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# VARIANT VERSIONS OF THE LEWENT TYPE DETERMINANTAL INEQUALITY

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Abstract. In this paper, we present a refinement of the Lewent determinantal inequality and show that the following inequality holds

$$\det \frac{I_{\mathcal{H}} + A_1}{I_{\mathcal{H}} - A_1} + \det \frac{I_{\mathcal{H}} + A_n}{I_{\mathcal{H}} - A_n} - \sum_{j=1}^{n} \lambda_j \det \left( \frac{I_{\mathcal{H}} + A_j}{I_{\mathcal{H}} - A_j} \right)$$
$$\geq \det \left[ \left( \frac{I_{\mathcal{H}} + A_1}{I_{\mathcal{H}} - A_1} \right) \left( \frac{I_{\mathcal{H}} + A_n}{I_{\mathcal{H}} - A_n} \right) \prod_{j=1}^{n} \left( \frac{I_{\mathcal{H}} + A_j}{I_{\mathcal{H}} - A_j} \right)^{-\lambda_j} \right],$$

where $A_j \in \mathbb{B}(\mathcal{H})$, $0 \leq A_j < I_{\mathcal{H}}$, $A_j$'s are trace class operators and $A_1 \leq A_j \leq A_n$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0$ $(j = 1, \ldots, n)$. In addition, we present some new versions of the Lewent type determinantal inequality.

## 1. Introduction and Preliminaries

Using some properties of power-series, in 1908 Lewent [4] proved the inequality

$$\frac{1 + \sum_{j=1}^{n} \lambda_j x_j}{1 - \sum_{j=1}^{n} \lambda_j x_j} \leq \prod_{j=1}^{n} \left( \frac{1 + x_j}{1 - x_j} \right)^{\lambda_j},$$

where $x_j \in [0, 1)$ and $\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0$ $(j = 1, \ldots, n)$ are scaler weights.

Several versions of this inequality are known which include inequalities involving determinant. In this paper we give some versions of the Lewent type


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determinantal inequalities based on the following inequality due to Mercer [7]

$$f\left(x_1+x_n-\sum_{j=1}^{n}\lambda_j x_j\right)\leq f(x_1)+f(x_n)-\sum_{j=1}^{n}\lambda_j f(x_j)\,, \tag{1.1}$$

where $x_j$'s also satisfy in the condition $0<x_1\leq x_2\leq\cdots\leq x_n$. To introduce the details we give some necessary definitions and notations.

In what follows we assume that $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$ are $C^*$-algebras of all bounded linear operators on the appropriate Hilbert space with identities $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$, and $\mathbb{B}_h(\mathcal{H})$ denotes the set of all self-adjoint operators in $\mathbb{B}(\mathcal{H})$. When $\dim\mathcal{H}=k$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_k(\mathbb{C})$ of all $k\times k$ matrices with entries in the complex field. An operator $A\in\mathbb{B}_h(\mathcal{H})$ is called *positive* (positive-semidefinite for matrices) if $\langle Ax,x\rangle\geq 0$ holds for every $x\in\mathcal{H}$, and then we write $A\geq 0$. For $A,B\in\mathbb{B}_h(\mathcal{H})$, we say $A\leq B$ if $B-A\geq 0$. Let $f$ be a continuous real valued function defined on an interval $J$. The function $f$ is called *operator monotone* if $A\leq B$ implies $f(A)\leq f(B)$ for all $A,B$ with spectra in $J$. A function $f$ is said to be *operator concave* on $J$ if

$$\lambda f(A)+(1-\lambda)f(B)\leq f(\lambda A+(1-\lambda)B),$$

for all $A,B\in\mathbb{B}_h(\mathcal{H})$ with spectra in $J$ and all $\lambda\in[0,1]$, and $f$ is said to be *operator convex* on $J$ if $-f$ is operator concave on $J$. A map $\Phi:\mathbb{B}(\mathcal{H})\to\mathbb{B}(\mathcal{K})$ is called *positive linear map* if $\Phi$ is linear and $\Phi(A)\geq 0$ whenever $A\geq 0$. We denote by $\mathbf{P}[\mathbb{B}(\mathcal{H}),\mathbb{B}(\mathcal{K})]$ the set of all positive linear maps $\Phi:\mathbb{B}(\mathcal{H})\to\mathbb{B}(\mathcal{K})$. For more information see [1, Chapter 1].

For each positive operator $A$ there exists an unique positive operator $B$, know as square root of $A$ such that $B^2=A$; we write $B=A^{1/2}$. For any operator $A$, one defines its *absolute value* $|A|=(A^*A)^{1/2}$. We say that $A$ is *strictly contractive* if $\|A\|<1$, such an $A$ is called a *contraction*. An operator $A$ is said to be *trace class operator* if $\|A\|_1:=\sum_{x\in E}\langle|A|x,x\rangle<+\infty$, where $E$ is an orthonormal basis of $\mathcal{H}$ [2, 9].

In [6], Matković, Pečarić, and Perić extended (1.1) as follows: Let $A_1,\ldots,A_n\in\mathbb{B}(\mathcal{H})$ be self-adjoint operators with spectra in $[m,M]$ for some scalers $m<M$ and $\Phi_1,\ldots,\Phi_n\in\mathbf{P}[\mathbb{B}(\mathcal{H}),\mathbb{B}(\mathcal{K})]$ are positive linear maps with $\sum_{j=1}^{n}\Phi_j(I_{\mathcal{H}})=I_{\mathcal{K}}$. If $f$ is continuous convex function on $[m,M]$, then

$$f\left(mI_{\mathcal{K}}+MI_{\mathcal{K}}-\sum_{j=1}^{n}\Phi_j(A_j)\right)\leq f(m)I_{\mathcal{K}}+f(M)I_{\mathcal{K}}-\sum_{j=1}^{n}\Phi_j(f(A_j))\,. \tag{1.2}$$

In [5], Lin proved that if $A_j$ $(j=1,\ldots,n)$ are contractive trace class operators over a separable Hilbert space $\mathcal{H}$, then

$$\left|\det\left(\frac{I_{\mathcal{H}}+\sum_{j=1}^{n}\lambda_jA_j}{I_{\mathcal{H}}-\sum_{j=1}^{n}\lambda_jA_j}\right)\right|\leq\prod_{j=1}^{n}\det\left(\frac{I_{\mathcal{H}}+|A_j|}{I_{\mathcal{H}}-|A_j|}\right)^{\lambda_j}\,,$$

where $\sum_{j=1}^{k}\lambda_j=1,\ \lambda_j\geq 0$ $(j=1,\ldots,k)$.

Here, the ratio $\frac{I_{\mathcal{H}}+A}{I_{\mathcal{H}}-A}$ is understood as $(I_{\mathcal{H}}+A)(I_{\mathcal{H}}-A)^{-1}$, which is equivalent to $(I_{\mathcal{H}}-A)^{-1}(I_{\mathcal{H}}+A)$, provided the inverse exists.

## 2. Refinement of Lewent Inequality in Discrete Form

Let $\mu = (\mu_1, \ldots, \mu_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be two probability tuples; that is, $\mu_i,\ \lambda_j \geq 0$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$, $\sum_{i=1}^{m} \mu_i = 1$, and $\sum_{j=1}^{n} \lambda_j = 1$. By a (discrete) weight function (with respect to $\mu$ and $\lambda$), we mean a mapping $\omega : \{(i,j)\ :\ 1 \leq i \leq m,\ 1 \leq j \leq n\} \to [0, \infty)$, such that $\sum_{i=1}^{m} \omega(i,j)\mu_i = 1$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^{n} \omega(i,j)\lambda_j = 1$ $(i = 1, \ldots, m)$.

**Proposition 2.1.** [10, Lemma 2.1] *In above setting, if $C$ is a convex subset of a real linear space, $x_1, \ldots, x_n \in C$ and $f : C \to \mathbb{B}_h(\mathcal{K})$ is a convex, then*

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \leq \sum_{i=1}^{m} \mu_i f\left(\sum_{j=1}^{n} \omega(i,j)\lambda_j x_j\right) \leq \sum_{j=1}^{n} \lambda_j f(x_j)\,, \tag{2.1}$$

*where the convexity of $f$ means*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \qquad (\lambda \in [0,1], x, y \in C)\,.$$

The proof of the above proposition follows the same argument as in [10]. We mention that the result of Rooin in [10] is about the function $f : C \to \mathbb{R}$, but clearly the same result is valid when we replace $\mathbb{R}$ by $\mathbb{B}_h(\mathcal{K})$.

Let $f$ be an operator convex functions on $[m, M]$. Then

$$C = \{A \in \mathbb{B}(\mathcal{H})\ :\ mI_{\mathcal{H}} \leq A \leq MI_{\mathcal{H}}\}$$

is a convex subset of a real linear space $\mathbb{B}_h(\mathcal{H})$ and $f$ can be extended to a convex function from $C$ to $\mathbb{B}_h(\mathcal{H})$ by functional calculus of $f$. So, by using (2.1), we have

$$f\left(\sum_{j=1}^{n} \lambda_j A_j\right) \leq \sum_{i=1}^{m} \mu_i f\left(\sum_{j=1}^{n} \omega(i,j)\lambda_j A_j\right) \leq \sum_{j=1}^{n} \lambda_j f(A_j)\,, \tag{2.2}$$

where $A_j \in C$ $(j = 1, \ldots, n)$, see [8] for another proof of (2.2).

**Proposition 2.2.** *Let $A_j (j = 1, \ldots, n)$ be operators such that $0 < mI_{\mathcal{H}} \leq A_j \leq MI_{\mathcal{H}}$ for some scalers $m < M$; then*

$$\sum_{j=1}^{n} \log A_j^{\lambda_j} \leq \sum_{i=1}^{m} \log\left(\sum_{j=1}^{n} \omega(i,j)\lambda_j A_j\right)^{\mu_i} \leq \log\left(\sum_{j=1}^{n} \lambda_j A_j\right). \tag{2.3}$$

*Moreover, if $A_j$'s are relatively commute, then we have the following refinement of the arithmetic-geometric mean inequality:*

$$\prod_{j=1}^{n} A_j^{\lambda_j} \leq \prod_{i=1}^{m} \left(\sum_{j=1}^{n} \omega(i,j)\lambda_j A_j\right)^{\mu_i} \leq \sum_{j=1}^{n} \lambda_j A_j\,. \tag{2.4}$$

*Proof.* Since $f(t) = -\log t$ is operator convex on $(0, \infty)$, by using (2.2), we get (2.3). By commuting $A_j$'s, (2.4) immediately follows from (2.3). $\square$

A matrix $A = [a_{ij}]_{n\times n}$ is said to be *doubly stochastic*, if $a_{ij} \geq 0$, for every $i, j = 1, \ldots n$, and $\sum_{i=1}^{n} a_{ij} = 1$ and $\sum_{j=1}^{n} a_{ij} = 1$.

**Corollary 2.3.** *Assume that $x_j$ $(j = 1, \dots n)$ are positive real numbers and that $A = [a_{ij}]_{n\times n}$ is a doubly stochastic matrix. Then*

$$\sqrt[n]{\prod_{j=1}^{n} x_j} \leq \sqrt[n]{\prod_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}x_j\right)} \leq \frac{1}{n}\sum_{j=1}^{n} x_j \, .$$

*Proof.* In Proposition 2.2, set $A_j = [x_j]_{1\times 1}$, $m = n$, $\mu_i = \lambda_j = \dfrac{1}{n}$, and $\omega(i,j) = na_{ij}$ $(i, j = 1, \dots, n)$. Therefore we get the desired inequality. □

**Proposition 2.4** (refinement of Lewent inequality)**.** *Assume that $x_j \in [0, 1)$ and that $\lambda_j$, $\mu_i$, and $\omega(i,j)$ are the same as above. So*

$$\frac{1+\sum_{j=1}^{n}\lambda_j x_j}{1-\sum_{j=1}^{n}\lambda_j x_j} \leq \prod_{i=1}^{m}\left(\frac{1+\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j}{1-\sum_{j=1}^{n}\omega(i,j)\lambda_j x_j}\right)^{\mu_i} \leq \prod_{j=1}^{n}\left(\frac{1+x_j}{1-x_j}\right)^{\lambda_j} . \tag{2.5}$$

*Proof.* Using (2.2) and taking $A_j = [x_j]_{1\times 1}$ and $f(t) = \log\left(\frac{1+t}{1-t}\right)$, for $t \in [0, 1)$, we get inequality (2.5). □

## 3. Main Results

In the rest of the paper, assume that $\mathcal{H}$ is a separable Hilbert space and the convex set $C$ is defined as follows:

$$C = \left\{A \in \mathbb{B}(\mathcal{H}) \ : \ 0 \leq A < I_{\mathcal{H}} \, , A \text{ is of trace class}\right\} .$$

First, we state an important lemma for the main theorem.

**Lemma 3.1.** [5, Lemma 6] *The function $f(A) = \log\det(I_{\mathcal{H}}+A) - \log\det(I_{\mathcal{H}}-A)$ is convex on $C$.*

**Theorem 3.2** (refinement of Lewent inequality)**.** *With the above setting, we have*

$$\begin{aligned}\det\left(\frac{I_{\mathcal{H}}+\sum_{j=1}^{n}\lambda_j A_j}{I_{\mathcal{H}}-\sum_{j=1}^{n}\lambda_j A_j}\right) &\leq \prod_{i=1}^{m}\left(\det\frac{I_{\mathcal{H}}+\sum_{j=1}^{n}\omega(i,j)\lambda_j A_j}{I_{\mathcal{H}}-\sum_{j=1}^{n}\omega(i,j)\lambda_j A_j}\right)^{\mu_i} \\ &\leq \prod_{j=1}^{n}\left(\det\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)^{\lambda_j} ,\end{aligned} \tag{3.1}$$

*where $A_j \in C$ $(j = 1, \dots, n)$.*

*Proof.* By Lemma 3.1 and (2.2), we have

$$\begin{aligned} f\left(\sum_{j=1}^{n}\lambda_j A_j\right) &= \log\det\left(I_{\mathcal{H}}+\sum_{j=1}^{n}\lambda_j A_j\right) - \log\det\left(I_{\mathcal{H}}-\sum_{j=1}^{n}\lambda_j A_j\right) \\ &= \log\det\left(\frac{I_{\mathcal{H}}+\sum_{j=1}^{n}\lambda_j A_j}{I_{\mathcal{H}}-\sum_{j=1}^{n}\lambda_j A_j}\right) .\end{aligned}$$

Thus, by a similar computation,

$$\sum_{i=1}^{m}\mu_i f\left(\sum_{j=1}^{n}\omega(i,j)\lambda_j A_j\right)=\log\prod_{i=1}^{m}\left(\det\frac{I_{\mathcal{H}}+\sum_{j=1}^{n}\omega(i,j)\lambda_j A_j}{I_{\mathcal{H}}-\sum_{j=1}^{n}\omega(i,j)\lambda_j A_j}\right)^{\mu_i},$$

$$\sum_{j=1}^{n}\lambda_j f(A_j)=\log\prod_{j=1}^{n}\left(\det\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)^{\lambda_j}.$$

Since the function $g(t)=e^t$ is increasing, we get the desired result. □

**Corollary 3.3.** *Let $A_j$'s be contractions. If $A=[a_{ij}]_{n\times n}$ is a doubly stochastic matrix, then*

$$\det\left(\frac{\sum_{j=1}^{n}(I_{\mathcal{H}}+A_j)}{\sum_{j=1}^{n}(I_{\mathcal{H}}-A_j)}\right)\le\prod_{i=1}^{n}\left(\det\frac{I_{\mathcal{H}}+\sum_{j=1}^{n}a_{ij}A_j}{I_{\mathcal{H}}-\sum_{j=1}^{n}a_{ij}A_j}\right)^{\frac{1}{n}}\le\prod_{j=1}^{n}\det\left(\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)^{\frac{1}{n}}.$$

*Proof.* The desired inequalities follow from (3.1) by taking $m=n$, $\mu_i=\lambda_j=\frac{1}{n}$, $\omega(i,j)=na_{ij}$ $(i,j=1,\dots,n)$. □

In the next theorem, by using operator monotonicity of the function $g(t)=\frac{1+t}{1-t}$ on $(-\infty,1)$, we present two new version of Lewent type determinantal inequality as follows:

**Theorem 3.4.** *Let $A_j\in C$ with $A_1\le A_j\le A_n$ $(j=1,\dots,n)$. Then*

$$\begin{aligned}&\det\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}+\det\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_n}-\sum_{j=1}^{n}\lambda_j\det\left(\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)\\&\ge\det\left[\left(\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}\right)\left(\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_n}\right)\prod_{j=1}^{n}\left(\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)^{-\lambda_j}\right],\end{aligned}\tag{3.2}$$

*where $\sum_{j=1}^{n}\lambda_j=1$ and $\lambda_j\ge0$ $(j=1,\dots,n)$.*

*Proof.* Since $g(t)=\frac{1+t}{1-t}$ is operator monotone function on $(0,1)$, hence

$$\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}\le\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\le\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_n}\qquad\text{for all } j=1,\dots,n.$$

Also, by Theorem 3.2 of [2], we know that for any trace class operator $A$ there exists a sequence of finite rank operators $\{F_n\}_{n=1}^{\infty}$ which convergence in the trace class norm to $A$ and

$$\det(I_{\mathcal{H}}+A)=\lim_{n\to\infty}\det(I_{\mathcal{H}}+F_n).$$

So Theorem 7.8 of [11] gives

$$\det\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}\le\det\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\le\det\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_n}.$$

By choosing $f(t) = \log t$ and $\Phi_j(t) = \lambda_j t$ in (1.2) and by concavity $f$ on $(0, \infty)$, we get

$$\begin{aligned}
&\log\left(\det\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}+\det\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_n}-\sum_{j=1}^{n}\lambda_j\det\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)\\
&\geq \log\det\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}+\log\det\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_n}-\sum_{j=1}^{n}\lambda_j\log\det\left(\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right) \qquad (3.3)\\
&= \log\det\left[\left(\frac{I_{\mathcal{H}}+A_1}{I_{\mathcal{H}}-A_1}\right)\left(\frac{I_{\mathcal{H}}+A_n}{I_{\mathcal{H}}-A_{\mathcal{H}}}\right)\prod_{j=1}^{n}\left(\frac{I_{\mathcal{H}}+A_j}{I_{\mathcal{H}}-A_j}\right)^{-\lambda_j}\right].
\end{aligned}$$

Since the function $g(t) = e^t$ is increasing, (3.2) implies (3.3). $\square$

**Lemma 3.5.** [3, Theorem 7.6.7] *The function*

$$f : \{X \in \mathcal{M}_n(\mathbb{C}) : \ X = X^*,\ sp(X) \subset (0, \infty)\} \to \mathbb{R}$$

*defined by $f(X) = \log\det(X)$ is concave.*

**Proposition 3.6.** *Let $\mathcal{H}$ be a finite dimensional complex Hilbert space with $\dim\mathcal{H} = k$, and let $A_j \in C$ with $A_1 \leq A_j \leq A_n$ $(j = 1, \ldots, n)$. Then*

$$\begin{aligned}
&\det\left(\frac{I_k+A_1}{I_k-A_1}+\frac{I_k+A_n}{I_k-A_n}+\sum_{j=1}^{n}\lambda_j\frac{I_k-A_j}{I_k+A_j}\right)\\
&\geq \det\prod_{j=1}^{n}\left(\frac{I_k+A_1}{I_k-A_1}+\frac{I_k+A_n}{I_k-A_n}+\frac{I_k-A_j}{I_k+A_j}\right)^{\lambda_j}\\
&\geq \prod_{j=1}^{n}\left(\det\frac{I_k+A_1}{I_k-A_1}+\det\frac{I_k+A_n}{I_k-A_n}+\det\frac{I_k-A_j}{I_k+A_j}\right)^{\lambda_j},
\end{aligned}$$

*where $I_k$ is the identity matrix in $\mathcal{M}_k(\mathbb{C})$.*

*Proof.* Since $\dim\mathcal{H} = k$, thus $\mathbb{B}(\mathcal{H}) \cong \mathcal{M}_k(\mathbb{C})$. Therefore

$$\begin{aligned}
&\log\det\left(\frac{I_k+A_1}{I_k-A_1}+\frac{I_k+A_n}{I_k-A_n}+\sum_{j=1}^{n}\lambda_j\frac{I_k-A_j}{I_k+A_j}\right)\\
&= \log\det\left(\sum_{j=1}^{n}\lambda_j\left[\frac{I_k+A_1}{I_k-A_1}+\frac{I_k+A_n}{I_k-A_n}+\frac{I_k-A_j}{I_k+A_j}\right]\right)\\
&\geq \sum_{j=1}^{n}\lambda_j\log\det\left(\frac{I_k+A_1}{I_k-A_1}+\frac{I_k+A_n}{I_k-A_n}+\frac{I_k-A_j}{I_k+A_j}\right) \qquad \text{(by Lemma 3.5)}\\
&= \log\prod_{j=1}^{n}\det\left(\frac{I_k+A_1}{I_k-A_1}+\frac{I_k+A_n}{I_k-A_n}+\frac{I_k-A_j}{I_k+A_j}\right)^{\lambda_j}
\end{aligned}$$

$$\geq \sum_{j=1}^{n} \lambda_j \log \left( \det \frac{I_k + A_1}{I_k - A_1} + \det \frac{I_k + A_n}{I_k - A_n} + \det \frac{I_k - A_j}{I_k + A_j} \right) \quad \text{(by Theorem 7.7 [11])}$$

$$= \log \prod_{j=1}^{n} \left( \det \frac{I_k + A_1}{I_k - A_1} + \det \frac{I_k + A_n}{I_k - A_n} + \det \frac{I_k - A_j}{I_k + A_j} \right)^{\lambda_j},$$

and this completes the proof. □

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