda(a,-\lambda)^{-1}$$

and $S\big((a,-\lambda)^{-1},\frac{M}{|\lambda|}\big) = S\big(-\frac{|\lambda|}{\lambda}\left(-\frac{a}{\lambda},1\right)^{-1}, M\big)$, then $(a,-\lambda)^{-1} \in (A_1)_0$ what is impossible. Hence, $\left(\frac{a}{\lambda},0\right)_q^{-1} = \left(\left(\frac{a}{\lambda}\right)_q^{-1},0\right) \notin (A_1)_0$, because of which $\left(\frac{a}{\lambda}\right)_q^{-1} \notin A_0$; Thus, $\lambda \in \mathrm{sp}^r_A(a)$. Consequently, $\mathrm{sp}^r_{A_1}((a,0)) \subseteq \mathrm{sp}^r_A(a)$ for each $a \in A$. $\square$

Properties of the regular spectrum $\mathrm{sp}^r_A(a)$ of a, of the resolvent map R_a of $a \in A$, defined above, and of the map f_a, defined by

$$f_a(\lambda) = (e_A - \lambda a)^{-1}$$

for each $\lambda \in \Lambda_a = \{\lambda \in \mathbb{C} : e_A - \lambda a \in \mathrm{Inv}A\}$, have been studied in several papers (see, for example, [2], [4], and [16]) in case when A is a unital topological (mostly locally convex) algebra. Next we describe properties of the regular spectrum $\mathrm{sp}^r_A(a)$ by the resolvent map R_a of a, and by the *disolvent map* D_a of $a \in A$, defined by

$$D_a(\lambda) = \left(\frac{a}{\lambda}\right)_q^{-1}$$

for every $\lambda \in \mathbb{C} \setminus \mathrm{sp}_A(a)$ if A is not unital.

3. Properties of the regular spectrum

For describing the main properties of the regular spectrum of elements in topological algebras over $\mathbb{C}$, we need the following result.

Lemma 3.1. *If*

(a) *A is a unital Waelbroeck algebra*

or

(b) *A is a unital topological algebra with jointly continuous multiplication,*

then the power map $\mu_n : A \to A$ with fixed natural number $n \geqslant 2$, defined by $\mu_n(a) = a^n$ for every $a \in A$, is continuous in A.

Proof. We use here the idea from [7]. Let, first A be a unital Waelbroeck algebra. Since the set $\mathrm{Inv}A$ is open in A, then there is a balanced neighborhood O of zero such that $e_A + a, e_A - a \in \mathrm{Inv}A$ for every $a \in O$. Therefore,

$$a^2 = e_A - 2((e_A + a)^{-1} + (e_A - a)^{-1})^{-1}$$

for each $a \in O$. Let ε denote the inversion of elements in A (that is, $\varepsilon(a) = a^{-1}$ for each $a \in \mathrm{Inv}A$), let l be the addition of elements in A (that is, $l((a,b)) = a+b$ for each $a,b \in A$), let g_a with fixed $a \in A$ be the map, defined by $g_a(b) = a + b$ for each $b \in A$,let h_α with fixed $\alpha \in \mathbb{C}$ be the map, defined by $h_\alpha(a) = \alpha a$ for each $a \in A$, and let (f,g) be the map, defined by $(f,g)(a) = (f(a),g(a))$ for each $a \in A$ and maps $f : A \to A$ and $g : A \to A$. Since

$$\mu_2 = g_{e_A} \circ h_{-2} \circ \varepsilon \circ l \circ ((\varepsilon \circ g_{e_A}),(\varepsilon \circ g_{e_A} \circ h_{-1}))$$

and every component in this composition is continuous, then μ_2 is a continuous map.

Let now A be a topological algebra with jointly continuous multiplication, let $a_0 \in A$, and let O an arbitrary neighborhood of zero in A. Then there is another neighborhood O_1 of zero in A such that $O_1O_1 + O_1a_0 + a_0O_1 \subset O$. Since

$$\begin{aligned}\mu_2(a) - \mu_2(a_0) &= (a - a_0)(a + a_0) - aa_0 + a_0a \\ &= (a - a_0)((a - a_0) + 2a_0) - (a - a_0)a_0 + a_0(a - a_0) \\ &\in O_1O_1 + O_1a_0 + a_0O_1 \subset O\end{aligned}$$

for every $a \in a_0 + O_1$, then the map μ_2 is continuous.

To show that the map μ_n is continuous for $n \geqslant 3$, we use the induction, where $\mu_{n+1} = f_a \circ \mu_n$ for each $n \geqslant 2$ and f_a denotes the map, defined by $f_a(b) = ab$ for each $b \in A$. $\square$

Corollary 3.2 (Turpin [12])**.** *Let A be a commutative unital Waelbroeck algebra. Then the multiplication in A is jointly continuous.*

The next proposition gives a similar result as in [4, Lemma 3.6] for not necessary locally convex algebra over $\mathbb{C}$.

Proposition 3.3. *Let A be a unital topological algebra over $\mathbb{C}$ with jointly continuous multiplication. If*

(a) *the inversion $a \to a^{-1}$ is continuous on* $\mathrm{Inv}A$,

then

a) *for every $a \in A$ and $\lambda \in K = \mathbb{C} \setminus \mathrm{cl}_{\mathbb{C}}\mathrm{sp}_A(a)$, the complex derivative $R_a^{(n)}$ of the resolvent map R_a of a has the form*

$$R_a^{(n)}(\lambda) = n!R_a(\lambda)^{n+1} \tag{3.1}$$

for every natural number $n \geqslant 1$;

b) *for every $a \in A$ and $\lambda \in \Lambda_a$. the complex derivative $f_a^{(n)}$ of the map f_a* (defined in the introduction) *has the form*

$$f_a^{(n)}(\lambda) = n!a^n f_a(\lambda)^{n+1} \tag{3.2}$$

and

$$(R_a \circ \varepsilon)^{(n)}(\lambda) = -n!a^{n-1} f_a(\lambda)^{n+1} \tag{3.3}$$

for any natural number $n \geqslant 1$ (here $\varepsilon(\lambda) = \frac{1}{\lambda}$ if $\lambda \neq 0$).

Proof. Let $a \in A$ and $\lambda_0 \in K$. Then there is an open neighborhood $O(\lambda_0) \subset K$. Since

$$R_a(\lambda) - R_a(\lambda_0) = (\lambda - \lambda_0)R_a(\lambda)R_a(\lambda_0)$$

for every $\lambda \in O(\lambda_0) \setminus \{\lambda_0\}$, then

$$R_a'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{R_a(\lambda) - R_a(\lambda_0)}{\lambda - \lambda_0} = R_a(\lambda_0)^2$$

by the condition (a). Now

$$R_a'(\lambda) - R_a'(\lambda_0) = R_a(\lambda)^2 - R_a(\lambda_0)^2 = (R_a(\lambda) - R_a(\lambda_0))(R_a(\lambda) + R_a(\lambda_0)) + S_1,$$

where
$$S_1 = R_a(\lambda_0)R_a(\lambda) - R_a(\lambda)R_a(\lambda_0) = \theta_A,$$
because $R_a(\lambda)$ and $R_a(\mu)$ commute for every $\lambda, \mu \in K$. Therefore (due to the joint continuity of multiplication in A)
$$\begin{aligned} R''_a(\lambda_0) &= \lim_{\lambda - \lambda_0} \frac{R_a(\lambda) - R_a(\lambda_0)}{\lambda - \lambda_0} \lim_{\lambda - \lambda_0} (R_a(\lambda) + R_a(\lambda_0)) \\ &= 2R'_a(\lambda_0)R_a(\lambda_0) = 2R_a(\lambda_0)^3. \end{aligned}$$
To prove by the induction that equality (3.1) holds for every n, we assume that
$$R_a^{(n)}(\lambda_0) = n!R_a(\lambda_0)^{n+1}.$$
By the formula
$$a^{n+1} - b^{n+1} = (a-b)\sum_{k=1}^{n+1} a^{n+1-k}b^{k-1} + \sum_{k=1}^{n}[ba^{n+1-k}b^{k-1} - a^{n+1-k}b^k]$$
for each $n \geqslant 2$ and $a, b \in A$, we have
$$\begin{aligned} R_a^{(n)}(\lambda) - R_a^{(n)}(\lambda_0) &= n!(R_a(\lambda)^{n+1} - R_a(\lambda_0)^{n+1}) \\ &= n!(R_a(\lambda) - R_a(\lambda_0))\sum_{k=1}^{n+1} R_a(\lambda)^{n+1-k}R_a(\lambda_0)^{k-1} + S_n, \end{aligned}$$
where
$$S_n = n!\sum_{k=1}^{n}[R_a(\lambda_0)R_a(\lambda)^{n+1-k}R_a(\lambda_0)^{k-1} - R_a(\lambda)^{n+1-k}R_a(\lambda_0)^k] = \theta_A,$$
because $R_a(\lambda)$ and $R_a(\mu)$ commute for all $\lambda, \mu \in K$. Taking this, the joint continuity of the multiplication in A and the continuity of power maps into account,
$$\begin{aligned} R_a^{(n+1)}(\lambda_0) &= n! \lim_{\lambda\to\lambda_0} \frac{R_a(\lambda) - R_a(\lambda_0)}{\lambda - \lambda_0} \lim_{\lambda\to\lambda_0} \Big[\sum_{k=1}^{n-1} \mu_{n+1-k}(R_a(\lambda))R_a(\lambda_0)^{k-1} \\ &\qquad\qquad + R_a(\lambda)R_a(\lambda_0)^{n-1} + R_a(\lambda_0)^n\Big] \\ &= n!R_a(\lambda_0)^2(n+1)R_a(\lambda_0)^n \\ &= (n+1)!R_a(\lambda_0)^{n+2} \end{aligned}$$
by Lemma 3.1. Consequently, the equality (3.1) holds by the mathematical induction for all $n \in \mathbb{N}$.

Similarly, using the mathematical induction, it is not difficult to show that the equalities (3.2) and (3.3) hold for every $n \in \mathbb{N}$. □

By Corollary 3.2,we have the following result.

Corollary 3.4. *Let A be a commutative unital Waelbroeck algebra over $\mathbb{C}$. Then equalities (3.1)–(3.3) hold.*

To prove the next result, we need the following.

Lemma 3.5. *Let A be a nonunital topological algebra with nontrivial continuous linear functionals. Then every weakly bounded subset in the unitization A_1 of A is bounded if every weakly bounded subset in A is bounded.*

Proof. Let $f \in A^*$ (the topological dual of A) be nontrivial. Then F_f, defined by $F_f((a, \lambda)) = f(a) + \lambda$ for each $(a, \lambda) \in A_1$, is a nontrivial continuous linear functional on A_1. (Since f is nontrivial, then there is an element $a_0 \in A$ such that $f(a_0) \neq 0$. Therefore, $F_f(a_0, f(a_0)) = 2f(a_0) \neq 0$, that is, F_f is not trivial.)

Let S be a weakly bounded subset in A_1, and let $A_S = \{a \in A : (a, \lambda) \in S\}$ and $\mathbb{C}_S = \{\lambda \in \mathbb{C} : (a, \lambda) \in S\}$. Then $F_f(S)$ is a bounded subset in $\mathbb{C}$ for each $f \in A^*$. Therefore, $f(A_S)$ for each $f \in A^*$ and $\mathbb{C}_S$ are bounded in $\mathbb{C}$. Since every weakly bounded subset in A is bounded, then A_S is a bounded subset in A. Now, from $S \subseteq A_S \times \mathbb{C}_S$, it follows that S is bounded in A_1. □

Proposition 3.6. *Let A be a topological algebra over $\mathbb{C}$ with jointly continuous multiplication which satisfies the conditions*

(a') *the quasi-inversion $a \to a_q^{-1}$ on $\operatorname{Qinv}A$ is continuous;*

(b) *the topological dual space A^* of A contains at least one nonzero element and*

(c) *every weakly bounded set in A is bounded in A.*

Then

$$sp_A(a) \subseteq \operatorname{sp}^r_A(a) \subseteq \operatorname{cl}_{\mathbb{C}_\infty}(\operatorname{sp}_A(a)) \tag{3.4}$$

for every $a \in A$.

Proof. Let $a \in A$. The inclusion $\operatorname{sp}_A(a) \subseteq \operatorname{sp}^r_A(a)$ follows from the definition of $\operatorname{sp}^r_A(a)$. First we consider the case, when A is a unital topological algebra. If $\operatorname{cl}_{\mathbb{C}_\infty}(\operatorname{sp}_A(a)) = \mathbb{C}_\infty$, then $\operatorname{sp}^r_A(a) \subseteq \operatorname{cl}_{\mathbb{C}_\infty}(\operatorname{sp}_A(a))$. Let now $\operatorname{cl}_{\mathbb{C}_\infty}(\operatorname{sp}_A(a)) \neq \mathbb{C}_\infty$, and take a $\lambda_0 \in \mathbb{C}_\infty \setminus \operatorname{cl}_{\mathbb{C}_\infty}(\operatorname{sp}_A(a))$. If $\lambda_0 \neq \infty$; then there exists a neighborhood $O(\lambda_0)$ of λ_0 in $\mathbb{C}$ such that $O(\lambda_0) \subset \mathbb{C} \setminus \operatorname{cl}_{\mathbb{C}_\infty}(\operatorname{sp}_A(a))$. Hence, $R_a(\lambda)$ exists in A for all $\lambda \in O(\lambda_0)$. By the condition (b) there is at least one nonzero continuous linear functional f on A. Let φ_f be the nonzero continuous functional on $O(\lambda_0)$, defined by $\varphi_f(\lambda) = f(R_a(\lambda))$ for each $\lambda \in O(\lambda_0)$. Since A satisfies the conditions (a) (from (a') follows (a), because $(e_A - a)^{-1} = e_A - a_q^{-1}$), R_a is infinitely many times differentiable at λ_0 and

$$R_a^{(n)}(\lambda_0) = n! R_a(\lambda_0)^{n+1}$$

for every natural number $n \geqslant 1$ by Proposition 3.3, then φ_f is infinitely many times differentiable at λ_0 and

$$\varphi_f^{(n)}(\lambda_0) = n! f(R_a(\lambda_0)^{n+1})$$

for each $n \in \mathbb{N}$. Therefore, φ_f is an analytic function on some neighborhood $O_f(\lambda_0)$ of λ_0 and its Taylor series

$$\sum_{k=1}^{\infty} f[(R_a(\lambda_0)^{k+1}](\lambda - \lambda_0)^k. \tag{3.5}$$

converges in every point of $O_f(\lambda_0)$. Let $\rho_f > 0$ and

$$O_{\rho_f}(\lambda_0) = \{\lambda \in \mathbb{C}; |\lambda - \lambda_0| < \rho_f\} \subset O_f(\lambda_0).$$

Then $\rho_f < R_f$, where R_f denotes the radius of convergence of the power series (3.5). Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{|f[R_a(\lambda_0)^{n+1}]|} = \frac{1}{R_f} < \frac{1}{\rho_f} < \infty.$$

Hence, there exists a natural number N_f such that

$$\sqrt[n]{|f[R_a(\lambda_0)^{n+1}]|} \leqslant \frac{1}{\rho_f}$$

for every $n > N_f$. So, the set $S = \{(f[R_a(\lambda_0)])^{n+1} : n \in \mathbb{N}\}$ is bounded in $\mathbb{C}$ for every $f \in A^*$. Hence, $R_a(\lambda_0) \in A_0$ by the condition (c), because of which $\lambda_0 \notin \mathrm{sp}^r_A(a)$. Hence, from $\lambda_0 \in \mathbb{C}_\infty \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$ and $\lambda_0 \neq \infty$, it follows that $\lambda_0 \notin \mathrm{sp}^r_A(a)$ for each $a \in A$. It means that $\mathrm{sp}^r_A(a) \subseteq \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$.

Let now $\lambda_0 = \infty$. To show that φ_f, for $f \in A^*$, is analytic at ∞, we show that the function ψ_f, defined by $\psi_f(\lambda) = f\big[R_a\big(\frac{1}{\lambda}\big)\big]$, is analytic at zero. Since

$$\psi_f^{(n)}(0) = \lim_{\lambda\to 0} f\Big[(R_a \circ \varepsilon_A)^{(n)}(\lambda)\Big] = -n!f(a^{n-1})$$

for every $n \in \mathbb{N}$ by Proposition 3.3, then ψ_f is analytic at zero and the Taylor series

$$-\sum_{n=0}^{\infty} f(a^{n-1})\lambda^n$$

of ψ_f converges pointwise in every point of some neighborhood O_f of zero. Let now $\nu_f > 0$ be such that $O_{\nu_f} \subset O_f$. Then $\nu_f < R_f$ (the radius of convergence of this power series). As

$$\limsup_{n\to\infty} \sqrt[n]{|f(a^{n-1})|} = \frac{1}{R_f} < \frac{1}{|\nu_f|} < \infty,$$

then there are $N_f \in \mathbb{N}$ and $M > 0$ such that $|f((\lambda_1 a)^n))| \leqslant M$ for every $n > \mathbb{N}_f$. Hence, the set $\{|\lambda_1 f(a^n)| : n \in \mathbb{N}\}$ is bounded for each $f \in A^*$. Therefore, the set $\{|\lambda_1 a^n| : n \in \mathbb{N}\}$ is bounded in A by the condition (c). Similarly as above, we have that $a \in A_0$. So, $\infty \notin \mathrm{sp}^r_A(a)$. Consequently, from $\infty \in \mathbb{C}_\infty \setminus \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$, it follows that $\infty \notin \mathrm{sp}^r_A(a)$ for each $a \in A$. In this way, we have shown that the inequalities (3.4) hold.

Next we consider the case, when A has not the unit element. Instead of A, we consider the unitization $A_1 = A \times \mathbb{C}$ of A. Then (see, for example, [6, p. 156]), A_1 satisfies the condition (a) for every $(a, \lambda) \in \mathrm{Inv}A_1$ by the condition (a'). Moreover, by the condition (b), there is a nonzero continuous linear functional f on A. Let F_f be as in Lemma 3.5. Then F_f is a nonzero continuous $\mathbb{C}$-valued linear functional on A_1, that is, A_1 satisfies the condition (b), and every weakly bounded set in A_1 is bounded by Lemma 3.5.

Hence, by Proposition 2.3 and the part above, we have that

$$\mathrm{sp}^r_A(a) = \mathrm{sp}^r_{A_1}((a,0)) \subseteq \mathrm{cl}_{\mathbb{C}_\infty}[\mathrm{sp}_{A_1}((a,0))] = \mathrm{cl}_{\mathbb{C}_\infty}[\mathrm{sp}_A(a)]. \qquad \square$$

Next we need the following results which are partly known (at least in unital case)

Lemma 3.7. *Let A be a topological Hausdorff algebra, and let B be the maximal commutative subalgebra of A. Then*

a) *B is closed;*

b) $\mathrm{Qinv}B = B \cup \mathrm{Qinv}A$

and

c) $\mathrm{sp}_B(b) = \mathrm{sp}_A(b)$ *and* $\mathrm{sp}^r_B(b) = \mathrm{sp}^r_A(b)$ *for every* $b \in B$.

Proof. a) Let $a \in \mathrm{cl}_A B$. Then there is a net $(b_\lambda)_{\lambda\in\lambda}$ in B such that $(b_\lambda)_{\lambda\in\lambda}$ converges to a. Since A is a Hausdorff space, the multiplication in A is separately continuous and $bb_\lambda = b_\lambda b$ for each $b \in B$ and $\lambda \in \lambda$, then $ab = ba$ for each $b \in B$. Hence, the linear span of $\{a\} \cup B$ is a commutative subalgebra in A. Because B is the maximal commutative subalgebra in A, then $a \in B$. Hence, B is closed.

b) We show that $B \cup \mathrm{Qinv}A \subseteq \mathrm{Quinv}B$. For this, let $b \in B \cup \mathrm{Qinv}A$, and let b' be an arbitrary element of B. Since

$$b_q^{-1}\circ b' = (b_q^{-1}\circ b')\circ\theta_A = (b_q^{-1}\circ b')\circ(b\circ b_q^{-1}) = b_q^{-1}\circ(b'\circ b)\circ b_q^{-1} = b_q^{-1}\circ(b\circ b')\circ b_q^{-1}$$
$$= (b_q^{-1}\circ b)\circ(b'\circ b_q^{-1}) = \theta_A\circ(b'\circ b_q^{-1}) = b'\circ b_q^{-1},$$

then $b_q^{-1}b' = b'b_q^{-1}$. As in above, the linear span of $\{b_q^{-1}\} \cup B$ is a commutative subalgebra in A. Because B is the maximal commutative subalgebra in A, then $b_q^{-1} \in B$. Hence, $b \in \mathrm{Qinv}B$.

c) Let $b \in B$ and $\lambda \in \mathrm{sp}_A(b)$. If $\lambda \neq 0$, then $\frac{b}{\lambda} \notin \mathrm{Qinv}A$. Therefore, $\frac{b}{\lambda} \notin \mathrm{Qinv}B$ by equality b). Hence, $\lambda \in \mathrm{sp}_B(b)$. Thus, $\mathrm{sp}_A(b) \subseteq \mathrm{sp}_B(b)$. The converse inclusion is similar.

Let now $\lambda \in \mathrm{sp}^r_A(b)$. If $\lambda \in \mathrm{sp}_A(b)$, then $\lambda \in \mathrm{sp}_B(b)$ as above. Let now $\frac{b}{\lambda} \in \mathrm{Qinv}A$ but $(\frac{b}{\lambda})_q^{-1} \notin A_0$. Then $\frac{b}{\lambda} \in \mathrm{Qinv}B$ by the equality b) and $(\frac{b}{\lambda})_q^{-1} \notin B_0$. Hence, $\lambda \in \mathrm{sp}^r_B(b)$. Consequently, $\mathrm{sp}^r_A(b) \subseteq \mathrm{sp}^r_B(b)$. Converse inclusion is similar. □

Corollary 3.8. *Let A be a locally convex algebra over $\mathbb{C}$. If, in addition,*

a) *the multiplication in A is jointly continuous and the quasi-inversion is continuous,*

then the inequality (3.4) holds for every $a \in A$;

b) *A is a Waelbroeck and Hausdorff algebra, then $\mathrm{sp}^r_A(a) = \mathrm{sp}_A(a)$ for every $a \in A$.*

Proof. a) Since A is locally convex, then the conditions (b) and (c) in Proposition 3.6 have been fulfilled (see, for example, Corollary of Theorem 3.4 and Proposition 3.18 in [11]). Therefore, the statement holds by Proposition 3.6.

b) Let first A be a commutative unital locally convex Waelbroeck algebra. Then the condition a) is fulfilled and the multiplication in A is jointly continuous by Corollary 3.2. Moreover, since A is locally convex, then (b) and (c) have been fulfilled (see the proof of part a)). Therefore, every element in A is bounded by Theorem 1 in [3]. Hence, $\mathrm{sp}^r_A a) = \mathrm{sp}_A(a)$ for every $a \in A$ by the definition of the regular spectrum.

Let now A be an arbitrary unital Hausdorff locally convex Waelbroeck algebra, a be an element in A, B_a be the subalgebra of A, generated by a and e_A, and M_a be the maximal commutative unital subalgebra of A. Then $B_a \subset M_a$ and M_a is a commutative unital Hausdorff locally convex Waelbroeck algebra by Lemma 3.7 b). Hence,

$$\mathrm{sp}^r_A(a) = \mathrm{sp}^r_{M_a}(a) = \mathrm{sp}_{M_a}(a) = \mathrm{sp}_A(a)$$

for every $a \in A$ by the first part of the proof and Lemma 3.7 c).

Next we coincide the case when A has not unit. Instead of A we coincide the unitization A_1 of A. Since A is a Waelbroeck algebra, then A_1 is a Waelbroeck algebra with unit element by Proposition 3.6.28 in [6]. Moreover, A_1 is a Hausdorff space. Hence,

$$\mathrm{sp}^r_A(a) = \mathrm{sp}^r_{A_1}((a,0)) = \mathrm{sp}_{A_1}((a,0))] = \mathrm{sp}_A(a)$$

by Proposition 2.3. □

Corollary 3.8 a) has been given in [4, Theorem 4.1] for unital locally convex algebras. In [5] has been given a unital commutative locally convex Q-algebras in which the usual spectrum and the regular spectrum of elements coincide.

4. Properties of the disolvent map

For any topological algebra A over $\mathbb{C}$, let

$$\sigma^r_A(a) = \mathbb{C}_\infty \setminus \mathrm{sp}^r_A(a)$$

for each $a \in A$. Later on we need the following results.

Lemma 4.1. *Let A be a topological algebra over $\mathbb{C}$, and let $a \in A$ and $\lambda, \mu \in \mathbb{C} \setminus \mathrm{sp}_A(a)$. Then*

1) $aD_a(\lambda) = D_a(\lambda)a = a + \lambda D_a(\lambda)$

and

2) $D_a(\lambda)D_a(\mu) = D_a(\mu)D_a(\lambda)$.

Proof. 1) Since $\lambda \neq 0$ and

$$\frac{a}{\lambda} \circ D_a(\lambda) = \theta_A = D_a(\lambda) \circ \frac{a}{\lambda},$$

then

$$D_a(\lambda)\frac{a}{\lambda} = \frac{a}{\lambda} + D_a(\lambda) = \frac{a}{\lambda}D_a(\lambda).$$

Hence, the statement 1) holds.

2) Since $\lambda, \mu \notin \sigma_A(a)$, then $\lambda \neq 0$, $\mu \neq 0$ and $\frac{a}{\lambda} \in \mathrm{Qinv}A$. Moreover,

$$\frac{a}{\mu} \circ D_a(\lambda) = D_a(\lambda) \circ \frac{a}{\mu}.$$

Therefore, from

$$\begin{aligned} D_a(\lambda) \circ D_a(\mu) =& \Big(D_a(\mu) \circ \frac{a}{\mu}\Big) \circ (D_a(\lambda) \circ D_a\mu) = D_a(\mu) \circ \Big(\frac{a}{\mu} \circ D_a(\lambda)\Big) \circ D_a(\mu) \\ =& (D_a(\mu) \circ D_a(\lambda)) \circ \Big(\frac{a}{\mu} \circ D_a(\mu)\Big) = D_a(\mu) \circ D_a(\lambda) \end{aligned}$$

it follows that $D_a(\lambda)D_a(\mu) = D_a(\mu)D_a(\lambda)$. □

Proposition 4.2. *Let A be a topological algebra over $\mathbb{C}$, let $\lambda_0 \in \mathbb{C} \setminus \mathrm{cl}_{\mathbb{C}}(\mathrm{sp}_A(a))$, and let $O(\lambda_0)$ be a neighborhood of λ_0 such that $O(\lambda_0) \subset \mathbb{C} \setminus \mathrm{cl}_{\mathbb{C}}(\mathrm{sp}_A(a))$. Then*

$$\begin{aligned} D_a(\lambda) =& \frac{\lambda_0}{\lambda} D_a(\lambda_0) + \frac{\lambda_0}{\lambda} D_a(\lambda_0) \sum_{v=1}^{n} \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v) \\ &+ \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^{n+1} D_a^{n+1}(\lambda_0) D_a(\lambda) \end{aligned} \tag{4.1}$$

for each $\lambda \in O(\lambda_0)$ and $n \in \mathbb{N}$. Moreover, if O is a neighborhood of ∞ such that $O \cap \mathrm{sp}_A(a)$ is empty, then

$$D_a(\lambda) = -\sum_{v=1}^{n} \frac{a^v}{\lambda^v} + \Big(\frac{a}{\lambda}\Big)^n D_a(\lambda) \tag{4.2}$$

for each $\lambda \in O$ and $n \in \mathbb{N}$.

Proof. Dividing both sides of the equation

$$\frac{a}{\lambda} + D_a(\lambda) - \frac{a}{\lambda} D_a(\lambda) = \theta_A \tag{4.3}$$

by λ_0 and both sides of the equation

$$\frac{a}{\lambda_0} + D_a(\lambda_0) - \frac{a}{\lambda_0} D_a(\lambda_0) = \theta_A \tag{4.4}$$

by λ and then subtracting from the first new equation the second new equation, we obtain the equation

$$\frac{D_a(\lambda)}{\lambda_0} - \frac{D_a(\lambda_0)}{\lambda} = \frac{a(D_a(\lambda) - D_a(\lambda_0))}{\lambda\lambda_0}. \tag{4.5}$$

Now multiplying both sides of the equation (4.3) from the left by $\lambda D_a(\lambda_0)$ and both sides of the equation (4.4) from the right by $\lambda_0 D_a(\lambda)$ and then subtracting from the first new equation the second new equation, we obtain by Lemma 4.1 the equation

$$a(D_a(\lambda) - D_a(\lambda_0)) = (\lambda - \lambda_0) D_a(\lambda_0) D_a(\lambda). \tag{4.6}$$

Hence, from (4.5) and (4.6) follows that

$$D_a(\lambda) = \frac{\lambda_0}{\lambda} D_a(\lambda_0) + \Big(\frac{\lambda - \lambda_0}{\lambda}\Big) D_a(\lambda_0) D_a(\lambda). \tag{4.7}$$

By relation (4.7) one gets that relation (4.1) is true for $n = 1$.

Let now the formula (4.1) hold, when $n = m$. To show that this formula holds also when $n = m + 1$, we put the value of $D_a(\lambda)$ in (4.7) to the right part of the formula (4.1). Then

$$\begin{aligned}D_a(\lambda) =&\frac{\lambda_0}{\lambda}D_a(\lambda_0) + \frac{\lambda_0}{\lambda}D_a(\lambda_0)\sum_{v=1}^{m}\Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v)\\ &+\Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^{m+1}D_a^{m+1}(\lambda_0)\Big(\frac{\lambda_0}{\lambda}D_a(\lambda_0)+\Big(\frac{\lambda-\lambda_0}{\lambda}\Big)D_a(\lambda_0)D_a(\lambda)\Big)\\ =&\frac{\lambda_0}{\lambda}D_a(\lambda_0) + \frac{\lambda_0}{\lambda}D_a(\lambda_0)\sum_{v=1}^{m+1}\Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v)+\\ &+\Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^{m+2}D_a^{m+2}(\lambda_0)D_a(\lambda).\end{aligned}$$

Hence, by the induction, the formula (4.1) holds for every $n \in \mathbb{N}$.

Let now $\lambda \in O \cap (\mathbb{C} \setminus \mathrm{sp}_A(a))$. Then $\lambda \neq 0$ and, from (4.3), it follows that the equality (4.2) holds for $n = 1$. Similarly as above it is easy to show by induction that the equality (4.2) holds for each $n \in \mathbb{N}$. □

Corollary 4.3. *Let A be a topological algebra over $\mathbb{C}$ with continuous quasi-inversion, $a \in A$, and let $\lambda_0 \in \mathbb{C} \setminus \mathrm{sp}_A(a)$. Then*

$$\lim_{\lambda\to\lambda_0}\frac{\lambda D_a(\lambda)-\lambda_0 D_a(\lambda_0)}{\lambda-\lambda_0} = D_a^2(\lambda_0) \tag{4.8}$$

in a neighborhood of λ_0.

Proof. The equality (4.8) holds by (4.7) because the quasi-inversion in A is continuous. □

For any topological algebra, A let $\mathcal{B}_k$ denote the collection of all closed, bounded, idempotent, and absolutely k-convex subsets in A.

Proposition 4.4. *Let A be a locally complete topological Hausdorff algebra with idempotently pseudoconvex von Neumann bornology, and let $a \in A \setminus \{\theta_A\}$. Then*

(a) *for any $\lambda_0 \in \sigma_A^r(a) \cap \mathbb{C}$, there exist a number $k(\lambda_0) \in (0,1]$, an open neighborhood $O(\lambda_0) \subset \sigma_A^r(a)$ of λ_0, and a set $B(\lambda_0) \in \mathcal{B}_{k(\lambda_0)}$ such that $D_a(\lambda) \in A_{B(\lambda_0)}$ for each $\lambda \in O(\lambda_0)$;*

(b) *if $\infty \in \sigma_A^r(a)$, then there exist numbers $t > 0$, $k \in (0,1]$, an open neighborhood O of ∞, and a set $C \in \mathcal{B}_k$ such that $D_a(\lambda) \in A_C$ whenever $\lambda \in O \subset \sigma_A^r(a)$. Moreover,*

$$\lim_{|\lambda|\to\infty} D_a(\lambda) = \theta_A.$$

Proof. Let A be a (not necessarily unital) locally complete topological Hausdorff algebra with idempotently pseudoconvex von Neumann bornology, and let $a \in A\setminus\{\theta_A\}$ and $\lambda_0 \in \sigma_A^r(a)\cap\mathbb{C}$. Then $D_a(\lambda_0) \in A_0$. Hence, there exists a number $\mu > 0$ such that $S(D_a(\lambda_0), \mu)$ is a bounded and idempotent subset in A. Therefore, there is a number $k = k(\lambda_0) \in (0,1]$ such that $\Gamma_k(S(D_a(\lambda_0), \mu))$ is also bounded in A. Hence, $B = \mathrm{cl}_A(\Gamma_k(S(D_a(\lambda_0), \mu))$ is a closed, absolutely k-convex, bounded, and idempotent subset in A (see [8, p. 103] and [9, Lemma 1.3]). Thereby the subalgebra A_B of A, generated by B, is by [1, Proposition 2.2], a k-normed algebra with respect to the submultiplicative norm p_B, defined by

$$p_B(a) = \inf\{|\mu|^k : a \in \mu B\}$$

for every $a \in A_B$. Since $a \neq \theta$ and $\lambda_0 \neq 0$, then $D_a(\lambda_0) \neq \theta_A$. Therefore, $p_B(D_a(\lambda_0)) > 0$.

Let $\varepsilon \in (0,1)$ and

$$O(\lambda_0) = \Big\{\lambda \in \mathbb{C} \setminus \{0\} : \Big|\frac{\lambda - \lambda_0}{\lambda}\Big| < \frac{\varepsilon}{p_B(D_a(\lambda_0))^{\frac{1}{k}}}\Big\}.$$

Then $O(\lambda_0)$ is a neighborhood of λ_0 in $\mathbb{C}$. Indeed, $\lambda_0 \in O(\lambda_0)$ and $O(\lambda_0)$ is an open subset of $\mathbb{C}$ (if $\lambda' \in \mathrm{cl}_{\mathbb{C}}(\mathbb{C} \setminus O(\lambda_0))$, then there exists a sequence $(\lambda_n) \in \mathbb{C} \setminus O(\lambda_0)$ such that (λ_n) converges to λ'. Hence, if $\lambda' \neq 0$, then from

$$|\lambda_n - \lambda_0| \geqslant |\lambda_n| \frac{\varepsilon}{p_B(D_a(\lambda_0))^{\frac{1}{k}}}$$

for each $n \in \mathbb{N}$, it follows that $\lambda' \in \mathbb{C} \setminus O(\lambda_0)$ and if $\lambda' = 0$, then $\lambda' \in \mathbb{C} \setminus O(\lambda_0)$ by the definition of $O(\lambda_0)$. Hence $\lambda' \in \mathbb{C} \setminus O(\lambda_0)$).

Let $\lambda \in (O(\lambda_0) \cap \sigma_A^r(a)) \setminus \{\lambda_0\}$ and

$$T_n(a,\lambda) = \sum_{v=1}^{n} \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v$$

for each $n \in \mathbb{N}$. Then $T_n(a,\lambda) \in A_B$,

$$0 < q(\lambda) = \Big|\frac{\lambda - \lambda_0}{\lambda}\Big|^k p_B(D_a(\lambda_0)) < \varepsilon^k < 1,$$

$$D_a(\lambda) = \frac{\lambda_0}{\lambda} D_a(\lambda_0) + \frac{\lambda_0}{\lambda} D_a(\lambda_0) T_n(a,\lambda) + \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^{n+1} D_a^{n+1}(\lambda_0) D_a(\lambda)$$

for each $n \in \mathbb{N}$, by Proposition 4.2, and

$$\begin{aligned} T_{n+m}(a,\lambda) - T_n(a,\lambda) &= \sum_{v=n+1}^{n+m} \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v \\ &= \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^n D_a(\lambda_0)^n \sum_{v=1}^{m} \Big(\frac{\lambda - \lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v \end{aligned}$$

for each $n, m \in \mathbb{N}$. Therefore

$$\begin{aligned} p_B(T_{n+m}(a,\lambda) - T_n(a,\lambda)) &\leqslant \Big|\frac{\lambda - \lambda_0}{\lambda}\Big|^{kn} p_B(D_a(\lambda_0))^n \sum_{v=1}^{m} \Big|\frac{\lambda - \lambda_0}{\lambda}\Big|^{kv} p_B(D_a(\lambda_0))^v \\ &= q(\lambda)^n \sum_{v=1}^{m} q(\lambda)^v < \varepsilon^{kn} \sum_{v=1}^{\infty} \varepsilon^{kv} = \frac{1}{1-\varepsilon^k} \varepsilon^{kn} \end{aligned}$$

for each $n, m \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} p_B(T_{n+m}(a,\lambda) - T_n(a,\lambda)) = 0$$

for each $m \in \mathbb{N}$. It means that $(T_n(a,\lambda))$ is a Cauchy sequence in A_B. Thereby, $(T_n(a,\lambda))$ is a convergent (and bounded) sequence in A_B for each

$\lambda \in O(\lambda_0) \cap \sigma_A^r(a)$ (because A_B is complete by assumption). Hence, there is a number $M(\lambda) > 0$ such that

$$T_n(a,\lambda) \in M(\lambda)B$$

for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} T_a(a,\lambda) = \sum_{v=1}^{\infty} \Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v \in M(\lambda)B$$

for each $\lambda \in O(\lambda_0) \cap \sigma_A^r(a)$ because A is a Hausdorff space and B is closed. Since

$$p_B\Big[\Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^n D_a(\lambda_0)^n\Big] \leqslant q(\lambda)^n < \varepsilon^{kn}$$

for each $n \in \mathbb{N}$ and $\lambda \in O(\lambda_0) \cap \sigma_A^r(a)$, then the sequence $\Big(\big(\frac{\lambda-\lambda_0}{\lambda}\big)^n D_a(\lambda_0)^n\Big)$ vanishes in A_B for each fixed $\lambda \in O(\lambda_0) \cap \sigma_A^r(a)$, hence also in A, because the topology on A_B defined by p_B is not weaker than the subset topology on A_B (see [1, Proposition 2.2]). It means that

$$X_\lambda = \lim_{n\to\infty} \Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^n D_a(\lambda_0)^n D_a(\lambda) = \theta_A$$

for each fixed $\lambda \in O(\lambda_0) \cap \sigma_A^r(a)$. Taking this into account,

$$\begin{aligned} D_a(\lambda) = \lim_{n\to\infty} D_a(\lambda) &= \frac{\lambda_0}{\lambda} D_a(\lambda_0) + \frac{\lambda_0}{\lambda} D_a(\lambda_0) \lim_{n\to\infty} T_n(a,\lambda) + X_\lambda \\ &= \frac{\lambda_0}{\lambda} D_a(\lambda_0) + \frac{\lambda_0}{\lambda} D_a(\lambda_0) \sum_{v=1}^{\infty} \Big(\frac{\lambda-\lambda_0}{\lambda}\Big)^v D_a(\lambda_0)^v \in A_B \end{aligned}$$

for each $\lambda \in O(\lambda_0) \cap \sigma_A^r(a)$. Because $A_B \subset A_0$ (see [1, Proposition 2.3]), then $O(\lambda_0) \subset \sigma_A^r(a)$ by the definition of the regular spectrum.

Let now $\infty \in \sigma_A^r(a)$. Then $\infty \notin \mathrm{sp}_A^r(a)$ Therefore, $a \in A_0$. Hence, there are numbers $\rho > 0$ and $k \in (0,1]$ such that $\Gamma_k(S(a,\rho))$ is bounded in A by the assumption. Now $C = \mathrm{cl}(\Gamma_k(S(a,\rho))$ is a closed, absolutely k-convex, bounded, and idempotent set in A and $a \in A_C$. Again (A_C, p_C) is a k-normed subalgebra of A by Proposition 2.2 in [1].

Let $\varepsilon > 0$,

$$O = \{\lambda \in \mathbb{C} : |\lambda| > \max\{\varepsilon, p_C(a)^{\frac{1}{k}}\}\},$$

$\lambda \in O \cap \sigma_A^r(a)$,

$$\psi(\lambda) = \frac{p_C(a)}{|\lambda|^k}$$

, and

$$W_n(a,\lambda) = -\sum_{v=1}^{n} \frac{a^v}{\lambda^v}$$

for each $n \in \mathbb{N}$. Then $W_n(a,\lambda) \in A_C$ for each $n \in \mathbb{N}$,

$$D_a(\lambda) = W_n(a,\lambda) + \Big(\frac{a}{\lambda}\Big)^n D_a(\lambda)$$

for each $n \in \mathbb{N}$ by (4.2), and $p_C(D_a(\lambda)) > 0$ because $a \neq \theta_A$. Now

$$p_C[W_{n+m}(a,\lambda) - W_n(a,\lambda)] \leqslant \psi(\lambda)^n \sum_{n=0}^{\infty} \psi(\lambda)^v = \psi(\lambda)^n \frac{1}{1-\psi(\lambda)}$$
$$= \psi(\lambda)^{n-1} \frac{p_C(a)}{|\lambda|^k - p_C(a)}.$$

Since $0 < \psi(\lambda) < 1$ for each $\lambda \in O \cap \sigma_A^r(a)$, then $(W_n(a,\lambda))$ is a Cauchy (hence a bounded) sequence in A_C. Because A is locally complete, then the sequence $(W_n(a,\lambda))$ converges in A_C . Since $p_C\big(\big(\frac{a}{\lambda}\big)^n\big) \leqslant \psi(\lambda)^n$ for each $n \in \mathbb{N}$, then $\big(\frac{a}{\lambda}\big)^n$ vanishes in the topology of A_C (hence, also in the topology of A). Therefore, $\big(\big(\frac{a}{\lambda}\big)^n D_a(\lambda)\big)$ vanishes in A for each $\lambda \in O \cap \sigma_A^r(a)$. Hence

$$D_a(\lambda) = \lim_{n\to\infty} W_n(a,\lambda) + \lim_{n\to\infty} \left(\frac{a}{\lambda}\right)^n D_a(\lambda) = -\sum_{v=1}^{\infty} \frac{a^v}{\lambda^v}$$

for each $\lambda \in O \cap \sigma_A^r(a)$ since A is a Hausdorff space. As the sequence $(W_n(a,\lambda))$ is bounded in A_C, then there exists a number $N(\lambda) > 0$ such that $W_n(a,\lambda) \in N(\lambda)C$ for each $n \in \mathbb{N}$. Consequently, $D_a(\lambda) \in N(\lambda)C \in A_C$ (because C is closed). Similarly as above, $O \subset \sigma_A^r(a)$.

Since,

$$p_C(D_a(\lambda)) \leqslant \frac{p_C(a)}{|\lambda|^k} \frac{|\lambda|^k}{|\lambda|^k - p_C(a)} = \frac{p_C(a)}{|\lambda|^k - p_C(a)}$$

for every $\lambda \in O$, then

$$\lim_{|\lambda|\to\infty} D_a(\lambda) = \theta_A$$

in A_C. The topology on A_C, defined by the norm p_C, is not weaker than the subset topology on A_C (see [1, Proposition 2.2]). Therefore, this limit holds also in A. □

Similar result for the resolvent map in unital case has been proved in [2, Proposition 2.1].

Corollary 4.5. *Let A be a locally complete Hausdorff algebra over $\mathbb{C}$ with idempotent pseudoconvex von Neumann bornology. Then the regular spectrum $\mathrm{sp}_A^r(a)$ is closed in $\mathbb{C}_\infty$ and not empty for every $a \in A$.*

Proof. The regular spectrum $\mathrm{sp}_A^r(a)$ is closed in $\mathbb{C}_\infty$ by Proposition 4.3 and nonempty by the definition and, in unital case, by Proposition 2.2 in [2]. □

Corollary 4.6. *Let A be a locally complete locally convex Hausdorff algebra over $\mathbb{C}$ with jointly continuous multiplication and continuous quasi-inversion. Then*

$$\mathrm{sp}_A^r(a) = \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$$

for each $a \in A$.

Proof. This statement is true by Corollaries 3.8 and 4.5. □

Corollary 4.7. *Let A be a locally complete locally convex Hausdorff algebra over $\mathbb{C}$ with jointly continuous multiplication and continuous quasi-inversion. Then $a \in A_0$ if and only if $\mathrm{sp}_A(a)$ is bounded.*

Proof. If $a \in A_0$, then $\infty \notin \mathrm{sp}^r_A(a)$ by the definition of the regular spectrum. Therefore, $\infty \notin \mathrm{sp}_A(a))$. Consequently, $\mathrm{sp}_A(a)$ is bounded. Otherwise, every $n \in \mathrm{sp}_A(a)$ by what $\infty \in \mathrm{sp}_A(a)$ because the regular spectrum of A is closed by Corollary 4.5.

Let now $\mathrm{sp}_A(a)$ be bounded. Then $\infty \notin \mathrm{cl}_{\mathbb{C}_\infty}(\mathrm{sp}_A(a))$. Hence, $\infty \notin \mathrm{sp}^r_A(a))$ by Corollary 4.6. So, $a \in A_0$ by the definition of the regular spectrum. □

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