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ON TRIGONOMETRIC SERIES ESTIMATES OF DENSITIES

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It is pointed out that several results due to Walter and Blum do not hold strictly as they are stated. We find expansions for the mean square errors and mean integrated square errors of trigonometric series estimates of densities, and use them to compare the efficiencies of the estimates.

1. Introduction. In a recent paper in this journal, Walter and Blum (1979) examined a large class of estimates of density functions which includes the classical Fourier series, or Dirichlet kernel, estimates on $(-\pi, \pi)$. They concluded that "the rate [of convergence of the mean integrated square error] for the Dirichlet sequence estimator approaches $O(n^{-1})$ for sufficiently differentiable densities". This remark ignores the influence of "edge effects", or the Gibbs phenomenon, on the bias of trigonometric series estimates of a density. The bias converges to zero more quickly in the interior of $(-\pi, \pi)$ than towards the boundary, and the mean integrated square error is dominated by the bias near $\pm\pi$. The net result is that even if the unknown density f has an infinite number of bounded derivatives on $(-\pi, \pi)$ the fastest rate of convergence of the MISE may be as poor as $O(n^{-1/2})$. Technically speaking, the error in Walter and Blum's argument is that they did not declare their Sobolev spaces to be *periodic*; see Wahba (1975, pages 24–25) for a discussion of the difference between periodic and aperiodic spaces.

If we ask that the jth derivative of f, whenever it exists, satisfies $f^{(j)}(-\pi+) = f^{(j)}(\pi-)$ then Walter and Blum's statement is correct. However, in the majority of applications the values of f and its derivatives at $\pm\pi$ will be unknown, and the assumption of periodicity will be unrealistic.

We shall find the dominant terms in expansions of the MSE and MISE. Similar results are provided for the cosine and sine series estimates and the Fejér estimate, introduced by Kronmal and Tarter (1968). The results are reminiscent of expansions for kernel estimators, and minimal values of the MISE may be obtained by using Lemma 4a of Parzen (1962).

2. Fourier series estimators. Suppose f has its support confined to $(-\pi,\pi)$ and has a convergent Fourier series expansion at $x \in (-\pi,\pi)$. An estimate of $f(x)$ based on an independent sample $X_1, X_2, \cdots, X_n$ is given by

$$\hat{f}_n(x; m) \equiv (2\pi)^{-1}[1 + 2 \textstyle\sum_1^m (\hat{a}_j \cos jx + \hat{b}_j \sin jx)] = n^{-1} \sum_1^n D_m(x - X_i),$$

where $\hat{a}_j = n^{-1} \sum_1^n \cos jX_i$ and $\hat{b}_j = n^{-1} \sum_1^n \sin jX_i$ are estimates of the Fourier coefficients $a_j = \int_{-\pi}^{\pi} f(u) \cos ju\, du$ an $b_j = \int_{-\pi}^{\pi} f(u) \sin j\, du$, and $D_m(u) = \sin [(2m + 1)u/2]/2\pi \sin (u/2)$ is the Dirichlet kernel. If f is bounded on $(-\pi, \pi)$ and continuous at x then $\operatorname{Var} \hat{f}_n(x; m) = mf(x)/n\pi + o(m/n)$. Two integrations by parts show that if f has two derivatives on $(-\pi, \pi)$ then

$$a_j = (-1)^j j^{-2}[f'(\pi-) - f'(-\pi+)] - j^{-2}a_j'', \qquad b_j = (-1)^{j+1} j^{-1}[f(\pi-) - f(-\pi+)] - j^{-2}b_j'',$$

where a_j'' and b_j'' are the Fourier coefficients of f''. If f'' is of bounded variation then

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$|a_j''| + |b_j''| = O(j^{-1})$ (Whittaker and Watson (1927, page 172)), and so $b(x;\, m) \equiv f(x) - E[\hat{f}_n(x;\, m)] = \pi^{-1} g_m(x) + O(m^{-2})$ uniformly in $|x| < \pi$, where

$$g_m(x) = [f'(\pi-) - f'(-\pi+)] \sum_{m+1}^{\infty} (-1)^j j^{-2} \cos jx - [f(\pi-) - f(-\pi+)] \sum_{m+1}^{\infty} (-1)^j j^{-1} \sin jx.$$

Using Abel's transformation we see that $g_m(x) = O(m^{-1})$ for each (fixed) x, and if we choose $m = Cn^{1/3}$ then $E[\hat{f}_n(x;\, m) - f(x)]^2 = O(n^{-2/3})$. Generalizations may be obtained under the conditions $f^{(j)}(\pi-) = f^{(j)}(-\pi+)$ for $0 \le j \le r - 1$, say. However, if $f(\pi-) \neq f(-\pi+)$ this rate is not attained *uniformly* in x.

THEOREM 1. *Suppose f has $r + 1 \ge 1$ absolutely integrable derivatives on $(-\pi, \pi)$, and $f^{(j)}(\pi-) = f^{(j)}(-\pi+)$ for $0 \le j \le r - 1$. (If $r = 0$, this condition is null.) Then*

$$\int_{-\pi}^{\pi} E[\hat{f}_n(x;\, m) - f(x)]^2\, dx = m/n\pi + [f^{(r)}(\pi-) - f^{(r)}(-\pi+)]^2/(2r+1)\pi m^{2r+1} + o(m/n + m^{-2r-1}).$$

PROOF. By Parseval's equality, $\int_{-\pi}^{\pi} b(x;\, m)^2\, dx = \pi^{-1} \sum_{m+1}^{\infty} (a_j^2 + b_j^2)$. Also, writing $i = \sqrt{-1}$,

$$a_j + ib_j = \int_{-\pi}^{\pi} e^{iju} f(u)\, du = (-1)^r (ij)^{-r} \int_{-\pi}^{\pi} e^{iju} f^{(r)}(u)\, du$$
$$= (-1)^{r+j} (ij)^{-(r+1)} [f^{(r)}(\pi-) - f^{(r)}(-\pi+)] + o(j^{-r-1})$$

by the Riemann-Lebesgue lemma. Therefore

$$\int_{-\pi}^{\pi} b(x;\, m)^2\, dx = [f^{(r)}(\pi-) - f^{(r)}(-\pi+)]^2/(2r+1)\pi m^{2r+1} + o(m^{-(2r+1)}),$$

and since

$$n \int_{-\pi}^{\pi} \mathrm{Var}[\hat{f}_n(x;\, m)]\, dx = \int_{-\pi}^{\pi} D_m(u)^2\, du + O(1) = m/\pi + o(m),$$

the result follows.

If f has its support confined to $(0, \pi)$m, two estimators of $f(x)$ $(0 < x < \pi)$ are

$$\hat{f}_{1,n}(x;\, m) = \pi^{-1}[1 + 2 \textstyle\sum_1^m \hat{a}_j \cos jx] \qquad \text{and} \qquad \hat{f}_{2,n}(x;\, m) = 2\pi^{-1} \textstyle\sum_1^m \hat{b}_j \sin jx.$$

If f'' is of bounded variation then $\hat{f}_{1,n}(x;\, n^{1/5})$ has a MSE of $O(n^{-4/5})$, and if f' is of bounded variation then $\hat{f}_{2,n}(x;\, n^{1/3})$ has a MSE of $O(n^{-2/3})$. However, these MSE are not attained uniformly.

THEOREM 2. *Suppose f has $2r + 2 \ge 2$ absolutely integrable derivatives on $(0, \pi)$, and $f^{(2j+1)}(\pi-) = f^{(2j+1)}(0+) = 0$ for $0 \le j \le r - 1$. (If $r = 0$ this condition is null.) Then*

$$\int_0^{\pi} E[\hat{f}_{1,n}(x;\, m) - f(x)]^2\, dx = m/n\pi + 2[f^{(2r+1)}(\pi-)^2 + f^{(2r+1)}(0+)^2]/(4r+3)\pi m^{4r+3} + o(m/n + m^{-4r-3}).$$

If f has $2r + 1$ absolutely integrable derivatives, and $f^{(2j)}(\pi-) = f^{(2j)}(0+) = 0$ for $0 \le j \le r - 1$, then

$$\int_0^\pi E[\hat{f}_{2,n}(x; m) - f(x)]^2 \, dx = m/n\pi + 2[f^{(2r)}(\pi-)^2 + f^{(2r)}(0+)^2]/(4r + 1)\pi m^{4r+1} + o(m/n + m^{-4r-1}).$$

3. Cesàro means of Fourier estimators. Analogues of theorems 1 and 2 may be obtained for the first Cesàro means of the Fourier, cosine or sine series estimators. We consider only the first two, given by $\hat{f}_n^*(x; m) = (m + 1)^{-1}\sum_o^m \hat{f}_n(x; j) = n^{-1}\sum_1^n F_m(x - X_i)$, and $\hat{f}_{1,n}^* = (m + 1)^{-1} \sum_o^m \hat{f}_{1,n}(x; j)$, respectively, where $F_m(u) = \{\sin[(m + 1)u/2]/\sin(u/2)\}^2/2\pi(m + 1)$ is the Fejér kernel. Both these estimators are guaranteed to be nonnegative.

THEOREM 3. *Suppose f has its support confined to $(-\pi, \pi)$ and has an absolutely integrable derivative on $(-\pi, \pi)$. Then*

$$\int_{-\pi}^\pi E[\hat{f}_n^*(x; m) - f(x)]^2 \, dx = m/3n\pi + 2[f(\pi-) - f(-\pi+)]^2/m\pi + o(m/n + 1/m)$$

as m and $n \to \infty$. If $f(\pi-) = f(-\pi+)$, and if f' is square integrable and satisfies a uniform Lipshitz condition on $(-\pi, \pi)$ of order $\alpha > ½$, then

$$\int_{-\pi}^\pi E[\hat{f}_n^*(x; m) - f(x)]^2 \, dx = m/3n\pi + m^{-2} \int_{-\pi}^\pi |f'(x)|^2 \, dx + o(m/n + m^{-2}).$$

THEOREM 4. *Suppose f has its support confined to $(0, \pi)$, $|f''|$ is integrable on $(0, \pi)$, and $\tilde{f}'$ is continuous and of bounded variation on each interval $[\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$. If $0 < x < \pi$ then*

$$E[\hat{f}_{1n}^*(x; m) - f(x)]^2 = mf(x)/3n\pi + m^{-2}|\tilde{f}'(x)|^2 + o(m/n + m^{-2})$$

as m and $n \to \infty$. Furthermore,

$$\int_0^\pi E[\hat{f}_{1n}^*(x; m) - f(x)]^2 \, dx = m/3n\pi + m^{-2} \int_0^\pi |f'(x)|^2 \, dx + o(m/n + m^{-2}).$$

Here $\tilde{f}(x) = (2\pi)^{-1} \int_0^\pi [f(x - u) - f(x + u)]\cot(u/2)du$ is the Hilbert transform of f, which is defined by $f(u) = f(-u)$ on $(-\pi, 0)$ and extended from $(-\pi, \pi]$ to $(-\infty, \infty)$ by periodicity. Note that $\int_0^\pi |f'(x)|^2 \, dx = \int_0^\pi |\tilde{f}'(x)|^2 \, dx$.

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REFERENCES

KRONMAL, R. and TARTER, M. (1968). The estimation of probability densities and cumulatives by Fourier series methods. *J. Amer. Statist. Assoc.* **63** 925–952.

PARZEN, E. (1962). On the estimation of a probability density and mode. *Ann. Math. Statist.* **33** 1065–1076.

WAHBA, G. (1975). Optimal convergence properties of variable knot, kernel, and orthogonal series methods for density estimation. *Ann. Statist.* **1** 15–29.

WALTER, G. and BLUM, J. (1979). Probability density estimation using delta sequences. *Ann. Statist.* **7** 328–340.

WHITTAKER, E. T. and WATSON, G. N. (1977). *A Course of Modern Analysis.* 4th ed., Cambridge Univ. Press.

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