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JOINT ADMISSIBILITY OF THE SAMPLE MEANS AS ESTIMATORS OF THE MEANS OF FINITE POPULATIONS

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When samples are taken independently from different populations, the sample means are jointly admissible for the population means with the squared error as loss function. The result supplements a previous result that when there are many variate values associated with the population units, the sample means of the variate values are jointly admissible for the population means for general loss functions.

1. Introduction. It was shown in [3] that when there are $k(k > 1)$ variate values associated with the population units, the sample means of the variate values are jointly admissible for the population means. This was proved for the squared error loss function and also for more general loss functions considered in [2]. Thus, the clubbing together of the different estimates does not have an effect like Stein's inadmissibility result for the multivariate normal population. In this note, the question whether such an effect occurs if samples are taken independently from k different finite populations is considered. It is shown that in this case also, with the squared error as the loss function, the sample means are jointly admissible for the population means.

2. Notation. Let the populations be arranged in some order. For $i = 1, 2, \cdots, k$ let for the ith population, N_i be the population size, s_i a sample (a set of distinct units), S_i the set of all s_i, p_i the sampling design (any probability distribution on S_i), and $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{iN_i})$ the parametric vector. Let

$$s = \{s_1, s_2, \cdots, s_k\}, \tag{1-i}$$

$$p(s) = \Pi_{i=1}^{k} p_i(s_i), \tag{1-ii}$$

and

$$\mathbf{x} = \{x_{ir}, r = 1, 2, \cdots, N_i, \quad i = 1, 2, \cdots, k\}. \tag{1-iii}$$

R_N denotes the space of all points $\mathbf{x}$ and S the set of all s. The population totals are given by

$$T_i(\mathbf{x}) = \Sigma_{r=1}^{N_i} x_{ir}. \tag{2}$$

An estimator of $T_i(\mathbf{x})$ may be based on the pooled data of the k samples. Hence, it

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is denoted for $i = 1, 2, \cdots, k$ by $e_i(s, \mathbf{x})$, a function defined on $S \times R_N$ which for each s, depends on $\mathbf{x}$ through only those coordinates x_{ir}, for which $u_{ir} \in s_i$ and $s_i \in s$. It is assumed that for $i = 1, 2, \cdots, k$, the units in the ith population are labelled by subscripts $i1, i2, \cdots, iN_i$. The estimators of the population totals $T_i(\mathbf{x})$ based on the sample means are given by

$$e_i^*(s, \mathbf{x}) = \frac{N_i}{n(s_i)} \Sigma_{ir \in s_i} x_{ir}, \qquad i = 1, 2, \cdots, k \tag{3}$$

where $n(s_i)$ equals size of sample s_i, and $ir \in s_i$ is written shortly for $u_{ir} \in s_i$. Joint admissibility of the $e_i^*(s, \mathbf{x})$ for $T_i(\mathbf{x})$ is obviously equivalent to that of the sample means for the population means.

3. Main result. Suppose that with the squared error loss function the estimators $e_i^*(s, \mathbf{x})$ are not jointly admissible for the population totals $T_i(\mathbf{x})$. Then there exists a set of estimators $e_i(s, \mathbf{x})$ satisfying

$$\Sigma_{s \in S} \Sigma_{i=1}^k p(s)[e_i(s, \mathbf{x}) - T_i(\mathbf{x})]^2 \leqslant \Sigma_{s \in S} \Sigma_{i=1}^k p(s)[e_1^*(s, \mathbf{x}) - T_i(\mathbf{x})]^2 \tag{4}$$

for all $\mathbf{x} \in R_N$, with the sign of strict inequality holding for at least one $\mathbf{x} \in R_N$.

In the following it is shown that the strict inequality in (4) cannot hold for any $\mathbf{x} \in R_N$. For clarity of presentation, the proof is divided into independent lemmas.

FURTHER NOTATION. For $k = 1, 2, \cdots, P_k$ denotes the proposition that *if* for the designated fixed k the inequalities (4) hold for all $\mathbf{x} \in R_N$, then for all s such that $p(s) > 0$ and all $\mathbf{x} \in R_N$, $e_i(s, \mathbf{x}) = e_i^*(s, \mathbf{x})$ for $i = 1, 2, \cdots, k$. For $j = 1, 2, \cdots, N_k$, B_j denotes the subset of R_{N_k} such that the coordinate of $\mathbf{x}_k$ contain at most j distinct values. For $k = 1, 2, \cdots, j = 1, 2, \cdots, N_k P_{k,j}$ denotes the proposition that if the inequalities (4) hold for all $\mathbf{x} \in R_N$, then for $\mathbf{x}$ such that $\mathbf{x}_k \in B_j$ and all s such that $p(s) > 0$, $e_i(s, \mathbf{x}) = e_i^*(s, \mathbf{x})$ for $i = 1, 2, \cdots, k$.

LEMMA 3.1. *Let P_{k-1} be true for some k. Then $P_{k,1}$ is true.*

PROOF. Take any point $\mathbf{x} \in R_N$ such that $\mathbf{x}_k \in B_1$. As all the coordinates x_{kr}, $r = 1, 2, \cdots, N_k$ have the same value, by (3)

$$e_k^*(s, x) = T_k(x). \tag{5}$$

Now put for all $s \in S$, all $\mathbf{x} \in R_N$, and $i = 1, 2, \cdots, (k-1)$

$$\begin{aligned} s' &= \{s_1, s_2, \cdots, s_{k-1}\}, \\ \mathbf{x}' &= \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{k-1}\}, \\ p(s') &= \Sigma_{s_k} p(s', s_k) \\ p(s')e_i(s', \mathbf{x}') &= \Sigma_{s_k} p(s)e_i(s, \mathbf{x}), \\ e_i^*(s', \mathbf{x}') &= e_i^*(s, \mathbf{x}). \end{aligned} \tag{6}$$

Let S' denote the set of all s' and R_N' the space of the points $\mathbf{x}'$. On substitution by

(6) and (5), (4) reduces to

$$\begin{aligned}(7)\qquad &\Sigma_{s'\in S'}\Sigma_{i=1}^{k-1}p(s')[e_i(s',\mathbf{x}')-T_i(\mathbf{x}')]^2+\Sigma_{s\in S}\Sigma_{i=1}^{k-1}p(s)[e_i(s,\mathbf{x})-e_i(s',\mathbf{x}')]^2\\ &\qquad+\Sigma_{s\in S}p(s)[e_k(s,\mathbf{x})-T_k(\mathbf{x})]^2\\ &\leqslant \Sigma_{s'\in S'}\Sigma_{i=1}^{k-1}p(s')[e_i^*(s',\mathbf{x}')-T_i(\mathbf{x}')]^2\end{aligned}$$

which on omitting the nonnegative second and third terms in the left-hand side reduces to

$$\begin{aligned}(8)\quad &\Sigma_{s'\in S'}\Sigma_{i=1}^{k-1}p(s')[e_i(s',\mathbf{x}')-T_i(\mathbf{x}')]^2\\ &\leqslant \Sigma_{s'\in S'}\Sigma_{i=1}^{k-1}p(s')[e_i^*(s',\mathbf{x}')-T_i(\mathbf{x}')]^2.\end{aligned}$$

As the inequalities (8) hold for all $\mathbf{x}'$ by proposition P_{k-1} assumed to be true, it follows that for s' such that $p(s')>0$, all $\mathbf{x}'$, and $i=1,2,\cdots,k-1$

$$(9)\qquad e_i(s',\mathbf{x}')=e_i^*(s',\mathbf{x}')=e_i^*(s,\mathbf{x})$$

by (6). On substitution by (9) in (7), it is seen that the second and third terms in the left-hand side of (7) vanish. The vanishing of the second term implies that for s, such that $p(s)>0$, for $i=1,2,\cdots,(k-1)e_i(s,\mathbf{x})=e_i(s',\mathbf{x}')=e_i^*(s,\mathbf{x})$ by (9) and the vanishing of the third implies that for $p(s)>0$, $e_k(s,\mathbf{x})=T_k(\mathbf{x})=e_k^*(s,\mathbf{x})$ by (5). Hence if P_{k-1} is true, then $P_{k,1}$ is true.

LEMMA 3.2. *Let P_{k-1} and $P_{k,j-1}$ be true for some k and j. Then $P_{k,j}$ is true.*

PROOF. We can eliminate in (4) all terms for which $n(s_k)=N_k$ because, for such terms, we can put $e_k'(s,\mathbf{x})=T_k(\mathbf{x})$. Assuming, accordingly, that all such terms have been eliminated, put in (4),

$$\begin{aligned}(10)\qquad g_k(s,\mathbf{x})&=[N_k-n(s_k)]^{-1}[e_k(s,\mathbf{x})-\Sigma_{kr\in s_k}x_{kr}]\\ g_k^*(s,\mathbf{x})&=[N_k-n(s_k)]^{-1}[e_k^*(s,\mathbf{x})-\Sigma_{kr\in s_k}x_{kr}]\end{aligned}$$

which yields

$$\begin{aligned}(11)\qquad &\Sigma_{s\in S}p(s)\Big\{\Sigma_{i=1}^{k-1}[e_i(s,\mathbf{x})-T_i(\mathbf{x})]^2\\ &\qquad+[N_k-n(s_k)]^2\Big[g_k(s,\mathbf{x})-\frac{1}{N_k-n(s_k)}\Sigma_{kr\notin s_k}x_{kr}\Big]^2\Big\}\\ &\leqslant \Sigma_{s\in S}p(s)\Big\{\Sigma_{i=1}^{k-1}[e_i^*(s,\mathbf{x})-T_i(\mathbf{x})]^2\\ &\qquad+[N_k-n(s_k)]^2\Big[g_k^*(s,\mathbf{x})-\frac{1}{N_k-n(s_k)}\Sigma_{kr\notin s_k}x_{kr}\Big]^2\Big\}.\end{aligned}$$

Next take expectations of both sides of (11) with respect to a discrete distribution ω on R_{N_k} under which the x_{kr}, $r=1,2,\cdots,N_k$ are distributed independently and identically with common mean $\Theta(\omega)$ and common variance $\sigma^2(\omega)$. Writing

$$\left[g_k(s, \mathbf{x}) - \frac{1}{N_k - n(s_k)}\Sigma_{kr\notin s_k} x_{kr}\right]^2$$

as

$$\left\{[g_k(s, \mathbf{x}) - \Theta(\omega)] - \frac{1}{N_k - n(s_k)}\Sigma_{kr\notin s_k}[x_{kr} - \Theta(\omega)]\right\}^2$$

and similarly for the corresponding term in the right-hand side of (4), noting that the expectations of the product terms vanish on both sides due to the independence of the x_{kr}, and cancelling out the common term $\sigma^2(\omega)$ from both sides, we obtain

$$\begin{aligned}(12)\quad &\Sigma_{s\in S} p(s) E_\omega\{\Sigma_{i=1}^{k-1}[e_i(s, \mathbf{x}) - T_i(\mathbf{x})]^2 \\ &\qquad + [N_k - n(s_k)]^2[g_k(s, \mathbf{x}) - \Theta(\omega)]^2 \\ &\leqslant \Sigma_{s\in S} p(s) E_\omega\{\Sigma_{i=1}^{k-1}[e_i^*(\mathbf{x}) - T_i(\mathbf{x})]^2 \\ &\qquad + [N_k - n(s_k)|^2[g_k^*(s, x) - \theta(\omega)|^2\}.\end{aligned}$$

In (12) E_ω denotes expectation with respect to ω.

Since x_{kr} for $r = 1, 2, \cdots, N_k$ are distributed independently and identically, in (12) we can replace x_{kr} for $kr \in s_k$ in $e_i(s, \mathbf{x})$, $g_k(s, \mathbf{x})$ and $g_k^*(s, \mathbf{x})$ taken in some order by $x_{k1}, x_{k2}, \cdots, x_{kn(s_k)}$. Let

$$\begin{aligned}(13)\quad h_i(s, \mathbf{x}) &= \text{resulting value of } e_i(s, \mathbf{x}) \text{ for } i = 1, 2, \cdots, k-1, \\ &= \text{resulting value of } g_k(s, \mathbf{x}) \text{ for } i = k, \\ h_k^*(s, \mathbf{x}) &= \text{resulting value of } g_k^*(s, \mathbf{x}).\end{aligned}$$

Note that in (13), $h_i(s, \mathbf{x})$ for $i = 1, 2, \cdots, k$ depends for each s on $\mathbf{x}$ only through the coordinates $x_{ir} \in s_i$ for $i = 1, 2, \cdots, (k-1)$ and $x_{k1}, x_{k2}, \cdots, x_{kn(s_k)}$. Also, by (3), (10) and (13),

$$(14)\quad h_k^*(s, \mathbf{x}) = \frac{1}{n(s_k)}\Sigma_{r=1}^{n(s_k)} x_{kr}.$$

Hence, (12) is transformed into

$$\begin{aligned}(15)\quad &\Sigma_{s\in S} p(s) E_\omega\{\Sigma_{i=1}^{k-1}[h_i(s, \mathbf{x}) - T_i(\mathbf{x})]^2 \\ &\qquad + [N_k - n(s_k)]^2[h_k(s, \mathbf{x}) - \Theta(\omega)]^2\} \\ &\leqslant \Sigma_{s\in S} p(s) E_\omega\{\Sigma_{i=1}^{k-1}[e_i^*(s, \mathbf{x}) - T_i(\mathbf{x})]^2 \\ &\qquad + [N_k - n(s_k)]^2[h_k^*(s, \mathbf{x}) - \Theta(\omega)]^2\}.\end{aligned}$$

Let ω assign positive probabilities to only j specified values, viz., $\omega(t_i) = p_i$, $i = 1, 2, \cdots, j$, $p_i > 0$, $\Sigma_{i=1}^{j} p_i = 1$. The distribution assigns positive probabilities to only those points $\mathbf{x}_k \in R_{N_k}$ for which each coordinate x_{kr}, $r = 1, 2, \cdots, N_k$ has one of the values $t_1, t_2, \cdots, t_j$. Let $B_j(t_1, t_2, \cdots, t_j)$ be the set of all such points. Then

$$B_j(t_1, t_2, \cdots, t_j) \subset B_j.$$

Let $\mathbf{t}$ denote the vector $(t_1, t_2, \cdots, t_j)$, and let $g(t_i, \mathbf{x}_k)$ denote for $i = 1, 2, \cdots, j$ the number of coordinates of $\mathbf{x}_k$ which are equal to t_i. Also, put

$$h_k(s, \mathbf{x}) = h_k^*(s, \mathbf{x}) + v_k(s, \mathbf{x}). \tag{16}$$

Then (15) is reduced to

$$\begin{aligned}
&\Sigma_{s \in S} p(s) \Sigma_{\mathbf{x}_k \in B_j(\mathbf{t})} \{\Sigma_{i=1}^{k-1}[h_i(s, \mathbf{x}) - T_i(\mathbf{x})]^2\} \Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)} \\
&\quad + \Sigma_{s \in S} p(s) \Sigma_{\mathbf{x}_k \in B_j(\mathbf{t})} [N_k - n(s_k)]^2 v_k^2(s, \mathbf{x}) \Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)} \\
&\quad + 2\Sigma_{s \in S} p(s) \Sigma_{\mathbf{x}_k \in B_j(\mathbf{t})} [N_k - n(s_k)]^2 v_k(s, \mathbf{x}) \\
&\qquad [h_k^*(s, \mathbf{x}) - \Theta(\omega)] \Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)} \\
&\leqslant \Sigma_{s \in S} p(s) \Sigma_{\mathbf{x}_k \in B_j(\mathbf{t})} \{\Sigma_{i=1}^{k-1}[e_i^*(s, \mathbf{x}) - T_i(\mathbf{x})]\}^2 \Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)}.
\end{aligned} \tag{17}$$

Let

$$B_j(\mathbf{t}) = B_j^1(\mathbf{t}) + B_j^2(\mathbf{t}) \tag{18}$$

where $B_j^1(\mathbf{t})$ is the subset consisting of all points $\mathbf{x}_k \in B_j(\mathbf{t})$ such that each of the values $t_1, \cdots, t_j$ occurs at least once in the coordinates of $\mathbf{x}_k$.

By assumption $P_{k,j-1}$ is true. Hence (13) implies that, the two sides of (17) are equal for $\mathbf{x}_k \in B_j^2(\mathbf{t})$. Hence, in (17) summation over $\mathbf{x}_k \in B_j(\mathbf{t})$ can be replaced by summation over $\mathbf{x}_k \in B_j^1(\mathbf{t})$. Further the third term in the left-hand side of (17) can be rearranged as follows. Let S_m be the subset of S defined by

$$s \in S_m, \qquad \text{iff,} \quad n(s_k) \equiv m.$$

Then, denoting the third term in the left-hand side of (17) by T_3, we have

$$\begin{aligned}
T_3 = 2\Sigma_{m=1}^{N_k-1} \Sigma_{s \in S_m} p(s) \Sigma_{\mathbf{x}_k \in B_j^1(\mathbf{t})} (N_k - m)^2 v_k(s, \mathbf{x}) \\
[h_k^*(s, \mathbf{x}) - \Theta(\omega)] \Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)}.
\end{aligned} \tag{19}$$

In (19), $v_k(s, \mathbf{x})$, $h_k^*(s, \mathbf{x})$ depend on $\mathbf{x}_k$ only through the m-vector $\mathbf{x}_{km} = (x_{k1}, x_{k2}, \cdots, x_{km})$. Let R_{km} be the space of the vectors $\mathbf{x}_{km}$ and let $D_j(\mathbf{t})$ be the subset of R_{km} defined by

$$\mathbf{x}_{km} \in D_j(\mathbf{t}) \qquad \text{iff} \quad \mathbf{x}_k \in B_j(\mathbf{t}). \tag{20}$$

Then, in the right-hand side of (19), we can sum up the factor $\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)}$ over all those $\mathbf{x}_k$ which give the same value for $\mathbf{x}_{km}$. Let $g(t_i, \mathbf{x}_{km})$ be number of coordinates $x_{k1}, x_{k2}, \cdots, x_{km}$ having the value t_i. Note that with an obvious notation, $\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)} = p_\omega(\mathbf{x}_k)$, $\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_{km})} = P_\omega(\mathbf{x}_{km})$, and $P_\omega(\mathbf{x}_k) = P_\omega(\mathbf{x}_{km}) P_\omega(\mathbf{x}_k | \mathbf{x}_{km})$. Hence Σ denoting summation over $\mathbf{x}_k \in R_{N_k}$ with a given value of $\mathbf{x}$, $\Sigma P_\omega(\mathbf{x}_k) = P_\omega(\mathbf{x}_{km}) \Sigma P_\omega(\mathbf{x}_k | \mathbf{x}_{km}) = P_\omega(\mathbf{x}_{km})$. Hence on summation over $\mathbf{x}_k$ with a given value of

$\mathbf{x}_{km}$, (19) reduces to

$$T_3 = 2\Sigma_{m=1}^{N_k-1}\Sigma_{s\in S_m}p(s)\Sigma_{\mathbf{x}_{km}\in D_j(\mathbf{t})}(N_k - m)^2 v_k(s, \mathbf{x}) \tag{21}$$
$$[h_k^*(s, \mathbf{x}) - \Theta(\omega)]\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_{km})}.$$

Let

$$D_j(\mathbf{t}) = D_j^1(\mathbf{t}) + D_j^2(\mathbf{t}) \tag{22}$$

where $D_j^1(\mathbf{t})$ is the subset of $D_j(\mathbf{t})$ of all those points $\mathbf{x}_{km}$ for which each value t_i occurs at least once in the coordinates of $\mathbf{x}_{km}$. For $\mathbf{x}_{km} \in D_j^2(\mathbf{t})$ each term in the summation in (19) vanishes. Because for the given $\mathbf{x}$ choose $\mathbf{x}'$ such that, $\mathbf{x}_i' = \mathbf{x}_i$ for $i = 1, 2, \cdots, (k-1)$, $x_{ks}' = x_{ks}$ for $s = 1, 2, \cdots, m$ and for $s > m$, the coordinates x_{ks}' assume only the values belonging to the set of values of x_{ks} for $s = 1, 2, \cdots, m$. Then $\mathbf{x}_k' \in B_{j-1}$. Hence by the assumption $P_{k,j-1}$, (10) and (13), $h_k(s, \mathbf{x}') = h_k^*(s, \mathbf{x}')$; hence by (16), $v_k(s, \mathbf{x}') = 0$. By the note preceding (16), and noting that $n(s_k) = m$, $v(s, \mathbf{x}')$ depends on $\mathbf{x}_k'$ only through x_{ks}', $s = 1, 2, \cdots, m$. These coordinates have the same values for $\mathbf{x}'$ and $\mathbf{x}$; also $\mathbf{x}_i' = \mathbf{x}_i$ for $i = 1, 2, \cdots, (k-1)$; hence $v(s, \mathbf{x}) = v(s, \mathbf{x}') = 0$ if $\mathbf{x}_{km} \in D_j^2(\mathbf{t})$. Hence, it follows that each term in the summation in (19) vanishes for $\mathbf{x}_{km} \in D_j^2(\mathbf{t})$. Hence, summation over $\mathbf{x}_{km} \in D_j(\mathbf{t})$ can be replaced by summation over $\mathbf{x}_{km} \in D_j^1(\mathbf{t})$. Thus (17) is equivalent to

$$\Sigma_{s\in S}p(s)\Sigma_{\mathbf{x}_k\in B_j^1(\mathbf{t})}\{\Sigma_{i=1}^{k-1}[h_i(s, \mathbf{x}) - T_i(\mathbf{x})]^2\}\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)}$$
$$+\Sigma_{s\in S}p(s)\Sigma_{\mathbf{x}_k\in B_j^1(\mathbf{t})}[N_k - n(s_k)]^2 v_k^2(s, \mathbf{x})\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)}$$
$$+2\Sigma_{m=1}^{N_k-1}\Sigma_{s\in S_m}p(s)\Sigma_{\mathbf{x}_{km}\in D_j^1(\mathbf{t})}(N_k - m)^2 v_k(s, \mathbf{x}) \tag{23}$$
$$[h_k^*(s, \mathbf{x}) - \Theta(\omega)]\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_{km})}$$
$$\leqslant \Sigma_{s\in S}p(s)\Sigma_{\mathbf{x}_k\in B_j^1(\mathbf{t})}\{\Sigma_{i=1}^{k-1}[e_i^*(s, \mathbf{x}) - T_i(\mathbf{x})]^2\Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_m)}\}.$$

Now $g(t_i, \mathbf{x}_k) \geqslant 1$ for $\mathbf{x}_k \in B_j^1(\mathbf{t})$, and $g(t_i, \mathbf{x}_{km}) \geqslant 1$ for $\mathbf{x}_{km} \in D_j^1(\mathbf{t})$. Hence, we can divide both sides of (21) by $\Pi_{i=1}^j p_i$. The resulting expressions are integrated with respect to $p_1, p_2, \cdots, p_{j-1}$ over the domain

$$Q = [p_1, p_2, \cdots, p_j, p_i > 0, i = 1, 2, \cdots, j, \Sigma_{i=1}^j p_i = 1].$$

For each $s \in S_k'$, the sum of the integrals of the third term in the left-hand side of (23) over the points $\mathbf{x}_{km} \in D_j^1(\mathbf{t})$ vanishes (cf. equation (12*) in [2]). For $\mathbf{x}_k \in B_j^1(\mathbf{t})$, let

$$\int_Q \Pi_{i=1}^j p_i^{g(t_i, \mathbf{x}_k)-1}dp_1 dp_2 \cdots dp_{j-1} = f(\mathbf{x}_k). \tag{24}$$

Thus integration of (23) yields

$$\Sigma_{s\in S}p(s)\Sigma_{\mathbf{x}_k\in B_j^1(\mathbf{t})}f(\mathbf{x}_k)\Sigma_{i=1}^{k-1}[h_i(s, \mathbf{x}) - T_i(\mathbf{x})]^2$$
$$+\Sigma_{s\in S}p(s)\Sigma_{\mathbf{x}_k\in B_j^1(\mathbf{t})}f(\mathbf{x}_k)[N_k - n(s_k)]^2 v_k^2(s, \mathbf{x}) \tag{25}$$
$$\leqslant \Sigma_{s\in S}p(s)\Sigma_{i=1}^{k-1}[e_i^*(s, \mathbf{x}) - T_i(\mathbf{x})]^2\Sigma_{\mathbf{x}_k\in B_j^1(\mathbf{t})}f(\mathbf{x}_k).$$

The order of summation can be interchanged in the right-hand side as $e_i^*(s, \mathbf{x})$ and $T_i(\mathbf{x})$ for $i = 1, 2, \cdots, k-1$ are independent of $\mathbf{x}_k$. Let

$$K = \Sigma_{\mathbf{x}_k \in B_j^1(\mathbf{t})} f(\mathbf{x}_k).$$

Each $f(\mathbf{x}_k) > 0$ and hence K is a finite positive number as $B_j^1(\mathbf{t})$ is a finite set. Next put

$$e_i^1(s, \mathbf{x}') = \frac{1}{K} \Sigma_{\mathbf{x}_k \in B_j^1(\mathbf{t})} f(\mathbf{x}_k) h_i(s, \mathbf{x}) \tag{26-i}$$

where $\mathbf{x}' = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{k-1})$ and for each $s' = \{s_1, s_2, \cdots, s_{k-1}\}$

$$\bar{e}_i(s', \mathbf{x}) = \Sigma_{s_k \in S_k} p(s_k) e_i^1[(s', s_k), \mathbf{x}]. \tag{26-ii}$$

Then the first term in the left-hand side of (25)

$$\begin{aligned} &= K\Sigma_{s' \in S'} p(s') \Sigma_{i=1}^{k-1} [\bar{e}_i(s', \mathbf{x}) - T_i(\mathbf{x})]^2 \\ &\quad + K\Sigma_{s \in S} p(s) \Sigma_{i=1}^{k-1} [e_i^1(s, \mathbf{x}) - \bar{e}_i(s', \mathbf{x})]^2 \\ &\quad + \Sigma_{s \in S} p(s) \Sigma_{\mathbf{x}_k \in B_j^1(\mathbf{t})} f(\mathbf{x}_k) \Sigma_{i=1}^{k-1} [h_i(s, \mathbf{x}) - e_i^1(s, \mathbf{x})]^2. \end{aligned} \tag{27}$$

Hence, omitting nonnegative terms, we obtain from (25), cancelling out the factor K,

$$\begin{aligned} &\Sigma_{s' \in S'} p(s') \Sigma_{i=1}^{k-1} [\bar{e}_i(s', \mathbf{x}') - T_i(\mathbf{x}')]^2 \\ &\qquad \leqslant \Sigma_{s' \in S'} p(s') \Sigma_{i=1}^{k-1} [e_i^*(s', \mathbf{x}') - T_i(\mathbf{x}')]^2 \end{aligned} \tag{28}$$

where $p(s') = \Sigma_{s_k} p(s', s_k)$. For given $t_1, t_2, \cdots, t_j$, $e_i(s', \mathbf{x}')$ in (28) depend on $\mathbf{x}'$ only through $x'_{ir} \in s_i$, $i = 1, 2, \cdots, (k-1)$ and hence are estimators.

By the assumption that P_{k-1} is true, (28) implies that if $p(s') > 0$, $\bar{e}_i(s', \mathbf{x}') = e_i^*(s', \mathbf{x}') = e_i^*(s, \mathbf{x})$ for $i = 1, 2, \cdots, (k-1)$; then in (27), each of the nonnegative terms must vanish thus yielding by (13) that if $p(s) \neq 0$, $e_i(s, \mathbf{x}) = \bar{e}_i(s', \mathbf{x}') = e_i^*(s, \mathbf{x})$ for $i = 1, 2, \cdots, (k-1)$. Lastly by (25), for $p(s) \neq 0$, $v_k(s, \mathbf{x}) = 0$ for $\mathbf{x}_k \in B_j$ which by (16), (13) and (10) yields that for $p(s) \neq 0$, $e_k(s, \mathbf{x}) = e_k^*(s, \mathbf{x})$. Thus if P_{k-1} and $P_{k,j-1}$ are true, then $P_{k,j}$ is true.

LEMMA 3.3. *P_k is true for all k.*

PROOF. Lemmas 3.1 and 3.2 together yield by induction on j that if P_{k-1} is true, then $P_{k,j}$ is true for all $j \leqslant N_k$ and hence that P_k is true. By a previous result (cf. [1]) P_1 is true. Hence, P_k is true for all k. Lemma 3.3 implies that in (4) the strict inequality does not hold for any $\mathbf{x} \in R_N$. This proves the joint admissibility of the estimators $e^*(s_i, \mathbf{x})$ for the population totals (and hence of the sample means for the population means).

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