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ON SOME INEQUALITIES FOR THE APPROXIMATION NUMBERS IN BANACH ALGEBRAS

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Abstract. In this paper, we generalize some inequalities for the approximation numbers of an element in a normed (Banach) algebra X and, as an application, we present inequalities for the quasinorms of some ideals defined by means of the approximation numbers.

In particular, if $X = L(E)$ - the algebra of linear and bounded operators $T : E \longrightarrow E$, where E is a Banach space, we obtain inequalities for certain quasinorms of operators.

1. Introduction and auxiliary results

Let $(X, \|\cdot\|)$ be a unital normed algebra and let $\|\cdot\|^* : X \longrightarrow \mathbb{N} \cup \{\infty\}$ such that $\|x\|^* = 0$ iff $x = 0$, $\|x + y\|^* \leq \|x\|^* + \|y\|^*$, and $\|xy\|^* \leq \min\{\|x\|^*, \|y\|^*\}$.

For an arbitrary $x \in X$, the sequence of the approximation numbers $(a_n(x))_n$ is defined as follows

$$a_n(x) = \inf\{\|x - \overline{x}\| : \overline{x} \in X, \|\overline{x}\|^* < n\}, \; n \in \mathbb{N}, \tag{1.1}$$

and it is obvious that we have $\|x\| = a_1(x) \geq a_2(x) \geq \cdots \geq 0$.

Let $X = L(E)$ be the algebra of all linear and bounded operators $T : E \longrightarrow E$, where E is a Banach space. We denote $\|T\|^* = rank(T) = dim(T(E))$, for $T \in L(E)$.

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Remark 1.1. It is known that $\|aT\|^* = \|T\|^*$, where $a \neq 0$ is a scalar and $T \in L(E)$. The same result for ax, $x \in X$, using

$$\|ax\|^* = \|aex\|^* \leq \min\{\|ae\|^*, \|x\|^*\} \leq \|x\|^*$$

and

$$\|x\|^* = \left\|\frac{1}{a}eax\right\|^* \leq \min\left\{\left\|\frac{e}{a}\right\|^*, \|ax\|^*\right\} \leq \|ax\|^*,$$

where e is the unit of the algebra. (For normed (Banach) algebras it can see [9].)

Definition 1.2 ([1], [4], [3])**.** We denote by

$$K = \{x \,|\, x \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n > 0, \, n \in \mathbb{N}\}.$$

An application $\Phi : K \longrightarrow \mathbb{R}$ is called a symmetric norming function (also referred to as symmetric gauge function or Schatten function in the literature) if it satisfies the following conditions:

(1) $\Phi(x) > 0$, for all $x \in K$, $x \neq 0$;
(2) $\Phi(\lambda x) = \lambda\Phi(x)$, for all $\lambda \in \mathbb{R}_+$, and all $x \in K$;
(3) $\Phi(x + y) \leq \Phi(x) + \Phi(y)$, for all $x, y \in K$;
(4) $\Phi(1, 0, \cdots, 0) = 1$;
(5) If x, $y \in K$ and $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$, for all $k = 1, \ldots, n$, then $\Phi(x) \leq \Phi(y)$.

Example 1.3. Some examples of symmetric norming functions are indicated below.

(1) $\Phi_\infty(x) = x_1$, $\Phi_p(x) = \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$, $p \geq 1$, where $x = (x_1, \cdots, x_n)$;
(2) If Φ is a symmetric norming function and $p \geq 1$, then

$$\Phi_{(p)} : K \longrightarrow \mathbb{R}_+, \; \Phi_{(p)}(x) = (\Phi(x^p))^{\frac{1}{p}},$$

where $x^p = (x_1^p, \cdots, x_n^p)$, is also a symmetric norming function (see [3]).

In [7] it was shown that

(a): $\sum_{n=1}^{k} a_n(x_1 + x_2) \leq 2\sum_{n=1}^{k}(a_n(x_1) + a_n(x_2))$, $k \in \mathbb{N}$;

(b): $\sum_{n=1}^{k} a_n(x_1 x_2) \leq 2\sum_{n=1}^{k} a_n(x_1)\, a_n(x_2)$, $k \in \mathbb{N}$.

For $r \in \mathbb{N}$, $r \geq 2$, using mathematical induction it can be shown that

(a'): $\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r} x_i\right) \leq 2^{r-1}\sum_{n=1}^{k}\sum_{i=1}^{r} a_n(x_i)$, $k \in \mathbb{N}$;

(b'): $\sum_{n=1}^{k} a_n\left(\prod_{i=1}^{r} x_i\right) \leq 2^{r-1}\sum_{n=1}^{k}\prod_{i=1}^{r} a_n(x_i)$, $k \in \mathbb{N}$.

The factor 2^{r-1} in the above inequalities is not the best, and it can be improved as follows.

Proposition 1.4. *The following inequalities are true:*

(1) $\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r} x_i\right) \leq r \sum_{n=1}^{k}\sum_{i=1}^{r} a_n(x_i),\ k \in \mathbb{N};$

(2) $\sum_{n=1}^{k} a_n\left(\prod_{i=1}^{r} x_i\right) \leq r \sum_{n=1}^{k}\prod_{i=1}^{r} a_n(x_i),\ k \in \mathbb{N}.$

Proof. (1) For $\varepsilon > 0$ arbitrarily fixed, it follows from (1.1) that there exist $\overline{x}_i \in X$ such that $\|\overline{x}_i\|^* < n$ and $\|x_i - \overline{x}_i\| \leq a_n(x_i) + \frac{\varepsilon}{r}$, for $i = 1, \dots, r$.

Since

$$\left\|\sum_{i=1}^{r} \overline{x}_i\right\|^* \leq \sum_{i=1}^{r} \|\overline{x}_i\|^* \leq r(n-1) < r(n-1)+1,$$

we have

$$a_{nr-(r-1)}\left(\sum_{i=1}^{r} x_i\right) \leq \left\|\sum_{i=1}^{r} x_i - \sum_{i=1}^{r} \overline{x}_i\right\| \leq \sum_{i=1}^{r} \|x_i - \overline{x}_i\| \leq \sum_{i=1}^{r} a_n(x_i) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily fixed, the above inequality implies

$$a_{nr-(r-1)}\left(\sum_{i=1}^{r} x_i\right) \leq \sum_{i=1}^{r} a_n(x_i).$$

We have

$$\begin{aligned}
\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r} x_i\right) &\leq \sum_{n=1}^{k} \sum_{j=(n-1)r+1}^{nr} a_j\left(\sum_{i=1}^{r} x_i\right) \\
&\leq r\sum_{n=1}^{k} a_{nr-(r-1)}\left(\sum_{i=1}^{r} x_i\right) \\
&\leq r\sum_{n=1}^{k}\sum_{i=1}^{r} a_n(x_i).
\end{aligned}$$

(2) For $\varepsilon > 0$ arbitrarily fixed, from (1.1) it follows that there exist $\overline{x}_i \in X$ such that $\|\overline{x}_i\|^* < n$ and $\|x_i - \overline{x}_i\| \leq a_n(x_i) + \varepsilon$, for $i = 1, \dots, r$.

We have

$$\begin{aligned}
\prod_{i=1}^{r}(x_i - \overline{x}_i) = \prod_{i=1}^{r} x_i - \Big[\overline{x}_1 \prod_{i=2}^{r}(x_i - \overline{x}_i) \\
+ \sum_{i=2}^{r-1} x_1 \cdots x_{i-1}\overline{x}_i(x_{i+1} - \overline{x}_{i+1}) \cdots (x_r - \overline{x}_r) + \left(\prod_{i=1}^{r-1} x_i\right)\overline{x}_r\Big].
\end{aligned}$$

Since

$$\left\|\overline{x}_1 \prod_{i=2}^{r}(x_i - \overline{x}_i) + \sum_{i=2}^{r-1} x_1 \cdots x_{i-1}\overline{x}_i(x_{i+1} - \overline{x}_{i+1}) \cdots (x_r - \overline{x}_r) + \left(\prod_{i=1}^{r-1} x_i\right)\overline{x}_r\right\|^*$$
$$< r(n-1)+1,$$

we have

$$a_{nr-(r-1)}\left(\prod_{i=1}^{r} x_i\right) \leq \left\|\prod_{i=1}^{r}(x_i-\overline{x}_i)\right\| \leq \prod_{i=1}^{r}\|x_i-\overline{x}_i\| \leq \prod_{i=1}^{r}(a_n(x_i)+\varepsilon).$$

Since $\varepsilon>0$ was arbitrarily chosen, the above inequality implies

$$a_{nr-(r-1)}\left(\prod_{i=1}^{r} x_i\right) \leq \prod_{i=1}^{r} a_n(x_i).$$

We obtain

$$\begin{aligned}\sum_{n=1}^{k} a_n\left(\prod_{i=1}^{r} x_i\right) &\leq \sum_{n=1}^{k}\sum_{j=(n-1)r+1}^{nr} a_j\left(\prod_{i=1}^{r} x_i\right)\\ &\leq r\sum_{n=1}^{k} a_{nr-(r-1)}\left(\prod_{i=1}^{r} x_i\right)\\ &\leq r\sum_{n=1}^{k}\prod_{i=1}^{r} a_n(x_i).\end{aligned}$$

□

We remark that an inequality of type (1) is also true for the sequences of the errors in approximation spaces (see [2] for this notion).

The above raises the following problem.

Problem 1.5. Is r the best constant in the inequalities in Proposition 1.4?

Remark 1.6. In [6] it was shown that it may happen that

$$\sum_{n=1}^{k} a_n(S+T) \nleq \sum_{n=1}^{k}(a_n(S)+a_n(T)),$$

for $S,T\in L(E)$ and $k\in\mathbb{N}$.

It is known (the Ky Fan inequality, see [1], [3], [5], [6]) that if in addition E is a separable Hilbert space and $S,T\in L(E)$ are compact operators then for any $k\in\mathbb{N}$ we have

$$\sum_{n=1}^{k} a_n(S+T) \leq \sum_{n=1}^{k}(a_n(S)+a_n(T)).$$

Corollary 1.7. *For* $x_{ij}\in X$, $i=\overline{1,r}$, $j=\overline{1,s}$, r, $s\in\mathbb{N}$ *we have*

$$\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r}\prod_{j=1}^{s} x_{ij}\right) \leq rs\sum_{n=1}^{k}\sum_{i=1}^{r}\prod_{j=1}^{s} a_n(x_{ij}), \qquad k\in\mathbb{N}.$$

Proof. We have

$$\begin{aligned}\sum_{n=1}^{k} a_n\left(\sum_{i=1}^{r}\prod_{j=1}^{s} x_{ij}\right) &\leq r\sum_{n=1}^{k}\sum_{i=1}^{r} a_n\left(\prod_{j=1}^{s} x_{ij}\right)\\ &\leq r\sum_{n=1}^{k}\sum_{i=1}^{r}\left(s\prod_{j=1}^{s} a_n\left(x_{ij}\right)\right) = rs\sum_{n=1}^{k}\sum_{i=1}^{r}\prod_{j=1}^{s} a_n\left(x_{ij}\right).\end{aligned}$$

□

2. Applications to the ideals $A_\Phi(X)$

For $x \in X$ and Φ a symmetric norming function, following [7], [8] we define

$$A_\Phi(X) = \{x \in X : \Phi\left(\{a_n(x)\}\right) < \infty\},$$

where

$$\Phi\left(\{a_n(x)\}\right) = \lim_{n\to\infty} \Phi\left(a_1(x), \ldots, a_n(x)\right) = \sup_{n\in\mathbb{N}} \Phi\left(a_1(x), \ldots, a_n(x)\right).$$

We note that $A_\Phi(X)$ is a bilateral ideal in X and the application $x \mapsto \|x\|_\Phi = \Phi\left(\{a_n(x)\}\right)$, $x \in X$, is a quasinorm on the vector space $A_\Phi(X)$.

Proposition 2.1. *We have the following:*

(1) $\left\|\sum_{i=1}^{r} x_i\right\|_\Phi \leq r\sum_{i=1}^{r}\|x_i\|_\Phi$, $r \in \mathbb{N}$;

(2) $\left\|\prod_{i=1}^{r} x_i\right\|_{\Phi_{(p)}} \leq r\prod_{i=1}^{r}\|x_i\|_{\Phi_{(p_i)}}$, $r \in \mathbb{N}$, *where* $\frac{1}{p} = \sum_{i=1}^{r}\frac{1}{p_i}$.

Proof. The first statement follows from

$$\begin{aligned}\left\|\sum_{i=1}^{r} x_i\right\|_\Phi &= \Phi\left(\left\{a_n\left(\sum_{i=1}^{r} x_i\right)\right\}\right) \leq r\Phi\left(\left\{\sum_{i=1}^{r} a_n\left(x_i\right)\right\}\right)\\ &\leq r\sum_{i=1}^{r}\Phi\left(\{a_n\left(x_i\right)\}\right) = r\sum_{i=1}^{r}\|x_i\|_\Phi.\end{aligned}$$

For the second statement, we use the Hölder inequality for Φ (see [3]):

$$\Phi\left(\{x_i y_i\}\right) \leq \Phi_{(p)}\left(\{x_i\}\right)\Phi_{(q)}\left(\{y_i\}\right), \qquad 1 = \frac{1}{p} + \frac{1}{q},$$

which can be generalized as follows (see [6]):

$$\Phi_{(p)}\left(\prod_{i=1}^{r} z_i\right) \leq \prod_{i=1}^{r}\Phi_{(p_i)}\left(z_i\right), \qquad \frac{1}{p} = \sum_{i=1}^{r}\frac{1}{p_i}.$$

We have

$$\begin{aligned}\left\|\prod_{i=1}^{r} x_i\right\|_{\Phi_{(p)}} &= \Phi_{(p)}\left(\left\{a_n\left(\prod_{i=1}^{r} x_i\right)\right\}\right) \leq r\Phi_{(p)}\left(\left\{\prod_{i=1}^{r} a_n\left(x_i\right)\right\}\right)\\ &\leq r\prod_{i=1}^{r}\Phi_{(p_i)}\left(\left\{a_n\left(x_i\right)\right\}\right) = r\prod_{i=1}^{r}\left\|x_i\right\|_{\Phi_{(p_i)}},\end{aligned}$$

concluding the proof. □

Remark 2.2. If we consider the functions $\Phi_{(p)}$, $1 \leq p < \infty$, part (1) of the above proposition yields

$$\left\|\sum_{i=1}^{r} x_i\right\|_{\Phi_{(p)}} \leq r\sum_{i=1}^{r}\left\|x_i\right\|_{\Phi_{(p)}},$$

which is a Minkowski type inequality.

In the case of $0 < p < 1$, $\Phi_{(p)}$ is a quasinorm and in this case the Minkowski type inequality is

$$\left\|\sum_{i=1}^{r} x_i\right\|_{\Phi_{(p)}} \leq r^{\frac{2}{p}-1}\sum_{i=1}^{r}\left\|x_i\right\|_{\Phi_{(p)}}.$$

Remark 2.3. If $X = L(H)$, where H is a separable Hilbert space, then

$$A_\Phi(H) = \{T \in L(H) : \Phi\left(a_n(T)\right) < \infty\}$$

are the Schatten classes (see [1], [3], [4], [5]), and it is known that

$$\|S+T\|_\Phi \leq \|S\|_\Phi + \|T\|_\Phi .$$

Using mathematical induction it can be shown that

$$\left\|\sum_{i=1}^{r} T_i\right\|_{\Phi} \leq \sum_{i=1}^{r}\left\|T_i\right\|_{\Phi},$$

thus without the factor r in the right side.

3. The case of some special operators

We consider $B : X \times X \longrightarrow X$ a bilinear and bounded operator, i.e.

$$\begin{aligned}B\left(\lambda_1 x_1 + \lambda_2 x_2, y\right) &= \lambda_1 B\left(x_1, y\right) + \lambda_2 B\left(x_2, y\right),\\ B\left(x, \lambda_1 y_1 + \lambda_2 y_2\right) &= \lambda_1 B\left(x, y_1\right) + \lambda_2 B\left(x, y_2\right),\end{aligned}$$

where λ_1, λ_2 are scalars, and there exists $M \in \mathbb{R}$ such that for every $x, y \in X$ we have $\|B\left(x, y\right)\| \leq M \cdot \|x\| \cdot \|y\|$.

In addition, we assume that $\|B\left(x, y\right)\|^* < n^2$ if $\|x\|^* < n$ and $\|y\|^* < n$.

Proposition 3.1. *For $r, k \in \mathbb{N}$ and x_i, $y_i \in X$, $i = 1, \ldots, r$, we have*

$$\sum_{n=1}^{k}\frac{1}{n}a_n\left(\sum_{i=1}^{r} B\left(x_i, y_i\right)\right) \leq 3Mr\sum_{n=1}^{k}\sum_{i=1}^{r}\frac{a_n(x_i)\left\|y_i\right\| + 2\left\|x_i\right\| a_n(y_i)}{n}.$$

Proof. For $x, y \in X$ and $\varepsilon > 0$ exist $\overline{x}$, $\overline{y} \in X$ such that $\|x\|^* < n$, $\|y\|^* < n$ and $\|x - \overline{x}\| \leq a_n(x) + \varepsilon$, $\|y - \overline{y}\| \leq a_n(y) + \varepsilon$.

Since $\|B(x, y)\|^* < n^2$, we obtain

$$\begin{aligned} a_{n^2}(B(x,y)) &\leq \|B(x,y) - B(\overline{x},\overline{y})\| = \|B(x-\overline{x},y) + B(\overline{x}, y-\overline{y})\| \\ &\leq M(\|x-\overline{x}\| \cdot \|y\| + \|\overline{x}\| \cdot \|y-\overline{y}\|) \\ &\leq M[(a_n(x)+\varepsilon) \cdot \|y\| + (\|\overline{x}-x\| + \|x\|) \cdot (a_n(y)+\varepsilon)] \\ &\leq M[(a_n(x)+\varepsilon) \cdot \|y\| + 2\|x\| \cdot (a_n(y)+\varepsilon)], \end{aligned}$$

and passing to the limit with $\varepsilon \to 0$ we conclude

$$a_{n^2}(B(x,y)) \leq M[a_n(x)\|y\| + 2\|x\| a_n(y)].$$

We have

$$\begin{aligned} &\sum_{n=1}^{k} \frac{a_n(B(x,y))}{n} \leq \sum_{n=1}^{(k+1)^2-1} \frac{a_n(B(x,y))}{n} = \sum_{n=1}^{k} \sum_{i=n^2}^{(n+1)^2-1} \frac{a_i(B(x,y))}{i} \\ &\leq 3 \sum_{n=1}^{k} \frac{a_{n^2}(B(x,y))}{n} \leq 3M \sum_{n=1}^{k} \frac{a_n(x)\|y\| + 2\|x\| a_n(y)}{n}. \end{aligned}$$

Finally, since $\sum\limits_{n=1}^{k} u_n \leq \sum\limits_{n=1}^{k} v_n$ implies $\sum\limits_{n=1}^{k} \frac{u_n}{n} \leq \sum\limits_{n=1}^{k} \frac{v_n}{n}$, and using part (1) of Proposition 1.4 and the above inequality, we obtain

$$\begin{aligned} \sum_{n=1}^{k} \frac{1}{n} a_n\left(\sum_{i=1}^{r} B(x_i, y_i)\right) &\leq r \sum_{n=1}^{k} \sum_{i=1}^{r} \frac{a_n(B(x_i,y_i))}{n} \\ &\leq 3Mr \sum_{n=1}^{k} \sum_{i=1}^{r} \frac{a_n(x_i)\|y_i\| + 2\|x_i\| a_n(y_i)}{n}. \end{aligned}$$

□

As an application, we have the following.

Application 3.2. *Let Φ be a symmetric norming function. We consider the ideal $A_{\overline{\Phi}}(X)$, where*

$$\overline{\Phi}(\{a_n(x)\}) = \Phi\left(\left\{\frac{a_n(x)}{n}\right\}\right), \qquad x \in X.$$

For $x_i,\ y_i \in A_{\overline{\Phi}}(X)$, $i = 1, \ldots, r$, $r \in \mathbb{N}$, *and* B *a bilinear operator as above, we have*

$$\begin{aligned}
\overline{\Phi}\left(\left\{a_n\left(\sum_{i=1}^{r} B\left(x_i, y_i\right)\right)\right\}\right) &= \Phi\left(\left\{\frac{1}{n}a_n\left(\sum_{i=1}^{r} B\left(x_i, y_i\right)\right)\right\}\right) \\
&\leq 3Mr\Phi\left(\left\{\sum_{i=1}^{r}\frac{a_n(x_i)\left\|y_i\right\| + 2\left\|x_i\right\| a_n(y_i)}{n}\right\}\right) \\
&\leq 3Mr\sum_{i=1}^{r}\left(\left\|x_i\right\|_{\overline{\Phi}}\cdot\left\|y_i\right\| + 2\left\|x_i\right\|\cdot\left\|y_i\right\|_{\overline{\Phi}}\right), \\
&\leq 9Mr\sum_{i=1}^{r}\left\|x_i\right\|_{\overline{\Phi}}\cdot\left\|y_i\right\|_{\overline{\Phi}},
\end{aligned}$$

which shows that $\sum_{i=1}^{r} B\left(x_i, y_i\right) \in A_{\overline{\Phi}}(X)$ *and*

$$\left\|\sum_{i=1}^{r} B\left(x_i, y_i\right)\right\|_{\overline{\Phi}} \leq 9Mr\sum_{i=1}^{r}\left\|x_i\right\|_{\overline{\Phi}}\cdot\left\|y_i\right\|_{\overline{\Phi}}.$$

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