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ASYMPTOTIC OPTIMALITY OF CERTAIN SEQUENTIAL ESTIMATORS[1]

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We give a decision-theoretic justification of a heuristic principle suggested by Robbins. Suppose a sequence $X_1, X_2, \cdots$ of independent random variables can be observed; they are known to be identically distributed with unknown mean and variance u, $\sigma^2(u)$ respectively. We assume further that the common distribution is of exponential type, so that the sample mean $\bar{X}$ is a sufficient statistic. The problem is to estimate u taking into account the cost of sampling, which is assumed to be linear in the sample size, n.

For the case of a fixed-size sample, with squared-error loss function, the minimum risk (expected loss) is obtained, after rescaling if necessary, as $\min_n E(n + A(\bar{X} - u)^2) = \min_n(n + A\sigma^2(u)/n) = 2A^{\frac{1}{2}}\sigma(u)$, and is attained at $n_0 = A^{\frac{1}{2}}\sigma(u)$. Since u is unknown, this optimum value is not available; and for any fixed n, if $\sigma(u)$ is unbounded, the risk $n + A\sigma^2(u)/n$ can be much larger than $2A^{\frac{1}{2}}\sigma(u)$.

For calibration of the performance of sequential stopping rules, Robbins advocated consideration of the "regret"

$$E\left(n + A(\bar{X} - u)^2\right) - A^{\frac{1}{2}}\sigma(u),$$

and several authors have constructed procedures with uniformly bounded regret.

Two questions arise, which we settle here (in a certain asymptotic sense, and after making certain smoothness assumptions). First, can any procedure have strictly negative regret, for all u? Second, if a procedure has uniformly bounded regret, is it necessarily close to being optimum, in the sense that for each (suitably smooth) sequence of prior distributions on u, is it only boundedly worse than the corresponding sequence of Bayes procedures? Our answers are no, and yes, respectively.

Several examples are discussed, and analogies pointed out with the fixed-sample-size concepts of the asymptotic optimality of maximum-likelihood estimates, and the super-efficiency phenomenon.

1. Introduction. Let $X, X_1, X_2, \cdots$ be a sequence of i.i.d. rv's with density belonging to a one parameter Koopman-Darmois family $\mathfrak{F}$. Excluding only trivial cases, and reparametrizing $\mathfrak{F}$ if necessary, one can write: $\mathfrak{F} = \{f_u(\ );\ f_u(x) = \exp[q(u)x - G(u)],\ u \in U\}$ where all the densities $f_u(\ )$ are w.r.t. the same measure, $\nu(x)$ say;

$$(1.1)\qquad E_uX = u;$$

$$\sigma^2(u) \equiv \mathrm{Var}_u X = \left[E_u\left(\frac{\partial}{\partial u}\log f_u(X)\right)^2\right]^{-1} = \frac{1}{q'(u)} > 0,$$

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for all $u \in U$; U is an interval (maybe unbounded). We consider the problem of estimating u, with squared error loss and linear sampling cost. Specifically, we investigate the optimality of procedures that use $\bar{X}_n = \Sigma_1^n X_i/n$ for estimating u, whenever n observations are available. This reduces our problem to that of finding a good stopping time (ST). The restriction to the above class of procedures is justified by our showing below, that certain (unrestricted) Bayes procedures cannot be much better.

For fixed n we have

(1.2) (a) The loss $= L_n \equiv A(\bar{X}_n - u)^2 + n, \qquad A > 0$ known.

(b) The risk $= R_n \equiv E_u L_n = A\sigma^2(u)/n + n$.

We refer to the n that minimizes (1.2)(b) as the best fixed stopping time (BFST) even though, usually, it depends on u and thus is not available to the experimenter.

To determine the BFST we treat n as a continuous variable; this does not alter the results that follow. From (1.2)(b) the BFST is thus $n^*(u) = A^{\frac{1}{2}}\sigma(u)$ and $R_{n^*} = 2A^{\frac{1}{2}}\sigma(u)$.

Assuming that $\sigma(u) \not\equiv$ const. it is clear that no fixed ST (i.e., nonrandom ST) can realize $2A^{\frac{1}{2}}\sigma(u)$ as its risk function. Consequently, the statistician may be interested in finding a (random) ST, η say, for which

$$\sup_u\left(E_u L_\eta - 2A^{\frac{1}{2}}\sigma(u)\right) < \infty. \tag{1.3}$$

The interesting case is, obviously, the one where $\sup_u \sigma(u) = \infty$.

H. Robbins (1959) coined the term 'regret' (of using η rather than $n^*(u)$) for $(E_u L_\eta - R_{n^*})$, and using this terminology we can summarize the approach by saying that the statistician is interested in finding a ST with a uniformly bounded regret. The above considerations give rise to the following two interesting questions.

(a) Since the quantity $2A^{\frac{1}{2}}\sigma(u)$ evolved from restricting ourselves to the sample mean as a terminal decision and considering fixed ST's only, is it possible to have a statistical procedure, $(\eta, \delta(X_1, \cdots, X_\eta))$ say, such that

$$\sup_u\left\{E_u\left[A\left(\delta(X_1, \cdots, X_\eta) - u\right)^2 + \eta\right] - 2A^{\frac{1}{2}}\sigma(u)\right\} < 0?$$

If this were the case, the whole 'bounded regret approach' would be questionable.

(b) What optimum properties may a procedure $(\eta, \bar{X}_\eta)$ possess, if it has uniformly bounded regret?

In answering (b) we prove that bounded regret procedures are asymptotically Bayes relative to any sequence of a priori densities $\{\psi_i\}$ which spread their mass in a smooth manner. The answer to (a) is no, provided certain conditions are satisfied. More details are given after the proof of Theorem 2.

The above questions are of special interest for the cases $\sigma(u) = u^{\frac{1}{2}}$ and $\sigma(u) = u$ (The exponential and Poisson families, respectively) for which uniformly bounded regret procedures are known to exist (Starr and Woodroofe (1972), and Vardi (1978)).

2. Results. A theorem of M. Alvo (1977) is the tool for obtaining our results. Observing that his proof is independent of the actual parametrization of $\mathcal{F}$ and choosing $g(u) = u$, we get the following

THEOREM 1 *(M. Alvo). Let ψ be an* a priori *density* (w.r.t. *Lebesgue measure*) *for u, supported by an interval $J \subseteq U$. Let a_1 and a_2 be the lower and upper end points of J ($\pm\infty$ are not excluded). Denote by $(N^*, \delta^*(X_1, \cdots, X_{N^*}))$ the sequential Bayes procedure, assumed to observe at least one observation, and assume that conditions I–IV below are satisfied. Then*

$$\int_J\Big(E_u\big[A(\delta^*(X_1, \cdots, X_{N^*}) - u)^2 + N^*\big]\Big)\psi(u)\, du \tag{2.1}$$
$$\geqslant 2A^{\frac{1}{2}} E\sigma(u) - b_\psi^2,$$

where

$$b_\psi^2 = \int_J \sigma^2(u)\left[\frac{d}{du}\log \sigma(u)\psi(u)\right]^2 \psi(u)\, du. \tag{2.2}$$

The conditions are

I. $\int_J u^2\psi(u)\, du < \infty$.

II. $\lim_{u\to a_i}[u\sigma(u)\psi(u)] = 0$, $\lim_{u\to a_i}[\sigma(u)\psi(u)] = 0$; $\quad i = 1, 2$.

III. $\dfrac{\sigma(u)\psi(u)}{\int_{a_1}^u \sigma(y)\psi(y)\, dy}[\int_{a_1}^u \sigma^2(y)\psi(y)\, dy]^{\frac{1}{2}} = O(1)$ *as* $u \downarrow a_1$.

IV. $\dfrac{\sigma(u)\psi(u)}{\int_u^{a_2} \sigma(y)\psi(y)\, dy}[\int_u^{a_2} \sigma^2(y)\psi(y)\, dy]^{\frac{1}{2}} = O(1)$ *as* $u \uparrow a_2$.

Addressing question (b) of the introduction first, our purpose is to show that uniformly bounded regret procedures are asymptotically Bayes in the sense described by Theorem 2.

Let $\{\psi_i\}$ be a sequence of a priori densities for u. We denote by r_i^* the Bayes risk of the Bayes procedure (assumed to observe at least one observation) and by $r_i(\eta, \bar{X}_\eta)$ the Bayes risk of the statistical procedure $(\eta, \bar{X}_\eta)$. Then we have

THEOREM 2. *Let $(\eta, \bar{X}_\eta)$ be a statistical procedure with uniformly bounded regret; that is,*

$$\sup_u\Big(E_u L_\eta - 2A^{\frac{1}{2}}\sigma(u)\Big) < M, \qquad \text{for some} \quad M < \infty. \tag{2.3}$$

Suppose $\{\psi_i\}$ is any sequence of a priori *densities satisfying*

(i) *for each i, ψ_i satisfies conditions I–IV of Theorem* 1.

(ii) $\lim_{i\to\infty} \int \sigma(u)\psi_i(u)\, du = \infty$.

(iii) $\limsup_{i\to\infty} b_{\psi_i}^2 \leqslant \bar{M}$, *for some* $\bar{M} < \infty$.

Then

(A) $\lim_{i\to\infty} r_i^* = \infty$

(B) $\limsup_{i\to\infty}[r_i(\eta, \bar{X}_\eta) - r_i^*] \leqslant M + \bar{M}$.

PROOF. Conditions (ii) and (iii) combine with (2.1) to prove (A), while condition (2.3) and condition (iii) combine with (2.1) to prove (B).

Thus the result that answers our question (b) is simple in principle; it is the verification of the conditions in the specific examples where the work lies.

Focusing attention on question (a) of the introduction, it is not difficult to verify that if there exists a sequence $\{\psi_i\}$, of a priori densities, satisfying I–IV of Alvo's theorem and such that $b^2_{\psi_i}\downarrow 0$, then for any statistical procedure $(N, \delta(X_1, \cdots, X_N))$, we have

$$\sup_u\Big\{E_u\big[A(\delta(X_1, \cdots, X_N) - u)^2 + N\big] - 2A^{\frac{1}{2}}\sigma(u)\Big\} \geqslant 0.$$

Therefore (1.5) is impossible, and the bounded regret approach is justified in that sense too.

DISCUSSION. In the applications considered here, as in Robbins (1959), Starr and Woodroofe (1969, 1972) and Vardi (1978), the interest is in optimality as $\sigma(u)\uparrow\infty$; this is reflected in condition (ii) of Theorem 2. To illustrate our point, assume that $U = (0, \infty)$ and $\sigma(u)\uparrow\infty$ as $u\uparrow\infty$; then our procedure is "asymptotically optimal" as $u\uparrow\infty$. This last statement means that our procedure is asymptotically Bayes w.r.t. any sequence of a priori densities $\{\psi_i\}$ that tend to spread their mass on $(0, \infty)$ in a suitably smooth manner. Elaborating further we see that if the ψ_i's satisfy

(2.4) $$P_{\psi_i}(u > c)\uparrow 1 \quad \text{as} \quad i\uparrow\infty \quad \text{for all} \quad c < \infty,$$

then condition (ii) is immediately satisfied, and by adjusting (if necessary) the tails of the ψ_i's, condition (i) is also satisfied. At this stage one might be tempted to think that Theorem 2 can be strengthened and bounded regret procedures would be asymptotically Bayes w.r.t. the sequence $\{\psi_i\}$ if the ψ_i's satisfy (2.4) alone. This is by no means the case. It is possible to choose a monotone sequence $\{u_j\}$ such that $0 < u_j$ and $u_j \to \infty$, and a sequential procedure, specifically tailored to this sequence, with risk function $R(u)$ satisfying:

(2.5) $$\lim_{j\to\infty}\Big(R(u_j) - 2A^{\frac{1}{2}}\sigma(u_j)\Big) = -\infty.$$

It is now clear that if the ψ_i's would concentrate their mass along the sequence $\{u_j\}$ then procedures with risk function $2A^{\frac{1}{2}}\sigma(u) + 0(u)$ cannot be asymptotically Bayes w.r.t. $\{\psi_i\}$. It is for this very reason that only smooth ψ_i's are candidates for fulfilling the condition of the theorem; these smoothness conditions are (implicitly) imposed by condition (iii).

An actual example of a sequential procedure satisfying (2.5) is given in Vardi (1977). This type of procedure can be considered as a sequential analogue of the "super-efficient" sequence of estimators, introduced by Hodges (see LeCam (1953)). Though they do not have much value from a practical viewpoint they are important in showing that one should not expect a significantly stronger optimum result than the one described in Theorem 2.

Two more points are worth mentioning:

(a) The surprising fact that a heuristically derived procedure, $(\eta, \bar{X}\eta)$, is asymptotically Bayes w.r.t. any sequence of a priori laws, satisfying some smoothness conditions, resembles the asymptotic optimum property of the maximum likelihood estimators (MLE's) for nonsequential procedures, as the sample size approaches ∞. This similarity should not be overemphasized at this stage because it holds only for the squared error loss function, while the MLE's optimum property is not restricted to this loss function alone.

(b) Another motivation for the optimality of sequential procedures, which is different than the one we present here, is to consider the risk as the cost of observation ($1/A$ in our case) approaches 0. Examples can be found, among other places, in Bickel and Yahav (1968) and Wald (1951), for point estimation problems, and in Schwartz (1962) and Kiefer and Sacks (1963), for hypotheses testing problems.

EXAMPLE. This example will include all families $\mathfrak{F}$ for which $U = (0, \infty)$ and $\sigma(u) = cu^t$ for some $t \in (0, 1]$ and $c > 0$. Note that the case $t = 1$ includes the gamma family with known index; if $c = 1$ we get the exponential family. The case $t = \frac{1}{2}$, $c = 1$ includes the Poisson family. Let

$$\psi_{(\alpha,\beta)}(u) = \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-\beta u} \qquad \text{if } u > 0 \tag{2.6}$$
$$= 0 \qquad \text{otherwise,}$$

where $\beta > 0$ and $\alpha > 2(1 - t)$.
We then get

$$cE_{(\alpha,\beta)} u^t = c\Gamma(\alpha + t)/\beta^t \Gamma(\alpha), \tag{2.7}$$

$$b^2_{(\alpha,\beta)} = c^2 \beta^{2(1-t)} \Gamma(\alpha + 2t - 2)(t^2 + \alpha - 1)/\Gamma(\alpha). \tag{2.8}$$

For the case $t < 1$ we hold α fixed and let β approach 0, while for the case $t = 1$ we let α and β approach 0 in such a way that $\alpha/\beta \uparrow \infty$. In both cases, (2.7) approaches ∞ while (2.8) approaches 0. The conditions of Theorem 2 are satisfied and bounded regret procedures (when they exist) are asymptotically Bayes.

3. Concluding remarks.

1. Even though the normal distribution with unknown mean and unknown variance is not an $\mathfrak{F}$ of the type we have considered, a conditioning argument shows that a bounded regret procedure (see Starr and Woodroofe (1969) for such a procedure) is asymptotically Bayes. The actual computations are easily derived from Alvo's paper and hence will be omitted.

2. While Theorem 2 justifies the use of bounded regret procedures, the question of existence of such procedures has not yet been answered satisfactorily. In the cases where $\mathfrak{F}$ is a gamma family (with known index), or a Poisson family, bounded regret procedures are known to exist (Starr and Woodroofe (1972) and Vardi (1978)); similar results for other $\mathfrak{F}$'s (and for classes of $\mathfrak{F}$'s) are yet to be derived.

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