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$T1$ THEOREM FOR INHOMOGENEOUS TRIEBEL–LIZORKIN AND BESOV SPACES ON RD-SPACES AND ITS APPLICATION

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ABSTRACT. Using Calderón's reproducing formulas and almost orthogonal estimates, the $T1$ theorem for the inhomogeneous Triebel–Lizorkin and Besov spaces on RD-spaces is obtained. As an application, new characterizations for these spaces with "half" the usual conditions of the approximate to the identity are presented.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The main purpose of this paper is to characterize the inhomogeneous Triebel–Lizorkin and Besov spaces on RD-spaces with "half" the usual conditions of the approximation to the identity. For this purpose, we prove a new $T1$ theorem of these spaces, where the inhomogeneous Calderón–Zygmund kernel satisfies "half" smoothness conditions.

To state the main results, let us first recall spaces of homogeneous type which were introduced by Coifman and Weiss [1]. A quasi-metric ρ on a set X is a function ρ: $X \times X \to [0, \infty)$ satisfying (i) $\rho(x, y) = 0$ if and only if $x = y$; (ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$; (iii) There exists a constant $A \in [1, \infty)$ such that, for all x, y and $z \in X$,

$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].$$

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Any quasi-metric defines a topology, for which the balls

$$B(x,r) = \{y \in X : \rho(y,x) < r\},$$

for all $x \in X$ and all $r > 0$, form a basis. We say that (X, ρ, μ) is a space of homogeneous type in the sense of Coifman and Weiss if ρ is a quasi-metric and μ is a nonnegative Borel regular measure on X satisfying the doubling condition; that is, for all $x \in X, r > 0$, then $0 < \mu(B(x,r)) < \infty$ and

$$\mu(B(x,2r)) \leq C\mu(B(x,r)), \tag{1.1}$$

where μ is assumed to be defined on a σ-algebra which contains all Borel sets and all balls $B(x,r)$, and the constant $0 < C < \infty$ is independent of $x \in X$ and $r > 0$.

Macías and Segovia [14] showed that the quasi-metric ρ can be replaced by another quasi-metric d such that the topologies induced on X by ρ and d coincide, and d has the regularity property; there exist constants $C > 0$ and $0 < \theta < 1$ such that, for all $x, x', y \in X$,

$$|d(x,y) - d(x',y)| \leq Cd(x,x')^{\theta}[d(x,y) + d(x',y)]^{1-\theta}.$$

Moreover, if $B(x,r)$, the ball defined by the quasi-metric d, then

$$\mu(B(x,r)) \approx r. \tag{1.2}$$

Note that the condition (1.2) is much stronger than the doubling property (1.1).

In [15], Nagel and Stein developed the product theory on Carnot–Carathéodory spaces with a smooth quasi-metric d and a measure μ satisfying the condition (1.1) and the "reverse" doubling condition; that is, there exist constants a_0 and $C \in (1,\infty)$ such that, for all $x \in X$ and all $0 < r < \sup_{x,y \in X} d(x,y)/a_0$,

$$C\mu(B(x,r)) \leq \mu(B(x,a_0 r)). \tag{1.3}$$

Such "reverse" doubling condition was further extended by Han, Müller, and Yang in [6] and [7] into metric measure spaces.

We point out that the doubling condition (1.1) and "reverse" doubling condition (1.3) imply that there exist positive constants ω (the *upper dimension* of μ), $\kappa \in (0,\omega]$ (the *lower dimension* of μ), $c \in (0,1]$, and $C \geq 1$, such that, for all $x \in X$, $0 < r < \sup_{x,y\in X} d(x,y)/2$, and $1 \leq \lambda < \sup_{x,y \in X} d(x,y)/2r$,

$$c\lambda^{\kappa}\mu(B(x,r)) \leq \mu(B(x,\lambda r)) \leq C\lambda^{\omega}\mu(B(x,r)).$$

Such spaces of homogeneous type satisfying the "reverse" doubling condition are called RD-spaces, which was originally introduced in [7]. See also [6] and [18] for more equivalent characterizations of RD-spaces.

Throughout this paper, (X, d, μ) denotes an RD-space with $\mu(X) = \infty$. We use C to denote a positive constant, whose value may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. Let θ be the regularity exponent of X, and let M be the Hardy–Littlewood maximal operator. We denote by $f \sim g$ if there exists a constant $C > 0$ independent of the main parameters such that $C^{-1}g \leq f \leq Cg$. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For all $x \in X$ and all $r > 0$, we use the abbreviations

$$V_r(x) = \mu(B(x,r)) \text{ and } (x,y) = \mu(B(x,d(x,y))).$$

In order to introduce the inhomogeneous Triebel–Lizorkin and Besov spaces, we need the following definitions of approximates to the identity and spaces of test functions.

Definition 1.1. [7] A sequence $\{S_k\}_{k\in\mathbb{Z}_+}$ of linear operators is said to be an approximation to the identity if there exists a constant $C > 0$ such that, for all $k \in \mathbb{Z}_+$ and all $x, x', y, y' \in X$, $S_k(x,y)$, the kernel of S_k, satisfies the following conditions:
(i) $S_k(x,y) = 0$ if $d(x,y) \geq C2^{-k}$ and $|S_k(x,y)| \leq C\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(ii) $|S_k(x,y) - S_k(x',y)| \leq C2^{k\theta}d(x,x')^\theta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(iii) $|S_k(x,y) - S_k(x,y')| \leq C2^{k\theta}d(y,y')^\theta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}$;
(iv)

$$\begin{aligned}|[S_k(x,y) - S_k(x,y')] - [S_k(x',y) - S_k(x',y')]|\\ \leq C2^{2k\theta}d(x,x')^\theta d(y,y')^\theta \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)};\end{aligned}$$

(v) $\int_X S_k(x,y)\,d\mu(y) = 1$;
(vi) $\int_X S_k(x,y)\,d\mu(x) = 1$.

Definition 1.2. [7] Suppose that $0 < \beta, \gamma \leq \theta$. A function f defined on X is said to be a test function of type (β,γ) centered at $x_0 \in X$ with width $r > 0$ if f satisfies
(i) $|f(x)| \leq C\frac{1}{V_r(x_0)+V(x_0,x)}\frac{r^\gamma}{(r+d(x,x_0))^\gamma}$;
(ii) $|f(x) - f(y)| \leq C\left(\frac{d(x,y)}{r+d(x,x_0)}\right)^\beta \frac{1}{V_r(x_0)+V(x_0,x)}\frac{r^\gamma}{(r+d(x,x_0))^\gamma}$ for $d(x,y) \leq \frac{1}{2A}(r + d(x,x_0))$.

If f is a test function of type (β,γ) centered at x_0 with width $r > 0$, we write $f \in \mathcal{M}(x_0,r,\beta,\gamma)$, and the norm of f in $\mathcal{M}(x_0,r,\beta,\gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} = \inf\{C > 0 : (i) - (ii) \text{ hold }\}.$$

We denote by $\mathcal{M}(\beta,\gamma)$ the class of all $f \in \mathcal{M}(x_0,1,\beta,\gamma)$. It is easy to see that $\mathcal{M}(x_1,d,\beta,\gamma) = \mathcal{M}(\beta,\gamma)$ with the equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$.

Let $\widetilde{\mathcal{M}}(\beta,\gamma)$ be the completion of the space $\mathcal{M}(\theta,\theta)$ in $\mathcal{M}(\beta,\gamma)$ when $0 < \beta,\gamma \leq \theta$. If $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$, we define $\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)} = \|f\|_{\mathcal{M}(\beta,\gamma)}$.

We define the distribution space $(\widetilde{\mathcal{M}}(\beta,\gamma))'$ by all linear functionals $\mathcal{L}$ from $\widetilde{\mathcal{M}}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant $C \geq 0$ such that, for all $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$,

$$|\mathcal{L}(f)| \leq C\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}.$$

Now we recall definitions of the inhomogeneous Triebel–Lizorkin and Besov spaces, and the authors [7] showed that these spaces are well defined.

Definition 1.3. Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ be an approximation to the identity, and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Let $-\theta < \alpha < \theta$, and let $1 < p, q < \infty$. The inhomogeneous Triebel–Lizorkin space is the collection of $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$ with $0 < \beta, \gamma < \theta$ such that

$$\|f\|_{F_p^{\alpha,q}} = \|D_0(f)\|_{L^p} + \Big\|\Big\{\sum_{k=1}^{\infty}\Big(2^{k\alpha}|D_k(f)|\Big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^p} < \infty.$$

The inhomogeneous Besov space is the collection of all $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$ with $0 < \beta, \gamma < \theta$ such that

$$\|f\|_{B_p^{\alpha,q}} = \|D_0(f)\|_{L^p} + \Big\{\sum_{k=1}^{\infty}\Big(2^{k\alpha}\|D_k(f)\|_{L^p}\Big)^q\Big\}^{\frac{1}{q}} < \infty.$$

To formulate the main results of this paper, we need the following definitions.

For $\eta \in (0,\theta]$, let $C_0^\eta(X)$ be the set of all continuous functions f on X with compact support such that

$$\|f\|_{C_0^\eta(X)} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology, and let $(C_0^\eta(X))'$ be its dual space.

The following is the inhomogeneous Calderón–Zygmund kernel which was introduced by Meyer [12]. See, for example, [8], [16], [5], and references therein.

Definition 1.4. A continuous complex-valued function K on $\Omega = \{(x,y) \in X \times X : x \neq y\}$ is called an inhomogeneous Calderón–Zygmund kernel of type (ϵ,σ) if there exist constants $\epsilon \in (0,\theta], \sigma > 0$, and $C_1 > 0$ such that
(i) $|K(x,y)| \le C_1\frac{1}{V(x,y)}$;
(ii) $|K(x,y)| \le C_1\frac{1}{d(x,y)^\sigma}\frac{1}{V(x,y)}$ for $d(x,y) \ge 1$;
(iii) $|K(x,y) - K(x',y)| \le C_1\frac{d(x,x')^\epsilon}{d(x,y)^\epsilon}\frac{1}{V(x,y)}$ for $d(x,x') \le d(x,y)/2A$;
(iv) $|K(x,y) - K(x,y')| \le C_1\frac{d(y,y')^\epsilon}{d(x,y)^\epsilon}\frac{1}{V(x,y)}$ for $d(y,y') \le d(x,y)/2A$.

We now define Calderón–Zygmund singular integral operators with inhomogeneous kernels.

Definition 1.5. A continuous linear operator T: $C_0^\eta \to (C_0^\eta)'$ is an inhomogeneous Calderón–Zygmund singular integral operator if there exists an inhomogeneous kernels K such that

$$\langle Tf, g\rangle = \int_X\int_X K(x,y)f(y)g(x)\,d\mu(x)\,d\mu(y)$$

for all $f, g \in C_0^\eta$ with disjoint supports.

We also need the notion of the weak boundedness property.

Definition 1.6.[3] A Calderón-Zygmund singular integral operator T is said to have the weak boundedness property, if there exist constants $C_2 > 0$ and $\eta \in (0, \theta]$ such that, for all $x_0 \in X$ and $r > 0$,

$$|\langle Tf, g\rangle| \leq C_2 V_r(x_0) r^{2\eta} \|g\|_{C_0^\eta} \|f\|_{C_0^\eta},$$

where $f, g \in C_0^\eta$ with supp f, supp $g \subset B(x_0, r)$, $\|f\|_\infty \leq 1$, $\|g\|_\infty \leq 1$, $\|f\|_{C_0^\eta} \leq r^{-\eta}$, and $\|g\|_{C_0^\eta} \leq r^{-\eta}$. If T satisfies the weak boundedness property, then we denote it by $T \in WBP$.

The $T1$ theorem of the inhomogeneous Triebel–Lizorkin and Besov spaces can be stated as follows. We use T^* to denote the adjoint operator of T.

Theorem 1.7. *Suppose that T is a singular integral operator with the kernel which satisfies (i), (ii) and (iii), of Definition 1.4, $T(1) = 0$, and $T \in WBP$. Then T can be extended to a bounded linear operator on $F_p^{\alpha,q}$ and $B_p^{\alpha,q}$, for $0 < \alpha < \epsilon$ and $1 < p, q < \infty$.*

Theorem 1.8. *Suppose that T is a singular integral operator with the kernel which satisfies (i), (ii), and (iv) of Definition 1.4, $T^*(1) = 0$, and $T \in WBP$. Then T can be extended to a bounded linear operator on $F_p^{\alpha,q}$ and $B_p^{\alpha,q}$, for $-\epsilon < \alpha < 0$ and $1 < p, q < \infty$.*

Han and Sawyer [9] established the $T1$ theorem for the Triebel–Lizorkin and Besov spaces on spaces of homogeneous type. Moreover, they obtained new characterizations of the Triebel–Lizorkin and Besov spaces with "half" smoothness and cancellation conditions. See [13], [16], [17], [10], and [7] for the related results.

As an application of the $T1$ theorem, we give new characterizations of the inhomogeneous Triebel–Lizorkin and Besov spaces which only need "half" the usual conditions on the approximate to the identity.

Theorem 1.9. *Let $0 < \alpha < \theta$, and let $1 < p, q < \infty$. Suppose that S_k $(k \in \mathbb{Z}_+)$ is an approximation to the identity satisfying (i), (ii), and (v) of Definition 1.1, and that $E_k = S_k - S_{k-1}$ when $k \in \mathbb{N}$ and $E_0 = S_0$.*
(i) For $f \in F_p^{\alpha,q}$, then

$$\Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f)|\Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \sim \|f\|_{F_p^{\alpha,q}}. \tag{1.4}$$

(ii) For $f \in B_p^{\alpha,q}$, then

$$\Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} \|E_k(f)\|_{L^p}\Big)^q \Big\}^{\frac{1}{q}} \sim \|f\|_{B_p^{\alpha,q}}. \tag{1.5}$$

Theorem 1.10. *Let $-\theta < \alpha < 0$, and let $1 < p, q < \infty$. Assume that S_k $(k \in \mathbb{Z}_+)$ is an approximation to the identity satisfying (i), (iii), and (vi) of Definition 1.1, and that $E_k = S_k - S_{k-1}$ when $k \in \mathbb{N}$ and $E_0 = S_0$, then (1.4) and (1.5) also hold.*

2. Proof of the $T1$ Theorem

In this section, the $T1$ theorem of the inhomogeneous Triebel–Lizorkin and Besov spaces on RD-spaces is presented. We first formulate Calderón's reproducing formulas which are main tools in this paper.

Lemma 2.1. [7] *Let S_k $(k \in \mathbb{Z}_+)$ be as in Definition 1.1. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then there exist families of linear operators $\widetilde{D}_k$ and $\widetilde{\widetilde{D}}_k$, for $k \in \mathbb{Z}_+$, such that, for all $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ with $0 < \beta, \gamma < \theta$,*

$$f = \sum_{k=0}^{\infty} \widetilde{D}_k D_k(f) = \sum_{k=0}^{\infty} D_k \widetilde{\widetilde{D}}_k(f),$$

where the series converges in the both norm of $\widetilde{\mathcal{M}}(\beta', \gamma')$ with $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$, and norm of L^p with $1 < p < \infty$. When $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$, the series converges in the norm of $(\widetilde{\mathcal{M}}(\beta', \gamma'))'$ with $\beta < \beta' < \theta, \gamma < \gamma' < \theta$. Moreover, for any $\theta' \in (0, \theta)$, $\widetilde{D}_k(x, y)$, and $\widetilde{\widetilde{D}}_k(x, y)$, the kernels of $\widetilde{D}_k$ and $\widetilde{\widetilde{D}}_k$, satisfy the similar estimates but with x and y interchanged in (ii):
(i)

$$|\widetilde{D}_k(x, y)| \le C \frac{1}{V_{2^{-k}}(x) + V(x, y)} \frac{2^{-k\theta}}{(2^{-k} + d(x, y))^{\theta}};$$

(ii)

$$|\widetilde{D}_k(x, y) - \widetilde{D}_k(x', y)| \le C \Big(\frac{d(x, x')}{2^{-k} + d(x, y)}\Big)^{\theta'} \frac{1}{V_{2^{-k}}(x) + V(x, y)} \frac{2^{-k\theta}}{(2^{-k} + d(x, y))^{\theta}}$$

for $d(x, x') \le \frac{1}{2A}(2^{-k} + d(x, y))$;
(iii)

$$\int_X \widetilde{D}_0(x, y)\, d\mu(y) = \int_X \widetilde{D}_0(x, y)\, d\mu(x) = 1;$$

and for $k \in \mathbb{N}$

$$\int_X \widetilde{D}_k(x, y)\, d\mu(y) = \int_X \widetilde{D}_k(x, y)\, d\mu(x) = 0.$$

We describe a fundamental estimate before we give the proof of Theorem 1.7. In what follows, we denote $\min\{a, b\}$ by $a \wedge b$ for any $a, b \in \mathbb{R}$.

Proposition 2.2. *Suppose that $0 < \alpha < \epsilon$ and that T satisfies the hypotheses of Theorem 1.7 and that D_k $(k \in \mathbb{Z}_+)$ is the same as in Definitions 1.3. Then there exists a constant $C > 0$ such that*

$$|D_l T D_k(x, y)| \le C \Big(2^{(k-l)\epsilon} \wedge 1\Big) \frac{1 + (l - l \wedge k)}{V_{2^{-(k\wedge l)}}(x) + V(x, y)} \frac{2^{-(k\wedge l)\sigma'}}{(2^{-(k\wedge l)} + d(x, y))^{\sigma'}}, \quad (2.1)$$

where $\sigma' = \sigma$ if $l = 0$, otherwise $\sigma' = \epsilon$.

Proof. We only consider the cases: $l = 0, k \in \mathbb{Z}_+$ and $l > 0, k = 0$. When $k, l > 0$, the proof of (2.1) is just [11, Proposition 2.1]. In what follows, we fix a smooth cut-off function $\phi_0 \in C_0^\infty(\mathbb{R})$ with $\phi_0(x) = 1$ when $|x| \le 1$ and $\phi_0(x) = 0$ when $|x| > 2$, and set $\phi_1 = 1 - \phi_0$. When $l = 0, k \in \mathbb{Z}_+$, we consider two cases. When $d(x, y) \le 3A^2C$, we have

$$\begin{aligned} D_0TD_k(x,y) =& \langle TD_k(\cdot,y)(u), D_0(x,\cdot)(u)\rangle \\ =& \Big\langle T\Big(D_k(\cdot,y)\phi_0\Big(\frac{d(x,\cdot)}{2AC}\Big)\Big)(u), D_0(x,\cdot)(u)\Big\rangle \\ &+ \Big\langle T\Big(D_k(\cdot,y)\phi_1\Big(\frac{d(x,\cdot)}{2AC}\Big)\Big)(u), D_0(x,\cdot)(u)\Big\rangle \\ :=& I_{1.1} + I_{1.2}. \end{aligned}$$

For $I_{1.1}$, let $\psi(u) = D_0(x, u)$, and let $\varphi(u) = D_k(u, y)\phi_0\big(\frac{d(x,u)}{2AC}\big)$. Since $T \in WBP$, then

$$|I_{1.1}| = |\langle T\varphi, \psi\rangle| \le CV_1(x)\|\varphi\|_{C_0^\eta}\|\psi\|_{C_0^\eta} \le C\frac{1}{V_1(x)}.$$

Now we consider $I_{1.2}$. For any given x, since $\operatorname{supp}(D_0(x,\cdot)) \bigcap \operatorname{supp}\big(D_k(\cdot,y)\phi_1\big(\frac{d(x,\cdot)}{2AC}\big)\big) = \emptyset$, so we can write $I_{1.2}$ as

$$I_{1.2} = \int_X\int_X D_0(x,u)K(u,v)D_k(v,y)\phi_1\Big(\frac{d(x,v)}{2AC}\Big)d\mu(u)d\mu(v).$$

Notice that $d(x, u) \le C$ and by the support of ϕ_1, $d(x, v) \ge 2AC$, then we obtain that $d(x, v) \le Cd(x, u)$ and

$$V(u, v) \ge V_1(u) \sim CV_1(x).$$

By the above fact, we have

$$\begin{aligned} |I_{1.2}| \le& \int_X\int_X |D_0(x,u)K(u,v)D_k(v,y)\phi_1\Big(\frac{d(x,v)}{2AC}\Big)|d\mu(u)d\mu(v) \\ \le& C\int_X\int_X |D_0(x,u)|\frac{1}{V(u,v)}|D_l(v,y)|d\mu(u)d\mu(v) \\ \le& C\frac{1}{V_1(x)}. \end{aligned}$$

When $d(x, y) > 3A^2C$, note that $d(x, u) \le C$ and $d(v, y) \le C$, we have $d(u, v) \ge \frac{d(x,y)}{3A^2}$.

Thus, in this case, we obtain

$$\begin{aligned} |D_0TD_k(x,y)| \le& C\int_X\int_X |D_0(x,u)|\frac{1}{d(u,v)^\sigma}\frac{1}{V(u,v)}|D_k(v,y)|\,d\mu(u)\,d\mu(v) \\ \le& C\frac{1}{V(x,y)}\frac{1}{d(x,y)^\sigma} \le C\frac{1}{V(x,y)+V_1(x)}\frac{1}{(1+d(x,y))^\sigma}, \end{aligned}$$

where σ be as in Definition 1.4. Therefore, the above estimates enable us to get

$$|D_0TD_k(x,y)| \le C\frac{1}{V(x,y)+V_1(x)}\frac{1}{(1+d(x,y))^\sigma}.$$

We now estimate the case that $l>0$ and $k=0$, and consider it by two cases. When $d(x,y)\le 3A^2C$, since $T(1)=0$, then

$$\begin{aligned}
D_lTD_0(x,y) =&\langle TD_0(\cdot,y)(u), D_l(x,\cdot)(u)\rangle\\
=&\Big\langle T\Big(D_0(\cdot,y)-D_0(x,y)\Big)(u), D_l(x,\cdot)(u)\Big\rangle\\
=&\Big\langle T\Big([D_0(\cdot,y)-D_0(x,y)]\phi_0\Big(\frac{d(x,\cdot)}{2AC2^{-l}}\Big)\Big)(u), D_l(x,\cdot)(u)\Big\rangle\\
&+\Big\langle T\Big([D_0(\cdot,y)-D_0(x,y)]\phi_1\Big(\frac{d(x,\cdot)}{2AC2^{-l}}\Big)\Big)(u), D_l(x,\cdot)(u)\Big\rangle\\
:=&I_{2.1}+I_{2.2}.
\end{aligned}$$

Let $\psi(u)=D_l(x,u)$, and let $\varphi(u)=[D_0(u,y)-D_0(x,y)]\phi_0\big(\frac{d(x,u)}{2AC2^{-l}}\big)$. By the fact $T\in WBP$, then

$$|I_{2.1}| \le CV_{2^{-l}}(x)2^{-2l\eta}[V_{2^{-l}}(x)]^{-1}2^{l\eta}[V_{2^{-l}}(y)]^{-1}2^{-l\epsilon}2^{l\eta}\le C2^{-l\epsilon}[V_{2^{-l}}(y)]^{-1},$$

where $\eta\in(0,\epsilon]$. To estimate $I_{2.2}$, by $\int_X D_l(x,u)d\mu(u)=0$, we have

$$\begin{aligned}
&|I_{2.2}|\\
&\le \int_X\int_{d(x,v)>C2^{-l}}\Big|D_l(x,u)[K(u,v)-K(x,v)][D_0(v,y)-D_0(x,y)]\Big|\,d\mu(u)\,d\mu(v)\\
&\le C\int_X\int_{d(x,v)>C}|D_l(x,u)|\frac{d(x,u)^\epsilon}{d(x,v)^\epsilon}\frac{1}{V(x,v)}\frac{1}{V_1(y)}\,d\mu(u)\,d\mu(v)\\
&+\int_X\int_{C\ge d(x,v)>C2^{-l}}|D_l(x,u)|\frac{d(x,u)^\epsilon}{d(x,v)^\epsilon}\frac{1}{V(x,v)}\frac{1}{V_1(y)}\Big(\frac{d(x,v)}{1+d(v,y)}\Big)^\epsilon\,d\mu(u)\,d\mu(v)\\
&\le C(1+l)2^{-l\epsilon}\frac{1}{V_1(y)}.
\end{aligned}$$

When $d(x,y)\ge 3A^2C$, since $\int_X D_l(x,u)\,d\mu(u)=0$, we obtain

$$\begin{aligned}
|D_lTD_0(x,y)| =&\Big|\int_X\int_X D_l(x,u)[K(u,v)-K(x,v)]D_0(v,y)\,d\mu(u)\,d\mu(v)\Big|\\
\le& C\int_X\int_{d(x,u)\le C2^{-l}}|D_l(x,u)|\frac{d(x,u)^\epsilon}{d(u,v)^\epsilon}\frac{1}{V(u,v)}|D_0(v,y)|\,d\mu(u)\,d\mu(v)\\
\le& C2^{-l\epsilon}\frac{1}{V(x,y)+V_1(x)}\frac{1}{(1+d(x,y))^\epsilon},
\end{aligned}$$

which completes the proof of Proposition 2.2. □

Now we prove the Theorem 1.7.

Proof of Theorem 1.7. By an analogue argument to [2], we obtain that T can be extended to a continuous linear operator from $\widetilde{\mathcal{M}}(\beta,\gamma)$ to $(\mathcal{C}_0^\eta)'$. For $f\in\widetilde{\mathcal{M}}(\beta,\gamma)\cap F_p^{\alpha,q}$, then

$$T(f)=\sum_{k\in\mathbb{Z}_+}TD_k\widetilde{\widetilde{D}}_k(f)$$

in $(\mathcal{C}_0^\eta)'$. When $f \in \widetilde{\mathcal{M}}(\beta,\gamma) \cap F_p^{\alpha,q}$, applying Lemma 2.1, Proposition 2.2, Fefferman-Stein's vector-valued maximal inequality(see, for example, [4]), and notice that $D_k(x,\cdot) \in \mathcal{C}_0^\eta$, we have

$$\begin{aligned}
\|Tf\|_{F_p^{\alpha,q}} \leq & C\Big\|\Big\{\sum_{l=0}^{\infty}\Big(2^{l\alpha}|D_l(Tf)|\Big)^q\Big\}^{1/q}\Big\|_{L^p} \\
\leq & C\Big\|\Big\{\sum_{l=0}^{\infty}\Big(\sum_{k=0}^{\infty}2^{l\alpha}\Big|D_lTD_k\widetilde{\widetilde{D}}_k(f)\Big|\Big)^q\Big\}^{1/q}\Big\|_{L^p} \\
\leq & C\Big\|\Big\{\sum_{l=0}^{\infty}\Big(\sum_{k=0}^{\infty}2^{l\alpha}\Big(2^{(k-l)\epsilon}\wedge 1\Big)M\Big(\widetilde{\widetilde{D}}_k(f)\Big)\Big)^q\Big\}^{1/q}\Big\|_{L^p} \\
\leq & C\|f\|_{F_p^{\alpha,q}},
\end{aligned}$$

where $0 < \alpha < \epsilon$. Since $\widetilde{\mathcal{M}}(\beta,\gamma) \cap F_p^{\alpha,q}$ is dense in $F_p^{\alpha,q}$ with $0 < \alpha < \epsilon, 1 < p, q < \infty$ ([7, Proposision 5.46]). Therefore, when $f \in F_p^{\alpha,q}$ with $0 < \alpha < \epsilon$, $1 < p, q < \infty$, we have

$$\|Tf\|_{F_p^{\alpha,q}} \leq C\|f\|_{F_p^{\alpha,q}}.$$

The proof of the case $f \in B_p^{\alpha,q}$ is similar, and we conclude the proof of Theorem 1.7. □

By an analogous argument to Proposition 2.2, then we have the following proposition.

Proposition 2.3. *Suppose that $-\epsilon < \alpha < 0$, T satisfies the hypotheses of Theorem 1.8 and that D_k $(k \in \mathbb{Z}_+)$ is the same as in the Definition 1.3. Then there exists a constant $C > 0$ such that*

$$|D_lTD_k(x,y)| \leq C\Big(2^{(l-k)\epsilon}\wedge 1\Big)\frac{1+(k-k\wedge l)}{V_{2^{-(k\wedge l)}}(x)+V(x,y)}\frac{2^{-(k\wedge l)\epsilon}}{(2^{-(k\wedge l)}+d(x,y))^{\epsilon}},$$

where $\sigma' = \sigma$ if $k = 0$, otherwise $\sigma' = \epsilon$.

As an immediate result of Lemma 2.1 and Proposition 2.3, we can obtain Theorem 1.8. Here we omit the details.

3. New Characterizations of the Triebel–Lizorkin and Beosv spaces

In this section, we will use the $T1$ theorem to prove Theorems 1.9 and 1.10. We first give some estimates.

Proposition 3.1. (i) *Suppose that E_k is the same as in Theorem 1.9 and that $E_k(x,y)$ is the kernel of E_k for $k \in \mathbb{Z}_+$. Then $E_k(x,y) \in F_p^{\alpha,q}$ and $E_k(x,y) \in B_p^{\alpha,q}$ for any fixed x, $-\theta < \alpha < 0$, and $1 < p, q < \infty$.*

(ii) *Suppose that E_k is the same as in Theorem 1.10 and that $E_k(x,y)$ is the kernel of E_k for $k \in \mathbb{Z}_+$. Then $E_k(x,y) \in F_p^{\alpha,q}$ and $E_k(x,y) \in B_p^{\alpha,q}$ for any fixed x, $0 < \alpha < \theta$, and $1 < p, q < \infty$.*

Proof. For given $x \in X$, we claim that

$$|D_k(E_l(x,\cdot))(y)| \le C\Big(2^{(k-l)\theta} \wedge 1\Big) \frac{1}{V_{2^{-(l\wedge k)}}(x) + V(x,y)} \frac{2^{-(l\wedge k)\theta}}{(2^{-(l\wedge k)} + d(x,y))^\theta}. \tag{3.1}$$

When $l > k \ge 0$, notice that

$$|D_k(E_l(x,\cdot))(y)| = \Big| \int_X [D_k(y,z) - D_k(y,x)] E_l(x,z) d\mu(z) \Big|, \tag{3.2}$$

then the proof of (3.1) is the same as (3.7) of [7, Lemma 3.2]. When $k \ge l \ge 0$, we just use the size conditions of D_k and E_l, and consider $d(x,y) \le 4A2^{-k}$ and $d(x,y) > 4A2^{-k}$, respectively; the estimate in (3.2) follows easily. Here we omit the details.

(i) Given $x \in X$, when $0 < \alpha < \theta$ and $1 < p, q < \infty$, we have

$$\begin{aligned}
&\Big\| \Big\{ \sum_{k\in\mathbb{Z}_+} \Big(2^{k\alpha} |D_k(E_l(x,\cdot))|\Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
\le& C \sum_{k\in\mathbb{Z}_+} 2^{k\alpha} \Big(2^{(k-l)\theta} \wedge 1\Big) \Big(\int_X \Big(\frac{1}{V_{2^{-(l\wedge k)}}(x) + V(x,y)} \frac{2^{-(l\wedge k)\theta}}{(2^{-(l\wedge k)} + d(x,y))^\theta} \Big)^p d\mu(y) \Big)^{1/p} \\
\le& C \sum_{k\in\mathbb{Z}_+} 2^{k\alpha} \Big(2^{(k-l)\theta} \wedge 1\Big) \frac{1}{V_{2^{-(l\wedge k)}}(x)^{1-\frac{1}{p}}} \\
\le& C \sum_{0\le k\le l} 2^{(k-l)(\theta+\alpha)} 2^{l\alpha} \frac{1}{V_{2^{-l}}(x)^{1-\frac{1}{p}}} + \sum_{0<l\le k} 2^{(k-l)\alpha} 2^{l\alpha} \frac{1}{V_{2^{-l}}(x)^{1-\frac{1}{p}}} \\
\le& C_l < \infty,
\end{aligned}$$

where $-\theta < \alpha < 0$. Therefore, we get $E_k(x,y) \in F_p^{\alpha,q}$ for any fixed x. We can similarly conclude the case $E_k(x,y) \in B_p^{\alpha,q}$ for any fixed x.

(ii) Given $x \in X$, when $0 < \alpha < \theta$ and $1 < p, q < \infty$, we only consider the case $E_k(x,y) \in B_p^{\alpha,q}$, and the case $E_k(x,y) \in F_p^{\alpha,q}$ can be handled similarly. For given $x \in X$, by an analogous argument to (3.1), then

$$|D_k(E_l(x,\cdot))(y)| \le C\Big(2^{(l-k)\theta} \wedge 1\Big) \frac{1}{V_{2^{-(l\wedge k)}}(x) + V(x,y)} \frac{2^{-(l\wedge k)\theta}}{(2^{-(l\wedge k)} + d(x,y))^\theta}. \tag{3.3}$$

Using (3.3), we obtain

$$\begin{aligned}
&\Big\{ \sum_{k\in\mathbb{Z}_+} \Big(2^{k\alpha} \Big\|D_k(E_l(\cdot,y))\Big\|_{L^p}\Big)^q \Big\}^{1/q} \\
\le& \sum_{k\in\mathbb{Z}_+} 2^{k\alpha} \Big(2^{(l-k)\theta} \wedge 1\Big) \Big(\int_X \Big(\frac{1}{V_{2^{-(l\wedge k)}}(x) + V(x,y)} \frac{2^{-(l\wedge k)\theta}}{(2^{-(l\wedge k)} + d(x,y))^\theta} \Big)^p d\mu(x) \Big)^{1/p} \\
\le& C_l < \infty,
\end{aligned}$$

where $0 < \alpha < \theta$. The proof of (ii) is finished. $\square$

We now prove the following proposition.

Proposition 3.2. *Let $0 < \alpha < \theta$, and let $1 < p, q < \infty$, and suppose that $E_k (k \in \mathbb{Z}_+)$ is the same as in Theorem 1.9. Then there exists a constant $C > 0$ such that*

$$\Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \leq \|f\|_{F_p^{\alpha,q}} \tag{3.4}$$

and

$$\Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} \|E_k(f)\|_{L^p} \Big)^q \Big\}^{\frac{1}{q}} \leq \|f\|_{B_p^{\alpha,q}}. \tag{3.5}$$

Proof. Suppose that $f \in F_p^{\alpha,q}$ for $0 < \alpha < \theta$ and $1 < p, q < \infty$. By [7, Theorems 7.4 and 8.18] and Proposition 3.1, we can write

$$E_k(f) = E_k \Big(\sum_{l=0}^{\infty} \widetilde{D}_l D_l(f) \Big) = \sum_{l=0}^{\infty} E_k \widetilde{D}_l D_l(f). \tag{3.6}$$

When $0 < \alpha < \theta$; then (3.6), Hölder's inequality, and Fefferman-Stein's vector-valued maximal inequality imply that

$$\begin{aligned}
\Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \leq & \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(\sum_{l=0}^{\infty} 2^{k\alpha} |E_k D_l \widetilde{\widetilde{D}}_l(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
\leq & C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(\sum_{l=0}^{\infty} 2^{k\alpha} \Big(2^{(l-k)\theta} \wedge 1 \Big) M(\widetilde{\widetilde{D}}_l((f)) \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
\leq & C \|f\|_{F_p^{\alpha,q}}.
\end{aligned}$$

We can verify (3.5) similarly. Thus, we finish the proof of Proposition 3.2. □

To finish the proofs of Theorem 1.9 and Theorem 1.10, we need to show the converse inequalities of (3.4) and (3.5). We use Coifman's idea to achieve the goal. Let I be the identity operator, and let E_k be the same as in Theorem 1.9, for $k \in \mathbb{Z}_+$; then $I = \sum_{k=0}^{\infty} E_k$ in L^2. In the norm of L^2, we rewrite

$$\begin{aligned}
I = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} E_l E_k &= \sum_{k=0}^{\infty} \sum_{|l| \leq N} E_{l+k} E_k + \sum_{k=0}^{\infty} \sum_{|l| > N} E_{l+k} E_k \\
&= \sum_{k=0}^{\infty} E_k^N E_k + \sum_{k=0}^{\infty} \sum_{|l| > N} E_{l+k} E_k := T_N + R_N,
\end{aligned}$$

where $E_k^N = \sum_{|l| \leq N} E_{l+k}$ and $E_k = 0$ for $-l \in \mathbb{N}$.

We now show that R_N is bounded on $F_p^{\alpha,q}$ and $B_p^{\alpha,q}$ with an operator norm less than 1. We write $R_N = \sum_{|l-k| > N} E_l E_k$, and consider the sums for $k - l > N$ and $l - k > N$, respectively.

Proposition 3.3. *Let $0 < \alpha < \theta$, and let $1 < p, q < \infty$, and suppose that E_l $(l \in \mathbb{Z}_+)$ is the same as in Theorem 1.9. Then there exists a constant $C > 0$ such that*

$$\Big\| \sum_{k-l>N} E_l E_k(f) \Big\|_{F_p^{\alpha,q}} \le C 2^{-N\alpha} \|f\|_{F_p^{\alpha,q}} \tag{3.7}$$

and

$$\Big\| \sum_{k-l>N} E_l E_k(f) \Big\|_{B_p^{\alpha,q}} \le C 2^{-N\alpha} \|f\|_{B_p^{\alpha,q}}. \tag{3.8}$$

Proof. We only prove (3.7), and the proof of (3.8) is similar. By the definition of $F_p^{\alpha,q}$, (3.1), and Fefferman-Stein's vector-valued maximal inequality, then

$$\begin{aligned}
&\Big\| \sum_{k-l>N} E_l E_k(f) \Big\|_{F_p^{\alpha,q}} \\
&\quad = \Big\| \Big\{ \sum_{j=0}^{\infty} \Big(2^{j\alpha} \Big| D_j \Big(\sum_{k-l>N} E_l E_k(f) \Big) \Big| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
&\quad \le C \Big\| \Big\{ \sum_{j=0}^{\infty} \Big(\sum_{k-l>N} 2^{j\alpha} \Big(2^{(l-j)\theta} \wedge 1 \Big) M(E_k(f)) \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
&\quad \le C \Big\| \Big\{ \sum_{j=0}^{\infty} \Big(\sum_{k-l>N} \Big(2^{(l-j)\theta} \wedge 1 \Big) 2^{(j-l)\alpha} 2^{(l-k)\alpha} 2^{k\alpha} M(E_k(f)) \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
&\quad \le C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} M(E_k(f)) \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
&\quad \le C 2^{-N\alpha} \|f\|_{F_p^{\alpha,q}},
\end{aligned}$$

which is a desired result. □

Now we need to prove the following proposition.

Proposition 3.4. *Let $0 < \alpha < \theta' < \theta$, and let $1 < p, q < \infty$, and assume that E_l $(l \in \mathbb{Z}_+)$ is the same as in Theorem 1.9. Then there exist constants $C > 0$ and $\delta > 0$ such that*

$$\Big\| \sum_{l-k>N} E_l E_k(f) \Big\|_{F_p^{\alpha,q}} \le C 2^{-N\delta} \|f\|_{F_p^{\alpha,q}}$$

and

$$\Big\| \sum_{l-k>N} E_l E_k(f) \Big\|_{B_p^{\alpha,q}} \le C 2^{-N\delta} \|f\|_{B_p^{\alpha,q}}.$$

Proof. Let $\widetilde{R}_N(x,y)$ be the kernel of $\widetilde{R}_N := \sum_{l-k>N} E_l E_k(f)$. We claim that there exist $C > 0$ and $\delta, \sigma > 0$ such that $\widetilde{R}_N(x,y)$ satisfies $\widetilde{R}_N(1) = 0$, and

$$|\widetilde{R}_N(x,y)| \le C 2^{-N\delta} \frac{1}{V(x,y)}; \tag{3.9}$$

$$|\widetilde{R}_N(x,y)| \leq C2^{-N\delta}\frac{1}{d(x,y)^{\sigma}}\frac{1}{V(x,y)} \tag{3.10}$$

for $d(x,y) > 1$;

$$|\widetilde{R}_N(x,y) - \widetilde{R}_N(x',y)| \leq C2^{-N\delta}\frac{d(x,x')^{\epsilon}}{d(x,y)^{\epsilon}}\frac{1}{V(x,y)} \tag{3.11}$$

for $d(x,x') \leq \frac{1}{2A}d(x,y)$;

$$|\langle \widetilde{R}_N\phi, \psi\rangle| \leq C2^{-N\delta}V_r(x_0) \tag{3.12}$$

for all $\phi, \psi \in \mathcal{C}_0^{\eta}$ with supp ϕ, supp $\psi \in B(x_0, r)$, $\|\phi\|_\infty \leq 1$, $\|\psi\|_\infty \leq 1$, $\|\phi\|_{\mathcal{C}_0^{\eta}} \leq r^{-\eta}$, and $\|\psi\|_{\mathcal{C}_0^{\eta}} \leq r^{-\eta}$.

We rewrite $\widetilde{R}_N = \sum_{l-k>N} E_l E_k = \sum_{l>N}\sum_{k\in\mathbb{Z}_+} E_{k+l}E_k$ and $\widetilde{\widetilde{R}}_N := \sum_{k\in\mathbb{Z}_+} E_{k+l}E_k$. Obviously, $\widetilde{R}_N(1) = 0$. Applying [7, Lemma 3.21], by an analogous arguments to [11, Proposition 3.6], we can get (3.9), (3.11), and (3.12). Now we verify (3.10). When $d(x,y) > 1$, we have

$$\begin{aligned}
\left|\widetilde{\widetilde{R}}_N(x,y)\right| \leq & C\sum_{k\in\mathbb{Z}_+} 2^{-l\theta}\frac{1}{V_{2^{-k}}(x) + V(x,y)}\frac{2^{-k\epsilon}}{(2^{-k} + d(x,y))^{\theta}} \\
\leq & C\sum_{k\in\mathbb{Z}_+} 2^{-l\theta}\frac{1}{V(x,y)}\frac{2^{-k}}{d(x,y)^{\theta}} \\
\leq & C2^{-l\theta}\frac{1}{V(x,y)}\frac{1}{d(x,y)^{\theta}};
\end{aligned} \tag{3.13}$$

then $|\widetilde{R}_N(x,y)| \leq C2^{-N\theta}\frac{1}{V(x,y)}\frac{1}{d(x,y)^{\theta}}$. Thus, when $d(x,y) > 1$, from (3.9) and (3.13), it is easy to get

$$|\widetilde{\widetilde{R}}_N(x,y)| \leq C2^{-N\delta}\frac{1}{V(x,y)}\frac{1}{d(x,y)^{\sigma}}$$

for some $\delta, \sigma > 0$.

By Theorem 1.7, for all $f \in F_p^{\alpha,q}$, then

$$\Big\|\sum_{l-k>N} E_l E_k(f)\Big\|_{F_p^{\alpha,q}} \leq C2^{-N\delta}\|f\|_{F_p^{\alpha,q}}.$$

For all $f \in B_p^{\alpha,q}$, we also have

$$\Big\|\sum_{l-k>N} E_l E_k(f)\Big\|_{B_p^{\alpha,q}} \leq C2^{-N\delta}\|f\|_{B_p^{\alpha,q}}.$$

We finish the proof of Proposition 3.4. □

By the fact that $T_N^{-1} = (I - R_N)^{-1} = \sum_{m=0}^{\infty} R_N^m$ and Proposition 3.4, we can obtain that T_N^{-1} exists and is bounded on $F_p^{\alpha,q}$ and $B_p^{\alpha,q}$ for large integer N.

Similar to the proof of (3.1), given a large integer N, let E_k and E_l^N be as in Definition 1.3 and (3.7), for $k, l \in \mathbb{Z}_+$, respectively, then we have

$$|D_k E_l^N(x,y)| \leq C_N \Big(2^{(l-k)\theta} \wedge 1\Big) \frac{1}{V_{2^{-(k\wedge l)}}(x) + V(x,y)} \frac{2^{-(k\wedge l)^\theta}}{(2^{-(k\wedge l)} + d(x,y))^\theta}, \quad (3.14)$$

where the constant C_N depends only on N.

Using (3.14), by analogous arguments to (3.5) and (3.6), for $f \in L^2 \cap F_p^{\alpha,q}$, we have

$$\begin{aligned}
\|T_N(f)\|_{F_p^{\alpha,q}} \leq & \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(\sum_{l=0}^{\infty} 2^{k\alpha} |D_k E_l^N E_l(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
\leq & \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(\sum_{l=0}^{\infty} 2^{k\alpha} \Big(2^{(l-k)\theta} \wedge 1\Big) M(E_l((f))) \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} \\
\leq & \Big\| \Big\{ \sum_{l=0}^{\infty} \Big(2^{l\alpha} |E_l(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p},
\end{aligned}$$

and then

$$\|f\|_{F_p^{\alpha,q}} = \|T_N^{-1} T_N(f)\|_{F_p^{\alpha,q}} \leq C \|T_N(f)\|_{F_p^{\alpha,q}} \leq \Big\| \Big\{ \sum_{l=0}^{\infty} 2^{l\alpha} |E_l(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p}.$$

Since $L^2 \cap F_p^{\alpha,q}$ is dense in $F_p^{\alpha,q}$, and let $f \in F_p^{\alpha,q}$ with

$$\Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f)| \Big)^q \Big\}^{\frac{1}{q}} \Big\|_{L^p} < \infty;$$

we can choose a sequence $\{f_n\}_{n=1}^\infty$ with $f_n \in L^2 \cap F_p^{\alpha,q}$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{F_p^{\alpha,q}} = 0.$$

Thus, using the above fact and (3.4), then

$$\begin{aligned}
\|f\|_{F_p^{\alpha,q}} = \lim_{n\to\infty} \|f_n\|_{F_p^{\alpha,q}} \leq C \lim_{n\to\infty} & \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f_n)| \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
\leq & C \lim_{n\to\infty} \|f_n - f\|_{F_p^{\alpha,q}} + C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f)| \Big)^q \Big\}^{1/q} \Big\|_{L^p} \\
= & C \Big\| \Big\{ \sum_{k=0}^{\infty} \Big(2^{k\alpha} |E_k(f)| \Big)^q \Big\}^{1/q} \Big\|_{L^p},
\end{aligned}$$

which finishes the proof of the converse inequality of (3.4). The case $f \in B_p^{\alpha,q}$ can be dealt similarly. We conclude the proof of Theorem 1.9.

Applying Proposition 3.1, the proof of Theorem 1.10 is similar to Theorem 1.9 with necessary modifications. We leave the details to the interested reader.

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