The Annals of Statistics
1990, Vol. 18, No. 3, 1340–1357

AN OMNIBUS TEST FOR DEPARTURES FROM CONSTANT MEAN[1]

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Observations y_i are made at points x_i according to the model $y_i = F(x_i) + e_i$, where the e_i are independent normals with constant variance. In order to decide whether or not $F(x)$ is constant, a likelihood ratio test is constructed, comparing $F(x) \equiv \mu$ with $F(x) = \mu + Z(x)$, where $Z(x)$ is a Brownian motion. The ratio of error variance to Brownian motion variance is chosen to maximize the likelihood, and the resulting maximum likelihood statistic B is used to test departures from constant mean. Its asymptotic distribution is derived and its finite sample size behavior is compared with five other tests. The B-statistic is comparable or superior to each of the tests on the five alternatives considered.

1. Introduction and summary. Let (x_i, y_i), $i = 1, 2, \dots, n+1$, satisfy

$$y_i = F(x_i) + e_i,$$

where $x_i \in [0, 1]$ for each i, F is a regression function and the errors $\{e_i\}$ are uncorrelated with mean zero and variance v. Parametric estimation of the regression function $F(x)$ assumes a particular functional form for F depending on a small number of unknown parameters. Nonparametric estimation assumes only that F is a smooth function of x [see Prakasa Rao (1983) for an extensive review of nonparametric regression estimation].

In this paper we consider testing the hypothesis that F is a constant function, i.e., that the X and Y variables are independent. If the independence hypothesis is rejected, the procedure provides a spline estimate of F; rejection of independence indicates that attempts at parametric modelling (possibly involving the collection of more data) might prove successful.

Schoenberg (1964) proposed estimating F by $\hat{F}$, which minimizes

$$\sum \left(y_i - \hat{F}(x_i)\right)^2 + c\int_0^1 \hat{F}'(x)^2\,dx,$$

whose c is a nonnegative constant to be specified. $\hat{F}$ is a piecewise linear function; $c = \infty$ implies that $\hat{F}$ is constant, $c = 0$ implies that $\hat{F}$ interpolates the data and intermediate values of c imply a compromise between these two extremes. This is one of a class of spline estimates.

Wahba (1978) has shown that the estimate $\hat{F}$ is equivalent to the Bayes estimate of F where the errors $\{e_i\}$ are assumed i.i.d. $N(0, v)$ and the prior

Received January 1988; revised July 1989.
[1]Supported in part by NSF grant DMS-86-17919.
AMS 1980 *subject classifications*. Primary 62G10; secondary 62E20, 62M10.
Key words and phrases. Nonparametric regression, Brownian motion, nonstandard likelihood asymptotics, tests for constant mean.

specifies that F is drawn from a Brownian motion with unknown mean μ and scale v_1, where $v_1 = v/c$.

We describe a data based procedure for choosing a value for c, first proposed in Barry (1983) and later considered in Wahba (1985). The smaller the value of c chosen by the data, the stronger is the evidence against independence. Our test is a likelihood ratio test of the hypothesis H_0: $v_1 = 0$; unusual theoretical features of the test arise from the presence of an infinite number of nuisance parameters in the fitted function F. In Section 2 we describe a similar idea proposed by Yanagimoto and Yanagimoto (1987) to test the adequacy of the fit of a simple linear regression model. The advance in this paper is that the asymptotic distribution of the test statistic is derived, and that comparisons by simulation with a range of alternative test statistics are presented. Cox and Koh (1986) and Cox, Koh, Wahba and Yandell (1986), introduce the theory of locally most powerful tests of H_0: $v_1 = 0$ and extend the method to a broad class of spline models.

The assumption of i.i.d. normal errors is certainly open to question, and the technique may well be seriously affected by one or two outliers in the data; the results would be to estimate $F(x)$ to be very different at the outlying points and to conclude that F deviates from constancy. It is suggested that, before applying the technique, some method of identifying and downsizing outlying values be used.

The test statistic B is described in Section 2. In Section 3 we derive the asymptotic distribution of the test statistic under the hypothesis that F is constant for equally spaced x-values. Section 4 contains a simulation study comparing the power of the test developed in Section 2 with that of some well-known tests of independence, for a variety of departures from independence. The B test is comparable or superior to its competitors for each of the alternative hypotheses.

NOTE. When the range of notation is from 1 to n, it will be suppressed for ease of notation.

2. The test statistic. Suppose we have data (x_i, y_i), $i = 1, 2, \dots, n+1$, where

$$0 \le x_1 < x_2 < \cdots < x_{n+1} \le 1$$

and

$$y_i = F(x_i) + e_i, \qquad i = 1, 2, \dots, n+1,$$

where F: $[0, 1] \to \mathbb{R}$ is the regression function of Y on X. We wish to test the hypothesis that F is a constant function.

Let us assume the following probability model:

1. $F(x) = \mu + Z(x)$, where (a) μ is unknown; (b) $Z(x)$ is a zero-mean Gaussian process with $\mathrm{Cov}[Z(x), Z(y)] = v_1 \min(x, y)$;

2. The errors $\{e_i\}$ are i.i.d. $N(0, v)$ independent of F.

For $i = 1, 2, \dots, n$, define

$$w_i = y_{i+1} - y_i.$$

Then, under the probability model specified in (1) and (2), $\mathbf{w} = (w_1, w_2, \dots, w_n)$ has a multivariate normal distribution with mean 0 and

$$\mathrm{Var}(w_i) = v_1(x_{i+1} - x_i) + 2v$$

and

$$\mathrm{Cov}(w_i, w_j) = \begin{cases} -v, & |i-j| = 1, \\ 0, & |i-j| > 1. \end{cases}$$

For $\alpha = v_1/v$, let $L_n(\alpha, v)$ be the log likelihood for α and v based on $\mathbf{w}$. Let $\hat{v}_n(\alpha)$ be the MLE of v given α. Define

$$Q_n(\alpha) = 2L_n(\alpha, \hat{v}_n(\alpha))$$

and let $\hat{\alpha}_n$ be the value for which $Q_n(\alpha)$ is a maximum.

Clearly,

$$\begin{aligned} \alpha = 0 \quad &\Leftrightarrow \quad v_1 = 0 \\ &\Rightarrow \quad F \text{ is constant,} \end{aligned}$$

and so, to test the hypothesis that F is constant, we propose the test statistic

$$B_n = Q_n(\hat{\alpha}_n) - Q_n(0).$$

Yanagimoto and Yanagimoto (1987) follow a similar course but start from a model which specifies

$$F(x) = \alpha + \beta x + Z_1(x),$$

where α and β are unknown and $Z_1(x)$ is an integrated Brownian motion process. Their paper also includes simulation of the null distribution of the likelihood ratio test statistic and an example of the use of the test in practice.

In this paper we show, for x_i evenly distributed, that B_n is asymptotically distributed as

$$\sup_K \sum_{r=1}^{\infty} \frac{K}{K + \pi^2 r^2} Z_r^2 + \log\left[\frac{\sqrt{K}}{\sinh(\sqrt{K})}\right],$$

where the Z_r are i.i.d. $N(0, 1)$.

3. The distribution of B_n when F is constant. It follows as a special case of the calculations in Wahba (1985) that

$$Q_n(\alpha) = -n \log\{\mathbf{y}^T(I - A)\mathbf{y}\} + \log|I - A|^+,$$

where

$$y = (y_1, y_2, \ldots, y_{n+1})^T,$$

$$A = \left(I + \frac{1}{\alpha}H\right)^{-1}$$

where $H = (h_{ij})$ is an $(n+1) \times (n+1)$ symmetric tridiagonal matrix with

$$h_{i\,i+1} = -1/(x_{i+1} - x_i), \qquad 1 \le i \le n,$$

$$h_{ii} = \begin{cases} 1/(x_i - x_{i-1}) + 1/(x_{i+1} - x_i), & 1 < i < n, \\ 1/(x_2 - x_1), & i = 1, \\ 1/(x_{n+1} - x_n), & i = n+1, \end{cases}$$

and $|I - A|^+$ is the product of the positive eigenvalues of $I - A$.

We now specialize to the case where

$$x_{i+1} - x_i = \frac{1}{n+1}, \qquad i = 1, 2, \ldots, n.$$

The model may be expressed as the ARIMA model,

$$y_{i+1} - y_i = e_{i+1} - e_i + \eta_i, \quad \text{where } e_i \sim N(0, v),\ \eta_i \sim N\left(0, \frac{v_1}{n+1}\right).$$

$$Q_n(\alpha) = -n \log\left\{\sum_{r=1}^{n} \frac{(n+1)\lambda_r}{(n+1)\lambda_r + \alpha} Z_r^2\right\} + \sum_{r=1}^{n} \log\left\{\frac{(n+1)\lambda_r}{(n+1)\lambda_r + \alpha}\right\},$$

where

$$\lambda_r = 2\left(1 - \cos\left(\frac{\pi r}{n+1}\right)\right),$$

$$Z_r = \left(\frac{2}{n+1}\right)^{1/2} \sum_{i=1}^{n+1} \cos\left[\frac{\pi r}{n+1}\left(i - \frac{1}{2}\right)\right] y_i, \qquad r = 1, 2, \ldots, n.$$

The Z_r are Fourier coefficients for the series $y_{i+1} - y_i$ and are independent with mean 0 and variances $v(1 + \alpha/[(n+1)\lambda_r])$; the expression $Q_n(\alpha)$ arises from maximizing twice the log likelihood of the Z_r over v. We will consider the behavior of $\sup_\alpha Q_n(\alpha) - Q_n(0)$ when $\alpha = 0$, and the Z_r are independent $N(0, v)$.

For convenience, we express Q_n in terms of $K = (n+1)\alpha$ and write

$$Q_n(K) = -n \log\left\{\sum_{r=1}^{n} \frac{u_r}{u_r + K} Z_r^2\right\} + \sum_{r=1}^{n} \log\left\{\frac{u_r}{u_r + K}\right\},$$

where $u_r = (n+1)^2\lambda_r$. Note that the sequence u_r is positive and increasing.

THEOREM 1. *Let $\hat{K}_n$ be the value which maximizes $Q_n(K)$. Then $\hat{K}_n = O_p(1)$, that is,*

$$\limsup_{n\to\infty} P\{\hat{K}_n \geq K_0\} \to 0 \quad \text{as } K_0 \to \infty.$$

PROOF. We show separately that

$$\limsup_{n\to\infty} P\{\hat{K}_n > 5n^2\} = 0 \tag{1}$$

and

$$\limsup_{n\to\infty} P\{K_0 \leq \hat{K}_n \leq 5n^2\} \to 0 \quad \text{as } K_0 \to \infty. \tag{2}$$

PROOF OF (1). It suffices to show that

$$\limsup_{n\to\infty} P\Big\{\sup_{K>5n^2} Q_n(K) - Q_n(0) > 0\Big\} = 0.$$

Now,

$$\begin{aligned} Q_n(K) - Q_n(0) &= -n \log\left\{\left[\sum \frac{u_r}{u_r + K} Z_r^2\right]\Big/ \sum Z_r^2\right\} + \sum \log\left(\frac{u_r}{u_r + K}\right) \\ &= -n \log\left\{\left[\sum \frac{u_r K}{u_r + K} Z_r^2\right]\Big/ \sum Z_r^2\right\} + \sum \log\left(\frac{u_r K}{u_r + K}\right). \end{aligned}$$

Since $u_r K/(u_r + K) \leq u_r$ and $K/(u_r + K) \geq K/(u_n + K) \geq 5n^2/(u_n + 5n^2)$,

$$Q_n(K) - Q_n(0) \leq -n \log\left(\frac{5n^2}{u_n + 5n^2}\right) - n \log\left\{\frac{\sum u_r Z_r^2}{\sum Z_r^2}\right\} + \sum \log(u_r),$$

$$\sup_{K>5n^2} \frac{1}{n}[Q_n(K) - Q_n(0)] \leq -\log\left(\frac{5n^2}{u_n + 5n_2^2}\right) - \log\frac{\sum \lambda_r Z_r^2}{\sum Z_r^2} + \frac{1}{n}\sum \log \lambda_r.$$

Since $u_n/n^2 \to 4$,

$$\log\frac{5n^2}{u_n + 5n^2} \to \log\frac{5}{9}.$$

Also

$$\frac{1}{n}\sum \log \lambda_r \to \frac{1}{\pi}\int_0^{\pi} \log(2 - 2\cos x)\, dx = 0.$$

Finally, $V_n = \sum \lambda_r Z_r^2 / \sum Z_r^2 \to 2$ in probability as $n \to \infty$, since

$$V_n = 2\left(1 - \left[\sum \cos\frac{\pi r}{n+1} Z_r^2\right]\Big/ \sum Z_r^2\right) \quad \text{and} \quad \sum \cos\frac{\pi r}{n+1} = 0.$$

Thus,

$$P\left\{\sup_{K>5n^2}\frac{1}{n}(Q_n(K)-Q_n(0))\le\log\frac{9}{10}<0\right\}\to 1\quad\text{as } n\to\infty. \qquad\square$$

PROOF OF (2). It suffices to show that

$$\limsup_{n\to\infty}P\left\{\sup_{K_0\le K\le 5n^2}Q_n'(K)>0\right\}\to 0\quad\text{as } K_0\to\infty.$$

Now,

$$Q_n'(K)=n\left[\sum\frac{u_r}{(u_r+K)^2}Z_r^2\right]\Big/\left[\sum\frac{u_r}{u_r+Z}K_r^2\right]-\sum\frac{1}{u_r+K}>0$$

is equivalent to

$$\sum W_r(K)Z_r^2>0,$$

where

$$W_r(K)=\frac{u_r}{(u_r+K)^2}-\frac{u_r}{u_r+K}\left[\frac{1}{n}\sum\frac{1}{u_r+K}\right].$$

Hence,

$$\begin{aligned}P\left\{\sup_{K_0\le K\le 5n^2}Q_n'(K)>0\right\}&\le\sum_{K=K_0}^{5n^2}P\left\{\sup_{K\le l\le K+1}\sum W_r(l)Z_r^2>0\right\}\\&\le\sum_{K=K_0}^{5n^2}P\left\{\sum\Delta_r(K)Z_r^2>0\right\},\end{aligned}$$

where

$$\Delta_r(K)=\sup_{K\le l\le K+1}W_r(l).$$

We will prove the following lemma later.

LEMMA 1. *There exist positive constants D_1, D_2 and M_0 such that*

$$\sum\Delta_r(K)<-D_1K^{-1/2}\tag{3}$$

and

$$\sum\Delta_r^2(K)<D_2K^{-3/2}\tag{4}$$

for $K_0\le K\le 5n^2$, whenever $K_0>M_0$.

Hence, by Chebyshev's inequality, we have for any even integer s that

$$P\left\{\sum\Delta_r(K)Z_r^2>0\right\}\le\frac{E\left|\Sigma\Delta_r(K)(Z_r^2-1)\right|^s}{(\Sigma\Delta_r(K))^s}.$$

Whittle (1960) shows that

$$E\left|\sum \Delta_r(K)(Z_r^2-1)\right|^s \le C\left(\sum \left(E\left|\Delta_r(K)(Z_r^2-1)\right|^s\right)^{2/s}\right)^{s/2},$$

for some constant C depending on s. Hence,

$$P\left\{\sum \Delta_r(K)Z_r^2 > 0\right\} \le C_1\left[\frac{\Sigma\Delta_r^2(K)}{(\Sigma\Delta_r(K))^2}\right]^{s/2},$$

for some constant C_1 depending on s.

Taking $s = 8$ and using Lemma 1 gives

$$P\left\{\sum \Delta_r(K)Z_r^2 > 0\right\} \le C_2/K^2,$$

whose C_2 is a constant. The required result now follows, since

$$\sum_{K=K_0}^{\infty} \frac{1}{K^2} \to 0 \quad \text{as } K_0 \to \infty. \qquad \square$$

PROOF OF LEMMA 1.

PROOF OF (3). Let $N = n + 1$;

$$\sum W_r(K) = \sum \frac{1}{u_r+K} - \sum \frac{K}{(u_r+K)^2} - \frac{1}{n}\left[\sum \frac{u_r}{u_r+K}\right]\left[\sum \frac{1}{u_r+K}\right].$$

Since u_r increases with r,

$$\int_0^N \frac{1}{u_r+K}\,dr - \frac{1}{K} \le \sum \frac{1}{u_r+K} \le \int_0^N \frac{1}{u_r+K}\,dr = \frac{N}{(K+4N^2K)^{1/2}}.$$

Also,

$$\sum \frac{1}{(u_r+K)^2} \ge \frac{N}{\pi}\int_0^{\pi} \frac{dx}{\left[2N^2(1-\cos x)+K\right]^2} - \frac{1}{K^2}$$
$$= \frac{2N^3+NK}{(K^2+4N^2K)^{3/2}} - \frac{1}{K^2}.$$

Hence,

$$\sum W_r(K) \le \frac{N}{(K^2+4N^2K)^{1/2}} - \frac{(2N^3K+NK^2)}{(K^2+4N^2K)^{3/2}} + \frac{1}{K}$$
$$- \frac{1}{N}\left[N - \frac{NK}{(K^2+4N^2K)^{1/2}}\right]\left[\frac{N}{(K^2+4N^2K)^{1/2}} - \frac{1}{K}\right]$$
$$\le \frac{-1}{\sqrt{K}}\left[\frac{2N^3+NK}{(K+4N^2)^{3/2}} + \frac{2}{\sqrt{K}} - \frac{N\sqrt{K}}{K+4N^2}\right]$$
$$= \frac{-1}{\sqrt{K}}\left\{\frac{2+A}{(4+A)^{3/2}} - \frac{\sqrt{A}}{4+A} + \frac{2}{\sqrt{K}}\right\}, \quad \text{where } A = K/N^2.$$

Since $(2 + A)/(4 + A)^{3/2} - \sqrt{A}/4 + A$ is bounded above 0 for $0 \le A \le 5$, we have that, for K_0 large enough, there exists a constant $C_1 > 0$ such that

$$\sum W_r(K) \le -C_1/\sqrt{K}, \qquad K_0 \le K \le 5n^2.$$

For $0 \le \delta \le 1$,

$$W_r(K) - W_r(K+\delta) = \delta W_r'(K+\delta^*), \quad \text{where } 0 \le \delta^* \le \delta,$$

where

$$W_r'(l) = \frac{-2u_r}{(u_r + l)^3} + \frac{u_r}{n}\sum_s \frac{2l + u_r + u_s}{(u_r + l)^2(u_s + l)^2}.$$

Hence,

$$\begin{aligned} &|W_r(K) - W_r(K+\delta)| \\ &\quad \le |W_r'(K+\delta^*)| \\ &\quad \le \frac{2u_r}{(u_r + K)^3} + \frac{u_r}{n}\sum_s \frac{2l + u_r + u_s}{(u_r + l)^2(u_s + l)^2} \qquad (5) \\ &\quad \le \frac{2}{(u_r + K)^2} + \frac{1}{n}\sum \frac{1}{(u_s + K)^2} + \frac{1}{u_r + K}\left[\frac{1}{n}\sum \frac{1}{u_s + K}\right]. \end{aligned}$$

For $r = 1, 2, \ldots, n$,

$$\Delta_r(K) = W_r(K + \delta_r) \quad \text{for some } \delta_r \in [0, 1].$$

Hence,

$$\begin{aligned} \left|\sum \Delta_r(K) - \sum W_r(K)\right| &\le \sum |W_r(K) - W_r(K + \delta_r)| \\ &\le 3\sum \frac{1}{(u_r + K)^2} + \frac{1}{n}\left(\sum \frac{1}{u_r + K}\right)^2 \\ &\le \frac{3(N^3 + NK)}{(K^2 + 4N^2K)^{3/2}} + \frac{N}{(K^2 + 4N^2K)} \\ &= \frac{1}{K^{3/2}}\left\{\frac{3(1+A)}{(4+A)^{3/2}} + \frac{A^{3/2}}{A^2 + 4A}\right\} \quad \text{where } A = K/n^2 \\ &\le \frac{4}{K^{3/2}} \quad \text{for } 0 \le A \le 5. \end{aligned}$$

This proves (3).

PROOF OF (4).

$$\sum \Delta_r^2(K) \le 2\sum W_r^2(K) + 2\sum (W_r(K) - \Delta_r(K))^2.$$

From (5),

$$|W_r(K) - \Delta_r(K)| \le 4.$$

Therefore,

$$\sum (W_r(K) - \Delta_r(K))^2 \le 4 \sum |W_r(K) - \Delta_r(K)| \le \frac{16}{K^{3/2}}.$$

Also,

$$\begin{aligned}\sum W_r^2(K) &\le \sum \frac{u_r^2}{(u_r + K)^4} + \frac{u_r^2}{(u_r + K)^2}\left[\frac{1}{n}\sum \frac{1}{u_r + K}\right]^2 \\ &\le \sum \frac{1}{(u_r + K)^2} + \frac{1}{n}\left[\sum \frac{1}{u_r + K}\right]^2 \\ &\le \frac{4}{K^{3/2}}.\end{aligned}$$

Hence,

$$\sum \Delta_r^2(K) \le \frac{40}{K^{3/2}},$$

and this combined with (3) proves (4). □

THEOREM 2. *Let* $Z_1, Z_2, \dots, Z_n, \dots$ *be a sequence of* i.i.d. $N(0,1)$ *random variables. Define*

$$Q_n(K) = -n \log\left\{\sum_{r=1}^{n} \frac{u_r}{u_r + K} Z_r^2\right\} + \sum_{r=1}^{n} \log\left(\frac{u_r}{u_r + K}\right)$$

and

$$M_n(K) = Q_n(K) - Q(0).$$

Define

$$M(K) = \sum_{r=1}^{\infty} \frac{K}{K + \pi^2 r^2} Z_r^2 + \log\left[\frac{\sqrt{K}}{\sinh(\sqrt{K})}\right] \quad \textit{for } K > 0,\ M(0) = 0.$$

Then for any fixed $K_0 < \infty$,

$$\sup_{0 \le K \le K_0} |M_n(K) - M(K)| \to_P 0 \quad \textit{as } n \to \infty.$$

PROOF. Define

$$H_n(K) = -n \log\left\{\left[\sum \frac{\pi^2 r^2}{K + \pi^2 r^2} Z_r^2\right] \Big/ \sum Z_r^2\right\} + \sum \log\left[\frac{\pi^2 r^2}{K + \pi^2 r^2}\right].$$

We show

(a) $$\sup_{0 \le K \le K_0} |M_n(K) - H_n(K)| \to_P 0 \quad \text{as } n \to \infty,$$

(b) $$\sup_{0 \le K \le K_0} |H_n(K) - M(K)| \to_P 0 \quad \text{as } n \to \infty.$$

PROOF OF (a).

$$\begin{aligned} M_n(K) - H_n(K) &= -n \log\left\{\left[\sum \frac{\pi^2 r^2}{K + \pi^2 r^2} Z_r^2\right] \Big/ \left[\sum \frac{u_r}{K + u_r} Z_r^2\right]\right\} \\ &\quad + \sum \log\left(\frac{u_r}{u_r + K}\right) - \log\left(\frac{\pi^2 r^2}{K + \pi^2 r^2}\right) \\ &= C_n + D_n, \text{ say.} \end{aligned}$$

Introduce positive constants R_1 and R_2 such that

$$|\pi^2 r^2 - u_r| \le R_1 \pi^4 r^4 / n^2$$

and

$$\pi^2 r^2 \le R_2 u_r.$$

Then,

$$\begin{aligned} |C_n| &\le n \left| \sum \left(\frac{\pi^2 r^2}{\pi^2 r^2 + K} - \frac{u_r}{u_r + K} \right) Z_r^2 \right| \min\left[\sum \frac{\pi^2 r^2}{\pi^2 r^2 + K} Z_r^2, \sum \frac{u_r}{u_r + K} Z_r^2 \right] \\ &\le \frac{K R_1}{n} \sum \frac{\pi^4 r^4}{(\pi^2 r^2 + K)(u_r + K)} Z_r^2 \min\left[\sum \frac{\pi^2 r^2}{\pi^2 r^2 + K} Z_r^2, \sum \frac{u_r}{u_r + K} Z_r^2 \right] \\ &\le \frac{K_0 R_1 R_2}{n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

For D_n, let $\phi(u) = \log(u/(u + K))$. Then, for $\delta > 0$,

$$\phi(u + \delta) - \phi(u) = \delta \phi^1(u_0) \quad \text{for some } u_0 \in (u, u + \delta),$$

where

$$\phi^1(u) = \frac{K}{u(u + K)} \quad \Rightarrow \quad |\phi(u + \delta) - \phi(u)| \le \frac{K\delta}{u(u + K)}.$$

Hence,

$$\begin{aligned} |D_n| &\le K \sum |\pi^2 r^2 - u_r| \min[\pi^2 r^2(\pi^2 r^2 + K), u_r(u_r + K)] \\ &\le \frac{K R_1}{n^2} \sum \pi^4 r^4 \min[\pi^2 r^2(\pi^2 r^2 + K), u_r(u_r + K)] \\ &\le \frac{K_0 R_1 R_2^2}{n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence (a) is proved.

PROOF OF (b). From Jolley (1961),

$$\log\left[\sqrt{K}\sinh(\sqrt{K})\right] = \sum_{r=1}^{\infty}\log\left(\frac{\pi^2 r^2}{\pi^2 r^2 + K}\right).$$

Hence,

$$\sum \log\left(\frac{\pi^2 r^2}{\pi^2 r^2 + K}\right) - \log\left\{\frac{\sqrt{K}}{\sinh(\sqrt{K})}\right\} = \sum_{r=n+1}^{\infty}\log\left(1 + \frac{K}{\pi^2 r^2}\right)$$
$$\leq \frac{K_0}{\pi^2}\sum_{r=n+1}^{\infty}\frac{1}{r^2} \to 0 \quad \text{as } n \to \infty.$$

Also,

$$\log\left\{\left[\sum \frac{\pi^2 r^2}{\pi^2 r^2 + K} Z_r^2\right]\Big/ \sum Z_r^2\right\}$$
$$= \log\left\{1 - \left[\sum \frac{K}{\pi^2 r^2 + K} Z_r^2\right]\Big/ \sum Z_r^2\right\}$$
$$= \left[\sum \frac{K}{\pi^2 r^2 + K} Z_r^2\right]\Big/ \sum Z_r^2 + O_p\left(\frac{1}{n^2}\right) \quad \text{for } K \leq K_0$$
$$= \left[\sum \frac{K}{\pi^2 r^2 + K} Z_r^2\right]\Big/ n + O_p\left(\frac{1}{n^{3/2}}\right).$$

Hence,

$$n\log\left\{\left[\sum \frac{\pi^2 r^2}{\pi^2 r^2 + K} Z_r^2\right]\Big/ \sum Z_r^2\right\} = \sum \frac{K}{K + \pi^2 r^2} Z_r^2 + O_p\left(\frac{1}{\sqrt{n}}\right)$$
$$= \sum_{r=1}^{\infty} \frac{K}{K + \pi^2 r^2} Z_r^2 + O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{n}}\right),$$

since

$$\sum_{r=n+1}^{\infty} \frac{K}{K + \pi^2 r^2} Z_r^2 \leq \frac{K}{\pi^2}\sum_{r=n+1}^{\infty}\frac{Z_r^2}{r^2} = O_p\left(\frac{1}{n}\right) \quad \text{for } K \leq K_0.$$

This proves (b). □

THEOREM 3. *Let $\hat{K}$ be the value which maximizes*

$$M(K) = \sum_{r=1}^{\infty} \frac{K}{K + \pi^2 r^2} Z_r^2 + \log\left[\frac{\sqrt{K}}{\sinh(\sqrt{K})}\right].$$

Then

$$P\{\hat{K} \geq K_0\} \to 0 \quad \textit{as } K_0 \to \infty.$$

PROOF. Since $M(0) = 0$,

$$P\{\hat{K} \geq K_0\} \leq P\Big\{\sup_{K \geq K_0} M(K) > 0\Big\} \leq \sum_{K=K_0}^{\infty} P\Big\{\sup_{K \leq l \leq K+1} M(l) > 0\Big\}.$$

Now

$$M(K) > 0 \quad \Leftrightarrow \quad \sum_{r=1}^{\infty} \frac{K}{K + \pi^2 r^2} Z_r^2 > \log\left\{\frac{\sinh(\sqrt{K})}{\sqrt{K}}\right\},$$

and since $K/(K + \pi^2 r^2)$ and $\log\{\sinh(\sqrt{K})/\sqrt{K}\}$ are increasing functions of K, we have

$$P\{\hat{K} \geq K_0\} \leq \sum_{K=K_0}^{\infty} P\left\{\sum_{r=1}^{\infty} \frac{K+1}{K + 1 + \pi^2 r^2} Z_r^2 > \log\left\{\frac{\sinh(\sqrt{K})}{\sqrt{K}}\right\}\right\}.$$

Using Chebyshev and Whittle as in the proof of Theorem 1 gives that, for any even integer $S \geq 2$,

$$\begin{aligned} P&\left\{\sum_{r=1}^{\infty} \frac{K+1}{K + 1 + \pi^2 r^2} Z_r^2 > \log\left\{\frac{\sinh(\sqrt{K})}{\sqrt{K}}\right\}\right\} \\ &\leq B\left\{\left[\sum_{r=1}^{\infty}\left(\frac{K+1}{K+1+\pi^2 r^2}\right)^2\right] \Big/ \right. \\ &\qquad \left.\left[\log\left\{\frac{\sinh(\sqrt{K})}{\sqrt{K}}\right\} - \sum_{r=1}^{\infty} \frac{K+1}{K+1+\pi^2 r^2}\right]\right\}^{S/2} \end{aligned}$$

for some constant B,

$$\begin{aligned} B = \frac{B}{(K+1)^{S/4}} &\left\{\left[\sum_{r=1}^{\infty} \frac{(K+1)^{3/2}}{(K+1+\pi^2 r^2)^2}\right] \Big/ \right. \\ &\left.\left[\frac{1}{\sqrt{K+1}} \log\left\{\frac{\sinh(\sqrt{K})}{\sqrt{K}}\right\} - \sum_{r=1}^{\infty} \frac{(K+1)^{1/2}}{(K+1+\pi^2 r^2)}\right]^2\right\}^{S/2}. \end{aligned}$$

Standard arguments show that

$$\lim_{K \to \infty} \frac{1}{\sqrt{K+1}} \log\left\{\frac{\sinh(\sqrt{K})}{\sqrt{K}}\right\} = 1$$

$$\lim_{K \to \infty} \sum_{r=1}^{\infty} \frac{(K+1)^{3/2}}{(K+1+\pi^2 r^2)^2} \leq \frac{1}{4}$$

and

$$\lim_{K\to\infty}\sum_{r=1}^{\infty}\frac{(K+1)^{1/2}}{(K+1+\pi^2r^2)}\le\frac{1}{2}.$$

Hence, for K_0 large enough, there exists a constant C such that

$$P\{\hat{K}\ge K_0\}\le C\sum_{K=K_0}^{\infty}\frac{1}{K^{S/4}}.$$

Taking $S = 8$ gives the result. □

THEOREM 4. *Let* $\hat{K}_n$ *maximize* $M_n(K)$. *Let* $\hat{K}$ *maximize* $M(K)$. *Then* $B_n = M_n(\hat{K}_n)$ *and*

$$B_n - M(\hat{K}) \to_P 0 \quad \text{as } n\to\infty.$$

PROOF. If $M(\hat{K}) \ge M_n(\hat{K}_n)$, then

$$\begin{aligned} M(\hat{K}) - M_n(\hat{K}_n) &= M(\hat{K}) - M_n(\hat{K}) + M_n(\hat{K}) - M_n(\hat{K}_n)\\ &\le M(\hat{K}) - M_n(\hat{K}). \end{aligned}$$

Hence,

$$|M(\hat{K}) - M_n(\hat{K}_n)| \le |M(\hat{K}) - M_n(\hat{K})|.$$

Similarly, if $M(\hat{K}) \le M_n(\hat{K}_n)$, then

$$|M(\hat{K}) - M_n(\hat{K}_n)| \le |M(\hat{K}_n) - M_n(\hat{K}_n)|.$$

Hence,

$$|M(\hat{K}) - M_n(\hat{K}_n)| \le |M(\hat{K}) - M_n(\hat{K})| + |M(\hat{K}_n) - M_n(\hat{K}_n)|.$$

Given $\varepsilon > 0$, we have for each $K_0 > 0$ that

$$\begin{aligned} P\{|M(\hat{K}) - M_n(\hat{K}_n)| > \varepsilon\} &\le P\left\{\sup_{0\le K\le K_0}|M(K) - M_n(K)| > \frac{\varepsilon}{2}\right\}\\ &\quad + P\{\hat{K} > K_0\} + P\{\hat{K}_n > K_0\}. \end{aligned}$$

Taking limits first as $n\to\infty$ and then as $K_0\to\infty$ gives the theorem by Theorems 1, 2 and 3. □

Table 1 gives estimates of the 10, 5 and 1 percentiles of the null distribution of B_n for various sample sizes. The estimates for $n = \infty$ were calculated using $M(K)$. All the estimates are based on 10,000 iterations. The table also shows the number of iterations for which $\hat{K}_n = 0$. The asymptotic distribution is achieved quite closely at $n = 40$.

The maximum likelihood estimate of K was obtained by a repeated bisection method search. Almost all the likelihood curves have a single maximum. Each p-value required a computation of $M(K)$ of order $n \log n$, for each of approximately 40 K-values, for 10,000 repetitions.

TABLE 1
The null distribution of B_n, showing, for various sample sizes, estimates of the 10, 5, and 1 percentiles and the proportion of zero values based on 10,000 repetitions.

Sample size (n)	% of zero values	10%	5%	1%
5	58.9	1.47	2.25	3.89
10	61.3	1.35	2.37	5.05
20	62.9	1.16	2.18	4.87
40	64.6	1.05	1.99	4.65
60	64.9	0.99	1.93	4.35
100	65.0	1.00	1.92	4.32
200	64.6	1.02	1.88	4.30
400	65.0	1.02	1.92	4.36
Infinity	64.9	1.01	1.91	4.36
Standard errors	0.50	0.04	0.06	0.20

In general, the x_i's will not be equally spaced; in this case the log likelihood has the same form except that the λ_r's depend on the spacing. Similar asymptotic calculations should be possible in this case also, under suitable conditions on the spacings of the x_i's.

If the x's are not very unequally spaced, we would expect the test statistic calculated as if the data were equally spaced to perform about the same as the likelihood based test statistic. We have not examined this conjecture in detail.

4. Simulation study. In this section we report on the results of a simulation study comparing the power of six tests of independence:

1. The B test (B), as described in Section 2 and using critical values obtained in the simulation study of Section 3.
2. The runs test (R). Let $d_i = y_i - \bar{y}$, $i = 1, 2, \dots, n+1$. Let D be the number of runs in the sequence $d_1, d_2, \dots, d_{n+1}$. Small values of D constitute evidence against the independence hypothesis. Critical values were obtained from Table 18 of Lindley and Scott (1984).
3. Pearson's correlation coefficient (P). p-values were calculated, conditional on the x's.
4. Spearman's correlation coefficient (S).
5. Von Neumann's ratio (VN). Define

$$\mathrm{VN} = \frac{\sum_{i=1}^{n}(y_{i+1} - y_i)^2}{\sum_{i=1}^{n+1}(y_i - \bar{y})^2}$$

Small values of VN constitute evidence against independence. Critical values were obtained using the t-approximation suggested by Bingham and Nelson (1981). It can be shown that

$$Q_n(\infty) - Q_n(0) = -n \log[(n+1)\mathrm{VN}] + \sum \log[(n+1)\lambda_r].$$

6. Nyblom and Makelainen (NM). In the context of testing for autocorrelation, Nyblom and Makelainen (1983) propose the test statistic

$$L = \frac{\sum_{i=1}^{n+1}\left[\sum_{j=i}^{n+1}(y_j - \bar{y})\right]^2}{\sum_{i=1}^{n+1}(y_i - \bar{y})^2}$$

and give critical values for rejecting independence for large values of L.

The following five test functions were used:

(a) $$F(x) = x, \qquad 0 \le x \le 1;$$

(b) $$F(x) = 2e^{5x}/(1 + e^{5x}) - 1, \qquad 0 \le x \le 1;$$

(c) $$F(x) = 64x^3(1 - x)^3, \qquad 0 \le x \le 1;$$

(d) $$F(x) = (1 + \sin(3\pi x))/2, \qquad 0 \le x \le 1;$$

(e) $$F(x) = \begin{cases} (15x/6), & 0 \le x \le 0.2, \\ (5 - 10x)/6, & 0.2 \le x \le 0.4, \\ (-9 + 25x)/6, & 0.4 \le x \le 0.6, \\ (-18 + 20x)/6, & 0.6 \le x \le 0.8, \\ (-2 + 5x)/6, & 0.8 \le x \le 1. \end{cases}$$

Each function is continuous and ranges from a minimum of 0 to a maximum of 1; only (a) and (b) are monotonic.

Two values for $n + 1$ were used: $n + 1 = 20$ and 100. Four values for the standard deviation of the error distribution, (SD) were used: 0.1, 0.5, 1.0 and 2.0.

For each combination of F, n and SD, 100 datasets were generated by setting

$$x_i = \frac{2i - 1}{2(n + 1)},$$

$$y_i = F(x_i) + e_i, \qquad 1 \le i \le n + 1,$$

where $\{e_i\}$ are i.i.d. $N(0, \mathrm{SD}^2)$. The proportion of results significant at the 5% level for each of the six test procedures is given in Table 2.

The standard errors for differences between two percentages in the table never exceed 2%. (Note that the test statistics were each computed on the same data sample in order to reduce the variance of the difference between the percentages exceeding critical values.)

Reviewing the different alternatives, Runs and Von Neumann are distinctly inferior for detecting linear departures, with the other methods comparable. The same result holds for the logistic function. For the beta function, the B test is markedly superior. For the sine function, the B test is comparable to

TABLE 2

Showing, for various values of N and SD and various choices of F, the percentage of 1000 iterations for which the 5% critical values were exceeded using the B test (B), the runs test (R), Pearson's correlation (P), Spearman's correlation (S), Von Neumann's ratio test (VN), and the Nyblom–Makelainen procedure (NM).

	SD				
N	0.10	0.50	1.0	2.0	Test
		(a) $F(x) = x$			
20	100.0	58.2	18.8	8.5	B
	99.2	13.7	4.4	3.5	R
	100.0	69.3	22.0	7.9	P
	100.0	65.1	21.2	9.1	S
	100.0	29.5	10.0	7.2	VN
	100.0	67.8	21.2	8.3	NM
100	100.0	100.0	75.3	27.0	B
	100.0	52.7	13.6	6.0	R
	100.0	100.0	82.1	31.6	P
	100.0	100.0	78.2	28.7	S
	100.0	80.5	21.2	9.5	VN
	100.0	100.0	80.4	29.9	NM
		(b) $F(x) = 2\exp(5x)/(1 + \exp(5x)) - 1$			
20	100.0	53.3	16.8	7.8	B
	98.7	14.3	3.8	2.3	R
	100.0	57.4	18.2	7.7	P
	100.0	54.4	16.8	8.1	S
	100.0	32.0	9.5	5.2	VN
	100.0	58.4	17.8	8.2	NM
100	100.0	99.8	70.8	22.4	B
	100.0	46.4	12.4	7.0	R
	100.0	99.8	73.5	25.2	P
	100.0	99.7	69.5	21.7	S
	100.0	77.3	21.3	7.4	VN
	100.0	99.8	72.0	23.7	NM
		(c) $F(x) = 64x^3(1 - x)^3$			
20	100.0	58.1	14.7	7.3	B
	99.8	18.4	4.9	1.8	R
	0.0	1.5	3.2	5.4	P
	0.0	1.4	4.1	5.4	S
	100.0	48.9	14.3	7.4	VN
	97.1	6.4	4.7	5.6	NM
100	100.0	100.0	64.1	16.8	B
	100.0	69.5	18.0	8.4	R
	0.0	2.2	3.0	4.5	P
	0.0	3.0	3.0	4.2	S
	100.0	91.5	28.2	9.6	VN
	100.0	92.1	17.9	7.0	NM

Table 2 (*Continued*)

	SD				
N	**0.10**	**0.50**	**1.0**	**2.0**	**Test**
		(d) $F(x) = 0.5(1 + \sin(3\pi x))$			
20	100.0	35.2	10.2	5.8	B
	99.6	20.0	4.1	2.8	R
	0.0	1.4	3.8	4.4	P
	0.0	1.8	3.9	4.3	S
	100.0	38.1	12.2	5.9	VN
	0.1	3.5	4.4	4.6	NM
100	100.0	100.0	79.6	24.1	B
	100.0	78.4	18.9	7.5	R
	0.0	1.3	4.0	4.9	P
	0.0	1.9	3.6	4.0	S
	100.0	96.7	32.4	9.5	VN
	100.0	99.9	36.0	8.7	NM
		(e) $F(x)$ = sawtooth			
20	100.0	15.6	9.0	6.3	B
	58.7	6.3	3.1	2.4	R
	32.9	10.6	7.0	5.0	P
	43.9	10.7	6.8	5.8	S
	100.0	16.4	7.4	6.1	VN
	74.3	12.5	7.3	5.8	NM
100	100.0	85.1	27.5	10.5	B
	100.0	26.3	10.6	6.0	R
	100.0	44.6	14.5	8.7	P
	100.0	41.3	14.2	8.1	S
	100.0	51.0	14.3	8.1	VN
	100.0	63.1	18.3	8.4	NM

Von Neumann's test, and superior to the rest; it beats von Neumann's test for larger sample sizes. A similar conclusion holds for the sawtooth function.

Acknowledgment. We thank one of the referees, who noticed a significant error in our original definition of the sawtooth function.

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