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IDENTIFIABILITY OF FINITE MIXTURES OF VON MISES DISTRIBUTIONS

BY MARTIN D. FRASER, YU-SHENG HSU, AND JOSEPH J. WALKER

Georgia State University

Identifiability of finite mixtures of von Mises distributions is proved.

1. Introduction. This note addresses the problem of identifiability of finite mixtures of von Mises distributions. Gumbel (1954), Jones and James (1969), Mardia (1972), and Mardia and Sutton (1975) are among those who suggest mixtures of circular distributions to model multimodal measurements. Yet, to our knowledge, the question of identifiability (Teicher, 1963) in this case has not been settled in the literature. The property of identifiability is an important consideration when estimating the parameters of a mixture (e.g., Maritz, 1970).

2. Identifiability. Let $\{f(\theta; \boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \Omega\}$ be a class of continuous density functions, where Ω is a parameter space. This class is *identifiable* (Yakowitz and Spragins (1968), Theorem A) if for any distinct set of parameters $\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_k$, $k \geq 1$, and real $\lambda_1, \cdots, \lambda_k$, the relation $\sum_{i=1}^{k} \lambda_i f(\theta; \boldsymbol{\alpha}_i) \equiv 0$ implies that $\lambda_i = 0$, $i = 1, \cdots, k$. In this note we consider the identifiability of the class of von Mises distributions, having density

$$f(\theta; \mu,\kappa) = [2\pi I_0(\kappa)]^{-1} \exp[\kappa \cos(\theta - \mu)],$$

for $0 \leq \theta < 2\pi$ and 0 elsewhere, where $0 \leq \mu < 2\pi$, $\kappa > 0$ and $I_n(\kappa)$ is the modified Bessel function of the first kind.

THEOREM. *The class of von Mises distributions is identifiable.* The proof of the theorem uses the following two lemmas, which can be easily proved.

LEMMA 1. *For any positive integer n, there exist coefficients $a_n(i)$, $b_n(i)$, such that*

$$\cos^{2n-1} x = \sum_{j=1}^{n} a_n(n-j+1)\cos((2n-2j+1)x),$$

$$\cos^{2n} x = \sum_{j=0}^{n} b_n(n-j)\cos(2(n-j)x).$$

Furtherfore these coefficients satisfy

(i) $a_n(1) \geq \cdots \geq a_n(n) \geq 0, \qquad b_n(1) \geq \cdots \geq b_n(n) \geq 0, \qquad$ *and* $\quad b_n(1) \geq b_n(0) \geq 0,$

(ii) $$\sum_{j=1}^{n} a_n(j) = \sum_{j=0}^{n} b_n(j) = 1,$$

and

(iii) $$\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} b_n(1) = 0.$$

LEMMA 2. *Let $g(n)$ be a real-valued sequence and $\{a_n(i) : i = 1, \cdots, n\}$, $n = 1, 2, \cdots$, be a triangular sequence. If*

(i) $$a_n(1) \geq \cdots \geq a_n(n) \geq 0$$

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for all n,

(ii) $$\lim_{n\to\infty} a_n(1) = 0,$$

(iii) $$\sum_{i=1}^{n} a_n(i)$$

is bounded, and

(iv) $$\lim_{n\to\infty} g(n) = 0,$$

then

$$\lim_{n\to\infty} \sum_{i=1}^{n} a_n(i) g(i) = 0.$$

PROOF OF THEOREM. Without loss of generality, we can assume that $\mu_i \neq \pi/2$ or $3\pi/2$ for all i and that $\mu_i + \mu_j \neq 2\pi$ for all $i \neq j$. (Otherwise, consider alternatively the densities $f(\theta; \mu_i^*, \kappa_i)$, $i = 1, \cdots, k$, where $\mu_i^* = \mu_i + e$, $i = 1, \cdots, k$, with $0 < \epsilon < \min\{\min_{\mu_i \neq \pi/2} |\mu_i - \pi/2|, \min_{\mu_i \neq 3\pi/2} |\mu_i - 3\pi/2|, \min_{\mu_i + \mu_j < 2\pi} [\pi - (\mu_i + \mu_j)/2], \min_i (2\pi - \mu_i)\}$.) Suppose $\sum_{i=1}^{k} \lambda_i f(\theta; \mu_i, \kappa_i) = 0$. Upon multiplying by $\cos(n\theta)$ and integrating over $0 \leq \theta < 2\pi$, we obtain for any $n \geq 1$

(1) $$\sum_{i=1}^{k} \lambda_i \cos(n\mu_i) I_n(\kappa_i)/I_0(\kappa_i) = 0.$$

We may assume that $\kappa_1 \geq \cdots \geq \kappa_k$. If $\kappa_1 = \cdots = \kappa_k$, $\sum_{i=1}^{m} \lambda_i \cos(n\mu_i) = 0$ follows trivially from (1) with $m = k$. Otherwise let m be such that $\kappa_1 = \cdots = \kappa_m > \kappa_r$ for all $r > m$. Then $\mu_1, \cdots, \mu_m$ are distinct and $\lim_{n\to\infty}[I_n(\kappa_r)/I_0(\kappa_r)]/[I_n(\kappa_1)/I_0(\kappa_1)] = 0$ for all $r > m$. In either case, $\lim_{n\to\infty} \sum_{i=1}^{m} \lambda_i \cos(n\mu_i) = 0$. If $m = 1$, then $\lambda_1 = 0$ trivially, so assume $m \geq 2$. Using Lemmas 1 and 2 with $g(n) = \sum_{i=1}^{m} \lambda_i \cos(n\mu_i)$, we have $\lim_{n\to\infty} \sum_{i=1}^{m} \lambda_i \cos^n \mu_i = 0$. Because $\mu_i + \mu_j \neq 2\pi$ for any $i \neq j$, there are at most two μ_j, $j = 1, \cdots, m$, such that $|\cos\mu_j| \geq |\cos\mu_i|$ for all $i = 1, \cdots, m$ (say $j = 1$ or $j = 1$ or 2).

Case 1: $|\cos\mu_1| > |\cos\mu_i|$ for all $i = 2, \cdots, m$. Then $\lim_{n\to\infty}(\cos^n \mu_i)/(\cos^n \mu_1) = 0$ for all $i = 2, \cdots, m$, which implies $\lambda_1 = 0$.

Case 2: $\cos\mu_1 = -\cos\mu_2$. Then for $m \geq 3$, $\lim_{n\to\infty}(\cos^n \mu_i)/(\cos^n \mu_1) = 0$ for all $i = 3, \cdots, m$, so for $m \geq 2$, $\lim_{n\to\infty}[\lambda_1 + \lambda_2(\cos\mu_2/\cos\mu_1)^n] = 0$. Thus $\lambda_1 + \lambda_2 = \lambda_1 - \lambda_2 = 0$ which implies $\lambda_1 = \lambda_2 = 0$.

By induction, we conclude that $\lambda_i = 0$ for all $i = 1, \cdots, k$.

NOTE. The theorem remains true when the uniform distribution ($\kappa = 0$) is included.

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DEPARTMENT OF MATHEMATICS
GEORGIA STATE UNIVERSITY
ATLANTA, GEORGIA 30303