The Annals of Statistics
1984, Vol. 12, No. 4, 1500–1509

GENERATING THE INTRABLOCK AND INTERBLOCK SUBGROUPS FOR CONFOUNDING IN GENERAL FACTORIAL EXPERIMENTS

BY BRUCE JAY COLLINGS

Montana State University

A simple method of directly obtaining generators for the intrablock subgroup and for the interblock subgroup of a confounding plan is presented. The method is applicable to completely general factorial experiments and can be used either to construct the design which confounds a specified set of treatment effects or to determine the confounding pattern of a specified design. This procedure avoids the usual trial and error approach to confounding and involves nothing more complicated than addition and multiplication modulo a prime number.

1. Introduction.

The problem of confounding in factorial experiments has a long history in the statistical literature. The classical approach is to identify each element in **T**, the set of all treatment combinations, with an element of some suitable algebraic structure, **A**. The principal block of the confounding plan is then obtained as the set of solutions to an appropriate system of linear equations with addition and multiplication as defined in **A**. The early work employing Galois fields, by Yates (1937), Bose and Kishen (1940), Fisher (1942), Bose (1947) and Kempthorne (1947), is limited to symmetrical prime power factorials. An extension to asymmetrical prime power factorials by White and Hultquist (1965), Raktoe (1969, 1970) and Banerjee (1970) employs finite rings. Worthley (1973) and Worthley and Banerjee (1974) further extend these finite ring methods to completely general factorials. Bailey (1977) and Giovagnoli (1977) provide an alternative extension by considering **T** as a module over some suitable finite ring. Each of these approaches allows the experimenter to control which interactions are confounded. However, the methods for solving the system of equations determining the principal block of the design are largely trial and error and frequently require the computation of the addition and multiplication tables for the particular algebraic structure being used.

In addition to the above "classical" methods, the generalized cyclic designs of John (1973) have been employed for the construction of confounding plans in general factorials by John and Dean (1975) and Dean and John (1975). Although applicable to any factorial experiment, this method does not allow the experimenter to specify in advance which interactions are to be confounded. Also, the procedure given by these authors for determining what has been confounded by a particular design is somewhat cumbersome.

Received June 1983; revised June 1984.

AMS 1980 *subject classifications*. Primary 62K15; secondary 62K10.

Key words and phrases. Confounding, factorial experiments, generators, interblock subgroup, intrablock subgroup, mixed factorial experiments.

Two other interesting procedures for confounding m^n experiments in blocks of size m^r have been suggested by Das (1964) and Cotter (1974). The essence of each method is to append $n - r$ columns to an $r \times r$ identity matrix and generate the initial block by taking all possible linear combinations of the rows of the resulting $r \times n$ matrix. Das places certain restrictions on the appended columns to control the order of the interactions which are confounded. Cotter provides an auxillary table of coefficients used to determine the appended columns. In essence, Cotter has chosen a particular system of $n - r$ equations in n variables and provided the instructions for computing the $n - r$ constrained components of the solutions in terms of the r unconstrained components.

The purpose of this paper is to describe another systematic approach to constructing confounding plans for general factorial experiments. Based on Collings (1977), vector spaces are used to describe the structure of **T**. This procedure requires nothing more complicated than modular arithmetic. This method produces two sets of generators which specify the intrablock subgroup and the interblock subgroup for the design which confounds any given set of treatment effects. This approach is also easily applied to cases involving fractional replication or multiple levels of blocking.

The term interblock subgroup is introduced and the concept of generators is reviewed in Section 2. The basic procedure is given in Section 3 for the case where each factor has the same prime number of levels. General factorial experiments are discussed in Section 4. Extensions to designs with fractional replication and multiple levels of blocking are considered in Section 5.

2. The interblock subgroup. The essence of the classical method and its various extensions and the methods of Das (1964) and Cotter (1974) is to identify **T**, the set of treatment combinations, with a suitable abelian group or other more complex algebraic structure. Each confounding plan is then based on some subgroup of **T**. Similarly, Dean and Lewis (1980, Theorem 1) prove that the initial block of any single replicate generalized cyclic design must also be a subgroup. If **S** is the principal block, or intrablock subgroup, the blocks of the design are the distinct cosets of **S** in **T**, namely, $\mathbf{S} + \mathbf{t}_0, \cdots, \mathbf{S} + \mathbf{t}_b$. Letting **R** denote the set $\{\mathbf{t}_0, \mathbf{t}_1, \cdots, \mathbf{t}_b\}$, then **T** can be represented as

$$\mathbf{T} = \mathbf{R} \oplus \mathbf{S},$$

where $\oplus$ indicates the direct sum. If the set **R** is a subgroup, it is called the interblock subgroup of the design. By giving **T** the structure described in Sections 3 and 4, it will always be possible to choose the $\mathbf{t}_i$ such that **R** is a subgroup of **T**.

Given **S** and $\mathbf{t}_0, \cdots, \mathbf{t}_{i-1}$, the ith block can always be obtained by selecting any treatment combination which does not appear in any preceding block as $\mathbf{t}_i$. In this sense, the interblock subgroup is not necessary for the construction of the design. However, the direct construction of **R** avoids the necessity of (and potential for error in) searching for each successive $\mathbf{t}_i$. Although not formally recognized as such, the existence of an interblock subgroup for certain types of designs is implicit in the procedure for constructing additional blocks from the principal block called "ringing in the changes" (see, e.g. Bailey, 1959).

Generators have often been employed in an effort to reduce the computational effort involved in constructing designs (e.g. Bailey, 1977). A subset of the elements of a group $\mathbf{G}$ is called a set of generators of $\mathbf{G}$ if every element of $\mathbf{G}$ can be expressed as a sum (allowing possible repetitions) of elements of the subset. Since any subgroup may be specified by a set of generators, the design based on $\mathbf{S}$ may be completely specified by producing a set of generators for $\mathbf{S}$ and a set of generators for $\mathbf{R}$ (if $\mathbf{R}$ is a subgroup). The only systematic approach in the literature for producing such a set of generators for $\mathbf{S}$ is that provided by Katti (1960). Katti's result only applies to the p^n case and depends on the inversion of a suitably chosen $n \times n$ matrix which does not depend on p. Clearly, this need not be a simple task. However, the procedure described in Sections 3 and 4 insures that $\mathbf{R}$ is indeed a subgroup and provides a simple, direct method for obtaining sets of generators for $\mathbf{R}$ and $\mathbf{S}$ in completely general factorial settings.

3. The p^n case. Consider an n factor experiment where each factor has p levels for some prime p. Each element $\mathbf{t} \in \mathbf{T}$ can be represented as $\mathbf{t} = (t_1, \cdots, t_n)$ with t_i an integer between 0 and $p - 1$ indicating the level of the ith factor. Clearly, $\mathbf{T}$ can be viewed as an n-dimensional vector space over the field $\mathbf{GF}(p)$. It seems reasonable, therefore, to exploit some of the properties of vector spaces (e.g. bases, orthogonal subspaces and annihilator subspaces) in the construction of confounding plans.

To more fully exploit the vector space structure, define the effect space $\mathbf{E}$ to be the set of all linear mappings of $\mathbf{T}$ into $\mathbf{GF}(p)$. ($\mathbf{E}$ is the dual space of $\mathbf{T}$ (Herstein 1964, page 146).) Any element of $\mathbf{E}$ can be represented as $\mathbf{e} = (e_1, e_2, \cdots, e_n)$ with each e_i an integer between 0 and $p - 1$. The image of any $\mathbf{t} \in \mathbf{T}$ under the mapping represented by $\mathbf{e}$ is given by $\mathbf{te}' \pmod p$. Two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are said to be equivalent if $\mathbf{e}_1 = c\mathbf{e}_2$ for $c \in \mathbf{GF}(p)$, $c \neq 0$. This equivalence relation partitions the $p^n - 1$ nonzero vectors of $\mathbf{E}$ into equivalence classes of $p - 1$ vectors each. These equivalence classes are the usual "treatment effects" (see, e.g. Kempthorne, 1979) and may be conveniently represented by the member of the equivalence class whose first nonzero component is unity.

For any subspace $\mathbf{D}$ contained in $\mathbf{E}$, the annihilator of $\mathbf{D}$ is a subspace of $\mathbf{T}$ defined by

$$\mathbf{D}^0 = \{\mathbf{t} \in \mathbf{T}\colon \mathbf{td}' = 0 \quad \text{for all } \mathbf{d} \in \mathbf{D}\}.$$

The annihilator of a subspace of $\mathbf{T}$ can be defined similarly as a subspace of $\mathbf{E}$. Since $\mathbf{D}^0$ is a subspace of $\mathbf{T}$, it can be used as the intrablock subgroup of a confounding plan for $\mathbf{T}$. The effects confounded by the design with $\mathbf{S} = \mathbf{D}^0$ as the intrablock subgroup are precisely the effects contained in the subspace $\mathbf{D}$ (see Bailey, 1977). Furthermore, there exist subspaces $\mathbf{R}$ of $\mathbf{T}$ and $\mathbf{C}$ of $\mathbf{E}$ such that

$$\mathbf{T} = \mathbf{R} \oplus \mathbf{S}, \quad \text{and} \tag{3.1}$$

$$\mathbf{E} = \mathbf{C} \oplus \mathbf{D}, \tag{3.2}$$

with $\mathbf{C}$ isomorphic to $\mathbf{S}$, $\mathbf{D}$ isomorphic to $\mathbf{R}$ and $\mathbf{C} = \mathbf{R}^0$. A basis (set of generators) for $\mathbf{T}$ which partitions into a basis for $\mathbf{R}$ and a basis for $\mathbf{S}$ can be obtained by either of the following methods.

METHOD 1. Suppose $\{\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m\}$ represent m linearly independent effects in $\mathbf{E}$. Let $\mathbf{D}_0^* = \{\mathbf{0}\}$ and for each $i = 1, \cdots, m$, let $\mathbf{D}_i^*$ be the subspace generated by $\{\mathbf{d}_1, \cdots, \mathbf{d}_i\}$ with $\mathbf{S}_i^* = \mathbf{D}_i^{*0}$ and $\mathbf{R}_i^*$ such that $\mathbf{T} = \mathbf{R}_i^* \oplus \mathbf{S}_i^*$.

THEOREM 1. *Suppose $\{\mathbf{r}_1, \cdots, \mathbf{r}_h\}$ is a basis for $\mathbf{R}_h^*$ and $\{\mathbf{s}_{h+1}, \cdots, \mathbf{s}_n\}$ is a basis for $\mathbf{S}_h^*$. There exists some j, $h + 1 \le j \le n$ and $c_i \in \mathbf{GF}(p)$, $i = h + 1, \cdots, n$, such that $\{\mathbf{r}_1, \cdots, \mathbf{r}_h, \mathbf{s}_j\}$ is a basis for $\mathbf{R}_{h+1}^*$ and $\{\mathbf{s}_{h+1}^*, \cdots, \mathbf{s}_{j-1}^*, \mathbf{s}_{j+1}^* \cdots, \mathbf{s}_n^*\}$, with $\mathbf{s}_i^* = \mathbf{s}_i + c_i\mathbf{s}_j$, is a basis for $\mathbf{S}_{h+1}^*$.*

PROOF. Since $\mathbf{d}_{h+1} \notin \mathbf{D}_h^* = \mathbf{S}_h^{*0}$, $\mathbf{s}_j\mathbf{d}'_{h+1} \neq 0 \pmod p$ for some j, $h + 1 \le j \le n$. Let $b = \mathbf{s}_j\mathbf{d}'_{n+1} \pmod p$ and define $c_i = -b^{-1}\mathbf{s}_i\mathbf{d}'_{h+1} \pmod p$ for each i, $h + 1 \le i \le n$. Then the $\mathbf{s}_i^*$ $(i \neq j)$ as given in the theorem are linearly independent and each is such that $\mathbf{s}_i^*\mathbf{d}'_g = 0$ for $g = 1, \cdots, h + 1$. Hence,

$$\{\mathbf{s}_{h+1}^* \cdots, \mathbf{s}_{j-1}^*, \mathbf{s}_{j+1}^*, \cdots, \mathbf{s}_n^*\}$$

is a basis for $\mathbf{S}_{h+1}^*$. The set $\{\mathbf{r}_1, \cdots, \mathbf{r}_h, \mathbf{s}_{h+1}^*, \cdots, \mathbf{s}_{j-1}^*, \mathbf{s}_j, \mathbf{s}_{j+1}^*, \cdots, \mathbf{s}_n^*\}$ is a basis for $\mathbf{T}$, hence, $\{\mathbf{r}_1, \cdots, \mathbf{r}_h, \mathbf{s}_j\}$ must be a basis for $\mathbf{R}_{h+1}^*$.

The design confounding $\{\mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m\}$ can be obtained by setting $\mathbf{R}_0^* = \{\mathbf{0}\}$ and $\mathbf{S}_0^* = \mathbf{T}$ and selecting any basis for $\mathbf{S}_0^*$. Applying Theorem 1 m times produces a basis for the intrablock subgroup and a basis for the interblock subgroup. This procedure involves only arithmetic modulo p and produces the desired sets of generators directly without explicitly solving any system of linear equations.

EXAMPLE 1. (Kempthorne, 1979, page 325). Consider an experiment with four factors A, B, C and D, each at 3 levels. To obtain blocks of size 9, confound the effects ABC and AC^2D^2. Thus, $\mathbf{d}_1 = (1, 1, 1, 0)$ and $\mathbf{d}_2 = (1, 0, 2, 2)$. For convenience, any basis of $\mathbf{T}$ can be represented as a 4×4 matrix with the rows as basis vectors. Beginning with the identity matrix, the first application of Theorem 1 gives $j = 1$, $b = b^{-1} = 1$, $c_2 = 2$, $c_3 = 2$ and $c_4 = 0$. Thus,

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \hline 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \tag{3.3}$$

represents an intermediate basis for $\mathbf{T}$; the first row generates $\mathbf{R}_1^*$ and the last three rows generate $\mathbf{S}_1^*$.

Applying Theorem 1 to (3.3) gives $j = 2$, $b = b^{-1} = 2$, $c_3 = 1$ and $c_4 = 2$. Hence, a set of generators for $\mathbf{T} = \mathbf{R}_2^* \oplus \mathbf{S}_2^* = \mathbf{R} \oplus \mathbf{S}$ is obtained as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \tag{3.4}$$

where the first two rows generate $\mathbf{R}$ and the last two rows generate $\mathbf{S}$, i.e. the interblock subgroup can be generated by (1, 0, 0, 0) and (2, 1, 0, 0) while the intrablock subgroup can be generated by (1, 1, 1, 0) and (1, 2, 0, 1).

It is interesting to note that in this particular case the vectors which generate the intrablock subgroup are identical to vectors representing two of the effects which are confounded, namely, ABC and AB^2D. This will not, of course, be true in general. Also, there is nothing unique about (3.4); the same design could be specified by several different sets of generators.

REMARK. The amount of effort required by the above procedure depends largely on the choice of the basis for $\mathbf{S}_0$. Any choice will work; however, a judicious choice for this basis may substantially reduce the effort required in constructing the design.

METHOD 2. The following theorems provide an alternative to the above procedure. Theorem 1 provides a simple, general proof of a result of Das (1964). Theorem 2 indicates how to proceed for arbitrary sets of independent effects. Let $\{\mathbf{d}_1, \cdots, \mathbf{d}_m\}$ be as above. $\mathscr{I}$ denotes a suitable identity matrix.

THEOREM 2. *Let $\mathscr{A}$ be the $m \times n$ matrix whose i*th *row is $\mathbf{d}_i$. If $\mathscr{A}$ has the form $[\mathscr{I} \mid \mathscr{B}]$, then the $n \times n$ matrix*

$$\begin{bmatrix} \mathscr{I} & 0 \\ -\mathscr{B}' & \mathscr{I} \end{bmatrix} \tag{3.5}$$

is a basis for $\mathbf{T}$ such that the first r rows generate $\mathbf{R}$ and the last $n - r$ rows generate $\mathbf{S}$. The $-\mathscr{B}'$ in (3.5) *indicates that each element of $\mathscr{B}'$ is replaced by its additive inverse modulo p.*

PROOF. It is clear that the rows of (3.5) represent a basis for $\mathbf{T}$. Furthermore, since $[-\mathscr{B}' \mid \mathscr{I}]\, \mathscr{A}' = 0$, it follows that the last $n - m$ rows do indeed generate $\mathbf{S} = \mathbf{D}^0$. Again, since the rows of (3.5) form a basis for $\mathbf{T}$, the first m rows of (3.5) must be a basis for $\mathbf{R}$.

THEOREM 3. *If $\{\mathbf{d}_1, \cdots, \mathbf{d}_m\}$ is a basis for $\mathbf{D}$, there exists a basis $\{\mathbf{d}_1^*, \cdots, \mathbf{d}_m^*\}$ such that the $m \times n$ matrix $\mathscr{A}^*$ whose i*th *row is $\mathbf{d}_i^*$ has the form $[\mathscr{I} \mid \mathscr{B}^*]$ for some $\mathscr{B}^*$.*

PROOF. The matrix $\mathscr{A}^*$ is obtained by converting the matrix $\mathscr{A}$ into row-echelon form and possibly interchanging columns (reordering the factors).

EXAMPLE 1 (continued). The effects ABC and AC^2D^2 do not satisfy the conditions of Theorem 2. However, AC^2D^2 and BC^2D form an alternative basis for **D** which does satisfy the conditions of Theorem 2. This gives $\mathscr{A}$ equal to

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Substituting into (3.5) gives

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}. \tag{3.6}$$

Clearly, (3.6) is equivalent to (3.4) with the first two rows generating **R** and the last two rows generating **S**.

REMARK. The choice between Method 1 and Method 2 depends largely on the amount of effort involved in applying Theorem 3. Method 2 will generally be preferred when m is small. However, in situations involving multiple blocking as discussed in Section 5, the sequential nature of Method 1 generally makes it easier to use.

Either of the above methods may, of course, also be used to determine which effects will be confounded by a given design. Since **T** and **E** are dual spaces, their roles, as well as the roles of **D** and **S**, may be interchanged. If $\{\mathbf{s}_1, \cdots, \mathbf{s}_m\}$ is a basis for the intrablock subgroup, the above procedures will produce sets $\{\mathbf{c}_1, \cdots, \mathbf{c}_m\}$ and $\{\mathbf{d}_{m+1}, \cdots, \mathbf{d}_n\}$ which generate **C** and **D**, respectively. Replacing each $\mathbf{d}_i$ by the standard representative of the equivalence class to which $\mathbf{d}_i$ belongs yields a set of independent effects which, along with their generalized interactions, will be confounded by the design based on **S**.

4. General factorials. Suppose first that each of the n factors has p^k levels for some prime p. The procedures described in Section 3 could, of course, be carried out as indicated with all arithmetic operations being performed in $\mathbf{GF}(p^k)$. However, the fact that $\mathbf{GF}(p^k)$ is a vector space over $\mathbf{GF}(p)$ suggests the computationally simpler approach of describing **T** as a vector space of dimension nk over $\mathbf{GF}(p)$. This is easily accomplished by replacing each factor with k pseudofactors, each with p levels. The problem can then be dealt with as in Section 3 with all computations performed modulo p.

In addition to computational simplicity, the use of pseudofactors provides more flexibility in the choice of block sizes and confounding patterns (see, e.g. Giovagnoli, 1977). This greater flexibility occurs because the effect space is partitioned into sets with $p - 1$ degrees of freedom rather than $p^k - 1$ degrees of freedom.

The only disadvantage to the use of pseudofactors is that some formally high order interactions in the pseudofactors may represent main effects or low order interactions in the actual factors. Any difficulties posed by this can easily be avoided by denoting each actual factor by a capital letter and denoting the corresponding pseudofactors by the same letter suitably subscripted. Under this convention, the actual order of any pseudofactor interaction is simply the number of distinct letters it contains, ignoring subscripts, e.g. $A_1A_3B_2B_4C_2$ is a three-factor interaction.

In general, suppose that the ith factor of the experiment has m_i levels where m_i is any positive integer. Each factor can be represented by a set of pseudofactors, each of which has a prime number of levels. Hence, $\mathbf{T}$ can be viewed as isomorphic to the direct sum of vector spaces, namely,

$$\mathbf{T} \cong \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \cdots \oplus \mathbf{T}_k, \tag{4.1}$$

where $\mathbf{T}_i$ is a vector space over $\mathbf{GF}(p_i)$ for some prime p_i with all the primes distinct. As before, viewing $\mathbf{T}$ as an additive group, if $\mathbf{S}$ is any subgroup of $\mathbf{T}$, then there exists a subgroup $\mathbf{R}$ such that $\mathbf{T} = \mathbf{R} \oplus \mathbf{S}$. Furthermore, from (4.1), it follows that

$$\mathbf{S} \cong \mathbf{S}_1 \oplus \mathbf{S}_2 \oplus \cdots \oplus \mathbf{S}_k,$$

and

$$\mathbf{R} \cong \mathbf{R}_1 \oplus \mathbf{R}_2 \oplus \cdots \oplus \mathbf{R}_k,$$

where $\mathbf{T}_i = \mathbf{R}_i \oplus \mathbf{S}_i$. The definitions of the sets $\mathbf{E}$, $\mathbf{C}$ and $\mathbf{D}$ given in Section 3 easily extend to the general case with decompositions similar to (3.1) and (3.2) holding. The problem of confounding any subset, $\mathbf{D}$, of the effect set, $\mathbf{E}$, reduces to confounding each $\mathbf{D}_i$ relative to the corresponding $\mathbf{T}_i$, as in Section 3. By using the obvious correspondence between elements of $\mathbf{S}_i$ and elements of $\mathbf{S}$, a set of generators for $\mathbf{S}$ may be obtained as the union of the elements of $\mathbf{S}$ which correspond to generators of the individual $\mathbf{S}_i$'s. A set of generators for $\mathbf{R}$ can be obtained similarly.

EXAMPLE 2. Consider an experiment with three factors: A at 3 levels, B at 4 levels and C at 6 levels. Suppose that the available block size is 12. Using pseudofactors A, B_1, B_2, C_1 and C_2, where A and C_2 have 3 levels each and the other three pseudofactors have 2 levels each, a suitable design can be obtained by confounding the pseudofactor effects AC_2 and $B_1B_2C_1$.

Confounding AC_2 in a 3^2 experiment corresponds to computing the annihilator of the vector (1, 1). Simple application of either of the methods of Section 3 yields the rows of

$$\left[\begin{array}{cc} 1 & 0 \\ \hdashline 2 & 1 \end{array}\right] \tag{4.2}$$

as generators for $\mathbf{R}_1$ and $\mathbf{S}_1$, respectively. Similarly, confounding $B_1B_2C_1$ in a 2^3 experiment leads to the rows of

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hdashline 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \tag{4.3}$$

as sets of generators for $\mathbf{R}_2$ and $\mathbf{S}_2$, respectively. Using the natural embeddings into **T**, (4.2) and (4.3) imply **R** may be generated by (1, 0, 0, 0, 0) and (0, 1, 0, 0, 0) while **S** may be generated by (2, 0, 0, 0, 1), (0, 1, 1, 0, 0) and (0, 1, 0, 1, 0).

5. Extensions to multiple blocking. The methods described above are particularly convenient to use for many types of designs with complicated block structure. Most of the designs described by Nelder (1965) and discussed by Bailey (1977) can be handled in this fashion.

In particular, consider the common example of a fractional replicate design with blocks within the fraction. In this case the fraction used, say **F**, is a subgroup of **T** and the principal block, **S**, is a subgroup of **F**. The set **T** can be represented as $\mathbf{Q} \oplus \mathbf{F}$ and **F** can be further represented as $\mathbf{R} \oplus \mathbf{S}$. Here **S** is the intrablock-intrafraction subgroup, **R** is the interblock-intrafraction subgroup and **Q** is the interfraction subgroup.

If the experiment consists of n factors, each with the same prime number of levels, generators for **Q** and **F** may be obtained from any independent set of defining contrasts using either method described in Section 3. Any independent set of generators for the effects to be confounded with blocks within the replicate can then be used to convert the generators for **F** into generators for **S** and generators for **R** in the same manner. For general factorials, decompose **T** as in (4.1), then proceed as above within each $\mathbf{T}_i$.

EXAMPLE 3. (Kempthorne, 1977, page 426, and Bailey, 1959). A one-ninth replicate of a 3^7 experiment is to be constructed in blocks of size 27. With factors A, B, C, D, E, F and G, effects $ABCD^2E$ and $CD^2E^2F^2G^2$ are suggested as defining contrasts with AB^2F^2G and $BCDF$ to be confounded with blocks in the fraction. Applying Method 1 with $\mathbf{d}_1 = (1, 1, 1, 2, 1, 0, 0)$ and $\mathbf{d}_2 = (0, 0, 1, 2, 2, 2, 2)$ gives the rows of

$$\left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{5.1}$$

as a basis for **T**. The first two rows generate **Q** and the last five rows generate **F**.

Applying Method 1 to the last five rows of (5.1) with $\mathbf{d}_3 = (1, 2, 0, 0, 0, 2, 1)$ and $\mathbf{d}_4 = (0, 1, 1, 1, 0, 1, 0)$ gives the rows of

$$\left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \hdashline 2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \tag{5.2}$$

as the final basis for $\mathbf{T}$. The first two rows generate $\mathbf{Q}$, the next two rows generate $\mathbf{R}$ and the last three rows generate $\mathbf{S}$ for the desired design.

Method 2 of Section 3 is somewhat more difficult to apply in the case of multiple levels of blocking. The form of (3.5) suggests that one approach is to start by generating the innermost subgroup, $\mathbf{S}$, and then work out. For example, the defining contrasts and the effects to be confounded can be used to obtain a set of generators for $\mathbf{S}$ and $\mathbf{Q} \oplus \mathbf{R}$. Sets of generators for $\mathbf{Q}$ and $\mathbf{R}$ can then be obtained by using the defining contrasts restricted to the factors present in the set $\mathbf{Q} \oplus \mathbf{R}$.

EXAMPLE 3 (continued). The 4 x 7 matrix $\mathscr{A}$, whose ith row is $\mathbf{d}_i$, is clearly not in the form required by Theorem 3. However, it can be transformed into

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 & 2 \end{bmatrix}. \tag{5.3}$$

Now, (5.3) implies, via Theorem 2, that $\mathbf{S}$ is generated by the last three rows of (5.2) while $\mathbf{Q} \oplus \mathbf{R}$ is generated by the rows of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Generators for $\mathbf{R}$ and $\mathbf{Q}$ may now be obtained by confounding $ABCD^2$ and CD^2 (i.e. the defining contrasts restricted to the first four factors) in a 3^4 experiment. This is equivalent to confounding AB and CD^2. Reordering the factors as A, C, B and D, these effects can be represented by

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

Thus, (2, 0, 1, 0) and (0, 1, 0, 1) are generators of the "intrablock subgroup" with (1, 0, 0, 0) and (0, 1, 0, 0) as generators of the "interblock subgroup" in the restricted, reordered experiment. Returning to the original experiment gives the rows of

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

as generators of $\mathbf{R}$ with the rows of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

as generators of $\mathbf{Q}$. Clearly, these are equivalent to the sets of generators obtained in (5.2). In this particular case, however, the derivation by Method 1 is probably easier, especially considering the effort involved in obtaining (5.3).

REFERENCES

BAILEY, N. T. J. (1959). The use of linear algebra in deriving prime power factorial designs with confounding and fractional replication. *Sankhya* **21** 345–354.

BAILEY, R. A. (1977). Patterns of confounding in factorial designs. *Biometrika* **64** 597–603.

BANERJEE, K. S. (1970). Some observations on the White-Hultquist procedure for the construction of confounding plans for mixed factorial designs. *Ann. Math. Statist.* **41** 1008–1113.

BOSE, R. C. (1947). Mathematical theory of the symmetrical factorial design. *Sankhya* **8** 107–166.

BOSE, R. C. and KISHEN, K. (1940). On the problem of confounding in general symmetrical factorial designs. *Sankhya* **5** 21–36.

COLLINGS, B. J. (1977). Subgroup-equivalent confounding for general factorial experiments. Unpublished masters thesis. Brigham Young University.

COTTER, S. C. (1974). A general method for confounding for symmetrical factorial experiments. *J. Roy. Statist. Soc. B* **37** 267–276.

DAS, M. N. (1964). A somewhat alternative approach for construction of symmetrical factorial designs and obtaining maximum number of factors. *Calcutta Statist. Assoc. Bull.* **13** 1–17.

DEAN, A. M. and JOHN, J. A. (1975). Single replicate factorial experiments in generalized cyclic design: II. Asymmetrical arrangements. *J. Roy. Statist. Soc. B* **37** 72–76.

DEAN, A. M. and LEWIS, S. M. (1980). A unified theory for generalized cyclic designs. *J. Statist. Plann. Inference* **4** 13–23.

FISHER, R. A. (1942). The theory of confounding in factorial experiments in relation to the theory of groups. *Ann. Eugenics* **11** 341–353.

GIOVAGNOLI, A. (1977). On the construction of factorial designs using abelian group theory. *Rend. Sem. Mat. Univ. Padova* **58** 195–206.

HERSTEIN, I. N. (1964). *Topics in Algebra.* Ginn, Waltham, Mass.

JOHN, J. A. (1973). Generalized cyclic designs in factorial experiments. *Biometrika* **60** 55–63.

JOHN, J. A. and DEAN, A. M. (1975). Single replicate factorial experiments in generalized cyclic designs: I. Symmetrical arrangements. *J. Roy. Statist. Soc. B* **37** 63–71.

KATTI, S. K. (1960). A direct method for constructing the intrablock subgroup. *Biometrics* **16** 691–694.

KEMPTHORNE, O. (1947). A simple approach to confounding and fractional replication in factorial experiments. *Biometrika* **34** 255–272.

KEMPTHORNE, O. (1979). *The Design and Analysis of Experiments.* Krieger, Huntington, N.Y.

NELDER, J. A. (1965). The analysis of randomized experiments with orthogonal block structure. 1. Block structure and the null analysis of variance. *Proc. Royal Statist. Soc. A* **283** 147–162.

RAKTOE, B. L. (1969). Combining elements from distinct finite fields in mixed factorials. *Ann. Math. Statist.* **40** 498–504.

RAKTOE, B. L. (1970). Generalized combining of elements from finite fields. *Ann. Math. Statist.* **41** 1763–1767.

WHITE, D. and HULTQUIST, R. A. (1965). Construction of confounding plans for mixed factorial designs. *Ann. Math. Statist.* **36** 1256–1271.

WORTHLEY, R. G. (1973). On the use of finite linear algebra in the derivation of factorial experiments with confounding. *Sankhya* **35** 89–94.

WORTHLEY, R. and BANERJEE, K. S. (1974). A general approach to confounding plans in mixed factorials when the number of levels of a factor is any positive integer. *Ann. Statist.* **2** 579–585.

YATES, F. (1937). The design and analysis of factorial experiments. *Imperial Bureau of Soil Sciences, Technical Communication,* No. 35.

DEPARTMENT OF MATHEMATICAL SCIENCES
MONTANA STATE UNIVERSITY
BOZEMAN, MONTANA 59717