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POMPEIU–ČEBYŠEV TYPE INEQUALITIES FOR SELFADJOINT OPERATORS IN HILBERT SPACES

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ABSTRACT. In this work, generalizations of some inequalities for continuous h-synchronous (h-asynchronous) functions of selfadjoint linear operators in Hilbert spaces are proved.

1. INTRODUCTION

Let $\mathcal{B}(H)$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(H; \langle\cdot,\cdot\rangle)$ with the identity operator 1_H in $\mathcal{B}(H)$. Let $A \in \mathcal{B}(H)$ be a selfadjoint linear operator on $(H; \langle\cdot,\cdot\rangle)$. Let $C(\mathrm{sp}(A))$ be the set of all continuous functions defined on the spectrum of A ($\mathrm{sp}(A)$) and let $C^*(A)$ be the C^*-algebra generated by A and the identity operator 1_H.

Let us define the map $\mathcal{G} : C(\mathrm{sp}(A)) \to C^*(A)$ with the following properties ([4], p.3):

(1) $\mathcal{G}(\alpha f + \beta g) = \alpha\mathcal{G}(f) + \beta\mathcal{G}(g)$, for all scalars α, β.
(2) $\mathcal{G}(fg) = \mathcal{G}(f)\mathcal{G}(g)$ and $\mathcal{G}\left(\overline{f}\right) = \mathcal{G}(f)^*$; where $\overline{f}$ denotes to the conjugate of f and $\mathcal{G}(f)^*$ denotes to the Hermitian of $\mathcal{G}(f)$.
(3) $\|\mathcal{G}(f)\| = \|f\| = \sup\limits_{t\in\mathrm{sp}(A)} |f(t)|$.
(4) $\mathcal{G}(f_0) = 1_H$ and $\mathcal{G}(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for all $t \in \mathrm{sp}(A)$.

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Accordingly, we define the continuous functional calculus for a selfadjoint operator A by

$$f(A) = \mathcal{G}(f) \text{ for all } f \in C(\mathrm{sp}(A)).$$

If both f and g are real valued functions on $\mathrm{sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for all } t \in \mathrm{sp}(A) \text{ implies } f(A) \geq g(A), \tag{1.1}$$

in the operator order of $\mathcal{B}(H)$.

In [2], Dragomir studied the Čebyšev functional

$$C(f, g; A, x) := \langle f(A) g(A) x, x\rangle - \langle g(A) x, x\rangle \langle f(A) x, x\rangle, \tag{1.2}$$

for any selfadjoint operator $A \in \mathcal{B}(H)$ and $x \in H$ with $\|x\| = 1$.

To study the positivity of (1.2), Dragomir [2] introduced the following two results concerning continuous synchronous (asynchronous) functions of selfadjoint linear operators in Hilbert spaces.

Theorem 1.1. *Let A be a selfadjoint operator with $\mathrm{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and synchronous (asynchronous) on $[\gamma, \Gamma]$, then*

$$\langle f(A) g(A) x, x\rangle \geq (\leq) \langle g(A) x, x\rangle \langle f(A) x, x\rangle$$

for any $x \in H$ with $\|x\| = 1$.

Theorem 1.2. *Let A be a selfadjoint operator with $\mathrm{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$.*

(1) *If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and synchronous on $[\gamma, \Gamma]$, then*

$$\begin{aligned}\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle &\\ \geq [\langle f(A) x, x\rangle - f(\langle Ax, x\rangle)]&\\ \times [g(\langle Ax, x\rangle) - \langle g(A) x, x\rangle]&\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

(2) *If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and asynchronous on $[\gamma, \Gamma]$, then*

$$\begin{aligned}\langle f(A) x, x\rangle \cdot \langle g(A) x, x\rangle - \langle f(A) g(A) x, x\rangle &\\ \geq [\langle f(A) x, x\rangle - f(\langle Ax, x\rangle)]&\\ \times [\langle g(A) x, x\rangle - g(\langle Ax, x\rangle)]&\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

For more related results, we refer the reader to [3], [5] and [6].

Let $a, b \in \mathbb{R}$, $a < b$. Let $f, g, h : [a, b] \to \mathbb{R}$ be three integrable functions, the Pompeiu–Čebyšev functional was introduced in [1] such as:

$$\widehat{\mathcal{P}}_h(f, g) = \int_a^b h^2(t)\, dt \int_a^b f(t) g(t)\, dt - \int_a^b f(t) h(t)\, dt \int_a^b h(t) g(t)\, dt. \tag{1.3}$$

If we consider $h(x) = 1$, then

$$\widehat{\mathcal{P}}_1(f,g) = (b-a)\int_a^b f(t)\,g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt = (b-a)^2\,\mathcal{T}(f,g),$$

which is the celebrated Čebyšev functional.

The corresponding version of Pompeiu–Čebyšev functional (1.3) for continuous functions of selfadjoint linear operators in Hilbert spaces can be formulated such as:

$$\begin{aligned}\mathcal{P}(f,g,h;A,x) := &\left\langle h^2(A)\,x,x\right\rangle \langle f(A)\,g(A)\,x,x\rangle \\ &- \langle h(A)\,g(A)\,x,x\rangle \langle h(A)\,f(A)\,x,x\rangle \end{aligned} \tag{1.4}$$

for $x \in H$ with $\|x\| = 1$. This naturally, generalizes the Čebyšev functional (1.2).

In this work, we introduce the h-synchronous (h-asynchronous) where $h : [\gamma,\Gamma] \to \mathbb{R}_+$ is a nonnegative function defined on $[\gamma,\Gamma]$ for some real numbers $\gamma < \Gamma$. Accordingly, some inequalities for continuous h-synchronous (h-asynchronous) functions of selfadjoint linear operators in Hilbert spaces of the Pompeiu–Čebyšev functional (1.4) are proved. The proof Techniques are similar to that ones used in [3].

2. Main results

In [1], the author of this paper generalized the concept of monotonicity as follows:

Definition 2.1. A real valued function f defined on $[a,b]$ is said to be increasing (decreasing) with respect to a positive function $h : [a,b] \to \mathbb{R}_+$ or simply h-increasing (h-decreasing) if and only if

$$h(x)\,f(t) - h(t)\,f(x) \geq (\leq)\ 0,$$

whenever $t \geq x$ for every $x,t \in [a,b]$. In special case if $h(x) = 1$ we refer to the original monotonicity. Accordingly, for $0 < a < b$ we say that f is t^r-increasing (t^r-decreasing) for $r \in \mathbb{R}$ if and only if

$$x \leq t \Longrightarrow x^r f(t) - t^r f(x) \geq (\leq)\ 0$$

for every $x,t \in [a,b]$.

Example 2.2. Let $0 < a < b$ and define $f : [a,b] \to \mathbb{R}$ given by

(1) $f(s) = 1$, then f is t^r-decreasing for all $r > 0$ and t^r-increasing for all $r < 0$.
(2) $f(s) = s$, then f is t^r-decreasing for all $r > 1$ and t^r-increasing for all $r < 1$.
(3) $f(s) = s^{-1}$, then f is t^r-decreasing for all $r > -1$ and t^r-increasing for all $r < -1$.

Lemma 2.3. *Every h-increasing function is increasing. The converse need not be true.*

Proof. If $h=0$ nothing to prove. For $h\neq 0$, if f is h-increasing on $[a,b]$, then

$$x\le t\Longrightarrow 0\le h\left(x\right)f\left(t\right)-h\left(t\right)f\left(x\right)\le h\left(t\right)\left(f\left(t\right)-f\left(x\right)\right)\Longrightarrow f\left(x\right)\le f\left(t\right),$$

which means that f increases on $[a,b]$. □

There exists h-increasing (h-decreasing) function which is not increasing (decreasing). For example, consider the function $f:(0,1)\to\mathbb{R}$, given by $f(s)=s(1-s)$, $0<s<1$. Clearly, $f(s)$ is increasing on $(0,1/2)$ and decreasing on $(1/2,1)$. While if $1>t\ge x>0$, then

$$xt(1-t)-tx(1-x)=xt(x-t)\le 0,$$

i.e., f is t-decreasing on $(0,1)$. As a special case of Lemma 2.3, for $a,b\in\mathbb{R}$, $0<a<b$ and a positive function $h:[a,b]\to\mathbb{R}_+$, if $f:[a,b]\to\mathbb{R}$ is t^r-increasing for $r>0$ (t^r-decreasing for $r<0$), then f is increasing (decreasing) on $[a,b]$.

The concept of synchronization has a wide range of usage in several areas of mathematics. Simply, two functions $f,g:[a,b]\to\mathbb{R}$ are called synchronous (asynchronous) if and only if the inequality

$$\left(f\left(t\right)-f\left(x\right)\right)\left(g\left(t\right)-g\left(x\right)\right)\ge(\le)\ 0,$$

holds for all $x,t\in[a,b]$.

Next, we define the concept of h-synchronous (h-asynchronous) functions.

Definition 2.4. The real valued functions $f,g:[a,b]\to\mathbb{R}$ are called synchronous (asynchronous) with respect to a non-negative function $h:[a,b]\to\mathbb{R}_+$ or simply h-synchronous (h-asynchronous) if and only if

$$\left(h\left(y\right)f\left(x\right)-h\left(x\right)f\left(y\right)\right)\left(h\left(y\right)g\left(x\right)-h\left(x\right)g\left(y\right)\right)\ge(\le)\ 0$$

for all $x,y\in[a,b]$.

In other words if both f and g are either h-increasing or h-decreasing then

$$\left(h\left(y\right)f\left(x\right)-h\left(x\right)f\left(y\right)\right)\left(h\left(y\right)g\left(x\right)-h\left(x\right)g\left(y\right)\right)\ge 0.$$

While, if one of the function is h-increasing and the other is h-decreasing then

$$\left(h\left(y\right)f\left(x\right)-h\left(x\right)f\left(y\right)\right)\left(h\left(y\right)g\left(x\right)-h\left(x\right)g\left(y\right)\right)\le 0.$$

In special case if $h(x)=1$ we refer to the original synchronization. Accordingly, for $0<a<b$ we say that f and g are t^r-synchronous (t^r-asynchronous) for $r\in\mathbb{R}$ if and only if

$$\left(x^r f\left(t\right)-t^r f\left(x\right)\right)\left(x^r g\left(t\right)-t^r g\left(x\right)\right)\ge(\le)\ 0$$

for every $x,t\in[a,b]$.

Remark 2.5. In Definition (2.4), if $f=g$ then f and g are always h-synchronous regardless of h-monotonicity of f (or g). In other words, a function f is always h-synchronous with itself.

Example 2.6. Let $0<a<b$ and define $f,g:[a,b]\to\mathbb{R}$ given by

(1) $f(s)=1=g(s)$, then f and g are t^r-synchronous for all $r\in\mathbb{R}$.
(2) $f(s)=1$ and $g(s)=s$, then f is t^r-synchronous for all $r\in(-\infty,0)\cup(1,\infty)$ and t^r-asynchronous for all $0<r<1$.

(3) $f(s) = 1$ and $g(s) = s^{-1}$, then f is t^r-synchronous for all $r \in (-\infty, -1) \cup (0, \infty)$ and t^r-asynchronous for all $-1 < r < 0$.
(4) $f(s) = s$ and $g(s) = s^{-1}$, then f is t^r-synchronous for all $r \in (-\infty, -1) \cup (1, \infty)$ and t^r-asynchronous for all $-1 < r < 1$.

Let us start with the following result regarding the positivity of $\mathcal{P}(f, g, h; A, x)$.

Theorem 2.7. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}_+$ be a non-negative and continuous function. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then*

$$\left\langle h^2(A)x, x\right\rangle \langle f(A)g(A)x, x\rangle \geq (\leq) \langle h(A)g(A)x, x\rangle \langle h(A)f(A)x, x\rangle \tag{2.1}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f and g are h-synchronous then

$$(h(s)f(t) - h(t)f(s))(h(s)g(t) - h(t)g(s)) \geq 0,$$

and this is allow us to write

$$h^2(s)f(t)g(t) + h^2(t)f(s)g(s) \geq h(s)h(t)f(t)g(s) + h(s)h(t)g(t)f(s) \tag{2.2}$$

for all $t, s \in [a, b]$. We fix $s \in [a, b]$ and apply property (1.1) for inequality (2.2), then we have for each $x \in H$ with $\|x\| = 1$, that

$$\begin{aligned}&\left\langle \left(h^2(s)f(A)g(A) + h^2(A)f(s)g(s)\right)x, x\right\rangle \\ &\qquad \geq \langle (h(A)f(A)h(s)g(s) + h(A)g(A)h(s)f(s))x, x\rangle,\end{aligned}$$

and this equivalent to write

$$\begin{aligned}&h^2(s)\langle f(A)g(A)x, x\rangle + f(s)g(s)\left\langle h^2(A)x, x\right\rangle \\ &\qquad \geq h(s)g(s)\langle h(A)f(A)x, x\rangle + h(s)f(s)\langle h(A)g(A)x, x\rangle.\end{aligned} \tag{2.3}$$

Applying property (1.1) again for inequality (2.3), then we have for each $y \in H$ with $\|y\| = 1$, that

$$\begin{aligned}&\left\langle \left(h^2(A)\langle f(A)g(A)x, x\rangle + f(A)g(A)\left\langle h^2(A)x, x\right\rangle\right)y, y\right\rangle \\ &\qquad \geq \langle (h(A)g(A)\langle h(A)f(A)x, x\rangle + h(A)f(A)\langle h(A)g(A)x, x\rangle)y, y\rangle,\end{aligned}$$

which gives

$$\begin{aligned}&\left\langle h^2(A)y, y\right\rangle \langle f(A)g(A)x, x\rangle + \left\langle h^2(A)x, x\right\rangle \langle f(A)g(A)y, y\rangle \\ &\geq \langle h(A)g(A)y, y\rangle \langle h(A)f(A)x, x\rangle + \langle h(A)g(A)x, x\rangle \langle h(A)f(A)y, y\rangle\end{aligned} \tag{2.4}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which gives more than we need, so that by setting $y = x$ in (2.4) we get the '$\geq$' case in (2.1). The revers case follows trivially, and this completes the proof. $\square$

Corollary 2.8. *Let A be a selfadjoint operator with* $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ *for some real numbers* γ, Γ *with* $\gamma < \Gamma$. *Let* $h : [\gamma, \Gamma] \to \mathbb{R}_+$ *be a non-negative and continuous function. If* $f : [\gamma, \Gamma] \to \mathbb{R}$ *is continuous and* h*-synchronous on* $[\gamma, \Gamma]$, *then*

$$\langle h(A) f(A) x, x\rangle^2 \le \left\langle h^2(A) x, x\right\rangle \left\langle f^2(A) x, x\right\rangle \tag{2.5}$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. Setting $f = g$ in (2.1) we get the desired result. □

Remark 2.9. It is easy to check that the function $f(s) = s^{1/2}$ is $t^{-1/2}$-synchronous for all $s, t > 0$. Applying (2.5) the for $0 < \gamma < \Gamma$ we get that

$$1 \le \left\langle A^{-1}x, x\right\rangle \langle Ax, x\rangle .$$

Also, since $\gamma \cdot 1_H \le A \le \Gamma \cdot 1_H$, then the Kanotrovich inequality reads

$$\left\langle A^{-1}x, x\right\rangle \langle Ax, x\rangle \le \frac{(\gamma + \Gamma)^2}{4\gamma\Gamma},$$

combining the above two inequalities we get

$$1 \le \left\langle A^{-1}x, x\right\rangle \langle Ax, x\rangle \le \frac{(\gamma + \Gamma)^2}{4\gamma\Gamma}.$$

Corollary 2.10. *Let A be a selfadjoint operator with* $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ *for some real numbers* γ, Γ *with* $0 < \gamma < \Gamma$. *If* $f, g : [\gamma, \Gamma] \to \mathbb{R}$ *are continuous and* t*-synchronous* (t*-asynchronous*) *on* $[\gamma, \Gamma]$, *then*

$$\left\langle A^2 x, x\right\rangle \langle f(A) g(A) x, x\rangle \ge (\le) \langle A g(A) x, x\rangle \langle A f(A) x, x\rangle$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. Setting $h(t) = t$ in (2.1) we get the desired result. □

Before we state our next remark, we interested to give the following example.

Example 2.11.
(1) If $f(s) = s^p$ and $g(s) = s^q$ $(s > 0)$, then f and g are t^r-synchronous for all $p, q > r > 0$ and t^r-asynchronous for all $p > r > q > 0$.
(2) If $f(s) = s^p$ and $g(s) = \log(s)$ $(s > 1)$, then f is t^r-synchronous for all $p < r < 0$ and t^r-asynchronous for all $r < p < 0$.
(3) If $f(s) = \exp(s) = g(s)$, then f is t^r-synchronous for all for all $r \in \mathbb{R}$.

Remark 2.12. From the proof of the above theorem we observe, that, if A and B are selfadjoint operators such that $\operatorname{sp}(A), \operatorname{sp}(B) \in [\gamma, \Gamma]$; and $h : [\gamma, \Gamma] \to \mathbb{R}_+$ is non-negative continuous, then for any continuous functions $f, g : [\gamma, \Gamma] \to \mathbb{R}$ which are both h-synchronous (h-asynchronous)

$$\begin{aligned} \left\langle h^2(B) y, y\right\rangle \langle f(A) g(A) x, x\rangle + \left\langle h^2(A) x, x\right\rangle \langle f(B) g(B) y, y\rangle \\ \ge (\le) \langle h(B) g(B) y, y\rangle \langle h(A) f(A) x, x\rangle \\ + \langle h(A) g(A) x, x\rangle \langle h(B) f(B) y, y\rangle \end{aligned} \tag{2.6}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. Using Example 2.11 we can observe the following special cases:

(1) If $f(s)=s^{p}$ and $g(s)=s^{q}$ $(s>0)$, then f and g are t^{r}-synchronous for all $p,q>r>0$, so that we have

$$\left\langle B^{2r}y,y\right\rangle\left\langle A^{p+q}x,x\right\rangle+\left\langle A^{2r}x,x\right\rangle\left\langle B^{p+q}y,y\right\rangle \geq\left\langle B^{q+r}y,y\right\rangle\left\langle A^{p+r}x,x\right\rangle+\left\langle A^{q+r}x,x\right\rangle\left\langle B^{p+r}y,y\right\rangle .$$

If $p>r>q>0$, then f and g are t^{r}-asynchronous and thus the reverse inequality holds.

(2) If $f(s)=s^{p}$ and $g(s)=\log s$ $(s>1)$, then f and g are t^{r}-synchronous for all $p<r<0$ we have

$$\left\langle B^{2r}y,y\right\rangle\left\langle A^{p}\log(A)x,x\right\rangle+\left\langle A^{2r}x,x\right\rangle\left\langle B^{p}\log(B)y,y\right\rangle \geq\left\langle B^{r}\log(B)y,y\right\rangle\left\langle A^{p+r}x,x\right\rangle+\left\langle A\log(A)x,x\right\rangle\left\langle B^{p+r}y,y\right\rangle .$$

If $r<p<0$, then f and g are t^{r}-asynchronous and thus the reverse inequality holds.

(3) If $f(s)=\exp(s)=g(s)$, then f and g are t^{r}-synchronous for all $r\in\mathbb{R}$, so that we have

$$\left\langle B^{2r}y,y\right\rangle\left\langle\exp(2A)x,x\right\rangle+\left\langle A^{2r}x,x\right\rangle\left\langle\exp(2B)y,y\right\rangle \geq 2\left\langle A^{r}\exp(A)x,x\right\rangle\left\langle B^{r}\exp(B)y,y\right\rangle .$$

Therefore, by choosing an appropriate function h such that the assumptions in Remark 2.12 are fulfilled then one may generate family of inequalities from (2.6).

Corollary 2.13. *Let A be a selfadjoint operator with $\operatorname{sp}(A)\subset[\gamma,\Gamma]$ for some real numbers γ,Γ with $0<\gamma<\Gamma$. If $f:[\gamma,\Gamma]\to\mathbb{R}$ is continuous and f is t-synchronous on $[\gamma,\Gamma]$, then*

$$\left\langle A^{2}x,x\right\rangle\left\langle f(A)x,x\right\rangle\geq\left\langle Ax,x\right\rangle\left\langle Af(A)x,x\right\rangle$$

for each $x\in H$ with $\|x\|=1$. In particular, if $f(s)=s^{p}$ $(p>1)$ for all $s\in[\gamma,\Gamma]$, then

$$\left\langle A^{2}x,x\right\rangle\left\langle A^{p}x,x\right\rangle\geq\left\langle Ax,x\right\rangle\left\langle A^{p+1}x,x\right\rangle .$$

Proof. Setting $f=g$ in Corollary 2.10 we get the desired result. □

Corollary 2.14. *Let A be a selfadjoint operator with $\operatorname{sp}(A)\subset[\gamma,\Gamma]$ for some real numbers γ,Γ with $\gamma<\Gamma$. Let $h:[\gamma,\Gamma]\to\mathbb{R}$ be a non-negative continuous. If $f:[\gamma,\Gamma]\to\mathbb{R}$ is continuous and h-synchronous, then*

$$\left\langle h^{2}(A)x,x\right\rangle\left\langle f(A)x,x\right\rangle\geq\left\langle h(A)x,x\right\rangle\left\langle h(A)f(A)x,x\right\rangle \tag{2.7}$$

for each $x\in H$ with $\|x\|=1$. In particular, if $f(s)=s^{p}$ is h-synchronous for all $s\in[\gamma,\Gamma]$, then we have

$$\left\langle h^{2}(A)x,x\right\rangle\left\langle A^{p}x,x\right\rangle\geq\left\langle h(A)x,x\right\rangle\left\langle h(A)A^{p}x,x\right\rangle .$$

Remark 2.15. Setting $f(s) = s^{-1}$, $\forall s \in [\gamma, \Gamma]$ in (2.7) (in this case we assume $0 < \gamma < \Gamma$) then for each $x \in H$ with $\|x\| = 1$, we have

$$\left\langle h^2(A)x, x\right\rangle \geq \left\langle h(A)A^{-1}x, x\right\rangle \langle Ah(A)x, x\rangle,$$

provided that s^{-1} is h-synchronous on $[\gamma, \Gamma]$.

Theorem 2.16. *Let A be a selfadjoint operator with* $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ *for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then*

$$\begin{aligned} & h^2(\langle Ax, x\rangle)\langle f(A)g(A)x, x\rangle - \langle h(A)f(A)x, x\rangle \cdot \langle h(A)g(A)x, x\rangle \\ & \quad \geq (\leq) \left[h(\langle Ax, x\rangle)\langle h(A)f(A)x, x\rangle - \left\langle h^2(A)x, x\right\rangle f(\langle Ax, x\rangle)\right] \cdot g(\langle Ax, x\rangle) \\ & \qquad + \left[h(\langle Ax, x\rangle) f(\langle Ax, x\rangle) - \langle h(A)f(A)x, x\rangle\right] \cdot \langle h(A)g(A)x, x\rangle \end{aligned} \tag{2.8}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f, g are synchronous and $\gamma \leq \langle Ax, x\rangle \leq \Gamma$ for any $x \in H$ with $\|x\| = 1$, we have

$$\begin{aligned} & (h(\langle Ax, x\rangle) f(t) - h(t) f(\langle Ax, x\rangle)) \\ & \qquad \times (h(\langle Ax, x\rangle) g(t) - h(t) g(\langle Ax, x\rangle)) \geq 0 \end{aligned} \tag{2.9}$$

for any $t \in [a, b]$ for any $x \in H$ with $\|x\| = 1$.

Employing property (1.1) for inequality (2.9) we have

$$\begin{aligned} & \langle [h(\langle Ax, x\rangle) f(B) - h(B) f(\langle Ax, x\rangle)] \\ & \qquad \times [h(\langle Ax, x\rangle) g(B) - h(B) g(\langle Ax, x\rangle)] y, y\rangle \geq 0 \end{aligned} \tag{2.10}$$

for any bounded linear operator B with $\operatorname{sp}(B) \subseteq [\gamma, \Gamma]$ and $y \in H$ with $\|y\| = 1$.

Now, since

$$\begin{aligned} & \langle [h(\langle Ax, x\rangle) f(B) - h(B) f(\langle Ax, x\rangle)] \\ & \qquad \times [h(\langle Ax, x\rangle) g(B) - h(B) g(\langle Ax, x\rangle)] y, y\rangle \\ & = h^2(\langle Ax, x\rangle)\langle f(B)g(B)y, y\rangle - h(\langle Ax, x\rangle) f(\langle Ax, x\rangle)\langle h(B)g(B)y, y\rangle \\ & \qquad - h(\langle Ax, x\rangle) g(\langle Ax, x\rangle)\langle h(B)f(B)y, y\rangle \\ & \qquad + \left\langle h^2(B)y, y\right\rangle f(\langle Ax, x\rangle) g(\langle Ax, x\rangle), \end{aligned}$$

then from (2.10) we get

$$\begin{aligned} & h^2(\langle Ax, x\rangle)\langle f(B)g(B)y, y\rangle + \left\langle h^2(B)y, y\right\rangle f(\langle Ax, x\rangle) g(\langle Ax, x\rangle) \\ & \qquad \geq h(\langle Ax, x\rangle) f(\langle Ax, x\rangle)\langle h(B)g(B)y, y\rangle \\ & \qquad + h(\langle Ax, x\rangle) g(\langle Ax, x\rangle)\langle h(B)f(B)y, y\rangle, \end{aligned}$$

and this is equivalent to write

$$\begin{aligned} & h^2(\langle Ax, x\rangle)\langle f(B)g(B)y, y\rangle - \langle h(A)f(A)x, x\rangle \cdot \langle h(A)g(A)x, x\rangle \\ & \geq g(\langle Ax, x\rangle)\left[h(\langle Ax, x\rangle)\langle h(B)f(B)y, y\rangle - \left\langle h^2(B)y, y\right\rangle f(\langle Ax, x\rangle)\right] \\ & \quad + h(\langle Ax, x\rangle) f(\langle Ax, x\rangle)\langle h(B)g(B)y, y\rangle - \langle h(A)f(A)x, x\rangle \cdot \langle h(A)g(A)x, x\rangle \end{aligned} \tag{2.11}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. Setting $B = A$ and $y = x$ in (2.11) we get the required result in (2.8). The reverse sense follows similarly. $\square$

Remark 2.17. Let $0 < \gamma < \Gamma$ and choose $f(s) = s$ and $g(s) = s^{-1}$, $s > 0$ in Theorem 2.16. So that, if f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then

$$\begin{aligned} &h^2(\langle Ax, x\rangle) - \langle Ah(A)x, x\rangle \cdot \left\langle A^{-1}h(A)x, x\right\rangle \\ &\quad \geq (\leq) \left[h(\langle Ax, x\rangle)\langle Ah(A)x, x\rangle - \left\langle h^2(A)x, x\right\rangle \langle Ax, x\rangle\right] \cdot \langle Ax, x\rangle^{-1} \\ &\qquad + \left[h(\langle Ax, x\rangle)\langle Ax, x\rangle - \langle Ah(A)x, x\rangle\right] \cdot \left\langle A^{-1}h(A)x, x\right\rangle \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$. In special case, if $h(t) = 1$ for all $t \in [\gamma, \Gamma]$, then s and s^{-1} are asynchronous so that we have

$$1 \leq \langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle$$

Corollary 2.18. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f : [\gamma, \Gamma] \to \mathbb{R}$ is continuous and h-synchronous on $[\gamma, \Gamma]$, then*

$$\begin{aligned} &h^2(\langle Ax, x\rangle)\left\langle f^2(A)x, x\right\rangle - \langle h(A)f(A)x, x\rangle^2 \\ &\quad \geq \left[h(\langle Ax, x\rangle)\langle h(A)f(A)x, x\rangle - \left\langle h^2(A)x, x\right\rangle f(\langle Ax, x\rangle)\right] \cdot f(\langle Ax, x\rangle) \\ &\qquad + \left[h(\langle Ax, x\rangle)f(\langle Ax, x\rangle) - \langle h(A)f(A)x, x\rangle\right] \cdot \langle h(A)f(A)x, x\rangle \end{aligned} \tag{2.12}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Setting $f = g$ in (2.8), respectively, we get the required results. $\square$

Corollary 2.19. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $0 < \gamma < \Gamma$. If $f : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and t-synchronous on $[\gamma, \Gamma]$, then*

$$\begin{aligned} &\langle Ax, x\rangle^2 \left\langle f^2(A)x, x\right\rangle - \langle Af(A)x, x\rangle^2 \\ &\quad \geq \left[\langle Ax, x\rangle\langle Af(A)x, x\rangle - \left\langle A^2x, x\right\rangle f(\langle Ax, x\rangle)\right] \cdot f(\langle Ax, x\rangle) \\ &\qquad + \left[\langle Ax, x\rangle f(\langle Ax, x\rangle) - \langle Af(A)x, x\rangle\right] \cdot \langle Af(A)x, x\rangle \end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Setting $h(t) = t$ in (2.12), respectively, we get the required results. $\square$

Theorem 2.20. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then*

$$\begin{aligned} &h^2(\langle Ax, x\rangle) f\left(\left\langle A^{-1}x, x\right\rangle\right) g\left(\left\langle A^{-1}x, x\right\rangle\right) + h^2\left(\left\langle A^{-1}x, x\right\rangle\right) f(\langle Ax, x\rangle) g(\langle Ax, x\rangle) \\ &\geq (\leq) h(\langle Ax, x\rangle) h\left(\left\langle A^{-1}x, x\right\rangle\right) \\ &\quad \times \left[f\left(\left\langle A^{-1}x, x\right\rangle\right) g(\langle Ax, x\rangle) + f(\langle Ax, x\rangle) g\left(\left\langle A^{-1}x, x\right\rangle\right)\right] \end{aligned} \tag{2.13}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Since f, g are synchronous and $\gamma \le \langle Ax, x\rangle \le \Gamma$, $\gamma \le \langle By, y\rangle \le \Gamma$ for any $x, y \in H$ with $\|x\| = \|y\| = 1$, we have

$$\begin{aligned}&\left(h\left(\langle Ax,x\rangle\right) f\left(\langle By,y\rangle\right) - h\left(\langle By,y\rangle\right) f\left(\langle Ax,x\rangle\right)\right)\\ &\qquad \times \left(h\left(\langle Ax,x\rangle\right) g\left(\langle By,y\rangle\right) - h\left(\langle By,y\rangle\right) g\left(\langle Ax,x\rangle\right)\right) \ge 0 \quad (2.14)\end{aligned}$$

for any $t \in [a, b]$ for any $x \in H$ with $\|x\| = 1$.

Employing property (1.1) for inequality (2.14) we have

$$\begin{aligned}&h^2\left(\langle Ax,x\rangle\right) f\left(\langle By,y\rangle\right) g\left(\langle By,y\rangle\right)\\ &\qquad + h^2\left(\langle By,y\rangle\right) f\left(\langle Ax,x\rangle\right) g\left(\langle Ax,x\rangle\right)\\ &\qquad - h\left(\langle Ax,x\rangle\right) h\left(\langle By,y\rangle\right) f\left(\langle By,y\rangle\right) g\left(\langle Ax,x\rangle\right)\\ &\qquad - h\left(\langle By,y\rangle\right) h\left(\langle Ax,x\rangle\right) f\left(\langle Ax,x\rangle\right) g\left(\langle By,y\rangle\right) \ge 0\end{aligned}$$

for any bounded linear operator B with $\operatorname{sp}(B) \subseteq [\gamma, \Gamma]$ and $y \in H$ with $\|y\| = 1$.

Now, since

$$\begin{aligned}&h^2\left(\langle Ax,x\rangle\right) f\left(\langle By,y\rangle\right) \cdot g\left(\langle By,y\rangle\right) + h^2\left(\langle By,y\rangle\right) f\left(\langle Ax,x\rangle\right) \cdot g\left(\langle Ax,x\rangle\right)\\ &\ge h\left(\langle Ax,x\rangle\right) h\left(\langle By,y\rangle\right) \left[f\left(\langle By,y\rangle\right) g\left(\langle Ax,x\rangle\right) + f\left(\langle Ax,x\rangle\right) g\left(\langle By,y\rangle\right)\right] \quad (2.15)\end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$. Setting $B = A^{-1}$ and $y = x$ in (2.15) we get the required result in (2.13). The reverse sense follows similarly. □

Remark 2.21. Let $0 < \gamma < \Gamma$ and choose $f(s) = s$ and $g(s) = s^{-1}$, $s > 0$ in Theorem 2.20. So that, if f and g are h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then

$$\begin{aligned}&h^2\left(\langle Ax,x\rangle\right) + h^2\left(\left\langle A^{-1}x,x\right\rangle\right)\\ &\ge (\le) 2h\left(\langle Ax,x\rangle\right) h\left(\left\langle A^{-1}x,x\right\rangle\right) \left[\left\langle A^{-1}x,x\right\rangle \langle Ax,x\rangle^{-1} + \langle Ax,x\rangle \left\langle A^{-1}x,x\right\rangle^{-1}\right]\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Corollary 2.22. *Let A be a selfadjoint operator with $\operatorname{sp}(A) \subset [\gamma, \Gamma]$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f : [\gamma, \Gamma] \to \mathbb{R}$ is continuous and h-synchronous on $[\gamma, \Gamma]$, then*

$$\begin{aligned}&h^2\left(\langle Ax,x\rangle\right) f^2\left(\left\langle A^{-1}x,x\right\rangle\right) + h^2\left(\left\langle A^{-1}x,x\right\rangle\right) f^2\left(\langle Ax,x\rangle\right)\\ &\qquad \ge 2h\left(\langle Ax,x\rangle\right) h\left(\left\langle A^{-1}x,x\right\rangle\right) f\left(\left\langle A^{-1}x,x\right\rangle\right) f\left(\langle Ax,x\rangle\right)\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

Proof. Setting $f = g$ in (2.13), respectively; we get the required results. □

An n-operators version of Theorem 2.7 is embodied as follows:

Theorem 2.23. *Let A_j be a selfadjoint operator with $\operatorname{sp}(A_j) \subset [\gamma, \Gamma]$ for $j \in \{1, 2, \cdots, n\}$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both h-synchronous*

(h-asynchronous) on $[\gamma,\Gamma]$*, then*

$$\sum_{j=1}^{n}\left\langle h^{2}\left(A_{j}\right)x_{j},x_{j}\right\rangle \sum_{j=1}^{n}\left\langle f\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle \geq (\leq) \sum_{j=1}^{n}\left\langle h\left(A_{j}\right)g\left(A_{j}\right)x_{j},x_{j}\right\rangle \sum_{j=1}^{n}\left\langle h\left(A_{j}\right)f\left(A_{j}\right)x_{j},x_{j}\right\rangle \quad (2.16)$$

for each $x_j\in H$*,* $j\in\{1,2,\cdots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$*.*

Proof. As in ([4], p.6), if we put

$$\widetilde{A}:=\begin{pmatrix} A_1 & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n\end{pmatrix} \quad\text{and}\quad \widetilde{x}:=\begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}$$

then we have $\operatorname{sp}\left(\widetilde{A}\right)\subset[\gamma,\Gamma]$, $\|\widetilde{x}\|=1$, $\left\langle h^2\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\langle h^2\left(A_j\right)x_j,x_j\rangle$,
$\left\langle f\left(\widetilde{A}\right)g\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\langle f\left(A_j\right)g\left(A_j\right)x_j,x_j\rangle$,
$\left\langle h\left(\widetilde{A}\right)g\left(\widetilde{A}\right)x,x\right\rangle=\sum_{j=1}^{n}\langle h\left(A_j\right)g\left(A_j\right)x_j,x_j\rangle$,
and $\left\langle h\left(\widetilde{A}\right)f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\langle h\left(A_j\right)f\left(A_j\right)x_j,x_j\rangle$. Applying Theorem 2.7 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired result. □

Corollary 2.24. *Let* A_j *be a selfadjoint operator with* $\operatorname{sp}(A_j)\subset[\gamma,\Gamma]$ *for* $j\in\{1,2,\cdots,n\}$ *for some real numbers* γ,Γ *with* $\gamma<\Gamma$*. Let* $h:[\gamma,\Gamma]\to\mathbb{R}$ *be a non-negative continuous. If* $f:[\gamma,\Gamma]\to\mathbb{R}$ *is continuous and* h*-synchronous on* $[\gamma,\Gamma]$*, then*

$$\left(\sum_{j=1}^{n}\left\langle h\left(A_{j}\right)f\left(A_{j}\right)x_{j},x_{j}\right\rangle\right)^{2}\leq \sum_{j=1}^{n}\left\langle h^{2}\left(A_{j}\right)x_{j},x_{j}\right\rangle \sum_{j=1}^{n}\left\langle f^{2}\left(A_{j}\right)x_{j},x_{j}\right\rangle \quad (2.17)$$

for each $x_j\in H$*,* $j\in\{1,2,\cdots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$*.*

Proof. Setting $f=g$ in (2.16), we get the desired result. □

Corollary 2.25. *Let* A_j *be a selfadjoint operator with* $\operatorname{sp}(A_j)\subset[\gamma,\Gamma]$ *for* $j\in\{1,2,\cdots,n\}$ *for some real numbers* γ,Γ *with* $0<\gamma<\Gamma$*. If* $f:[\gamma,\Gamma]\to\mathbb{R}$ *is continuous and* t*-synchronous on* $[\gamma,\Gamma]$*, then*

$$\left(\sum_{j=1}^{n}\left\langle A_{j}f\left(A_{j}\right)x_{j},x_{j}\right\rangle\right)^{2}\leq \sum_{j=1}^{n}\left\langle A_{j}^{2}x_{j},x_{j}\right\rangle \sum_{j=1}^{n}\left\langle f^{2}\left(A_{j}\right)x_{j},x_{j}\right\rangle \quad (2.18)$$

for each $x_j\in H$*,* $j\in\{1,2,\cdots,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$*.*

Proof. Setting $h(t) = t$ in (2.17), we get the desired result. $\square$

Remark 2.26. Let A_j be a selfadjoint operator with $\mathrm{sp}(A_j) \subset [\gamma, \Gamma]$ for $j \in \{1, 2, \cdots, n\}$ for some real numbers γ, Γ with $0 < \gamma < \Gamma$. Let $f(s) = s^{1/2}$ for $s \in [\gamma, \Gamma]$ then f is $t^{-1/2}$-synchronous so that by (2.18) we have

$$n^2 \le \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \cdot \left(\sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \right). \tag{2.19}$$

The discrete version of Čebyšev inequality, reads that

$$\frac{1}{m} \sum_{i=1}^{m} a_i b_i \ge \left(\frac{1}{m} \sum_{i=1}^{m} a_i \right) \left(\frac{1}{m} \sum_{i=1}^{m} b_i \right)$$

for all similarly ordered n-tuples $(a_1, \cdots, a_m)$ and $(b_1, \cdots, b_m)$.

Let $\{A_j\}_{j=1}^n$ be a finite positive sequence of invertible self-adjoint operators and consider $a_j = \langle A_j x_j, x_j \rangle$ and $b_j = \left\langle A_j^{-1} x_j, x_j \right\rangle$ for all $j = 1, \cdots, n$. If $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ similarly ordered n-tuples. Then by employing the Čebyšev inequality on (2.19) we get

$$1 \le \frac{1}{n} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \frac{1}{n} \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \le \frac{1}{n} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \left\langle A_j^{-1} x_j, x_j \right\rangle.$$

On other hand, if $\gamma_j \cdot 1_H \le A_j \le \Gamma_j \cdot 1_H$, then by Kanotrovich inequality we have

$$\begin{aligned} 1 &\le \frac{1}{n} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \cdot \frac{1}{n} \sum_{j=1}^{n} \left\langle A_j^{-1} x_j, x_j \right\rangle \\ &\le \frac{1}{n} \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \left\langle A_j^{-1} x_j, x_j \right\rangle \le \frac{1}{n} \sum_{j=1}^{n} \frac{(\Gamma_j - \gamma_j)^2}{4 \gamma_j \Gamma_j}. \end{aligned}$$

In case $n = 2$, we have

$$\begin{aligned} 1 &\le \frac{1}{4} \left[\langle A_1 x_1, x_1 \rangle + \langle A_2 x_2, x_2 \rangle \right] \cdot \left[\left\langle A_1^{-1} x_1, x_1 \right\rangle + \left\langle A_2^{-1} x_2, x_2 \right\rangle \right] \\ &\le \frac{1}{2} \left[\langle A_1 x_1, x_1 \rangle \left\langle A_1^{-1} x_1, x_1 \right\rangle + \langle A_2 x_2, x_2 \rangle \left\langle A_2^{-1} x_2, x_2 \right\rangle \right] \\ &\le \frac{1}{8} \left[\frac{(\Gamma_1 - \gamma_1)^2}{\gamma_1 \Gamma_1} + \frac{(\Gamma_2 - \gamma_2)^2}{\gamma_2 \Gamma_2} \right]. \end{aligned}$$

An n-operators version of Theorem 2.16 is incorporated in the following result.

Theorem 2.27. *Let A_j be a selfadjoint operator with $\mathrm{sp}(A_j) \subset [\gamma, \Gamma]$ for $j \in \{1, 2, \cdots, n\}$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous. If $f, g : [\gamma, \Gamma] \to \mathbb{R}$ are continuous and both f and g are*

h-synchronous (h-asynchronous) on $[\gamma, \Gamma]$, then

$$\begin{aligned}
& h^2\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\sum_{j=1}^n\langle f(A_j)\, g(A_j)\, x_j, x_j\rangle \\
& \quad -\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle\sum_{j=1}^n\langle h(A_j)\, g(A_j)\, x_j, x_j\rangle \\
& \geq (\leq)\left[h\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle\right. \\
& \qquad \left.-\sum_{j=1}^n\langle h^2(A_j)\, x_j, x_j\rangle f\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\right]\cdot g\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right) \\
& \quad +\left[h\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right) f\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\right. \\
& \qquad \left.-\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle\right]\cdot\sum_{j=1}^n\langle h(A_j)\, g(A_j)\, x_j, x_j\rangle
\end{aligned} \tag{2.20}$$

for each $x_j \in H$, $j \in \{1, 2, \cdots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Proof. The proof is similar to the proof of Theorem 2.23 on employing Theorem 2.16. ☐

Corollary 2.28. *Let A_j be a selfadjoint operator with $\operatorname{sp}(A_j) \subset [\gamma, \Gamma]$ for $j \in \{1, 2, \cdots, n\}$ for some real numbers γ, Γ with $\gamma < \Gamma$. Let $h : [\gamma, \Gamma] \to \mathbb{R}$ be a non-negative continuous and convex on $[\gamma, \Gamma]$. If $f : [\gamma, \Gamma] \to \mathbb{R}$ is continuous and h-synchronous on $[\gamma, \Gamma]$, then*

$$\begin{aligned}
& h^2\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\sum_{j=1}^n\left\langle f^2(A_j)\, x_j, x_j\right\rangle-\left(\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle\right)^2 \\
& \geq (\leq)\left[h\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle\right. \\
& \qquad \left.-\sum_{j=1}^n\langle h^2(A_j)\, x_j, x_j\rangle f\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\right]\cdot f\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right) \\
& \quad +\left[h\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right) f\left(\sum_{j=1}^n\langle A_j x_j, x_j\rangle\right)\right. \\
& \qquad \left.-\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle\right]\cdot\sum_{j=1}^n\langle h(A_j)\, f(A_j)\, x_j, x_j\rangle
\end{aligned} \tag{2.21}$$

for each $x_j \in H$, $j \in \{1, 2, \cdots, n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Proof. Setting $f = g$ in (2.20), respectively; we get the required results. □

Corollary 2.29. *Let A_j be a selfadjoint operator with $\operatorname{sp}(A_j) \subset [\gamma, \Gamma]$ for $j \in \{1, 2, \cdots, n\}$ for some real numbers γ, Γ with $0 < \gamma < \Gamma$. If $f : [\gamma, \Gamma] \to \mathbb{R}$ is continuous and t-synchronous on $[\gamma, \Gamma]$, then*

$$
\begin{aligned}
&\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^2 \sum_{j=1}^{n} \left\langle f^2(A_j) x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \langle A_j f(A_j) x_j, x_j \rangle\right)^2 \\
&\geq (\leq) \left[\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle A_j f(A_j) x_j, x_j \rangle\right. \\
&\qquad \left. - \sum_{j=1}^{n} \langle A_j^2 x_j, x_j \rangle f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)\right] \cdot f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \\
&\quad + \left[\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)\right. \\
&\qquad \left. - \sum_{j=1}^{n} \langle A_j f(A_j) x_j, x_j \rangle\right] \cdot \sum_{j=1}^{n} \langle A_j f(A_j) x_j, x_j \rangle
\end{aligned}
$$

for each $x_j \in H$, $j \in \{1, 2, \cdots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. Setting $h(t) = t$ in (2.21), we get the desired results. □

Remark 2.30. By choosing $h(t) = 1$ for all $t \in [a, b]$, in Theorems 2.7, 2.16, 2.23 and 2.27, then we recapture all inequalities obtained [2] and their consequences.

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