The Annals of Statistics
1981, Vol. 9, No. 1, 190–193

ON THE ALMOST SURE CONVERGENCE OF THE PERMUTATION DISTRIBUTION FOR ALIGNED RANK TEST STATISTICS IN RANDOMIZED BLOCK DESIGNS[1]

BY SERGE TARDIF

Université de Montréal

With the ranking after alignment procedure, a conditional permutational argument is introduced which gives rise to a conditional permutation distribution for the test statistic in randomized block designs. In this paper, the almost sure and uniform convergence of this distribution is derived for a wide class of test statistics as the number of blocks goes to infinity.

1. Introduction and the main result. Consider N observations divided into n blocks in which complete randomization is employed. Let, for $1 \leq i \leq n$ and $1 \leq j \leq p$ (≥ 2), m_{ij} (≥ 1) be the number of observations in the ith block under the jth treatment and define $M_i = \sum_{j=1}^{p} m_{ij}$; then $\sum_{i=1}^{n} M_i = N$. Furthermore let each observation be described by the familiar model

$$X_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}, \qquad k = 1, \cdots, m_{ij}, \qquad j = 1, \cdots, p, \qquad i = 1, \cdots, n,$$

where the α_i's and the β_j's are the block and treatment effects respectively. It is assumed that the block vectors of error components $(\epsilon_{i11}, \cdots, \epsilon_{i1m_{i1}}, \cdots, \epsilon_{ip1}, \cdots, \epsilon_{ipm_{ip}})$, $1 \leq i \leq n$, are independent and possess continuous joint cdf which are symmetric in their M_i arguments. The hypothesis of interest, namely, that there is no difference among the treatment effects, can be expressed as $H_0: \beta_1 = \cdots = \beta_p$.

Hodges and Lehmann (1962) introduced the ranking after alignment procedure to deal with this problem. This procedure is based on the combined ranking of the aligned observations

$$Y_{ijk} = X_{ijk} - \tilde{X}_i, \qquad k = 1, \cdots, m_{ij}, \qquad j = 1, \cdots, p, \qquad i = 1, \cdots, n,$$

where $\tilde{X}_i$ is any translation invariant symmetric function of the ith block observations such that the probability of two aligned observations being equal is zero. The block average and the block median for even block sizes satisfy these conditions. However the vector of ranks is not distribution free, even under H_0. To overcome this difficulty a conditional permutational argument is introduced. Let the term configuration denote the set $\mathscr{C}_n = \{(Y_{i(1)}, \cdots, Y_{i(M_i)}): 1 \leq i \leq n\}$ where $Y_{i(k)}$ is the kth smallest observation among $Y_{i11}, \cdots, Y_{ipm_{ip}}$. Under the null hypothesis, for each $i = 1, \cdots, n$, the joint cdf of $(Y_{i11}, \cdots, Y_{ipm_{ip}})$ remains invariant under the $M_i!$ permutations of the coordinates among themselves. Thus there exists a group of $\pi_n = \prod_{i=1}^{n} (M_i!)$ intrablock permutations which maps the sample space onto itself and leaves the distribution of the sample point invariant. Hence, conditioned on the configuration, the distribution of $(Y_{i11}, \cdots, Y_{ipm_{ip}}, \cdots, Y_{n11}, \cdots, Y_{npm_{np}})$ over the π_n intrablock permutations will be equally likely, each realization having the common conditional probability π_n^{-1}. This conditional permutational probability measure is denoted by $\mathscr{P}_n[\cdot \mid \mathscr{C}_n]$. Letting $R_{N;ijk}$ be the rank of Y_{ijk} among the N aligned observations and defining, for a sequence of scores $(a_N(1), \cdots, a_N(N))$:

Received January 1979; revised June 1979.

[1] This paper is based on part of the author's Ph.D. thesis at Université de Montréal and was written with the partial support of the National Research Council of Canada, Grant Nos. A3038 and A3114.

AMS 1970 *subject classifications.* Primary 62G10; secondary 60F05.

Key words and phrases. Randomized block designs, ranking after alignment, linear rank statistics, conditional permutation distribution, almost sure and uniform convergence.

$$a_N(R_{N;i\cdot\cdot}) = M_i^{-1} \sum_{j=1}^{p} \sum_{k=1}^{m_{ij}} a_N(R_{N;ijk}), \qquad i = 1, \cdots, n, \tag{1.1}$$

$$\tau_{Ni}^2 = \sum_{j=1}^{p} \sum_{k=1}^{m_{ij}} [a_N(R_{N;ijk}) - a_N(R_{N;i\cdot\cdot})]^2, \qquad i = 1, \cdots, n, \tag{1.2}$$

$$\sigma_{N;jj'} = \sum_{i=1}^{n} [M_i(M_i - 1)]^{-1} m_{ij}(\delta_{jj'} M_i - m_{ij'})\tau_{Ni}^2, \quad j, j' = 1, \cdots, p-1,$$

$$S_{Nj} = \sum_{i=1}^{n} \sum_{k=1}^{m_{ij}} [a_N(R_{N;ijk}) - a_N(R_{N;i\cdot\cdot})], \qquad j = 1, \cdots, p-1,$$

where $\delta_{jj'}$ is the Kronecker delta, one easily sees that under $\mathscr{P}_n[\cdot \mid \mathscr{C}_n]$ the quantities (1.1) and (1.2) are fixed, each S_{Nj} is the sum of n independent random variables and the vector $\mathbf{S}_N = (S_{N1}, \cdots, S_{N,p-1})'$ has expectation zero and covariance matrix $\mathbf{\Sigma}_N = ((\sigma_{N;jj'}))$. Hence the quadratic form $\mathscr{Q}_N = \mathbf{S}_N' \mathbf{\Sigma}_N^{-1} \mathbf{S}_N$ is obtained and is used as a test statistic.

Restricting themselves to linear scores, Hodges and Lehmann (1962) and Mehra and Sarangi (1967) showed that, under very weak conditions on $\tilde{X}_i$, the conditional permutation distribution of $\mathscr{Q}_N$ tends almost surely and uniformly in the configuration to a chi-square distribution, as the number of blocks goes to infinity. The former considered the case $p = 2$ and the latter generalized this result for $p \geq 2$. On the other hand, Sen (1968) considered a large class of aligned rank tests. Assuming the observations are aligned on the mean, the cell sizes m_{ij} are independent of i and the scores are generated by a function satisfying a set of generalized Chernoff-Savage conditions, he established the convergence in probability of the conditional permutation distribution of $\mathscr{Q}_N$. Puri and Sen (1971) further generalized this result by lifting the restriction of the alignment on the mean.

It is the purpose of the present paper to generalize the earlier results of Hodges and Lehmann (1962) and Mehra and Sarangi (1967) by showing that

$$\lim_{n\to\infty} \sup_x | \mathscr{P}_n[\mathscr{Q}_n \leq x \mid \mathscr{C}_n] - P[\chi_{p-1}^2 \leq x]| = 0, \text{ uniformly in } \mathscr{C}_n, \tag{1.3}$$

under the following set of assumptions:

A1. there exists a constant η, independent of n, such that

$$M_i \leq \eta \quad \text{for } i = 1, \cdots, n \quad \text{and } n \geq 1;$$

A2. after alignment, each block contains at least one positive and one negative observation. The sequence of scores is generated by a square-integrable function ϕ in the sense that $\lim_{N\to\infty} \int_0^1 \{a_N(1 + [uN]) - \phi(u)\}^2\, du = 0$, where ϕ satisfies

A3. for a certain $0 < u_0 < 1$, $\phi(u) \leq (\geq)\phi(u_0)$ on $(0, u_0)$, $\phi(u) \geq (\leq)\phi(u_0)$ on $(u_0, 1)$ and ϕ is not equal a.e. to $\phi(u_0)$ on $(0, u_0)$ nor on $(u_0, 1)$.

Thus, as a special case, (1.3) will hold for a test statistic defined with either the exact or the approximate normal scores.

2. Proof of the main result. Let the characteristic roots of $\mathbf{\Sigma}_N$ be given by $l_{N1}, \cdots, l_{N,p-1}$ in order of magnitude. It is first seen that $N^{-1} l_{N1}$ is asymptotically bounded away from zero. Consider the finite class of matrices $\mathbf{\Sigma}(\mathbf{m})$, $\mathbf{m} = (m_1, \cdots, m_p)$, with entries $[M(M - 1)]^{-1} m_j(\delta_{jj'} M - m_{j'})$, $j, j' = 1, \cdots, p-1$, where $p \leq \sum_{j=1}^{p} m_j = M \leq \eta$, for η given by A1. These matrices are positive definite. In fact, for any nonnull $\mathbf{x} = (x_1, \cdots, x_{p-1})'$, one has

$$\mathbf{x}'\mathbf{\Sigma}(\mathbf{m})\mathbf{x} = (M - 1)^{-1} \{\textstyle\sum_{j=1}^{p-1} m_j x_j^2 - M^{-1}(\sum_{j=1}^{p-1} m_j x_j)^2\} \tag{2.1}$$

which is immediately seen to be positive if the x_j's are all equal; if they are not, then following the lines of Lemma 2.2 of Mehra and Sarangi (1967), (2.1) can be bounded below by the variance of some nondegenerate random variable. Let $L > 0$ be the minimum of the smallest characteristic root of each matrix $\mathbf{\Sigma}(\mathbf{m})$. Further let $\mathbf{P}_N$ be the orthogonal matrix diagonalizing $\mathbf{\Sigma}_N$ and denote by $\mathbf{P}_{N1}$ its first column. Since $\mathbf{P}_{N1}' \mathbf{P}_{N1} = 1$, one has

$$l_{N1} = \mathbf{P}_{N1}' \mathbf{\Sigma}_N \mathbf{P}_{N1} = \textstyle\sum_{i=1}^{n} \{\mathbf{P}_{N1}' \mathbf{\Sigma}(m_{i1}, \cdots, m_{ip}) \mathbf{P}_{N1}\} \tau_{Ni}^2 \geq L \sum_{i=1}^{n} \tau_{Ni}^2.$$

Hence it suffices to show that $N^{-1} \sum_i \tau_{Ni}^2$ is asymptotically bounded away from zero and for that matter one may restrict his attention to a particular sequence of scores. To see this, denote by $\tau_N^2(a)$ and $\tau_N^2(b)$ the variable $\sum_i \tau_{Ni}^2$ defined with a_N and b_N respectively, where the latter are two sequences of scores generated by ϕ. Further let, for $1 \le k \le M_i$ and $1 \le i \le n$, $R_{N;i(k)}$ be the rank of $Y_{i(k)}$ among $Y_{1(1)}, \cdots, Y_{1(M_1)}, \cdots, Y_{n(1)}, \cdots, Y_{n(M_n)}$ and define $A_{ik} = a_N(R_{N;i(k)})$, $B_{ik} = b_N(R_{N;i(k)})$. Then

$$\begin{aligned} |N^{-1}\tau_N^2(a) - N^{-1}\tau_N^2(b)| &= |N^{-1}\textstyle\sum_i\sum_k\{A_{ik} - A_{i\cdot} - B_{ik} + B_{i\cdot}\}\{A_{ik} - A_{i\cdot} + B_{ik} - B_{i\cdot}\}| \\ &\le N^{-1}\{\textstyle\sum_i\sum_k(A_{ik} - B_{ik})^2\}^{1/2}\{2\sum_i\sum_k(A_{ik} - A_{i\cdot})^2 + 2\sum_i\sum_k(B_{ik} - B_{i\cdot})^2\}^{1/2} \\ &\le \{N^{-1}\textstyle\sum_{h=1}^N (a_N(h) - b_N(h))^2\}^{1/2}O(1) \to 0. \end{aligned}$$

Consider therefore the sequence of scores defined by $b_N(h) = N\int_{(h-1)/N}^{h/N} \phi(u)\,du$, $1 \le h \le N$. Without loss of generality one may assume that $\phi(u_0) = 0$, $\phi(u) \le 0$ on $(0, u_0)$ and $\phi(u) \ge 0$ on $(u_0, 1)$. It follows that $b_N(h) \le 0$ for $1 \le h \le [u_0N]$ and $b_N(h) \ge 0$ for $2 + [u_0N] \le h \le N$, provided $N^{-1} < \min\{u_0, 1 - u_0\}$. Now examine the location of positive and negative scores $b_N(R_{N;i(k)})$ in the design. A block containing negatives scores only cannot coexist with a block containing positive scores only because one would then have, for $i \ne i'$, $R_{N;i(M_i)} \le 1 + [u_0N] \le R_{N;i'(1)}$ (where at least one inequality is strict) which is contradictory in view of A2. Consequently let $s = s(n)$, $0 \le s \le n - 1$, be the number of blocks containing either positive or negative scores only and assume without loss of generality that they are the first s ones. Therefore

$$\begin{aligned} \tau_N^2(b) &= \textstyle\sum_{i=1}^n(2M_i)^{-1}\sum_{k\ne k'}(B_{ik} - B_{ik'})^2 \ge (2\eta)^{-1}\sum_{i=1}^n(\max_k B_{ik} - \min_k B_{ik})^2 \\ &\ge (2\eta)^{-1}\textstyle\sum_{i=s+1}^n \max_k B_{ik}^2 \ge (2\eta^2)^{-1}\sum_{i=s+1}^n\sum_k B_{ik}^2. \end{aligned}$$

Furthermore, if the first s blocks contain positive scores only (negative scores resp), one has:

$$\begin{aligned} &N^{-1}\textstyle\sum_{i=s+1}^n\sum_k B_{ik}^2 \ge N^{-1}\sum_{h=1}^{[u_0N]} b_N^2(h)(N^{-1}\sum_{h=2+[u_0N]}^N b_N^2(h) \text{ resp}) \\ &\qquad \to \int_0^{u_0}\phi^2(u)\,du\left(\int_{u_0}^1\phi^2(u)\,du \text{ resp}\right) > 0. \end{aligned}$$

Hence $N^{-1}\sum_i\tau_{Ni}^2$ is asymptotically bounded away from zero. Now let $\mathbf{W}_N = \mathbf{L}_N^{-1/2}\ \mathbf{P}_N'\mathbf{S}_N$, where $\mathbf{L}_N^{-1/2}$ is the diagonal matrix with entries $l_{N1}^{-1/2}, \cdots, l_{N,p-1}^{-1/2}$ on its diagonal, and assume that for any vector $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_{p-1})'$ such that $\boldsymbol{\lambda}'\boldsymbol{\lambda} = 1$ one has

$$\lim_{n\to\infty}\sup_x|\mathscr{P}_n[\boldsymbol{\lambda}'\mathbf{W}_N \le x \mid \mathscr{C}_n] - \Phi(x)| = 0, \text{ uniformly in } \mathscr{C}_n, \tag{2.2}$$

where Φ is the standardized normal distribution. It follows that, under $\mathscr{P}_n[\cdot \mid \mathscr{C}_n]$, $\mathbf{W}_N$ converge in law to a multinormal $(\mathbf{0}, \mathbf{I})$ and a fortiori $\mathscr{Q}_N = \mathbf{W}_N'\mathbf{W}_N$ to a χ_{p-1}^2, both uniformly in the configuration, in view of Lemmas 2.3 and 2.4 of Mehra and Sarangi (1967). Hence it remains to prove (2.2). Defining $\boldsymbol{\lambda}_N = \mathbf{P}_N\mathbf{L}_N^{-1/2}\boldsymbol{\lambda}$ and

$$Z_{Ni} = \textstyle\sum_{j=1}^{p-1}\lambda_{Nj}\sum_{k=1}^{m_{ij}}[a_N(R_{N;ijk}) - a_N(R_{N;i\cdot\cdot})], \qquad i = 1, \cdots, n,$$

one notes that $\boldsymbol{\lambda}'\mathbf{W}_N = \sum_{i=1}^n Z_{Ni}$ where under $\mathscr{P}_n[\cdot \mid \mathscr{C}_n]$, the Z_{Ni}'s are independent with mean zero and such that $\sum_{i=1}^n \operatorname{Var}_{\mathscr{P}_n} Z_{Ni} = 1$. Finally the Lindeberg condition $\sum_{i=1}^n E_{\mathscr{P}_n} Z_{Ni}^2 I(|Z_{Ni}| > \epsilon) \to 0$, $\epsilon > 0$, is trivially verified since for a certain constant $K < \infty$:

$$\max_{1\le j\le p-1}\lambda_{Nj}^2 \le \boldsymbol{\lambda}_N'\boldsymbol{\lambda}_N \le l_{N1}^{-1} \le KN^{-1}$$

and

$$\begin{aligned} \max_{1\le i\le n} Z_{Ni}^2 &\le (\eta - 1)^2\max_{1\le j\le p-1}\lambda_{Nj}^2\cdot 2\max_{1\le h\le N} a_N^2(h) \\ &\le 2(\eta - 1)^2K\cdot N^{-1}\max_h a_N^2(h) \to 0 \end{aligned}$$

uniformly in the configuration.

Acknowledgments. The author wishes to thank his thesis advisor, Professor C. van Eeden, for suggesting the problem. He is also grateful to the referee for several helpful comments.

REFERENCES

[1] HODGES, J. L. and LEHMANN, E. L. (1962). Rank methods for combination of independent experiments in analysis of variance. *Ann. Math. Statist.* **33** 482–497.
[2] MEHRA, K. L. and SARANGI, J. (1967). Asymptotic efficiency of certain rank tests for comparative experiments. *Ann. Math. Statist.* **38** 90–107.
[3] PURI, M. L. and SEN, P. K. (1971). *Nonparametric Methods in Multivariate Analysis.* Wiley, New York.
[4] SEN, P. K. (1968). On a class of aligned rank order tests in two-way layouts. *Ann. Math. Statist.* **39** 1115–1124.

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE
UNIVERSITÉ DE MONTRÉAL
MONTRÉAL, QUÉBEC
CANADA, H3C 3J7