The Annals of Statistics
1981, Vol. 9, No. 6, 1187–1195

ON THE ASYMPTOTIC ACCURACY OF EFRON'S BOOTSTRAP[1]

BY KESAR SINGH[2]

Stanford University

In the non-lattice case it is shown that the bootstrap approximation of the distribution of the standardized sample mean is asymptotically more accurate than approximation by the limiting normal distribution. The exact convergence rate of the bootstrap approximation of the distributions of sample quantiles is obtained. A few other convergence rates regarding the bootstrap method are also studied.

1. Introduction and main results. Recently, Efron (1979) introduced a very general resampling procedure, called the bootstrap, for estimating the distributions of statistics based on independent observations. The procedure is more widely applicable and perhaps has more sound theoretical basis than the popular Quenouille-Tukey jackknife. Efron considered a number of statistical problems and demonstrated the feasibility of the bootstrap method. The purpose of the present investigation is to examine the convergence of the bootstrap approximation in some basic estimation problems.

A formal description of the bootstrap goes as follows. Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size n from a population with distribution F and let $T(X_1, \cdots, X_n; F)$ be the specified random variable of interest, possibly depending upon the unknown distribution F. Let F_n denote the e.d.f. (empirical distribution function) of $\{X_1, \cdots, X_n\}$, i.e., the distribution that puts mass $1/n$ at each of the points $X_1, \cdots, X_n$. The bootstrap method is to approximate the distribution of $T(X_1, \cdots, X_n; F)$ under F by that of $T(Y_1, \cdots, Y_n; F_n)$ under F_n where $\{Y_1, \cdots, Y_n\}$ denotes a random sample of size n from F_n.

For the present asymptotic study, we have selected only very basic cases of $T(X_1, \cdots, X_n; F)$, namely $(\bar{X}_n - \mu)$, $(\bar{X}_n - \mu)/\sigma$ and $F_n^{-1}(t) - F^{-1}(t)$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, $\mu = E_F(X)$, $0 < \sigma^2 = V_F(X)$, and $F_n^{-1}(t)$ and $F^{-1}(t)$ are the right-continuous versions of the inverses of F_n and F respectively, at some fixed $t \in (0, 1)$. The attempt in this paper is to present more or less complete asymptotic results for these basic random variables. The author would like to mention here that the present paper and Bickel and Freedman (1980, 1981), which also deals with asymptotics for the bootstrap, were prepared independently at around the same period.

The main findings of this work are contained in the two theorems stated in this section. The proofs are given in Sections 2 and 3. The statements are valid for almost all sample sequences, i.e., with probability one under $F^{*\infty}$. In what follows, $\bar{Y}_n = n^{-1}\sum_{i=1}^n Y_i$, $s_n^2 = n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$, $G_n(x) = \#\{Y_i \le x; 1 \le i \le n\}/n$, $\mu_3 = E_F(X-\mu)^3$, $\hat{\mu}_3 = n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^3$, and $\rho = E_F|X-\mu|^3$. P and P^* denote probabilities under F and F_n; E and E^* denote expectations under F and F_n, respectively. $\| \ \|_\infty$ has been used for $\sup_{x\in R}| \ |$.

Parts A and B of Theorem 1 study the uniform convergence to zero of the discrepancy between the actual distribution of $n^{1/2}(\bar{X}_n - \mu)$ and the bootstrap approximation of it. Parts C, D and E concern the same convergence problem for the distribution of $n^{1/2}(\bar{X}_n - \mu)/\sigma$.

Received July, 1980; revised February, 1981.

[1] This research is supported by a Ford Foundation Grant that came through the Indian Statistical Institute.

[2] Currently at Rutgers University.

AMS 1970 *subject classifications.* 62G05, 62G15.

Key words and phrases. Bootstrap, Berry-Esseén bound, lattice distributions, central limit theorem, law of iterated logarithm, zero-one law, Edgeworth expansion.

In particular, suppose the underlying distribution is non-lattice. Then (1.5) together with the two term Edgeworth expansion for $P(n^{1/2}(\bar{X}_n - \mu)/\sigma \le x)$ implies that the bootstrap method has an edge over the approximation by the limiting normal distribution in the case of the standardized sample mean. The leading term of the Edgeworth expansion for sample means suggests that the difference in accuracies of the two approximations decreases with decreasing skewness of the underlying distribution and is non-existent for symmetric distributions. It follows from Part E of the theorem that the convergence in (1.5) is not valid in the lattice case. However, as suggested by (1.7), the effect of discreteness caused by rounding of data at higher decimal points should be negligible for moderate sample sizes.

Theorem 2 establishes the consistency of the bootstrap approximation of the distribution of $n^{1/2}\{F_n^{-1}(t) - F^{-1}(t)\}$ and provides the exact rate at which the discrepancy converges to zero. The normal approximation for this distribution is better than the corresponding bootstrap approximation provided $F'(F^{-1}(t))$ is exactly known (see Reiss (1974)). However it is rare that $F'(F^{-1}(t))$ is known. In essence, the theorem says that in the case of quantiles, the bootstrap approximation is as good as the normal approximation, with $F'(F^{-1}(t))$ replaced by a sample estimate, such that the difference between $F'(F^{-1}(t))$ and the sample estimate is $O(n^{-1/4}(\log\log n)^{1/2})$ a.s.

We now state the theorems.

THEOREM 1.
A. *If* $EX^2 < \infty$, *then*

$$\|P\{n^{1/2}(\bar{X}_n - \mu) \le x\} - P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n) \le x\}\|_\infty \to 0 \text{ a.s.} \tag{1.1}$$

B. *If* $EX^4 < \infty$, *then*

$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}(\log\log n)^{-1/2}\|P\{n^{1/2}(\bar{X}_n - \mu) \le x\} \\ &\quad - P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n) \le x\}\|_\infty = (2\sigma^2\sqrt{2\pi e})^{-1}\sqrt{2V_F((X-\mu)^2)} \quad \text{a.s.}\end{aligned} \tag{1.2}$$

where $V_F((X-\mu)^2)$ *is the variance of* $(X-\mu)^2$ *under* F.
C. *If* $E|X|^3 < \infty$, *then*

$$\begin{aligned}&\limsup_{n\to\infty} \rho\sigma^{-3}n^{1/2}\|P\{n^{1/2}(\bar{X}_n - \mu)/\sigma \le x\} \\ &\quad - P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\}\|_\infty \le 2K \quad \text{a.s.,}\end{aligned} \tag{1.3}$$

where K is the universal appearing in the Berry-Esséen bound.
D. *If* $E|X|^3 < \infty$ *and F is non-lattice, then*

$$P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} = \Phi(x) + \{\mu_3(1-x^2)/(6\sigma^3 n^{1/2})\}\phi(x) + o(n^{-1/2}) \tag{1.4}$$

uniformly in x a.s. *where* $\Phi(x)$ *and* $\phi(x)$ *are the standard normal distribution function and density, respectively; therefore, in this case*

$$n^{1/2}\|P\{n^{1/2}(\bar{X}_n - \mu)/\sigma \le x\} - P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\}\|_\infty \to 0 \quad \text{a.s.} \tag{1.5}$$

E. *If* $E|X|^3 < \infty$ *and F is lattice with span h,*

$$\begin{aligned}P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} &= \Phi(x) + \{\mu_3(1-x^2)/(6\sigma^3 n^{1/2})\}\phi(x) \\ &\quad + \{h/(\sigma n^{1/2})\}g(n^{1/2}s_n h^{-1}x)\phi(x) + o(n^{-1/2})\end{aligned} \tag{1.6}$$

uniformly in x a.s. *where* $g(y) = [y] - y + 1/2$ *for all* $y \in \mathbb{R}$. *Also, in this case,*

$$\begin{aligned}&\limsup_{n\to\infty} n^{1/2}\|P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} \\ &\quad - P\{n^{1/2}(\bar{X}_n - \mu)/\sigma \le x\}\|_\infty = h/\sqrt{2\pi\sigma^2} \quad \text{a.s.}\end{aligned} \tag{1.7}$$

THEOREM 2. *If F has bounded second derivative in a neighborhood of $F^{-1}(t)$ and $F'(F^{-1}(t)) > 0$, then* a.s.

$$\limsup_{n\to\infty} n^{1/4}(\log\log n)^{-1/2}\| P[n^{1/2}\{F_n^{-1}(t) - F^{-1}(t)\} \le x] - P^*[n^{1/2}\{G_n^{-1}(t) - F_n^{-1}(t)\} \le x]\|_\infty = K_{t,F}, \tag{1.8}$$

a constant depending upon t and F only.

2. Proof of theorem 1. In this section we give the proofs of all five parts of Theorem 1, and towards the end some remarks concerning these conclusions.

Part A. If $E\,X^2 < \infty$, $s_n^2 \to \sigma^2 > 0$ a.s. Therefore, (1.1) follows if we show that

$$\| P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} - \Phi(x)\|_\infty \to 0 \qquad \text{a.s.} \tag{2.1}$$

By the Lindberg-Feller CLT, (2.1) holds provided

$$s_n^{-2}\, E^*(X - \bar{X}_n)^2 I(|X - \bar{X}_n| \ge \epsilon\, n^{1/2} s_n) \to 0 \qquad \text{a.s.}$$

for all $\epsilon > 0$. But, since $s_n^2 \to \sigma^2$ and $\bar{X}_n \to \mu$ a.s., this essentially amounts to showing that, for all $\epsilon > 0$,

$$\textstyle\sum_{i=1}^n (X_i - \bar{X}_n)^2 I(X_i^2 \ge \epsilon\, n) = o(n) \qquad \text{a.s.} \tag{2.2}$$

$E\,X^2 < \infty$ implies $\sum_{i=1}^\infty P(X_i^2 \ge \epsilon\, i) < \infty$, and hence $\{X_i^2 \ge \epsilon\, i\}$ happens only for finitely many i's a.s. Thus, the left hand side of (2.2) is in fact bounded a.s.

Another very interesting way of seeing (2.1) is as follows. By the Berry-Esséen theorem, the left hand side of (2.1) does not exceed

$$K\, s_n^{-3} n^{-1/2} E^*|Y - \bar{X}_n|^3 \le 4\, K\, s_n^{-3} n^{-1/2}(E^*|Y|^3 + |\bar{X}_n|^3) \to 0 \qquad \text{a.s.}$$

since, as a consequence of the Marcinkiewicz-Zygmund SLLN $n^{-3/2}\sum_{i=1}^n |X_i|^3 \to 0$ a.s. if $E\,X^2 < \infty$.

Part B. Writing $s_n^2 - \sigma^2 = E^*(Y - \mu)^2 - E(X - \mu)^2 - (\bar{X}_n - \mu)^2$, and applying the law of iterated logarithm, we see that, if $E\,X^4 < \infty$, then

$$\limsup_{n\to\infty} n^{1/2}(\log\log n)^{-1/2}|s_n^2 - \sigma^2| = \sqrt{2V_F((X - \mu)^2)} \qquad \text{a.s.} \tag{2.3}$$

Due to the Berry-Esséen bound,

$$\| P\{n^{1/2}(\bar{X}_n - \mu) \le x\} - \Phi(x/\sigma)\|_\infty \le K\,\rho\,\sigma^{-3} n^{-1/2}$$

and

$$\| P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n) \le x\} - \Phi(x/s_n)\|_\infty \le K\, s_n^{-3} n^{-1/2} E^*|Y - \bar{X}_n|^3.$$

Further, using the Taylor expansion and (2.3) it is found that

$$\| \Phi(x/s_n) - \Phi(x/\sigma) - x(s_n^{-1} - \sigma^{-1})\phi(x/\sigma)\|_\infty = O(n^{-1}\log\log n) \qquad \text{a.s.}$$

In view of the above bounds, (1.2) follows from the equality

$$\limsup_{n\to\infty} n^{1/2}(\log\log n)^{-1/2}\| x(s_n^{-1} - \sigma^{-1})\phi(x/\sigma)\|_\infty = \text{r.h.s. of (1.2)},$$

which is clearly so because of (2.3) and the identity

$$\| x\,\phi(x/\sigma)\|_\infty = \sigma(2\pi\, e)^{-1/2}.$$

Part C. This part is an immediate consequence of the Berry-Esséen theorem. We approximate both the probabilities appearing in (1.3) by the standard normal distribution and collect the error bounds provided by the Berry-Esséen theorem.

Part D. Let us write $\psi(t)$ for $E(e^{itX})$ and $\psi^*(t)$ for $E^*(e^{itY})$. Appealing to Esséen's lemma (see Feller, 1970, Lemma 2, page 538), we have

$$\| P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} - \Phi(x) - \{\hat{\mu}_3(1-x^2)/(6\, s_n^3 n^{1/2})\}\phi(x) \|_\infty$$

$$\le \int_{-a\sqrt{n}}^{a\sqrt{n}} | \psi^{*n}(t/s_n\sqrt{n}) - e^{-t^2/2}\{1 + \hat{\mu}_3(it)^3/(6\, s_n^3 \sqrt{n})\} | \, t^{-1}\, dt + b/a\sqrt{n}$$

for all $a > 0$, where b is an absolute constant. For technical reasons, the above integral is broken into two parts, one over $|t| \le \delta\, n^{1/2}$ and the other over the remaining region, where $\delta \in (0, a)$ is to be chosen sufficiently small.

To exploit the non-lattice nature of F for estimating the integral over $\delta\, n^{1/2} \le |t| \le an^{1/2}$, we show that, for any fixed $a > 0$,

$$\sup_{|t|\le a} | \psi(t) - \psi^*(t) | \to 0 \text{ a.s.} \tag{2.4}$$

To this end, let us note that if $E|X| < \infty$, then

$$\sup_{|t|\le a} | \psi(t) - \psi^*(t) | \le \max\{ | \psi(t) - \psi^*(t) |;\, t = \pm 1/n, \pm 2/n, \cdots, \pm[an]/n\} + O(1/n) \quad \text{a.s.} \tag{2.5}$$

Further, by an elementary exponential inequality, it follows that for all $|t| \le a$ and $\epsilon > 0$,

$$P(|\psi(t) - \psi^*(t)| \ge \epsilon) = 0(e^{-\lambda n})$$

where $\lambda > 0$ does not depend upon t. This bound, along with (2.5) and the Bonferroni inequality leads to (2.4). Clearly, (2.4) and the fact that $\psi(t) \ne 1$ for all $t \ne 0$ imply together that the integral over the region $\delta\, n^{1/2} \le |t| \le a\, n^{1/2}$ decays exponentially fast a.s. for all $0 < \delta < a$.

To bound the integral over $|t| \le \delta\, n^{1/2}$, we expand $\psi^*(t)$ up to three terms and estimate the remainder. To do that, we write $\exp(itY)$ as $\cos(tY) + i\sin(tY)$, and expand both terms by Taylor's expansion separately and take the expectation. It turns out that, if we write

$$\psi^*(t) = 1 - s_n^2 t^2/2 + (it)^3\hat{\mu}_3/6 + t^3 r(t)/6 \tag{2.6}$$

and $E|X^3| < \infty$, then

$$\lim_{\epsilon\to\infty} \limsup_{n\to\infty}(\text{a.s.}) \sup_{|t|\le\epsilon} |r(t)| = 0 \tag{2.7}$$

If $|t/\sqrt{n}| \le \delta$ and δ is sufficiently small, one can show by using (2.7) and the expansion $\log(1+x) = \sum_{i=1}^{\infty}(-1)^{i+1}x^i/i$ (valid for $|x| < 1$) that $|r_1(t)| \le \{t^2/n + |t|^3/n^{3/2}\}^2$ for all large n a.s. where

$$r_1(t) = \log\left[1 - \frac{t^2}{2n} + \frac{t^3}{6\, s_n^3 n^{3/2}}\left\{i^3\hat{\mu}_3 + r\left(\frac{t}{s_n n^{1/2}}\right)\right\}\right] + \frac{t^2}{2n} - \frac{t^3}{6\, s_n^3 n^{3/2}}\left\{i^3\hat{\mu}_3 + r\left(\frac{t}{s_n n^{1/2}}\right)\right\}.$$

Thus,

$$\psi^{*n}(t/s_n\sqrt{n}) = e^{-t^2/2}\exp[(t^3/6\, s_n^3 n^{1/2})\{i^3\hat{\mu}_3 + r(t/s_n n^{1/2})\} + n\, r_1(t)].$$

This expression is approximated by using the bound $|e^x - 1 - x| \le |x|^2 e^{|x|}/2$, valid for all complex numbers. Combining all these bounds together we have

$$\left| \psi^{*n}\left(\frac{t}{s_n\sqrt{n}}\right) - e^{-t^2/2}\left\{1 + \frac{(it)^3\hat{\mu}_3}{6\, s_n^3\sqrt{n}}\right\} \right| \le \frac{|t|^3}{6\, s_n^3\sqrt{n}}\, r\left(\frac{t}{s_n\sqrt{n}}\right) e^{-t^2/2} + \left\{\frac{|t|^3}{\sqrt{n}} + \left(\frac{t^2}{\sqrt{n}} + \frac{|t|^3}{n^{3/2}}\right)^2\right\}^2 e^{-t^2/4}$$

for all large n a.s. provided $|t/\sqrt{n}| \le \delta > 0$ for δ sufficiently small. The desired result (1.4) is concluded from this last bound and (2.7) as a can be chosen arbitrarily large and δ arbitrarily small.

Part E. From the Borel-Cantelli lemma it follows that if F is lattice with span h, then so is F_n for all large n a.s. So, for asymptotic purposes, we can and we do assume that F_n is lattice with span h. In the lattice case the proof given for Part D breaks down in the region $\delta\, n^{1/2} \le |t| \le a\, n^{1/2}$. To arrive at (1.6), we first establish that

$$\|\Lambda(x) * U_n\|_\infty = o(n^{-1/2}) \text{ a.s.,} \tag{2.8}$$

where $\Lambda(x) = P^*\{n^{1/2}(\bar{X}_n - \bar{Y}_n)/s_n \le x\} - \Phi(x) - \{\hat{\mu}_3(1 - x^2)/(6\, s_n^3 n^{1/2})\}\phi(x)$, U_n is the uniform distribution over $[-h/2s_n n^{1/2}, h/2s_n n^{1/2}]$ and $*$ stands for the convolution operator. For (2.8) we essentially have to show that, for all $0 < \delta < a < \infty$

$$\int_{|t| \in [\delta\sqrt{n}, a\sqrt{n}]} \psi^{*n}(t/s_n n^{1/2}) \sin(t\, h/2s_n n^{1/2})(t^2 h/2s_n n^{1/2})^{-1}\, dt = o(n^{-1/2})$$

which is equivalent to

$$\int_{\delta \le |t| \le a} \psi^{*n}(t/s_n) \sin(t\, h/2\, s_n)\, dt = o(n^{-1/2}) \text{ a.s.}$$

Both the functions $\psi^*(t/s_n)$ as well as $\sin(t\, h/2\, s_n)$ have period $2\pi\, s_n/h$; hence (2.8) follows if we have, for some $0 < \epsilon < 2\pi\, \sigma/h$,

$$\int_{-\epsilon}^{(2\pi s_n/h)-\epsilon} t\, \psi^{*n}(t/s_n)\, dt = o(n^{-1/2}) \text{ a.s.}$$

As seen from (2.6) and (2.7), for all t in a neighborhood of 0, $|\psi^*(t/s_n)| \le e^{-t^2/4}$ eventually with probability one. Also, for any $0 < \epsilon < \pi\, \sigma/h$, $|\psi^*(t/s_n)|$ is bounded away from 1 over $t \in [\epsilon, 2\pi\, s_n/h - \epsilon]$ for all large n a.s., in view of (2.4). These facts lead to (2.8).

Let us now derive (1.6) from (2.8). Since, a.s.

$$[\Phi(x) + \{\hat{\mu}_3(1 - x^2)/(6\, s_n^3 n^{1/2})\}\phi(x)] * U_n = \Phi(x) + \{\hat{\mu}_3(1 - x^2)/(6\, s_n^3 n^{1/2})\}\phi(x) + o(n^{-1/2}),$$

(2.8) is the same as

$$P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} * U_n = \Phi(x) + \{\mu_3(1 - x^2)/(6\, \sigma^3 n^{1/2})\}\phi(x) + o(n^{-1/2}) \text{ a.s.} \tag{2.9}$$

The distribution of $n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n$ is lattice with span $h/s_n n^{1/2}$ and 0 is one of the points with positive mass. Consequently, the expansion given in Part D holds uniformly at all points of the form $(2j + 1)h/2\, s_n n^{1/2}$, where j denotes integers; and also

$$P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n = j\, h/s_n n^{1/2}\} = (h/s_n n^{1/2})\phi(j\, h/s_n n^{1/2}) + o(n^{-1/2})$$

uniformly over all integers j a.s. As a result of these estimates of the jumps,

$$\begin{aligned} P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} * U_n &= P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} \\ &\quad - g(x\, n^{1/2}s_n/h) \quad \text{(the jump at the nearest lattice point from } x) \\ &= P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \le x\} - (h/\sigma n^{1/2})g(x\, n^{1/2}s_n/h)\phi(x) + o(n^{-1/2}) \end{aligned} \tag{2.10}$$

uniformly in x a.s. The proof of (1.6) clearly ends by substituting (2.10) into (2.9).

Turning to (1.7), according to Theorem 3 of Esséen (1945), if F is lattice with span h and x_0 is one of its discontinuity points, then

$$P\{n^{1/2}(\bar{X}_n - \mu)/\sigma \le x\} = \Phi(x) + \frac{\mu_3(1 - x^2)}{6\, \sigma^3\sqrt{n}}\phi(x) + \frac{h}{\sigma\sqrt{n}}\, g\left(\frac{(x - x_n)\sigma n^{1/2}}{h}\right)\phi(x) + o(n^{-1/2})$$

a.s., where $x_n = \{(n\, x_0/h) - [n\, x_0/h]\}(h/\sigma\, n^{1/2})$. Thus, (1.7) amounts to showing that

$$\limsup_{n\to\infty} \|\{g((x - x_n)\sigma\, n^{1/2}/h) - g(x\, s_n n^{1/2}/h)\}\phi(x)\|_\infty = 1/\sqrt{2\pi} \quad \text{a.s.} \tag{2.11}$$

Since the function g is bounded by 1/2 in absolute value, obviously the lim sup above is less than or equal to $\phi(0) = 1/\sqrt{2\pi}$. To get the inequality another way, note that the event $\{\sigma - s_n > n^{-1/2}(\log\log n)^{1/4}\}$ happens for infinitely many n's, a.s. $xs_n n^{1/2}/h$ takes integer values at $x = jh/s_n n^{1/2}$, $j = 1, 2, \cdots$, and at such a value of x, the difference $x\sigma n^{1/2}/h - xs_n n^{1/2}/h$ equals $j(\sigma - s_n)/s_n$. As j varies from 1 to $[\theta n^{1/2}]$ for $\theta > 0$, the difference $j(\sigma - s_n)/s_n$ grows to more than $[\theta n^{1/2}]n^{-1/2}(\log\log n)^{1/4}/s_n$, infinitely often a.s. Consequently, for any fixed $\epsilon > 0$ and $\delta \in (0, 1)$, there exists $z_n \in (0, \epsilon)$ for infinitely many n's a.s., such that $z_n s_n n^{1/2}/h$ is an integer and $(z_n - x_n)\sigma n^{1/2}/h - z_n s_n n^{1/2}/h \in (1 - \delta, 1)$. For such a z_n, obviously

$$|g((z_n - x_n)\sigma n^{1/2}/h) - g(z_n s_n n^{1/2}/h)| \geq 1 - \delta.$$

These facts amount to the conclusion that the lim sup in (2.11) is greater than or equal to $(1 - \delta)\inf_{x\in(0,\epsilon)}\phi(x)$ a.s. Since $\epsilon > 0$ and $\delta \in (0, 1)$ are arbitrary, the desired result follows.

REMARK 2.1. To get some idea about the effect of dependence on the bootstrap, let us consider the simple case when the X_i are m-dependent. Since $Y_1, Y_2, \cdots, Y_n$ are conditionally independent and s_n^2 still converges to σ^2 a.s.,

$$n^{1/2}(\bar{Y}_n - \bar{X}_n) \to_{\mathscr{L}} N(0, \sigma^2) \quad \text{a.s.,}$$

whereas according to the CLT for the m-dependent processes

$$n^{1/2}(\bar{X}_n - \mu) \to_{\mathscr{L}} N(0, \sigma^2 + 2\textstyle\sum_{i=1}^{m-1}\operatorname{cov}(X_1, X_{1+i})).$$

Thus, the bootstrap as such should not be expected to provide consistent approximation even in the case of weak dependent processes; however, if the X_i are derived from some independent sequence of r.v.'s and the exact generating procedure is known, then the bootstrap can be modified suitably to get rid of such inconsistencies. A detailed study on this line seems desirable.

REMARK 2.2. $P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \leq x\}$ can be expanded up to as many terms as one wants under the usual moment conditions, but we do need to impose the Cramér condition on F. In particular, the three term expansion easily leads us to conclude that

$$\| P^*\{n^{1/2}(\bar{Y}_n - \bar{X}_n)/s_n \leq x\} - P\{n^{1/2}(\bar{X}_n - \mu)/\sigma \leq x\} \|_\infty = O(n^{-1}(\log\log n)^{1/2}) \quad \text{a.s.}$$

provided $E|X|^6 < \infty$ and the Cramér condition about F holds.

3. Proof of Theorem 2. This proof is somewhat long, so we shall separate out the major steps and present them in the form of lemmas. Further details of the proofs can be found in Singh (1980). We start off with an exponential bound.

LEMMA 3.1. *If $\xi_1, \xi_2, \cdots, \xi_u$ are* i.i.d., *$\xi_i = 1 - p$ or $-p$ with respective probabilities p and $1 - p$, then for any $u \leq N$, $p \leq B$, $Z \leq D$ and $ZNB \leq D^2$ we have*

$$P\{|\textstyle\sum_{i=1}^{u}\xi_i| \geq (1 + e/2)D\} \leq 2e^{-Z}.$$

The proof is an elementary application of Markov's inequality and we omit it.

LEMMA 3.2. *Under the conditions of Theorem* 2

$$\begin{aligned}\sup_{|x|\leq\log n}(1 + |x|)^{-1/2}|F_n(F_n^{-1}(t) + xn^{-1/2}) \\ - F_n(F_n^{-1}(t)) - F(F_n^{-1}(t) + xn^{-1/2}) + F(F_n^{-1}(t))| \\ = O(n^{-3/4}(\log\log n)^{1/2}) \quad \text{a.s.}\end{aligned}$$

PROOF. Because of the law of interated logarithm for $F_n^{-1}(t)$, it suffices to show that

$$\sup_{|x|\le\log n,|y|\le\log n}\left|(1+|x|)^{-1/2}\left(F_n\left(F^{-1}(t)+\frac{x+y}{\sqrt{n}}\right)\right.\right.$$
$$\left.\left.-F_n\left(F^{-1}(t)+\frac{y}{\sqrt{n}}\right)-F\left(F^{-1}(t)+\frac{x+y}{\sqrt{n}}\right)+F\left(F^{-1}(t)+\frac{y}{\sqrt{n}}\right)\right)\right|$$
$$=O(n^{-3/4}(\log\log n)^{1/2})\quad \text{a.s.}$$

Let us adopt the following notations: $R_n(x, y)$ = the expression inside $|\ |$ above,

$$S_{m,n}(a, b) = \textstyle\sum_{i=m+1}^{n}[I(\min(a, b) \le X_i \le \max(a, b)) - |F(b) - F(a)|]$$

where m, n are integers, $n \ge m$, $n_r = \exp(r^{1/2})$ and $C_r = \{n : n_r \le n < n_{r+1}\}$. For this choice of n_r,

$$n_r/2(r+1)^{1/2} \le n_{r+1} - n_r \le n_r/r^{1/2}.$$

An elementary approximation shows that the difference

$$\sup\{|R_n(x, y)|;\ |x| \le \log n,\ |y| \le \log n\} - L_n = O(n^{-1})$$

where $L_n = \max\{|R_n(x, y)|;\ |x|, |y| = 1/n, 2/n, \cdots, [1 + n\log n]/n\}$. For all $n \in C_r$, nL_n does not exceed the sum $L_{n1} + L(r) + L_{n2} + L_n/r$ where the new statistics are as follows:

$$L_{n1} = \max\left\{\left|S_{[n_r],n}\left(F^{-1}(t)+\frac{x+y}{\sqrt{n}}, F^{-1}(t)+\frac{y}{\sqrt{n}}\right)\right|(1+|x|)^{-1/2};\right.$$
$$\left.|x|, |y| = \frac{1}{n}, \cdots, \frac{[1+n\log n]}{n}\right\}$$

$$L(r) = \max\left\{\left|S_{1,[n_r]}\left(F^{-1}(t)+\frac{x+y}{\sqrt{n_r}}, F^{-1}(t)+\frac{y}{\sqrt{n_r}}\right)\right|(1+|x|)^{-1/2};\right.$$
$$\left.|x|, |y| = \frac{1}{r}, \cdots, \frac{[r\log n_{r+1}]}{r}\right\}$$

and

$$L_{n2} = \max\left\{\left|S_{1,[n_r]}\left(F^{-1}(t)+\frac{z}{\sqrt{n}}, F^{-1}(t)+\frac{[zr(n_r/n)^{1/2}]}{r\sqrt{n_r}}\right)\right|,\right.$$
$$\left.|z| = \frac{1}{n}, \cdots, \frac{[1+2n\log n]}{n}\right\}.$$

Along with the Bonferroni inequality, Lemma 3.1 is applied with varying choices of its parameters to see that all the four statistics are $O(n^{1/4}(\log\log n)^{1/2})$ a.s.

LEMMA 3.3. *Under the conditions of Theorem* 2, a.s.,

$$\limsup_{n\to\infty} n^{3/4}(\log\log n)^{-1/2}\{F_n(F_n^{-1}(t) + n^{-1/2})$$
$$- F_n(F_n^{-1}(t)) - F(F_n^{-1}(t) + n^{-1/2}) + F(F_n^{-1}(t))\} > 0.$$

PROOF. Let $\eta_n = \{t - F_n(F^{-1}(t))\}/F'(F^{-1}(t))$. Using Bahadur's representation of quantiles (see Bahadur, 1966), the *LIL* for $t - F_n(F^{-1}(t))$ and some standard approximations, it is verified that the lemma follows if we have

$$\text{(3.1)}\quad \limsup_{n\to\infty} n^{-1/4}(\log\log n)^{-1/2}\{S_{1,n}(F^{-1}(t) + \eta_n + n^{-1/2}, F^{-1}(t) + \eta_n)\} > 0 \quad \text{a.s.}$$

Let us define $m_r = 2^{r^2}$. For the sake of brevity, we shall write $\eta^*_{m_r}$ for $m_r^{-1}(m_r\eta_{m_r} - m_{r-1}\eta_{m_{r-1}})$ and f_t for $F'(F^{-1}(t))$. Note that, for all $r \ge 2$, $m_{r-1} \le 3(\log 2)^{-2}m_r(\log m_r)^{-2}$. Now, for all $r \ge 2$, consider the following four events:

$$A_r = \{|S_{1,m_r}(F^{-1}(t) + \eta_{m_r} + m_r^{-1/2}, F^{-1}(t) + \eta^*_{m_r} + m_r^{-1/2})| \geq m_r^{1/4}\}.$$

$$A'_r = \{|S_{1,m_r}(F^{-1}(t) + \eta_{m_r}, F^{-1}(t) + \eta^*_{m_r})| \geq m_r^{1/4}\}.$$

$$A''_r = \{|S_{1,m_{r-1}}(F^{-1}(t) + \eta^*_{m_r} + m_r^{-1/2}, F^{-1}(t) + \eta^*_{m_r})| \geq m_r^{1/4}\}.$$

$$A^*_r = \{[S_{m_{r-1}m_r}(F^{-1}(t) + \eta^*_{m_r} + m_r^{-1/2}, F^{-1}(t) + \eta^*_{m_r})] \geq \delta m_r^{1/4}(\log\log m_r)^{1/2}\}.$$

By the CLT,

$$\text{(3.2)} \qquad P\{S_{m_{r-1},m_r}(-\infty, F^{-1}(t)) \in W_r\} \to \tfrac{1}{2} \quad \text{as} \quad r \to \infty$$

where $W_r = \{-1, -2, \cdots, -m_r^{1/2}r^2\}$. We have, by an elementary property of multinomial distributions, that uniformly in $k \in W_r$

$$P\{A^*_r \mid S_{m_{r-1},m_r}(-\infty, F^{-1}(t)) = k\} \sim \Phi(-\delta(\log\log m_r)^{1/2}/(f_t^{1/2})).$$

Because of this and (3.2), $\sum_{r=1,\infty} P(A^*_r) = \infty$ if $\delta^2 < f_t$. Since the events A^*_r are independent (3.1) would follow if the events A_r, A'_r, A''_r happen only finitely often a.s. The later claims are verified through probability estimates involving Lemma 3.1. The details are omitted.

PROOF OF THEOREM 2. Let us notice first that, according to Lemma 3.2

$$\begin{aligned} F_n(F_n^{-1}(t) + xn^{-1/2}) - t &= F_n(F_n^{-1}(t) + xn^{-1/2}) - F_n(F_n^{-1}(t)) + O(n^{-1}) \quad \text{a.s.} \\ &= \{F(F_n^{-1}(t) + xn^{-1/2}) - F(F_n^{-1}(t))\} \\ &\quad + \{F_n(F_n^{-1}(t) + xn^{-1/2}) - F_n(F_n^{-1}(t)) \\ &\quad - F(F_n^{-1}(t) + xn^{-1/2}) + F(F_n^{-1}(t))\} + O(n^{-1}) \quad \text{a.s.} \\ &= f_t xn^{-1/2} + O((1 + |x|)^{1/2}n^{-3/4}(\log\log n)^{1/2}) \quad \text{a.s.} \end{aligned}$$

Using this bound, some set inequalities on $F_n^{-1}(t)$, Lemma 3.1 and the Berry-Esseén bound it is found that

$$P^*\{|G_n^{-1}(t) - F_n^{-1}(t)| \geq \log n\} = O(n^{-1}) \quad \text{a.s.,}$$

and uniformly in $|x| \leq \log n$

$$\begin{aligned} &P^*\{G_n^{-1}(t) - F_n^{-1}(t) \leq xn^{-1/2}\} \\ &\quad = \Phi((F_n(F_n^{-1}(t) + xn^{-1/2}) - t)n^{1/2}/(F_n(F_n^{-1}(t) + xn^{-1/2}))(1 - F_n(F_n^{-1}(t) + xn^{-1/2}))) \\ \text{(3.3)} &\qquad + O(n^{-1/2}) \\ &\quad = \Phi(xf_t(t(1-t))^{-1/2} + O((1 + |x|)^{1/2}n^{-1/4}(\log\log n)^{1/2})) + O(n^{-1/2}) \quad \text{a.s.} \\ &\quad = \Phi(xf_t(t(1-t))^{-1/2}) + O(n^{-1/4}(\log\log n)^{1/2}) \quad \text{a.s.} \end{aligned}$$

Putting together the estimates found so far and the Berry-Esseén bound for $F_n^{-1}(t) - F^{-1}(t)$ (see Reiss, 1974; the rate $n^{-1/2}$ in Reiss's theorem can be established easily under our conditions), we have now that the left side of (1.8) is finite. In the next paragraph we shall see that the lim sup in (1.8) > 0 a.s. The proof of the theorem is concluded using the Hewitt-Savage zero-one law.

In the special case of $x = 1$, consider the expression (3.3) above. According to Lemmas 3.2 and 3.3,

$$\infty > \limsup_{n\to\infty} n^{1/4}(\log\log n)^{1/2}\{n^{1/2}(F_n(F_n^{-1}(t) + n^{-1/2}) - t) - f_t\} > 0 \quad \text{a.s.}$$

This also means that $F_n(F_n^{-1}(t) + n^{-1/2}) = t + O(n^{-1/2})$ a.s. Thus, the expression (3.3) gives us $\limsup_{n\to\infty} n^{1/4}(\log\log n)^{-1/2}[P^*\{n^{1/2}(G_n^{-1}(t) - F_n^{-1}(t)) \leq 1\} - \Phi(f_t(t(1-t))^{-1/2}] > 0$ a.s., which implies the result desired.

Acknowledgment. This research was inspired by Professor Bradley Efron. Referees' comments helped to improve the presentation.

REFERENCES

BAHADUR, R. R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

BICKEL, P. J. and FREEDMAN, D. A. (1980). On Edgeworth Expansions for the bootstrap. Unpublished report.

BICKEL, P. J. and FREEDMAN, D. A. (1981). Some asymptotic theory for the bootstrap. *Ann. Statist.* **9** 1196–1217.

EFRON, B. (1979). Bootstrap methods: Another look at the Jackknife. *Ann. Statist.* **7** 1–26.

ESSEÉN, C. G. (1945). Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. *Acta Math.* **77** 1–125.

FELLER, W. (1970). *An Introduction to Probability Theory and its Applications.* Vol. 2, 2nd Ed., Wiley, New York.

REISS, R. D. (1974). On the accuracy of normal approximation for quantiles. *Ann. Probability* **2** 741–744.

SINGH, K. (1980). On asymptotic accuracy of Efron's bootstrap. Tech. Rept. No. 158, July 1980. Dept. Statist., Stanford Univ.

DEPARTMENT OF STATISTICS
HILL CENTER FOR MATHEMATICAL SCIENCES
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903

STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION

4 LOCATION OF KNOWN OFFICE OF PUBLICATION (*Street, City County State and ZIP Code*) (*Not printers*)
3401 INVESTMENT BLVD. # 6 HAYWARD (ALAMEDA COUNTY) CA 94545

5 LOCATION OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS (*Not printers*)
3401 INVESTMENT BLVD. # 6 HAYWARD (ALAMEDA COUNTY) CA 94545

6 NAMES AND COMPLETE ADDRESSES OF PUBLISHER EDITOR AND MANAGING EDITOR

PUBLISHER (*Name and Address*)
INSTITUTE OF MATHEMATICAL STATISTICS, 3401 INVESTMENT BL. # 6, HAYWARD, CA 94545

EDITOR (*Name and Address*)
DAVID V. HINKLEY, DEPT OF APPLIED STATISTICS, UNIV OF MINNESOTA, ST. PAUL, MN 55108

MANAGING EDITOR (*Name and Address*)
J. S. RUSTAGI, DEPT. OF STATISTICS, OHIO STATE UNIVERSITY, COLUMBUS, OH 43210

7

NAME	ADDRESS
INSTITUTE OF MATHEMATICAL STATISTICS (UNINCORPORATED NON-PROFIT SOCIETY)	3401 INVESTMENT BLVD. # 6 HAYWARD, CA 94545

8 KNOWN BONDHOLDERS MORTGAGEES AND OTHER SECURITY HOLDERS OWNING OR HOLDING 1 PERCENT OR MORE OF TOTAL AMOUNT OF BONDS MORTGAGES OR OTHER SECURITIES (*If there are none so state*)

NAME	ADDRESS
NONE	

9 FOR COMPLETION BY NONPROFIT ORGANIZATIONS AUTHORIZED TO MAIL AT SPECIAL RATES (*Section 132 122, PSM*)
The purpose function and nonprofit status of this organization and the exempt status for Federal income tax purposes (*Check one*)

[X] HAVE NOT CHANGED DURING PRECEDING 12 MONTHS [ ] HAVE CHANGED DURING PRECEDING 12 MONTHS (*If changed, publisher must submit explanation of change with this statement.*)

A TOTAL NO COPIES PRINTED (*Net Press Run*)	6353	5682
B PAID CIRCULATION 1 SALES THROUGH DEALERS AND CARRIERS, STREET VENDORS AND COUNTER SALES	0	0
2 MAIL SUBSCRIPTIONS	4574	4424
C TOTAL PAID CIRCULATION (*Sum of 10B1 and 10B2*)	4574	4424
D FREE DISTRIBUTION BY MAIL CARRIER OR OTHER MEANS SAMPLES COMPLIMENTARY AND OTHER FREE COPIES	12	12
E TOTAL DISTRIBUTION (*Sum of C and D*)	4586	4436
F COPIES NOT DISTRIBUTED 1 OFFICE USE LEFT OVER, UNACCOUNTED SPOILED AFTER PRINTING	1767	1246
2 RETURNS FROM NEWS AGENTS	0	0
G TOTAL (*Sum of E F1 and 2—should equal net press run shown in A*)	6353	5682

11 I certify that the statements made by me above are correct and complete

SIGNATURE AND TITLE OF EDITOR, PUBLISHER BUSINESS MANAGER OR OWNER
TREASURER, I.M.S.

12 FOR COMPLETION BY PUBLISHERS MAILING AT THE REGULAR RATES (*Section 132 121, Postal Service Manual*)

39 U S C 3626 provides in pertinent part No person who would have been entitled to mail matter under former section 4359 of this title shall mail such matter at the rates provided under this subsection unless he files annually with the Postal Service a written request for permission to mail matter at such rates

In accordance with the provisions of this statute I hereby request permission to mail the publication named in item 1 at the phased postage rates presently authorized by 39 U S C 3626

SIGNATURE AND TITLE OF EDITOR PUBLISHER BUSINESS MANAGER OR OWNER
TREASURER, I.M.S.